

SELF-SIMILAR SOLUTIONS, HAVING JUMPS AND INTERVALS OF CONSTANCY,
OF A DIFFUSION-HEAT CONDUCTION EQUATION

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Introduction.

We discuss the self-similar solutions $u(x,t) = u(x/t^{1/2}) = u(\eta)$ of the equation $E(u)_t = \alpha(u)_{xx}$, where either $\eta > 0$ or $-\infty < \eta < +\infty$, assuming that $E(\cdot)$ and $\alpha(\cdot)$ are monotone non-decreasing functions. We may think of this equation as describing conservation of thermal energy in a heat conduction process taking place in an infinite body with one dimensional (planar) symmetry, or as representing conservation of mass in the case of a diffusion process. $E(u)$ represents energy per unit volume at level (temperature) u , and $\alpha'(u) \geq 0$ - a non-negative measure in general - is the (thermal) conductivity or diffusion coefficient.

Under mild hypotheses on the singularities of E and α - meaning jumps and intervals of constancy - we show that these solutions are monotone functions of η , whose jumps correspond to the intervals of constancy of α , and whose intervals of constancy correspond to the jumps of α . The "flux" of this solution, $\alpha(u(\eta))' = \alpha'(u(\eta))u'(\eta)$ also admits jumps and intervals of constancy that correspond to the jumps and intervals of constancy of E .

We shall prove the existence of the limits

$$\lim_{\eta \rightarrow \pm\infty} E(u(\eta)) = E_{\pm\infty}, \quad \text{and} \quad \lim_{\eta \rightarrow \pm\infty} \alpha(u(\eta))' = 0.$$

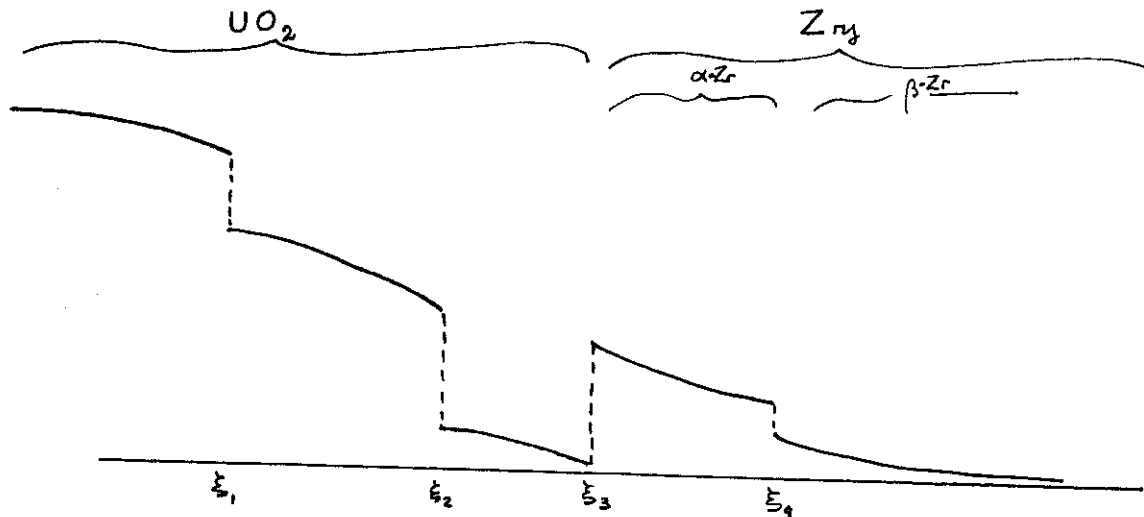
The values $E_{\pm\infty}$ constitute the proper initial values for a Cauchy problem in $-\infty < x < +\infty$, $t > 0$ for the PDE above, being the limits of $E(u(x/t^{1/2}))$ as $t \rightarrow 0^+$, both when $x > 0$ and $x < 0$. The value $u(0)$ of $u(x/t^{1/2})$ at $x = 0$ is the corresponding boundary value for a problem on $x > 0$, $t > 0$.

These questions seem to have more than a mere academic interest. In fact:

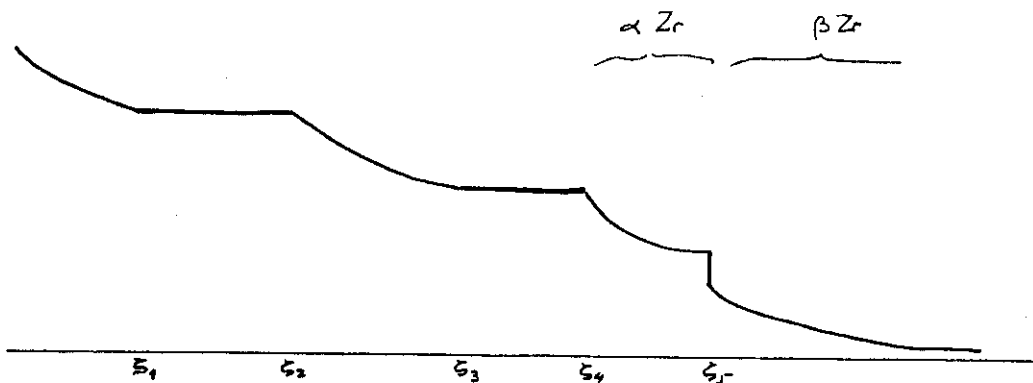
In the Seminar [B] D.A. Tarzia pointed out to us the paper [S-W-A], where an isothermal mushy region model is presented which admits a similarity solution (obtained via error functions). This model is based on the mushy zone width and the hypothesis that the zone contains a fixed fraction of the total latent heat of the phase change. The possibility of casting this problem in our general framework is discussed in Section 3.

The last motivation for these notes came from a Workshop at the Argentine National Atomic Energy Commission (CNEA), December 1985, organized by G. Marshall and M.C. Rey, and devoted to Stefan and related problems. Loosely speaking, we may admit that in the very first stages of a loss of coolant accident in a nuclear reactor, while the tubular surface of a fuel rod can be considered as being planar, reduction of the UO_2 fuel by the Zircalloy of the wall surface occurs. It is suggested [D-G 1]

that the oxygen concentration curve in the different cylindrical (taken to be planar) layers adjacent to the tube surface looks like



where the ξ_i are proportional to $t^{1/2}$, and the curved portions of the graph are sections of error functions satisfying jump conditions at the interfaces. On the other hand, on the outside of the fuel tube, oxidation of the Zircalloy by the coolant also takes place, and the oxygen concentration profile looks like ([D-G 2], [D-G 3])



the ξ_i also proportional to $t^{1/2}$. The flat portions are due to "short-circuits" in the oxygen diffusion due to a channeled structure of the material in that range of concentration. The question naturally arises of whether these pictures are not the graphs of a weak solution to a generalised equation like $E(u)_t = \alpha(u)_{xx}$, with suitable defined E and α .

The analysis in Sections 1,3 shows the conditions that must be satisfied for this to be true, but we have not attempted to relate these conditions to the actual diffusion phenomenon (let alone the accident!). In view of the results outlined at the beginning of the Section, the depressed portion of the first graph above is particularly baffling.

Acknowledgements.

We have revised and completed most of these notes -and lectured on these topics- during our residence at the IMA and the School of Mathematics, University of Minnesota, on Spring/Summer 1985. We wish to thank these Institutions for their hospitality, and CONICET and FCEyN, University of Buenos Aires, for the corresponding leave of absence.

1. WEAK SOLUTIONS

The analysis of a solution $u(\eta)$ to

$$\frac{1}{2}\eta E(u(\eta))' = \alpha(u(\eta))'' \quad , \quad ' = \frac{d}{d\eta} \quad , \quad (1.1)$$

relies on the study of both $u(\eta)$ and its "Kirchhoff variable" $\alpha(u(\eta))$: the latter needs to be defined at those values η for which α experiences a saltus at $u = u(\eta)$.

