REGULARITY FOR SOLUTIONS OF NONLINEAR VARIATIONAL INEQUALITIES WITH GRADIENT CONSTRAINTS

By

Hi Jun Choe

IMA Preprint Series # 801

May 1991

Regularity for Solutions of Nonlinear Variational Inequalities with Gradient Constraints

Hi Jun Choe [1]
Institute for Mathematics and Its Applications
University of Minnesota
Minneapolis, MN 55455

April, 1991

[1]Supported in part by NSF grant DMS-8704111 and NSF grant RII-8610671 and the Commonwealth of Kentucky through the Kentucky EPSCoR Program.

Suppose $A : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is C^1 and satisfies the monotonicity condition : for some $\lambda > 0$,

$$(A_i(P) - A_i(Q))(P_i - Q_i) \geq \lambda \mid P - Q \mid^2 \tag{1}$$

for all $P, Q \in \boldsymbol{R}^n$. Let $G : \boldsymbol{R}^n \to \boldsymbol{R}$ be a smooth nonnegative uniformly strictly convex function. Suppose $\Omega \subset \boldsymbol{R}^n$ is an open bounded domain with C^2 boundary and $u_0 \in C^2(\overline{\Omega})$ with $G(Du_0) \leq M$ in Ω for a fixed constant M. Moreover, assume that $G(0) = 0$ is the minimum.

Set $K = \{v \in \mathrm{Lip}(\overline{\Omega}) : v = u_0 \text{ on } \partial\Omega \text{ and } G(Dv) \leq M \text{ on } \Omega\}$ and let $u \in K$ be the solution of the following variational inequality :

$$\int_\Omega A_i(Du)(D_i v - D_i u)dx \geq 0 \tag{P1}$$

for all $v \in K$. The existence and uniqueness of the solution u to $(P1)$ are easy to prove from the monotonicity of the operator. We will show the following theorem.

Theorem 1

$$u \in W^{2,p}(\Omega)$$

for all $\infty > p \geq 1$ *and in particular that* $u \in C^{1,\alpha}(\overline{\Omega})$ *for all* $1 > \alpha > 0$. *Furthermore suppose that there is* $v \in K$ *such that*

$$G(Dv) < M,$$

then

$$u \in W^{2,\infty}_{loc}.$$

So we have the following lemma.

Lemma 1 *Suppose that* Ω *is a bounded domain with* C^2 *boundary and* u_0 *is* $C^2(\overline{\Omega})$. *Suppose that* $F; \boldsymbol{R}^n \to \boldsymbol{R}$ *is* C^2 *and satisfies the strong ellipticity condition*

$$F_{P_iP_j}(P)\xi_i\xi_j \geq \lambda \mid \xi \mid^2$$

for some λ *and for all* $\xi, P \in \boldsymbol{R}^n$. *Then the minimizer of*

$$I[v] = \int_\Omega F(Dv)dx$$

with respect to $\{v \in W^{1,\infty}_0(\Omega) + u_0; G(Dv) \leq 1\}$ *is* $C^{1,\alpha}(\overline{\Omega})$.

In [7] L. C. Evans considered the problem of solving a linear second order elliptic equation with the gradient constraint

$$G(Du) = \mid Du \mid \leq g$$

where g is $C^2(\overline{\Omega})$. He proved the solution u has $W^{2,\infty}_{loc}$ regularity. In [12] R. Jensen showed $W^{2,p}$ regularity for the solution to a linear second order elliptic equation with a constraint $F(x,u,Du) \leq 1$ where $F(x,z,p)$ is strictly convex in p. We also recall that L. Caffarelli and N.M. Riviere[3] proved $W^{2,\infty}$ regularity for elasto-plastic problems using a priori estimate near free boundary.

With $G(Du) = \mid Du \mid^2 \leq M$, H. Brezis and G. Stampacchia [2] and C. Gerhardt [8] proved $W^{2,p}$ regularity for quasilinear second order elliptic equations.

Here we extend $W^{2,p}$ regularity to quasilinear elliptic equations with gradient constraints such as

$$G(Du) \leq C$$

where G is a uniformly strictly convex function and C is a constant. Furthermore $W^{2,\infty}$ regularity under those general setting seems new.

We consider a double obstacle problem using a viscosity solution idea for Hamilton Jacobi equations as in [12]. Using a regularity result for viscosity solutions we prove $W^{2,p}$ regularity for the double obstacle problem. Then following the truncation idea of DeGiorgi we can prove $W^{2,\infty}$ regularity for the double obstacle problem in a relatively easy way compared to [1] or [3]. In fact we prove $W^{2,\infty}$ regularity for obstacle problems with semi concave or semi convex obstacles. Hence we even extend $W^{2,\infty}$ regularity results for obstacle problems in [1] to non-smooth obstacles. Finally considering a maximum principle we show that the solution to the double obstacle problem is the solution to (P1).