Definition: a locally bounded function $u(\eta)$ is a solution to (1.1) provided

$$\alpha(u(\eta)) \quad , \quad E(u(\eta)) \in L^1_{loc}(R) \quad , \quad \text{and}$$

$$\int_R \left\{ -\frac{1}{2}(\eta\rho(\eta))' E(u(\eta)) + \rho''(\eta)\alpha(u(\eta)) \right\} d\eta = 0 \quad , \quad \rho \in C^1_0(R) \quad (1.2)$$

It follows from (1.2) that, for a.e. η ,

$$\alpha(u(\eta)) = -\frac{1}{2} \int_{\eta_0}^{\eta} s E(u(s)) ds + \frac{1}{2} \int_{\eta_0}^{\eta} ds \int_{\eta_0}^s E(u(r)) dr + K\eta + B \quad , \quad (1.3)$$

for certain constants η_0 , K , B . Therefore, $\alpha(u(\eta)) \in$ Lebesgue class of $V(\eta)$, $V(\eta)$ absolutely continuous (AC) locally on R . (1.4i)

Furthermore, for a.e. η ,

$$V'(\eta) + \frac{1}{2} \eta E(u(\eta)) = \frac{1}{2} \int_{\eta_0}^{\eta} E(u(r)) dr + K \quad (1.4ii)$$

and thus

$$V'(\eta) + \frac{1}{2} \eta E(u(\eta)) \in \text{Lebesgue class of } h(\eta) \quad , \quad h(\eta) \text{ AC} \quad (1.4.iii)$$

Finally, for a.e. η ,

$$h'(\eta) = \frac{1}{2} E(u(\eta)) \quad (1.4.iv)$$

Conversely, the statements (1.4.i-iv), (1.3) and (1.2) are readily proven to be equivalent if $\alpha(u(\eta))$, $E(u(\eta)) \in L^1_{loc}(R)$.

It follows that if (1.1) has a solution (1.2) and u_c is one of its values at which α has a saltus, it is necessary that $u(\eta) \equiv u_c$ a.e. in certain interval

(η_1, η^1) , $\eta_1 < \eta^1$, and that $V(\eta)$ run (in fact, linearly, cf. Cor. 1.15) from one side limit of α at u_c to the other.

Theorem 1.5: If $E(u)$ and $\alpha(u)$ do not jump at the same value u , then $V(\eta)$ is a monotone function of η . In terms of the graphs v inverse to α , and E , this condition can be phrased: $E \circ v$ is a graph.

Proof. Assume for contradiction $V(a) = V(b) = C$, $a < b$, $V(\eta) > C$ in (a, b) . Upon integration by parts in (1.2), addition and subtraction of $E(v(C))$ one obtains for every smooth $\rho(\eta)$, support $\rho \subset [a, b]$:

$$0 = \int \rho'(\eta) V'(\eta) d\eta + \frac{1}{2} \int \rho'(\eta) \eta (E(u(\eta)) - E(v(C))) d\eta \\ + \frac{1}{2} \int \rho(\eta) (E(u(\eta)) - E(v(C))) d\eta \quad (1.6)$$

Consider the graph v inverse to α .

Put \bar{v} , \underline{v} for the monotone semicontinuous envelopes of v such that

$\underline{v} < v < \bar{v}$ with equality at each point of continuity of v .

Then for a.e. η , whenever $V(\eta) = \alpha(u(\eta))$,

$$\underline{v}(V(\eta)) < u(\eta) < \bar{v}(V(\eta)) \quad , \\ E(\underline{v}(V(\eta))) < E(u(\eta)) < E(\bar{v}(V(\eta))) \quad (1.7)$$

By assumption, C can be chosen so that it is a point of continuity of v and $v(C)$ is a point of continuity of E as well: in fact, as E and α do not jump at the same value u , the value u_c taken by v on an interval where $v = \text{constant}$ (meaning that α jumps at u_c) is such that E is continuous at u_c .

If E had a jump at $v(C)$ α would not, and v would not be constant near C : with a new C_1 arbitrarily close to C the desired situation would obtain in a new interval (a_1, b_1) : $V(a_1) = V(b_1) = C_1$, $V(\eta) > C_1$ in (a_1, b_1) , v continuous at C_1 and E continuous at $v(C_1)$. The proof would then follow by continuity of $V(\eta)$.

With the choice made above it follows that

$$E(\underline{v}(V(\eta))) - E(v(C)) < E(u(\eta)) - E(v(C)) < E(\bar{v}(V(\eta))) - E(v(C)).$$

Taking now $\rho(\eta) = \rho_n(\eta)$, $n > n_0$, so that ρ_n equals one in $(a + \frac{1}{n}, b - \frac{1}{n})$, zero outside of (a, b) and is piecewise linear everywhere it is not difficult to

show that for any $\epsilon > 0$, with n sufficiently large,

$$\int \rho_n'(\eta) V'(\eta) d\eta > -\epsilon,$$

$$\frac{1}{2} \int \rho_n(\eta) (E(u(\eta)) - E(v(C))) d\eta \rightarrow \frac{1}{2} \int (E(u(\eta)) - E(v(C))) d\eta$$

as $n \rightarrow \infty$ (cf. [B.S.V.]). On the other hand, the second integral in (1.6) tends to zero: on the left end-point, for instance,

$$\begin{aligned} \frac{n}{2} \int_a^{a+1/n} \eta (E(\underline{v}(V(\eta))) - E(v(C))) d\eta &\leq \frac{n}{2} \int_a^{a+1/n} \eta (E(u(\eta)) - E(v(C))) d\eta \\ &\leq \frac{n}{2} \int_a^{a+1/n} \eta (E(\bar{v}(V(\eta))) - E(v(C))) d\eta : \end{aligned}$$

the left and right-hand sides of these inequalities tend to zero as $n \rightarrow \infty$ due to the fact that $\eta = a$ is a point of continuity of their integrands.

From (1.6) it follows

$$\frac{1}{2} \int_a^b (E(u(\eta)) - E(v(C))) d\eta < \epsilon :$$

the integrand is non-negative by (1.7). It follows that $E(u(\eta)) = E(v(C))$ a.e. in (a,b) , and by (1.4ii)

$$V'(\eta) = \frac{1}{2} \int_{\eta_0}^a (E(u(s)) - E(v(C))) ds + K = \text{constant}, \eta \in (a,b). \quad (1.8)$$

Thus $V(\eta)$ is a linear function in (a,b) , $V(a) = V(b) = C$, whence $V(\eta) \equiv C$ in (a,b) , contradicting the assumptions. Theorem 1.5 is therefore proven.

$V(\eta)$ will be assumed to be monotone non-increasing. As $v \circ \alpha$ is not the identity map in general, $u(\eta)$ might not be a monotonic function for those η such that $u(\eta)$ belongs to an interval of constancy of α .

It follows from (1.4.ii) that $V'_{(0)} = \lim_{\eta \rightarrow 0} V'(\eta)$ exists

and $V'(\eta) - V'_{(0)} = \frac{1}{2} \int_0^\eta (E(u(s)) - E(u(\eta))) ds$. Therefore, by the assumption above,

$$0 \geq V'(\eta) \geq \frac{1}{2} \int_0^\eta (E(\underline{v}(V(s))) - E(\bar{v}(V(\eta)))) ds + V'_{(0)} ; \text{ take } \eta > 0 :$$

Now if $V(s) > V(\eta)$ for $s < \eta$, $\underline{v}(V(s)) > \bar{v}(V(\eta))$, the integrand is non-negative and $0 > V'(\eta) > V'_{(0)}$; otherwise $V(s) = V(\eta)$ for $0 < \eta_1 < s < \eta$,

implying $V'(s) = 0$ in (η_1, η) . Thus for $\sigma \in (\eta_1, \eta)$, $-V'(0) = \frac{1}{2} \int_0^\sigma E(u(s)) ds - \frac{1}{2} \sigma E(u(\sigma))$: it follows that $E(u(\sigma))$ is AC and $E(u(\sigma))' = 0$ a.e. in (η_1, η) , i.e. $E(u(\sigma))$ is constant in (η_1, η) . Hence in either case,

Lemma 1.9: $0 > V'(\eta) > V'(0)$. (1.9)

Corollary 1.10: If $V'(\eta) = 0$ in (a, b) , $E(u(\eta)) = \text{constant}$ a.e. in (a, b) .

$V'(\eta)$ being bounded by (1.9), the equation $h'(\eta) - h(\eta)/\eta = -V'(\eta)/\eta$ obtained from (1.4i-iv) integrates to give (cf. [B.S.V. Sect.5])

$$h(\eta) := V'(\eta) + \frac{1}{2} \eta E(u(\eta)) = \eta \int_\eta^\infty (V'(s)/s^2) ds + \frac{1}{2} \eta E_\infty. \quad (1.11)$$

$(E_\infty = \lim_{\eta \rightarrow \infty} E(u(\eta)))$ exists by virtue of the integrability of $V'(s)/s^2$, and is easily found to be independent of η .

These arguments can be reproduced for the negative half-line $(-\infty, 0)$, to show the existence of the $\lim_{\eta \rightarrow -\infty} E(u(\eta)) = E_{-\infty}$. This limits $E_{\pm\infty}$ must be the values taken by $E(u(x, t))$ on $\pm x > 0$ as $t \rightarrow 0^+$ (cf. Introduction), that is, the constant energy per unit volume of the initial state for $\pm x > 0$.

If $V'(\eta) = 0$ a.e. in (a, b) then, for $\eta \in (a, b)$

$$0 \geq \eta \int_\eta^\infty (V'(s)/s^2) ds = \eta \int_b^\infty (V'(s)/s^2) ds = \frac{1}{2} \eta (E(u(\eta)) - E_\infty)$$

and by (1.10) either $E(u(\eta)) = \text{constant} < E_\infty$ in (a, b) or $V'(\eta) = 0$ and $E(u(\eta)) = E_\infty$ for all $\eta > a$ (in which case $V(\eta)$ attains its g.l.b. at $\eta = a$).

Recall that $E_\infty = \lim_{\eta \rightarrow \infty} E(u(\eta))$ and $\underline{v}(V(\eta)) < u(\eta) < \bar{v}(V(\eta))$ so $u(\eta)$ can only grow within an interval of constancy of α as $\eta \rightarrow \infty$, because $V(\eta)$ is monotone non-increasing. So if the first alternative holds above, and $E(u(\eta)) < E_\infty$ in $a < \eta < b$, $u(\eta)$ must remain within an interval of constancy of α as $\eta \rightarrow \infty$. Therefore, $\alpha(u(\eta)) = V(\eta) = \text{constant}$ a.e. and reverting to (1.11) one obtains the contradiction $E(u(\eta)) = E_\infty$, $\eta > a$. Therefore we have shown the

Theorem 1.12: The function $V(\eta)$ is strictly monotone decreasing as long as $V(\eta) > \inf \{V(\eta), \eta > 0\}$ (or $-\infty$) (similar statements holds true in $\eta < 0$ as long as $V(\eta) < \sup \{V(\eta)\}$ and mutatis mutandis in the case of a monotone nondecreasing $V(\eta)$).

As a consequence, if α is constant in (u_1, u_2) , $\{\eta : u(\eta) \in (u_1, u_2)\}$ has measure zero.

Corollary 1.13:

- (i) $v(V(\eta)) = v(\alpha(u(\eta))) = u(\eta)$ a.e. η ;
- (ii) $u(\eta)$ is a non increasing function whose values skip the intervals where α is constant; to each of them there corresponds a jump of $u(\eta)$;
- (iii) $v \circ \alpha$ is the identity map acting on the possible solutions (1.2) to (1.1).

This property of the Kirchhoff variable $V(\eta)$ is not shared by the enthalpy variable $v = E(u(\eta))$; in this general context $E^{-1}(v(\eta))$ is not necessarily $u(\eta)$ ($u(\eta)$ crosses intervals where $E(u) = \text{constant}$, cf. (1.14ii) below).

Corollary 1.14:

- (i) Equation for $V(\eta)$: replacing $u(\eta)$ with $v(V(\eta))$ in (1.4), say, gives

$$V'(\eta) + \frac{1}{2} \eta E(v(V(\eta))) = \frac{1}{2} \int_0^\eta E(v(V(s))) ds + V'(0) .$$

- (ii) If $E(u(\eta)) = E(v(V(\eta))) = \text{constant}$, $\eta \in (a,b)$, then $V'(\eta) = \text{constant}$, $\eta \in (a,b)$. That is $V(\eta)$ is a linear function of $\eta \in (a,b)$. Therefore,

Corollary 1.15: If α has a jump at $u = u_c$, then $u(\eta) \equiv u_c$ for $\eta \in$ certain interval (η_1, η_2) , and $V(\eta)$ runs linearly from $\alpha(u_c^+)$ to $\alpha(u_c^-)$.

Proof. v is constant in $(\alpha(u_c^-), \alpha(u_c^+))$. Furthermore $E(v(V(\eta))) = \text{constant}$, and (1.14ii) applies.

Writing eq.(1.14i) as

$$\begin{aligned} 0 \geq V'(\eta) &= \frac{1}{2} \int_0^\eta \{E(v(V(s))) - E(v(V(\eta)))\} ds + V'(0) = \\ &= \frac{1}{2} \int_0^\eta \{E(u(s)) - E(u(\eta))\} ds + V'(0) , \end{aligned}$$

it follows that $V'(\eta)$ is monotone non-decreasing as $\eta \rightarrow \infty$, and $V'(\eta) \leq 0$; hence the limit $V'(+\infty) \leq 0$ exists, and from the equalities above,

$$V'(+\infty) - V'(0) = \frac{1}{2} \int_0^\infty (E(u(s)) - E_\infty) ds. \tag{1.16}$$

On the other hand, $u(\eta)$ is non-increasing, so $E(u(\eta)) \geq E_\infty$. Therefore if the value E_∞ is not taken by $E(u)$ along a half line $u \in (-\infty, u_*)$, $u(\eta) \geq \text{constant} =: u_\infty$ and a fortiori $V(\eta) = \alpha(u(\eta)) \geq \alpha(u_\infty) = \text{constant}$. Under this assumptions it is clear that $V'(+\infty) = 0$ and

$$-V'(0) = \frac{1}{2} \int_0^\infty (E(u(s)) - E_\infty) ds \quad (1.17)$$

indicating that all the energy which has entered the half line $x > 0$ has come through $x = 0$ as the flux $-V'(0)$.

Lemma 1.18:

(i) If E has a saltus at $u_f = u(\eta_f)$,

$$\begin{aligned} V'(\eta_f^-) - V'(\eta_f^+) &= \alpha(u(\eta_f^-))' - \alpha(u(\eta_f^+))' = \\ &= -\frac{1}{2} \eta_f \{E(u(\eta_f^-)) - E(u(\eta_f^+))\} \\ &= -\frac{1}{2} \eta_f \{E(u_f^+) - E(u_f^-)\} \end{aligned}$$

(the side limits must be reversed, in the last equality, if $u(\eta)$, $V(\eta)$ are non-decreasing functions);

(ii) If $E(u(\eta))$ has a saltus at $\eta = \eta_f$, the first two statements of (i) hold.