Set

$$w^+(x) = \sup_{v \in K} v(x)$$

and

$$w^-(x) = \inf_{v \in K} v(x)$$

and

$$J = \{v \in W^{1,2}(\Omega) : w^- \leq v \leq w^+\}.$$

We recall the following facts that

$$G(Dw^{\pm}) = M$$

almost every $x \in \Omega$ and

$$\frac{\partial^2 w^+}{\partial \zeta^2} \leq C \text{ and } -\frac{\partial^2 w^-}{\partial \zeta^2} \leq C \tag{2}$$

in the sense of distributions for any unit vector $\zeta \in \boldsymbol{R}^n$ for some constant C. (See [14, page 133].)

Let R be so large that $\{Q : G(Q) \leq 2M\} \subset B(0, R)$. We approximate A by $\overline{A} \in C^1$ such that

$$\overline{A}_i(P) = A_i(P)$$

if $| P | \leq R$ and

$$\overline{A}_i(P) = \mu P_i$$

for some large μ if $| P | \geq 2R$, and

$$[\overline{A}_i(P) - \overline{A}_i(Q)][P_i - Q_i] \geq C' \mid P - Q \mid^2$$

for all $P, Q \in \boldsymbol{R}^n$ and for some C'. Such an approximation was given by H. Brezis and G. Stampacchia in [2].

Theorem 2 *There exists a unique solution $w \in J$ of the double obstacle problem*

$$\int_\Omega \overline{A}_i(Dw)(D_i v - D_i w)dx \geq 0 \tag{P2}$$

for all $v \in J$. Moreover $w \in W^{2,p}(\Omega)$ for all $1 \leq p < \infty$.

We prove this by the penalty method. Let $\beta_1(t) \in C^\infty$ be a monotone increasing function such that $\beta_1 = 0$ if $t \leq 0$, $10t \geq \beta_1(t) > 0$ if $t > 0$, $10 \geq \beta_1'(t) \geq 0$ for all t and $\beta_1'' \geq 0$ for all t. Similarly let $\beta_2(t) \in C^\infty$ be a monotone increasing function such that $\beta_2(t) = 0$ if $t \geq 0$, $-10t \leq \beta_2(t) < 0$ if $t < 0$, $10 \geq \beta_2'(t) \geq 0$ for all t and $\beta_2'' \leq 0$ for all t.

Let u^ϵ be the $W^{1,2}$ solution of

$$L(u^\epsilon) = -D_i(\overline{A}_i(Du^\epsilon)) + \frac{1}{\epsilon}\beta_1(u^\epsilon - w^+) + \frac{1}{\epsilon}\beta_2(u^\epsilon - w^-) = 0 \tag{3}$$

and $u^\epsilon - u_0 \in W_0^{1,2}(\Omega)$. Existence and uniqueness of such a $W^{1,2}$ solution u^ϵ is proved by P. Hartman and G. Stampacchia [11] from the fact that β_1 and β_2 are monotone and $\overline{A}$ is coercive. We note that

$$\frac{1}{\nu} \mid \xi \mid^2 \leq \overline{A}_{i,P_j}(P)\xi_i\xi_j \leq \nu \mid \xi \mid^2$$

for all $P, \xi \in \boldsymbol{R}^n$ and for some $\nu > 0$. So by a difference quotient argument we see that $u^\epsilon \in W^{2,2}_{loc}(\Omega)$. Since $u^\epsilon_{x_k}$ is a solution of a linear strictly elliptic equation with bounded coefficients in Ω, we see that $u^\epsilon_{x_k}, k = 1, ..., n$ is $C^{0,\alpha}_{loc}(\Omega)$ by De Giorgi-Nash theorem. Since $\partial\Omega$ is C^2 and u_0 is C^2, from boundary regularity theory we have

$$u^\epsilon \in C^{1,\alpha}(\overline{\Omega})$$

for all $0 \leq \alpha < 1$.

Lemma 2

$$\| u^\epsilon \|_{W^{1,\infty}(\Omega)} \leq C$$

for some C independent of ϵ.

Proof. Suppose C is a constant such that

$$C \geq \| w^+ \|_{L^\infty(\Omega)} .$$

Since $C \geq w^+ \geq w^-$, we see that

$$L(C) = \frac{1}{\epsilon}\beta_1(C - w^+) \geq 0$$

and C is a supersolution to (3). Since $C \geq u^\epsilon$ on $\partial\Omega$, by the maximum principle we conclude that

$$C \geq u^\epsilon$$

for all ϵ. Similarly we see that

$$u^\epsilon \geq C'$$

for some constant C' and for all ϵ. So we have that

$$| u^\epsilon(x) | \leq C_0$$

for all $x \in \Omega$ and for some C_0 independent of ϵ.

Next we prove that

$$\| Du^\epsilon \|_{L^\infty(\Omega)} \leq C$$

for some C independent of ϵ.