Statement (1.18i) is precisely the jump condition for a (self similar) Stefan problem with two phases. If E had a saltus at the minimum value $E(u_f) < E(u_f^+)$, (i) above would read: $-\alpha(u(\eta_f^-)) = \frac{1}{2} \eta_f E(u_f^+)$.

Statement (1.18ii) is the interface condition (Rankine-Hugoniot condition) that includes the case of a jump of the solution $u(\eta)$ corresponding to an interval of constancy of α (cf. (1.13i)).

We summarize these results in the following

Theorem (1.19): Let $u(\eta)$ be a solution (1.2) to (1.1), for $-\infty < \eta < +\infty$ or $0 < \eta$; put $V(\eta) := \alpha(u(\eta))$. Then

(i) $u(\eta)$, $V(\eta)$ are functions monotone in the same sense;

(ii) $V(\eta)$ is AC. (1.4), strictly monotone where $\inf \{V(\cdot)\} < V(\eta) < \sup \{V(\cdot)\}$ (1.12); furthermore V has a definite concavity on $\eta > 0$ ($\eta < 0$) so the "flux" $V'(\eta)$ is monotone (in opposite sense to $V(\eta)$ and $u(\eta)$) there (1.16). If $V(\eta_*) =$

$\inf \{V(\cdot)\}$, $\eta^* > 0$, say, then $V'(\eta) = 0$ and $E(u(\eta)) = E_\infty$ for $\eta > \eta^*$.

(iii) If α is constant on (u_1, u_2) , $u(\eta)$ belongs to (u_1, u_2) for no η (1.13ii);

(iv) If α has a saltus at $u = u_c \in \text{Image of } u$, then $u(\eta) \equiv u_c$ a.e. if $\eta_1 < \eta < \eta_2$ for certain η_1, η_2 , $-\infty \leq \eta_1 < \eta_2 \leq +\infty$ (1.15) ;

(v) If $E(u(\eta))$ is constant on (η_1, η_2) , so is $V'(\eta)$ (from (1.4ii) ;

(vi) If $E(u(\eta))$ jumps at η_f , $V'(\eta)$ also jumps according to (1.18);

(vii) $E(u(\eta))$ has limits $E_\infty, E_{-\infty}$, as $\eta \rightarrow \pm\infty$ (1.11); if $E^{-1}[E_\infty]$ is a bounded interval u is bounded and $\lim_{\eta \rightarrow \infty} u(\eta) = u$ (same for $E_{-\infty}$), and so is $V(\eta) = \alpha(u(\eta))$ with corresponding limits. Furthermore, $\lim_{\eta \rightarrow \pm\infty} V'(\eta) = 0$ (1.17).

We shall always take $u(\eta)$, $V(\eta)$ to be monotone non increasing, hence $E(u(\eta)) \geq E_\infty$. The natural boundary values for $u(\eta)$ are the values $E_\infty, E_{-\infty}$ for $E(u(\eta))$ if $\eta \in \mathbb{R}$ (corresponding to a Cauchy problem), or $u(0^+)$, E_∞ for an initial-boundary value problem on $\eta > 0$.

W.l.o.g. we shall take $E_\infty = 0$ corresponding to a $\lim_{\eta \rightarrow \infty} u(\eta) = u_\infty = 0$ (cf. Theorem (1.19)). Also w.l.o.g. we put $\alpha(u_\infty) = \alpha(0) = 0$.

2. Solution to Initial-Boundary Value Problems.

We shall consider the problem

$$\begin{aligned} \text{(i)} \quad & \alpha(u(\eta))'' + \frac{1}{2} \eta E(u(\eta))' = 0 \quad , \quad \eta > 0 \quad , \\ & \hspace{20em} (2.1) \\ \text{(ii)} \quad & u(0^+) = 1 \quad , \quad \lim_{\eta \rightarrow \infty} E(u(\eta)) = 0 \quad . \end{aligned}$$

Here (i) is understood as in (1.2), $\alpha(0) = E(0) = 0$, α and E monotone non decreasing functions of u . If $E(u)$ is not constant zero near $u = 0$, it follows that $\lim_{\eta \rightarrow \infty} u(\eta) = 0$.

As it will turn out, if $E(u) \equiv 0$ on $[0, u_0]$ $u_0 < 1$, the solution $u(\eta)$ will remain $\geq u_0$ (This fact can be derived from the results in Sect.1: $V(\eta)$ is linear if $0 < u(\eta) < u_0$ and V' cannot have a jump at a value of η for which

$u(\eta) = 0$ ($E \equiv 0$ constant), leading to a contradiction). A similar situation arises if α has a jump at $u = 0$, for then $v \equiv 0$ near zero.

Theorem (2.2): Uniqueness of a solution of (2.1).

Proof. (cf. [B.S.V.]) Relies on the fact that, given solutions $u(\eta)$, $\tilde{u}(\eta)$ the corresponding Kirchhoff variables $V(\eta)$, $\tilde{V}(\eta)$ are strictly decreasing wherever positive. Two cases are possible

$$(i) \quad \tilde{V}(\eta) \geq V(\eta) \quad \text{in} \quad (a, b) \quad , \quad \tilde{V}(a) = V(a) \quad , \quad \tilde{V}(b) = V(b) > 0 \quad ,$$

$$(ii) \quad \tilde{V}(\eta) \geq V(\eta) \quad \text{in} \quad (0, +\infty) \quad .$$

In case (i), $u(a^+) = v(V(a^+)) = \tilde{u}(a^+) = v(\tilde{V}(a^+))$ (idem b^-) and $E(v(\tilde{V}(a^+))) = E(v(V(a^+)))$, $E(v(\tilde{V}(b^-))) = E(v(V(b^-)))$.

By subtracting the corresponding eqns. (1.14i) -or the corresponding weak formulations as in Thm.1.5- one obtains

$$\begin{aligned} 0 &\geq \tilde{V}'(b^-) - V'(b^-) - (\tilde{V}'(a^+) - V'(a^+)) = \\ &= \frac{1}{2} \int_a^b \{E(v(\tilde{V}(s))) - E(v(V(s)))\} ds \geq 0 \quad , \end{aligned}$$

and this gives $(\tilde{V} - V)' = \text{constant}$, then $\tilde{V} \equiv V$ in (a, b) as in Thm. 1.5.

In case (ii), employing the condition at infinity $E(u(+\infty)) = E(\tilde{u}(+\infty)) = E_\infty$ and subtracting the respective eqns.(1.16) give

$$0 \geq -\tilde{V}'(0) - (-V'(0)) = \frac{1}{2} \int_0^\infty (E(\tilde{u}(s)) - E(u(s))) ds \geq 0 \quad .$$

and the conclusions in the same.

This theorem actually states the monotone dependence of the solution $u(\eta)$ on its boundary value $u(0)$. If $u_\varepsilon(0) \downarrow u(0)$ or $u_\varepsilon(0) \uparrow u(0)$ the corresponding solutions $u_\varepsilon(\eta)$ (with the same value $E(u(+\infty)) = E_\infty$) converge to a solution $\underline{u}(\eta)$ with $\underline{u}(0) = u(0)$, and therefore $\underline{u}(\eta) \equiv u(\eta)$.

We shall introduce an integral equation for the solution to (2.1); its original version for the eqn. $(k(u)u')' + \frac{1}{2} \eta u' = 0$, $\int_0^1 \frac{k(u)}{u} du = +\infty$ is due to F.V. Atkinson and L.A. Peletier [A-P]. Our derivation begins with (1.11): recalling that $E_\infty = 0$,

$$h(\eta) = \eta \int_{\eta}^{\infty} \frac{V'(s)}{s^2} ds ,$$

as $h(\eta) := V'(\eta) + \frac{1}{2} \eta E(u(\eta))$, dividing by η and a few simple transformations give

$$\eta^2 = 4 \int_{\eta}^0 V'(\eta) d\eta / (E(u(\eta)) + 2 \int_{\infty}^{\eta} \frac{V'(s) ds}{s^2}) .$$

Now $u(\eta) = v(V(\eta))$ and changing variables of integration to V , $dV = V'(\eta) d\eta$ gives, for the function $\sigma(V) \geq 0$, inverse to $V(\eta)$, the integral equation

$$\sigma(V)^2 = 4 \int_V^{\alpha(1)} \frac{dV}{E(v(V)) + 2 \int_0^V \frac{d\alpha}{\sigma(\alpha)^2}} . \quad (2.3)$$

In the r.h.s. above we want to perform the change of variables $V = \alpha(u)$, turning it into the well-known equivalent Lebesgue-Stieltjes integral: a function $f \geq 0$ is sought such that

$$\sigma(\alpha) = f(v(\alpha)), \quad \int_0^{\alpha(u)} \frac{d\alpha}{f(v(\alpha))^2} = \int_0^u \frac{d\alpha(r)}{f(r)^2} \quad \text{and finally}$$

$$\sigma(V)^2 = 4 \int_V^{\alpha(1)} \frac{dV}{E(v(V)) + 2 \int_0^V \frac{ds}{\sigma(s)^2}} = 4 \int_u^1 \frac{d\alpha(u)}{E(u) + 2 \int_0^u \frac{d\alpha(r)}{f(r)^2}} = f(v(V))^2 = f(u)^2 . \quad (2.4)$$

which is the desired integral equation giving the function $f(u)$ inverse to $u(\eta)$. We put from now on $f(u) \equiv \eta(u)$, and therefore

$$\eta(u)^2 = 4 \int_u^1 \frac{d\alpha(s)}{E(s) + 2 \int_0^s \frac{d\alpha(r)}{\eta(r)^2}} \quad (2.5)$$

and

$$(i) \quad \sigma(V) = \eta(\omega(V)) \quad , \quad 0 \leq V \leq \alpha(1) \quad , \quad (2.6)$$

$$(ii) \quad \eta(u) = \sigma(\alpha(u)) \quad , \quad 0 \leq u \leq 1 \quad .$$

Formulas (2.6) relate the inverse functions to both solution $u(\eta)$ and its Kirchhoff variable $V(\eta)$.

The proof of existence of solution to (2.3) or (2.5) is based on the monotonicity of the r.h.s. with respect to $\sigma(V)$ or $\eta(u)$ (cf. [A-E]).

Consider (2.3), and put $(T\sigma(V))^2$ for the r.h.s., $T\sigma(V) \geq 0$, $0 < V < \alpha(1)$. It is clear that $0 \leq \sigma_1(V) \leq \sigma_2(V)$ implies $T\sigma_1(V) \leq T\sigma_2(V)$, and that for all $\sigma \geq 0$,

$$(T\sigma(V))^2 \leq 4 \int_V^{\alpha(1)} \frac{d\alpha}{E(v(\alpha))} \quad \text{if } V \in \text{support } E \circ v = [V_0, 1] \quad ,$$

$V_0 \geq 0$. To show the existence of a fixed point $\sigma = T\sigma$ it is enough to find a $\sigma_0(V)$ such that $\sigma_0 \leq T\sigma_0$. This function is sought for in the form $\sigma_0(V) = \max \{ \lambda(V_0 - V), \mu(\alpha(1) - V) \}$, $\lambda > \mu > 0$. Putting z_0 for the value V such that $\lambda(V_0 - z_0) = \mu(\alpha(1) - z_0)$, $z_0 = z_0(\lambda, \mu) < V_0$ one can show that there is a μ such that, for all $\lambda > \mu$,

$$(T\sigma_0(V))^2 \geq (\mu(\alpha(1) - V))^2 = (\sigma_0(V))^2 \quad , \quad \text{if } z_0 < V \leq \alpha(1) \quad , \quad \text{and}$$

$$(T\sigma_0(V))^2 > (\lambda(V_0 - V))^2 = (\sigma_0(V))^2 \quad , \quad \text{if } 0 < V < z_0 < V_0 \quad .$$

Two applications of the monotone convergence theorem to the sequence $\sigma_1 = T\sigma_0$, $\sigma_2 = T\sigma_1, \dots$ yield the existence of the fixed point $\sigma(V)$ to T .

As $\sigma(V) \geq \sigma_0(V)$ for every $\lambda > \mu$, it is clear that $\sigma(V) = +\infty$ in $[0, V_0)$ if $E \circ v \equiv 0$ in $[0, V_0]$, corresponding to $\eta(u) = +\infty$ if $0 < u < u_0$, or $u(\eta) \geq u_0 > 0$, as stated at the beginning of this Section ([B1]).

Consider the solutions $u(\eta)$, $u_1(\eta)$ to the equations $\alpha(u(\eta))'' + \frac{1}{2} \eta E(u(\eta))' = 0$ and $\alpha(u_1(\eta))'' + \frac{1}{2} \eta E_1(u_1(\eta))' = 0$ subject to the same boundary conditions (2.1.ii). The integral equation (2.3) can be used to supply a simple proof of

Theorem 2.7. If $E(u) \geq E_1(u)$, $0 < u < 1$, then $u_1(\eta) \geq u(\eta)$, $\eta > 0$.

(N.B. We have taken the solutions to be monotone non-increasing; as it will be plain from the proof, last inequality must be reversed in the other case).

Proof. We shall show $\eta_1(u) \geq \eta(u)$ for the inverse functions. Let $T\sigma(V)$ be the operator on the r.h.s. of (2.3), $T_1\sigma(V)$ the same integral with E_1 in place of E , $\sigma_0(V)$ the first function in the iteration for $\sigma = T\sigma$: then it follows

$$(T_1\sigma_0(V))^2 \geq (T\sigma_0(V))^2 \geq (\sigma_0(V))^2, \dots$$

$$T_1^n \sigma_0 \geq T^n \sigma_0, \quad (T_1^n \sigma_0(V))^2 \leq 4 \int_0^{\alpha(1)} \frac{d\alpha}{E_1(v(\alpha))},$$

$x \in \text{support } E_1 \circ v$, for every n . This proves that the fixed point $\sigma_1 = T_1 \sigma_1$ satisfies $\sigma_1(V) \geq \sigma(V)$, with $\sigma_1(V) = +\infty$ for $0 < V < V_1$ if $E_1 \circ v \equiv 0$ in $(0, V_1)$. By (2.6ii), $\eta_1(u) = \eta_1(\sigma(u)) \geq \sigma(\alpha(u)) = \eta(u)$, q.e.d.

The integral equation representation (2.3), (2.5) coupled to this antitone dependence of the solutions with respect to the coefficient E and the uniqueness theorem (2.2) for the problem (2.1) supply straightforward proofs of the convergence of solutions $u_n(\eta)$ to (2.1) with E replaced with E_n when $E_n(u) \nearrow E(u)$ a.e. $u \in (0, 1)$, $E_n(u) \searrow E(u)$ a.e. and finally $E_n(u) \rightarrow E(u)$ a.e., by employing the monotonically convergent sequences of solutions corresponding to the coefficients

$$E_{(n)}(u) = \inf \{E(u), E_k(u), k \geq n\}, \quad E^{(n)}(u) = \sup \{E(u), E_k(u), k \geq n\}$$

which satisfy $E_{(n)} \leq E_n$, $E \leq E^{(n)}$.

As we are dealing with pointwise limits of monotone functions, it is easy to see that the convergence is actually uniform on any compact interval contained in an open interval in which the limit function is continuous.

In the proof of Theorem 2.7 we have shown

Corollary 2.8 If $E(u) > E_1(u)$, $0 < u < 1$, then $\sigma_1(V) > \sigma(V)$ and $V_1(\eta) = V(\eta)$. Also $E_n(u) \rightarrow E(u)$ implies $\sigma_n(V) \rightarrow \sigma(V)$ (uniformly on compact sub intervals) etc.