First we show that there is a constant C independent of ϵ such that

$$| Du^\epsilon(x_0) | \leq C$$

for all $x_0 \in \partial\Omega$.

We construct a barrier φ^+ by using a distance function $d(x) = \mathrm{dist}(x, \partial\Omega)$. Define $\Omega_\delta = \{x \in \Omega; d(x) < \delta\}$ for all $\delta > 0$. Since $\partial\Omega \in C^2$, $d \in C^2(\Omega_\delta)$ for some small δ[10]. Moreover we see that

$$| Dd(x) | = 1$$

and

$$| \Delta d(x) | \leq C_1$$

for all $x \in \Omega_\delta$. We take $C_2 > 1$ so large that

$$C_0 + C_1 + \| w^\pm \|_{W^{1,\infty}(\Omega)} + \| u_0 \|_{W^{2,\infty}(\Omega)} \leq C_2$$

where C_2 is independent of ϵ. Let $\delta \leq \frac{1}{100C_2}$ and $\rho \geq 10R + 10\frac{C_2}{\delta}$. Define φ^+ by

$$\varphi^+ = u_0 + \rho d - \rho d^{\frac{3}{2}}.$$

So we see that

$$\varphi^+ \geq \rho\delta - \rho\delta^{\frac{3}{2}} - C_2 \geq \frac{9}{10}\rho\delta - C_2 \geq 8C_2$$

for all $x \in \partial\Omega_\delta \setminus \partial\Omega$. We have that

$$| D\varphi^+ | \geq \rho \mid Dd \mid - \mid Du_0 \mid -\frac{3}{2}\rho\delta^{\frac{1}{2}} \mid Dd \mid \geq 9\rho - C_2 \geq 10R$$

for all $x \in \Omega_\delta$ and

$$\Delta\varphi^+ = \Delta u_0 + \rho\Delta d - \frac{3}{2}\rho d^{\frac{1}{2}}\Delta d - \frac{3}{2}\rho\frac{1}{d^{\frac{1}{2}}} \mid Dd \mid^2$$

$$\leq -\frac{3}{2}\rho\frac{1}{\delta^{\frac{1}{2}}} + \frac{3}{2}\rho\delta^{\frac{1}{2}}C_2 + \rho C_2 + C_2 \leq 0$$

for all $x \in \Omega_\delta$ and φ^+ is superharmonic in Ω_δ. Now we see that since $| D\varphi^+ | \geq 2R$,

$$-D_i(\overline{A}_i(D\varphi^+)) = -\mu\Delta\varphi^+ \geq 0.$$

Let $\Omega_\epsilon^+ = \{x \in \Omega; u^\epsilon > w^+\}$, then

$$-D_i(\overline{A}_i(Du^\epsilon)) + \frac{\beta_1(u^\epsilon - w^+)}{\epsilon} = 0$$

in Ω_ϵ^+. Since $\beta_1 \geq 0$, we get

$$-D_i(\overline{A}_i(Du^\epsilon)) \leq 0$$

in Ω_ϵ^+. Since $u^\epsilon \leq \varphi^+$ on $\partial(\Omega_\epsilon^+ \cap \Omega_\delta)$, we conclude that from the maximum principle

$$u^\epsilon \leq \varphi^+$$

for all $x \in \Omega_\epsilon^+ \cap \Omega_\delta$. Since $w^+ \leq \varphi^+$ in Ω_δ,

$$u^\epsilon \leq \varphi^+$$

for all $x \in \Omega_\delta$.

In the similar way we can construct φ^- such that for some $\delta' > 0$

$$\varphi^- \leq u^\epsilon$$

for all $x \in \Omega_{\delta'}$ and

$$\varphi^-(x) = u_0(x)$$

for all $x \in \partial\Omega$.

So we have that
$$\frac{\partial \varphi^+}{\partial \eta}(x) \leq \frac{\partial u^\epsilon}{\partial \eta}(x) \leq \frac{\partial \varphi^-}{\partial \eta}(x)$$
for all ϵ and for all $x \in \partial\Omega$ where η is the outward normal vector at x and, hence
$$| Du^\epsilon(x) | \leq C$$
for all $x \in \partial\Omega$ and for some C independent of ϵ.