Theorem 2.9 If $\alpha_1(u) < \alpha(u) < \alpha^1(u)$, $\alpha_1(1) = \alpha(1) = \alpha^1(1)$, then

$$\sigma_1(\alpha(u)) < \sigma(\alpha(u)) = \eta(u) < \sigma^1(\alpha(u)).$$

Proof. Let $v_1 \succ v \succ v^1$ be the inverse graphs to $\alpha_1, \alpha, \alpha^1$. Then $E(v_1(V)) \succ E(v(V)) \succ E(v^1(V))$ in the integrands (2.3) defining T_1, T, T^1 with the same limits of integration. Therefore by Corollary 2.8, and $V = \alpha(u)$,

$$\sigma_1(\alpha(u)) \leq \sigma(\alpha(u)) = \eta(u) \leq \sigma^1(\alpha(u)), \text{ q.e.d.}$$

Again if $\alpha_n(u) \rightarrow \alpha(u)$, $\alpha_n(1) = \alpha(1)$ for all n , we can apply the considerations above to the respective v_n, v so $\sigma_n(V) \rightarrow \sigma(V)$, $V_n(n) \rightarrow V(n)$ hence from Theorem 2.9, similar results of convergence can be obtained for solutions corresponding to coefficients $\alpha_n(u) \rightarrow \alpha(u)$ with $\alpha_n(1) = \alpha(1)$, which are reminiscent of a classical convergence result for Stieltjes integrals.

On the other hand, the continuous dependence on α of the solution to $u_t = \Delta\alpha(u)$ is well known in the general case [B-C].

Theorem 2.9 has an application to numerical methods to compute the solution $u(n)$ to, say, $\alpha(u(n))'' + \frac{1}{2} \eta u' = 0$: we can introduce partitions, together with a coefficient $\alpha_i(u)$ in such a way that $\alpha_i(u_k) = \alpha(u_k)$, all k , and the (inverse of the) corresponding solution, $\eta_i(u)$, be a polygon, possibly admitting horizontal or vertical segments. The $\eta_i(u)$ are continuous functions except at jumps of α , and are constant on intervals where $\alpha = \alpha_i = \text{constant}$, all i , which are included in the partitions. Furthermore, the Kirchhoff variables $\underline{V}_i(n), \bar{V}_i(n)$ and their inverses $\underline{\sigma}_i(u), \bar{\sigma}_i(u)$, corresponding to the coefficients

$$\underline{\alpha}_i(u) = \alpha(u_{k-1}), \quad 0 < u < u_k, \quad k < n_i, \quad \underline{\alpha}_i(1) = \alpha(1);$$

$$\bar{\alpha}_i(u) = \alpha(u_k), \quad 0 < u < u_k, \quad \text{all } k > 0,$$

can also be obtained, with analogous discrete scheme, so that $\underline{\sigma}_i(V) + \sigma(V)$, $\overline{\sigma}_i(V) + \sigma(V)$, $\underline{v}_i(\eta) + V(\eta)$, $\overline{v}_i(\eta) + V(\eta)$ (uniformly on compact intervals, etc).

In the following sets of inequalities the inner ones are consequences of Theorem 2.9 and (2.6); the outer inequalities are due to the monotonicity of $\sigma(V)$, $V(\eta)$:

$$\underline{\sigma}_i(\alpha_i(u)) < \begin{array}{l} \underline{\sigma}_i(\alpha(u)) < \eta(u) < \overline{\sigma}_i(\alpha(u)) \\ \underline{\sigma}_i(\alpha_i(u)) < \eta_i(u) < \overline{\sigma}_i(\alpha_i(u)) \end{array} < \sigma_i(\underline{\alpha}_i(u));$$

$$\overline{v}_i(\underline{v}_i(\eta)) < \begin{array}{l} v(\underline{v}_i(\eta)) < u(\eta) < v(\overline{v}_i(\eta)) \\ v_i(\underline{v}_i(\eta)) < u_i(\eta) < v_i(\overline{v}_i(\eta)) \end{array} < v_i(\overline{v}_i(\eta)).$$

The functions on the extreme left and right are piecewise constant, taking as values the nodes of the polygonal lines $\underline{\sigma}_i(V)$, $\overline{\sigma}_i(V)$; $\underline{v}_i(\eta)$, $\overline{v}_i(\eta)$: they are sub- and superfunctions for both $\eta(u)$ and the approximation $\eta_i(u)$ (resp. $u(\eta)$, $u_i(\eta)$), and are useful to bound, in the numerical procedure, the error of the approximation. It is smooth noting that this method seems to be particularly effective in computing free boundaries, associated to intervals of constancy of α , α_i , and also of the solution $\eta(u)$.

In the actual computation the $\eta_i(u)$ were found as solutions to

$$\eta(1) = 0, \eta(u) \geq 0, \eta'(u) \int_0^u \eta(s) ds = -2 \alpha_i'(u), \quad 0 < u < 1;$$

the r.h.s. of the equation being a non-negative measure ([AdS - B], [AdS - B - M], [AdS 1]). A different discrete procedure, applicable also to the boundary condition $\int \eta(s) ds = 0$, which corresponds to the initial value problem in $-\infty < \eta < +\infty$, is given in [AdS 2].

We are now in a position in which

Initial-boundary value problems (2.1) can be solved, both for $\eta > 0$ and $\eta < 0$, for any $u(0) \in [0,1]$. We conclude Section 2 with

Theorem 2.10. The Cauchy problem $\alpha(u(\eta))'' + 1/2 \eta E(u(\eta))' = 0$, $-\infty < \eta < +\infty$,

$$\lim_{\eta \rightarrow -\infty} E(u(\eta)) = E_{-\infty}, \quad \lim_{\eta \rightarrow +\infty} E(u(\eta)) = E_{+\infty}$$

is uniquely solvable ([D-P]).

Proof. The uniqueness proof is similar to Thm. 2.2.

To show the existence, let

$\tilde{u}(\eta), \tilde{V}(\eta)$, $\eta < 0$ be the solution to

$$\alpha(\tilde{u}(\eta))'' + 1/2 \eta E(\tilde{u}(\eta))' = 0, \quad \eta < 0,$$

$$\tilde{u}(0) = z \in (0,1), \quad E(\tilde{u}(-\infty)) = E_{-\infty}, \quad u(\eta), V(\eta), \quad \eta > 0,$$

the solution to (2.1) with $u(0) = z \in [0,1]$ and assume $E_{-\infty} > E_{+\infty}$, $E(1) = E_{-\infty}$, $E(0) = E_{+\infty}$, according to the convention in Section 2.

By (1.17) and its analogue on $(-\infty, 0)$,

$$\begin{aligned} -V'_z(0) &= -V'(0) = \frac{1}{2} \int_0^{\infty} (E(u(\eta)) - E_{\infty}) d\eta; \\ -\tilde{V}'_z(0) &= -\tilde{V}'(0) = \frac{1}{2} \int_{-\infty}^0 (E_{-\infty} - (E(\tilde{u}(\eta)))) d\eta. \end{aligned}$$

By uniqueness and monotone dependence, Thm. 2.2, it follows that $u_z(\eta) \downarrow 0$ as $z \downarrow 0$, $\tilde{u}_z(\eta) \uparrow 1$ as $z \uparrow 1$, so that

$$-V'_z(0) \downarrow 0 \text{ as } z \rightarrow 0, \quad -\tilde{V}'_z(0) \downarrow 0 \text{ as } z \uparrow 1.$$

On the other hand $-V'_z(0) \uparrow V'_1(0) > 0$ as $z \uparrow 1$, $-\tilde{V}'_z(0) \uparrow -\tilde{V}'_0(0) > 0$, $z \downarrow 0$, and are continuous functions of z . Therefore a value $z_0 \in (0,1)$ must exist such that $-\tilde{V}'_{z_0}(0) = -V'_{z_0}(0)$, thus joining the functions $h(\eta) = V'(\eta) + 1/2 \eta E(u(\eta))$, $\eta > 0$ and $\tilde{h}(\eta) = \tilde{V}'(\eta) + 1/2 \eta E(\tilde{u}(\eta))$, $\eta < 0$ continuously at $\eta = 0$. The function

$$\begin{aligned}w(\eta) &= \tilde{u}_{z_0}(\eta) \quad , \quad \eta < 0 \quad , \\ &= u_{z_0}(\eta) \quad , \quad \eta > 0 \quad ,\end{aligned}$$

is the solution to the Cauchy problem. If $w(\eta)$ is continuous at $\eta = 0$, $w(0) = z_0$.