Now we estimate $| Du^\epsilon |$ in Ω. We assume that $| Du^\epsilon |$ attains its maximum at $x_0 \in \Omega$ and $| Du^\epsilon(x_0) | \geq 2C_2$. Otherwise $| Du^\epsilon |$ is bounded. We differentiate (3) with respect to x_k and we have that
$$-(\overline{A}_{i,P_j}(Du^\epsilon)u^\epsilon_{x_k x_j})_{x_i} + \frac{\beta_1'}{\epsilon}[u^\epsilon_{x_k} - w^+_{x_k}] + \frac{\beta_2'}{\epsilon}[u^\epsilon_{x_k} - w^-_{x_k}] = 0. \tag{4}$$
Let ζ be a nonnegative $C_0^\infty(\Omega)$ function. We apply $u^\epsilon_{x_k}\zeta$ as a test function to (4), then
$$\int_\Omega \overline{A}_{i,P_j}(Du^\epsilon)u^\epsilon_{x_k x_i}u^\epsilon_{x_k x_j}\zeta + \overline{A}_{i,P_j}u^\epsilon_{x_k x_j}u^\epsilon_{x_k}\zeta_{x_i}$$
$$+\frac{\beta_1'}{\epsilon}[| Du^\epsilon |^2 - Du^\epsilon \cdot Dw^+]\zeta + \frac{\beta_2'}{\epsilon}[| Du^\epsilon |^2 - Du^\epsilon \cdot Dw^-]\zeta dx = 0.$$
Since $u^\epsilon \in C^{1,\alpha}(\overline{\Omega})$ and $| Du^\epsilon(x_0) | > C_2 > | Dw^\pm(x_0) |$, we see that
$$| Du^\epsilon |^2 \geq Du^\epsilon \cdot Dw^\pm$$
for all $x \in B(x_0, r) \cap \Omega$ for some $r > 0$. Since $\beta_1', \beta_2' \geq 0$,
$$\int_\Omega \overline{A}_{i,P_j}(Du^\epsilon)[| Du^\epsilon |^2]_{x_j}\zeta_{x_i} dx \leq 0$$
for all $\zeta \in C_0^\infty(\Omega \cap B(x_0, r)), \zeta \geq 0$. So we can assume that $| Du^\epsilon |$ attains its maximum on $\partial(B(x_0, r) \cap \Omega)$. Continuing this process, we prove that $| Du^\epsilon |$ attains its maximum on $\partial\Omega$ and
$$| Du^\epsilon | \leq C$$
for all $x \in \Omega$ and for some C independent of ϵ. □

Once $\| u^\epsilon \|_{W^{1,\infty}}$ is bounded independently of ϵ, we show in the proof of theorem 2 that
$$\| \frac{\beta_1(u^\epsilon - w^+)}{\epsilon} \|_{L^p(\Omega)} + \| \frac{\beta_2(u^\epsilon - w^-)}{\epsilon} \|_{L^p(\Omega)} \leq C$$

for some constant C independent of ϵ. Consequently from L^p theory

$$\| u^\epsilon \|_{W^{2,p}(\Omega)} \leq C$$

for some constant C independent of ϵ.

Proof of Theorem 2 . Let $p \geq 1$ and $v = [\frac{1}{\epsilon}\beta_1(u^\epsilon - w^+)]^{p-1}$, then $v \in W_0^{1,2}(\Omega)$ and v is a legitimate test function. We have from (3)

$$(p-1)\int_\Omega \overline{A}_i(Du^\epsilon)(D_iu^\epsilon - D_iw^+)[\frac{1}{\epsilon}\beta_1(u^\epsilon - w^+)]^{p-2}\frac{1}{\epsilon}\beta_1'(u^\epsilon - w^+)dx$$

$$+\int_\Omega \mid \frac{1}{\epsilon}\beta_1(u^\epsilon - w^+) \mid^p dx = 0 \tag{5}$$

By subtracting

$$(p-1)\int_\Omega \overline{A}_i(Dw^+)(D_iu^\epsilon - D_iw^+) \mid \frac{\beta_1}{\epsilon} \mid^{p-2} \frac{\beta_1'}{\epsilon}dx$$

from both side of (5), we have

$$(p-1)\int_\Omega (\overline{A}_i(Du^\epsilon) - \overline{A}_i(Dw^+))(D_iu^\epsilon - D_iw^+) \mid \frac{\beta_1}{\epsilon} \mid^{p-2} \frac{\beta_1'}{\epsilon}dx + \int_\Omega \mid \frac{\beta_1}{\epsilon} \mid^p dx$$

$$= -(p-1)\int_\Omega \overline{A}_i(Dw^+)(D_iu^\epsilon - D_iw^+) \mid \frac{\beta_1}{\epsilon} \mid^{p-2} \frac{\beta_1'}{\epsilon}dx$$

Since $\beta_1' \geq 0$ and

$$(p-1)\int_\Omega (\overline{A}_i(Du^\epsilon) - \overline{A}_i(Dw^+))(D_iu^\epsilon - D_iw^+) \mid \frac{\beta_1}{\epsilon} \mid^{p-2} \frac{\beta_1'}{\epsilon}dx \geq 0,$$

we have

$$\int_\Omega \mid \frac{\beta_1}{\epsilon} \mid^p dx \leq -(p-1)\int_\Omega \overline{A}_i(Dw^+)(D_iu^\epsilon - D_iw^+) \mid \frac{\beta_1}{\epsilon} \mid^{p-2} \frac{\beta_1'}{\epsilon}dx$$

$$= \int_\Omega D_i(\overline{A}_i(Dw^+)) \mid \frac{\beta_1}{\epsilon} \mid^{p-1} dx. \tag{6}$$

Since

$$\frac{\partial^2 w^+}{\partial \zeta^2} \leq C$$

in the sense of distributions for any unit vector $\zeta \in \boldsymbol{R}^n$ for some constant C, we assume w^+ has second derivative for all $x \in \Omega$. Hence we have

$$\int_\Omega D_i(\overline{A}_i(Dw^+)) \mid \frac{\beta_1}{\epsilon} \mid^{p-1} dx = \int_\Omega \overline{A}_{i,P_j}(Dw^+)w^+_{x_ix_j} \mid \frac{\beta_1}{\epsilon} \mid^{p-1} dx.$$

Since $[w^+_{x_ix_j}]$ is a symmetric matrix, we can write

$$\overline{A}_{i,P_j}(Dw^+)w^+_{x_ix_j} = \overline{A}_{i,P_j}(Dw^+)Q_{il}(x)Q_{jk}(x)\Lambda_{lk}(x)$$

for some orthogonal matrix $Q(x)$ where $\Lambda_{lk}(x)$ is a diagonal matrix such that

$$\Lambda_{lk}(x) = \lambda_l(x)\delta_{lk}.$$

Since

$$\frac{\partial^2 w^+}{\partial \zeta^2} \leq C,$$

we see that

$$\lambda_l(x) \leq C$$

for each $l = 1, ..., n$ and for a.e. x. Also from the monotonicity condition (1), we have

$$0 < \lambda \leq \overline{A}_{i,P_j}(Dw^+)Q_{ik}(x)Q_{jk}(x) \leq C$$

for some C. So we have

$$\overline{A}_{i,P_j}(Dw^+)w^+_{x_ix_j} = \overline{A}_{i,P_j}(Dw^+)Q_{il}(x)Q_{jk}(x)\Lambda_{lk}(x) \leq C$$

for a.e. x and for some C. Hence using Young's inequality on (6) we have

$$\| \frac{\beta_1}{\epsilon} \|_p \leq C \| \overline{A}_{i,P_j}(Dw^+) \|_p .$$

Similarly we have

$$\| \frac{\beta_2}{\epsilon} \|_p \leq C \| \overline{A}_{i,P_j}(Dw^-) \|_p .$$

Hence we have

$$-D_i(\overline{A}_i(Du^\epsilon)) = f^\epsilon = -\frac{\beta_1}{\epsilon} - \frac{\beta_2}{\epsilon}$$

and

$$\| f^\epsilon \|_p \leq C(\| A_{i,P_j}(Dw^+) \|_p + \| A_{i,P_j}(Dw^-) \|_p) \tag{7}$$

for some C which is independent of ϵ. Hence by the L^p regularity theory [9] we have

$$\| u^\epsilon \|_{W^{2,p}(\Omega)} \leq C(\| f^\epsilon \|_p + \| u^\epsilon \|_{W^{1,p}(\Omega)})$$

$$\leq C(\| \overline{A}_{i,P_j}(Dw^+) \|_p + \| \overline{A}_{i,P_j}(Dw^-) \|_p + \| u^\epsilon \|_{W^{1,\infty}})$$

for all $p > n$ where C is independent of ϵ.

Since $\| u^\epsilon \|_{W^{2,p}}$ is bounded independently of ϵ,

$$u^{\epsilon_m} \rightharpoonup \tilde{u}$$

weakly in $W^{2,p}$ and

$$u^{\epsilon_m} \to \tilde{u}$$

strongly in $W^{1,p}$ for some sequence $\{u^{\epsilon_m}\}$ for each fixed p.

By standard arguments, we see that $\tilde{u}$ is the unique solution to (P2). □

The following remark is useful in proving $W^{2,\infty}$ regularity.

Remark 0.1 *Sending $p \to \infty$ in (7) we have*

$$||D_i(A_i(Du^\epsilon))||_{L^\infty_{loc}} \leq C$$

for some C independent of ϵ.

Now we prove that $\tilde{u} \in W^{2,\infty}$. Indeed we work under weaker assumption that $w^+(x) > w^-(x)$ for all $x \in \Omega$.

Theorem 3 *We have $\tilde{u} \in W^{2,\infty}_{loc}$.*

Proof. We follow the truncation idea of DeGiorgi. In fact we need only to prove that

$$||u^\epsilon||_{W^{2,\infty}_{loc}} \leq C$$

for some C independent of ϵ. Define

$$I^{+,\epsilon} = \{x \in \Omega : u^\epsilon(x) \geq w^+(x)\}$$

and

$$I^{-,\epsilon} = \{x \in \Omega : u^\epsilon(x) \leq w^-(x)\}.$$

First we assume that $B(x_0, 2r) \subset \Omega \setminus (I^{+,\epsilon} \cup I^{-,\epsilon})$. In this case we know that β_1 and β_2 are zero in $B(x_0, 2r)$. Hence from usual elliptic theory we have

$$||u^\epsilon||_{W^{2,\infty}(B(x_0,r))} \leq C$$

for some C independent of ϵ.