3. INTERVALS OF CONSTANCY. APPLICATIONS

Let assume that we have problem (2.1) with E and α such that the composition $E \circ v$, v the inverse to α , satisfies

$$(E \circ v)(V) = \text{constant} = E_c, \text{ for } 0 < V_1 < V < V^1, \quad (3.1)$$

$$(E \circ v)(V_1^-) < E_c < (E \circ v)(V^1+).$$

Let $u(\eta)$, $V(\eta)$ be the solution to (2.1) and $V(a) = V_1$, $V(b) = V^1$; by Thm(1.19), $V(\eta)$ is a linear function for $\eta \in (a,b)$, therefore

$$V'(a+) = V'(b-) = \text{constant} = -\frac{V^1 - V_1}{b - a}, \text{ or}$$

$$(b - a) \cdot V'(a+) = - (V^1 - V_1) \quad (3.2i)$$

while the jump conditions (1.18) at a and b read

$$V'(b+) - V'(b-) = \frac{1}{2} b ((E \circ v)(V(b-)) - (E \circ v)(V(b+)))$$

$$V'(a+) - V'(a-) = \frac{1}{2} a ((E \circ v)(V(a-)) - (E \circ v)(V(a+)))$$

and adding these we obtain, by (3.2)

$$V'(b+) - V'(a-) = \frac{1}{2} (b-a)E_c + \frac{1}{2} a(E \circ v)V(a-) - \frac{1}{2} b(E \circ v)V(b+) \quad (3.2ii)$$

The assumed shape of $E \circ v$ may arise, of course, from either constancy or discontinuity of E or v . Assume, for instance, that $\alpha(1) > \alpha(1-) = V_1$ and $E(u) \equiv u$: hence $v(V) \equiv 1$ for V in $(\alpha(1^-), \alpha(1)) = (V_1, \alpha(1))$, the Kirchhoff variable $V(\eta)$ is linear in certain interval $0 \leq \eta \leq b$, $V(0) = \alpha(1)$, $V(b) = V_1$, and in $[0, b]$ we have $u(\eta) = v(V(\eta)) \equiv 1 = u(0)$, as if the "impulsive diffusivity $\alpha'(1) = (\alpha(1) - \alpha(1^-)) \cdot \delta(u-1)$ " gave rise to an instantaneous propagation of the boundary concentration $u(0) = 1$ to all of $[0, b]$. Let us find an estimate for b : from (3.2i),

$$b \cdot V'(b^-) = -(\alpha(1) - V_1)$$

and from (2.3),

$$\frac{b^2}{2[\alpha(1) - \alpha(1^-)]} = \frac{1}{2} (-\sigma'(V)) \sigma(V) \Big|_{\substack{V_1^+ \\ \sigma=b^-}} = \frac{1}{v(V) + 2 \int_0^V dr/\sigma^2} \Big|_{\substack{V_1^+ \\ \sigma=b^-}} < \frac{1}{v(V_1^+)} = 1 .$$

Therefore $\alpha(1) - \alpha(1^-) \geq \frac{1}{2} b^2$.

Recalling that $V(\sigma)$ is convex, and thus $\sigma^2(r) \geq (b \frac{(\alpha(1)-r)}{\alpha(1)-\alpha(1^-)})^2$, we find,

after some calculation, using the same expression for $\sigma'(V)$,

$$\alpha(1) - \alpha(1^-) \geq \frac{b^2}{2} \geq \frac{(\alpha(1) - \alpha(1^-))^2}{\alpha(1)} .$$

However, the interest in the possibilities of playing with the composition $E \circ v$ can be exemplified by the following: in [S-W-A] a self-similar mushy zone model was introduced, which is based on the zone width, inversely proportional to flux through the solidus interface, and in which the isothermal mushy zone contains a fixed fraction of the total latent heat. The authors treatment of this variant of one-phase Stefan problem (a solidification problem, calling for a non decreasing solution as function of η) consisted in matching a segment of error function with the η -axis on the right: an interval of the latter adjoining the graph of the error function with the appropriate width is the mushy zone.

We shall describe this model -for a two phase, melting process- in the context of our solutions to $\alpha(u)'' + 1/2 \eta E(u)' = 0$, for suitable choices of α, E . In order to do this we shall employ formulas (3.2i-ii), and we wish to point out first a discrepancy of our result with that in [S-W-A] : in fact, (3.2i) gives the width of an isothermal ($u(\eta) \equiv \text{constant}$) zone as inversely proportional to the constant "flux" $V'(\eta)$ of the Kirchoff variable inside the zone, while in [S-W-A] the flux on the solidus front is taken.

Let $E(u)$ be the enthalpy function, exhibiting a saltus at a certain temperature $0 < u_c < 1$, let $\lambda = E(u_c^+) - E(u_c^-)$ be the latent heat (per unit volume) of a change of phase at temperature u_c . Suppose $E_c = E(u_c) = E(u_c^-) + \theta \lambda$, $0 < \theta < 1$, be the value corresponding to the fixed fraction of latent heat in the zone.

Let $k(u)$ be the thermal conductivity; in both $E(u)$ and $k(u)$, values $u > u_c$ correspond to properties of the liquid, values $u < u_c$ to the solid phase.

(to simplify the picture we may assume that $E(u)$ is smooth for $u \neq u_c$ and strictly increasing $k(u) > 0$). Assume a melting process for a medium occupying $x > 0$, with initial temperature $u(x,0) = 0 < u_c$ (solid), and put $u(0,t) = 1 > u_c$ for $t > 0$. The solution to this heat conduction problem for equation $(k(u)u')' + 1/2 \eta E(u)' = 0$ (cf.(2.1)) will not take into account the fraction of latent heat carried by the zone (actually a curve, the interphase) at temperature $u = u_c$.

Therefore, we introduce $\alpha(u) = \int_0^u k(s)ds + \gamma.H(u-u_c)$, H being the Heaviside function, $\gamma > 0$ the zone width constant, so that (cf. [S-W-A])

$$\text{zone width } \times \text{ temperature gradient } = \gamma .$$

$$\text{Put } \alpha(u_c^-) = v_1, \quad \alpha(u_c^+) = \alpha(u_c^-) + \gamma = v^1 .$$

with this choice of α , both E and α jump at $u = u_c$, the theory in Sect.1 is not applicable, and the r.h.s. of the integral equation (2.5), also (2.4), needs to be defined by the l.h.s. of (2.4) i.e., a Lebesgue integral. Putting

$(E \circ v)(y) \equiv E_c$ for $v_1 \leq y \leq v^1$, $E \circ v$ being the usual composition of the left-inverse to α , and E on the remaining $0 \leq y \leq \alpha(1)$, let $v(\eta)$ be the inverse to $\sigma(V)$, solution of (2.3). Let $u(\eta) = v(V(\eta))$. $v(\eta)$ will be linear, and $u(\eta) \equiv u_c$, on an interval $a \leq \eta \leq b$ (the mushy zone), such that, by (3.2i),

$$(b-a) \cdot (-v'(a^+)) = \gamma$$

and from (3.2ii) we find the condition of energy conservation across the zone $a \leq \eta \leq b$

$$\begin{aligned} v'(b^+) - v'(a^-) &= \frac{1}{2} (b-a) E_c + \frac{1}{2} a E(u_c^+) - \frac{1}{2} b E(u_c^-) \\ &= \frac{1}{2} (b-a) (E(u_c^-) + \theta \lambda) + \frac{1}{2} a E(u_c^+) - \frac{1}{2} b E(u_c^-) \end{aligned}$$

or

$$k(u_c^-)u'(b^+) - k(u_c^+)u'(a^-) = \lambda \cdot \left((1-\theta) \frac{a}{2} + \theta \frac{b}{2} \right) .$$

This is the two-phase version of [S-W-A (form.(2.1),(2.2))] (we could have used increasing solutions, as in a solidification process). For the case $u_c = 0$, last identity becomes

$$-v'(a^-) = -k(u_c^+)u'(a^-) = \lambda \left((1-\theta) \frac{a}{2} + \theta \frac{b}{2} \right) .$$

Remarks. 1) $V(\eta)$ is not obtained as a composition of $u(\eta) \equiv u_c$ and α for $a \leq \eta \leq b$; this would clearly not make sense. As $V(\eta)$ is linear on (a,b) , $V'(a+) = V'(b-)$ is the constant flux of heat across the zone.

2) The solution just found can be also obtained as a limit of solutions $u_\epsilon(\eta)$ eqns. $\alpha_\epsilon(u_\epsilon)'' + 1/2 \eta E_\epsilon(u_\epsilon) = 0$, with the same boundary values, α_ϵ and E_ϵ do not jump at the same value and $E_\epsilon \circ v_\epsilon \equiv E \circ v$ (to do this it is necessary to "expand" the value $E = E_c$ at $u = u_c$, so that $E_\epsilon(u) = E_c$ on $[u_c, u_c + \epsilon)$, and say, smoothen the graph of $\alpha(u)$ with a linear segment, joining $\alpha(u_c^-)$ and $\alpha(u_c^+)$, on $[u_c, u_c + \epsilon]$, and suitably modifying E and α on $[u_c + \epsilon, 1]$ to give $E_\epsilon \circ v_\epsilon \equiv E \circ v$ there). The (now unique) solution $u_\epsilon(\eta)$ to the equation above has $V_\epsilon(\eta) \equiv V(\eta)$, and $u_\epsilon(\eta) = v_\epsilon(V(\eta)) \dagger u(\eta)$ as $\epsilon \dagger 0$.

Part of these results were announced in [B2].

We turn now our attention to weak solutions that may not be obtained as weak solutions (1.2) to eqn. (1.1).

Assume $\alpha(u) = \int_0^u k(s)ds$ to fix ideas, and let

$$u_l(\eta), \quad \eta < \eta_c; \quad u_r(\eta), \quad \eta > \eta_c,$$

be (say, classical) solutions of $\alpha(u(\eta))'' + \frac{1}{2} \eta E(u(\eta)) = 0$ on their respective domains. It is easy to establish that, for the juxtaposition

$$\begin{aligned} w(\eta) &= u_l(\eta), & \eta < \eta_c, \\ &= u_r(\eta), & \eta > \eta_c \end{aligned}$$

to be a solution in the sense (1.2), the following conditions should be satisfied at $\eta = \eta_c$:

$$\left[\alpha'(w)w'(\eta) + \frac{1}{2} \eta E(w(\eta)) \right]_{\eta_c^-}^{\eta_c^+} = 0 = \left[\alpha(w(\eta)) \right]_{\eta_c^-}^{\eta_c^+} \quad (3.3)$$

(cf. (1.4i), (1.4iii)). However, if we assume $E(w(\eta))$ and the flux $k(w)w(\eta) = \alpha'(w)w(\eta) = \alpha(w(\eta))'$ to belong to L^1_{loc} and define weak solution in the conservation sense:

$$\rho \in C_0, \int \{k(w(\eta))w'(\eta)\rho'(\eta) + \frac{1}{2} (\eta\rho(\eta))'E(w(\eta))\}d\eta = 0, \quad (3.4)$$

only the l.h.s. of (3.3) needs to be satisfied, which amounts to a Rankine-Hugoniot condition at a jump of the energy $E(w(\eta))$.

Hence if we take (cf. [A-J]), for $c_0 < c_\infty < c_1$,

$$\begin{aligned} w(\eta) &\equiv c_0, & 0 < \eta < \eta_c, \\ &= u(\eta), & \eta > \eta_c, \end{aligned}$$

where $u(\eta)$ satisfies the free boundary problem: find $\eta_c, u(\eta), \eta > \eta_c$:

$$(k(u)u'(\eta))' + \frac{1}{2} \eta u'(\eta) = 0, \quad \eta > \eta_c,$$

$$u(\eta_c^+) = c_1; \quad u(+\infty) = c_\infty, \quad \text{and}$$

$$k(c_1)u'(\eta_c^+) = \frac{1}{2} \eta_c(c_0 - c_1),$$

then $w(\eta)$ in a weak solution (3.4) of $(k(w)w'(\eta))' + \frac{1}{2} \eta w'(\eta) = 0, \eta > 0$, $w(0) = c_0, w(+\infty) = c_\infty$. $w(\eta)$ is not monotone, and we may have a situation resembling that of the first graph in the Introduction. The problem above has a unique solution ($c_0 < c_\infty < c_1$ are given, as well as $w \equiv c_0, \eta < \eta_c$), but, if only the values $w(0), w(+\infty)$ were prescribed, we could expect an infinite number of solutions. (I owe this observation to M.K. Kortn).

We complete this Section with an example ([AdS 3]) of non-uniqueness for (2.1) in the case of the unbounded ∞ . Let

$$\begin{aligned} \alpha(u) &= \frac{1}{2} \log u, & u < \frac{1}{2} \\ &= u - \frac{1}{2} (1 - \log \frac{1}{2}), & u > \frac{1}{2}; \\ E(u) &= u - \frac{1}{2}, & u > \frac{1}{2}, \\ &= 0, & u < \frac{1}{2}. \end{aligned}$$

$E(u)$ is a monotone, continuous function; $\alpha(u)$ is continuously differentiable with $\alpha'(u) = \frac{1}{2}u$ for $u < \frac{1}{2}$, $\alpha'(u) \equiv 1$ for $u > \frac{1}{2}$. For every $B > \frac{1}{2}$,

$$u(\eta) = u_{\ell}(\eta) = 1 - \frac{B}{\pi^{1/2}} \int_0^{\eta} e^{-s^2/4} ds, \quad 0 < \eta < \eta_c$$

$$= u_r(\eta) = \frac{1}{2} \exp(-2C(\eta - \eta_1)), \quad \eta_c < \eta < \infty,$$

where $C = C(B) = \frac{B}{\pi^{1/2}} \exp(-\eta_c^2/4)$ and $\eta_c > 0$ is the unique solution (due to $B > 1/2$) of $1 - \frac{B}{\pi^{1/2}} \int_0^{\eta_c} e^{-s^2/4} ds = \frac{1}{2}$. Clearly

$$u_{\ell}(\eta_c^-) = u_r(\eta_c^+) = 1/2$$

$$u'_{\ell}(\eta_c^-) = u'_r(\eta_c^+) = -C:$$

as $\alpha \in C^1$ and $E \in C$, it is easily seen that $u(\eta)$ solves the equation, $u(0) = 1$, $u(+\infty) = E(u(+\infty)) = 0$.

Therefore we have one solution for each $B > 1/2$, satisfying, moreover, $\lim_{\eta \rightarrow \infty} \alpha(u(\eta))' = -C < 0$.

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