Now assume $x_0 \in I^{+,\epsilon}$. Since $w^{x_0} > w^-(x_0)$, there exists r independent of ϵ such that

$$B(x_0, 2r) \cap I^{-,\epsilon} = \emptyset.$$

Let $\zeta \in \boldsymbol{R}^n$ be a unit vector. Now if we differentiate (3) twice along ζ direction, we have

$$-D_i(\bar{A}_{i,P_j}(Du^\epsilon)(u_{\zeta\zeta})_{x_j}) - D_i(\frac{\partial}{\partial\zeta}(\bar{A}_{i,P_j})u_{\zeta x_j}) + \frac{\beta_1'}{\epsilon}[u^\epsilon_{\zeta\zeta} - w^+_{\zeta\zeta}]$$

$$+\frac{\beta_1''}{\epsilon}[u^\epsilon_\zeta - w^+_\zeta][u^\epsilon_\zeta - w^+_\zeta] = 0.$$

Now for simplification we set $v = u^\epsilon_{\zeta\zeta}$. From the fact that $\beta_1'' \geq 0$ we have

$$-D_i(\bar{A}_{i,P_j}v_{x_j}) \leq D_i(g_i) - \frac{\beta_1'}{\epsilon}[v - w^+_{\zeta\zeta}] \tag{8}$$

in $B(x_0, 2r)$, where we define

$$g_i = \frac{\partial}{\partial\zeta}(\bar{A}_{i,P_j})u_{\zeta x_j}.$$

Note that from theorem 2 we have

$$g \in L^p$$

for all $p \in (1, \infty)$. We also recall that

$$w^+_{\zeta\zeta} \leq C_0$$

for some C_0. Let $k > C_0$ be a large number and $\eta \in C_0^\infty(B(x_0, 2r))$ be a nonnegative cutoff function. Now we take $\phi = (v - k)^+\eta^2$ as a test function in (8). Hence we have

$$\int_{\{v>k\}\cap B(x_0,2r)} \bar{A}_{i,P_j}v_{x_j}v_{x_i}\eta^2 dx + 2\int_{\{v>k\}\cap B(x_0,2r)} \bar{A}_{i,P_j}v_{x_j}\eta_{x_i}(v-k)^+\eta dx$$

$$\leq -\int_{\{v>k\}\cap B(x_0,2r)} g_i((v-k)^+\eta^2)_{x_i}dx - \int_{\{v>k\}\cap B(x_0,2r)} \frac{\beta_1'}{\epsilon}[v-w_{\zeta\zeta}^+](v-k)^+\eta^2dx.$$

Now we estimate the last term. Since $v \geq k > C_0$ in the support of ϕ and $w_{\zeta\zeta}^+ \leq C_0$, we have

$$\int_{\{v>k\}\cap B(x_0,2r)} \frac{\beta_1'}{\epsilon}[v-w_{\zeta\zeta}^+](v-k)^+\eta^2dx \geq 0.$$

Thus after using Young's inequality we can write

$$\int |D((v-k)^+)|^2\eta^2dx \leq C\int |(v-k)^+D(\eta)|^2dx + C\int_{v>k}|g|^2\eta^2dx$$

for some C independent of ϵ. This is enough to prove that

$$v \leq C \tag{9}$$

in $B(x_0,r)$ for some C independent of ϵ(see lemma 5.4 in [13]). Now we need to show that v is bounded below. This is achieved by using equation. Since from remark 0.1 we note that

$$||D_i(A_i(Du^\epsilon))||_{L^\infty} < C$$

for some C independent of ϵ. Hence we have

$$A_{i,P_j}u_{x_i,x_j} \geq C$$

for some C. Note that A_{i,P_j} is a strictly positive matrix. After a suitable coordinate transformation we assume that $\{u_{x_ix_j}\}$ is a diagonal matrix. Hence we have

$$\sum_{i=1}^{n}\lambda_i u_{\zeta_i\zeta_i} \leq C \tag{10}$$

for some C independent of ϵ, some strictly positive $\lambda_i, i=1,...,n$ and some orthonormal vectors $\zeta_i, i=1,...,n$. From (9) and (10) we also have

$$v \geq C$$

in $B(x_0,r)$ for some C. We can argue in the same way when $x_0 \in I^{-,\epsilon}$ and complete the proof. □

With an obvious modification we have the following lemma which shows that a one sided bound for the second derivatives of the obstacles is enough for $W^{2,\infty}$ regularity for the solutions of obstacle problems.

Lemma 3 *Suppose that ψ is semi convex, that is,*

$$\frac{\partial^2}{\partial \zeta^2}\psi \geq C$$

in the sense of distribution for all unit vector $\zeta \in \boldsymbol{R}^n$ and for some C. We also assume $u_0 \geq \psi$ a.e. in Ω, where $u_0 \in W^{1,2}(\Omega)$. Suppose $u \in W^{1,2}(\Omega)$ is a solution to

$$\int_\Omega A_i(Du)(Dv - Du)dx \geq 0$$

for all $v \in \{v \in W_0^{1,2}(\Omega) + u_0 : v \geq \psi\}$. Then we have

$$u \in W_{loc}^{2,\infty}(\Omega).$$

Remark 0.2 *The proofs for theorem 2 and theorem 3 also show $W^{2,\infty}$ regularity for double obstacle problems with smooth obstacles.*

Remark 0.3 *Indeed we have an example that solutions of obstacle problems are more regular than the obstacles.*

We recall the following lemma which may not be well known[6].

Lemma 4 *Suppose $v \in W^{2,2}(\Omega)$ is a solution to*

$$D_i(A_i(Dv)) = 0,$$

then $G(Dv)$ is a subsolution to

$$D_i(A_{i,P_j}(Dv)D_j w) = 0.$$

We show the solution of (P2) is the solution of (P1) by using a maximum principle on $G(Dw)$.

Theorem 4 *The solution w of $(P2)$ is the solution of $(P1)$ so that $w \in K$ and*

$$\int_\Omega A_i(Dw)(D_i v - D_i w)dx \geq 0$$

for all $v \in K$. Thus $w = u$ and $u \in W^{2,p}(\Omega)$ for all $p \geq 1$.

Proof. To show that w is the solution of $(P1)$ it is enough to show $w \in K$, i.e. that

$$G(Dw) \leq M$$

since $K \subset J$.

Let I be the coincidence set, i.e., $I = \{x \in \overline{\Omega} : w(x) = w^{+}(x) \text{ or } w(x) = w^{-}(x)\}$. Then I is a closed subset of $\overline{\Omega}$ since w^{+}, w^{-} and w are continuous.

Hence w satisfies

$$\int_{\Omega} \overline{A}_i(Dw) D_i \phi dx = 0$$

in $\Omega \setminus I$ for all $\phi \in C_0^{\infty}(\Omega \setminus I)$.

Also we see that from Sobolev's embedding theorem $w \in C^{1,\alpha}(\overline{\Omega})$ for all $0 \leq \alpha < 1$.

Since $w^{+}(x) = w^{-}(x) = w(x)$ for all $x \in \partial\Omega$ and $w^{-}(x) \leq w(x) \leq w^{+}(x)$ for all $x \in \Omega$, we have that

$$\frac{\partial w^{+}(x)}{\partial \eta} \leq \frac{\partial w(x)}{\partial \eta} \leq \frac{\partial w^{-}(x)}{\partial \eta}$$

for all $x \in \partial\Omega$ where η is the outward normal vector to $\partial\Omega$ at $x \in \partial\Omega$. Clearly

$$\frac{\partial w^{+}(x)}{\partial \tau} = \frac{\partial w(x)}{\partial \tau} = \frac{\partial w^{-}(x)}{\partial \tau}$$

for all tangent vector τ to $\partial\Omega$ at $x \in \partial\Omega$. So we have that for each $x \in \partial\Omega$

$$Dw(x) = tDw^{+}(x) + (1-t)Dw^{-}(x)$$

for some $0 \leq t \leq 1$. Since G is convex, we have

$$G(Dw(x)) \leq tG(Dw^{+}(x)) + (1-t)G(Dw^{-}(x)) = M$$

for all $x \in \partial\Omega$.

Moreover we see that

$$G(Dw) = M$$

for all $x \in I$ since $w(x) = w^{+}(x)$ or $w(x) = w^{-}(x)$ for all $x \in I$ and $w^{\pm}$ is differentiable at x if $w^{\pm}(x) = w(x)$[4].

So by lemma 4 $G(Dw)$ has a maximum principle

$$\sup_{x \in \Omega \setminus I} G(Dw(x)) \leq \sup_{x \in \partial(\Omega \setminus I)} G(Dw) \leq M$$

and we prove that w is the solution of $(P1)$. $\square$

References

[1] H. Brezis and D. Kinderlehrer, The Smoothness of Solutions to Nonlinear Variational Inequalities, *Indiana University Mathematics Journal*, Vol. **23**, (1974), 831-844.

[2] H. Brezis and G. Stampacchia, Sur la regularite de la solution d'inequations elliptiques, Bull. Soc. Math. France **96**, (1968), 153-180.

[3] L. Caffarelli and N. M. Riviere, The Lipschitz Character of the Stress Tensor When Twisting an Elastic and Plastic Bar, Arch. Rat. Mech. and Anal. **69**, (1979), 31-36.

[4] P. Cannarsa and H. M. Soner, On the Singularities of the Viscosity Solutions to Hamilton-Jacobi-Bellman Equations, *Indiana University Mathematics Journal*, Vol. **36**, (1987), 501-524.

[5] M. Chipot, Variational Inequalities and Flow in Porous Media, Applied Mathematical Sciences **52**, Springer, 1984.

[6] H. J. Choe, Regularity for Minimizers of Certain Singular Functionals, Ph.D. thesis, Univ. of Kentucky, Lexington, 1989.

[7] L. C. Evans, A Second Order Elliptic Equation with Gradient Constraint, *Comm. in P.D.E.*, **4**, (1979), 555-572.

[8] C. Gerhardt, Regularity of Solutions of Nonlinear Variational Inequalities with a Gradient Bound as Constraint, *Archive for Rational Mechanics and Analysis*, **58**, (1975), 309-317.

[9] C. Gerhardt, Regularity of Solutions of Nonlinear Variational Inequalities, *Archive for Rational Mechanics and Analysis*, **52**, (1973), 389-393.

[10] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd edition, Springer-Verlag, 1983.

[11] P. Hartman and G. Stampacchia, On some nonlinear elliptic differential functional equations, Acta Math.**115**, (1971), 271-310.

[12] R. Jensen, Regularity for Elasto-Plastic Type Variational Inequality, *Indiana University Mathematics Journal*, **32**, (1983), 407-423.

[13] O. A. Ladyzhenskaya and N. N. Ural'tzeva, "Linear and Quasilinear Elliptic Equations," Academic Press, 1968.

[14] P. L. Lion, Generalized solutions of Hamilton-Jacobi equations, Research Notes in Mathematics **69**, Pitman Advanced Publishing Program, 1982.

Recent IMA Preprints

#	Author/s	Title
721	**Ian M. Anderson, Niky Kamran and Peter J. Olver**, Internal, external and generalized symmetries	
722	**C. Foias and J.C. Saut**, Asymptotic integration of Navier–Stokes equations with potential forces. I	
723	**Ling Ma**, The convergence of semidiscrete methods for a system of reaction-diffusion equations	
724	**Adelina Georgescu**, Models of asymptotic approximation	
725	**A. Makagon and H. Salehi**, On bounded and harmonizable solutions on infinite order arma systems	
726	**San-Yih Lin and Yan-Shin Chin**, An upwind finite-volume scheme with a triangular mesh for conservation laws	
727	**J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart**, On the dynamics of fine structure	
728	**KangPing Chen and Daniel D. Joseph**, Lubrication theory and long waves	
729	**J.L. Ericksen**, Local bifurcation theory for thermoelastic Bravais lattices	
730	**Mario Taboada and Yuncheng You**, Some stability results for perturbed semilinear parabolic equations	
731	**A.J. Lawrance**, Local and deletion influence	
732	**Bogdan Vernescu**, Convergence results for the homogenization of flow in fractured porous media	
733	**Xinfu Chen and Avner Friedman**, Mathematical modeling of semiconductor lasers	
734	**Yongzhi Xu**, Scattering of acoustic wave by obstacle in stratified medium	
735	**Songmu Zheng**, Global existence for a thermodynamically consistent model of phase field type	
736	**Heinrich Freistühler and E. Bruce Pitman**, A numerical study of a rotationally degenerate hyperbolic system part I: the Riemann problem	
737	**Epifanio G. Virga**, New variational problems in the statics of liquid crystals	
738	**Yoshikazu Giga and Shun'ichi Goto**, Geometric evolution of phase-boundaries	
739	**Ling Ma**, Large time study of finite element methods for 2D Navier–Stokes equations	
740	**Mitchell Luskin and Ling Ma**, Analysis of the finite element approximation of microstructure in micromagnetics	
741	**M. Chipot**, Numerical analysis of oscillations in nonconvex problems	
742	**J. Carrillo and M. Chipot**, The dam problem with leaky boundary conditions	
743	**Eduard Harabetian and Robert Pego**, Efficient hybrid shock capturing schemes	
744	**B.L.J. Braaksma**, Multisummability and Stokes multipliers of linear meromorphic differential equations	
745	**Tae Il Jeon and Tze-Chien Sun**, A central limit theorem for non-linear vector functionals of vector Gaussian processes	
746	**Chris Grant**, Solutions to evolution equations with near-equilibrium initial values	
747	**Mario Taboada and Yuncheng You**, Invariant manifolds for retarded semilinear wave equations	
748	**Peter Rejto and Mario Taboada**, Unique solvability of nonlinear Volterra èquations in weighted spaces	
749	**Hi Jun Choe**, Holder regularity for the gradient of solutions of certain singular parabolic equations	
750	**Jack D. Dockery**, Existence of standing pulse solutions for an excitable activator-inhibitory system	
751	**Jack D. Dockery and Roger Lui**, Existence of travelling wave solutions for a bistable evolutionary ecology model	
752	**Giovanni Alberti, Luigi Ambrosio and Giuseppe Buttazzo**, Singular perturbation problems with a compact support semilinear term	
753	**Emad A. Fatemi**, Numerical schemes for constrained minimization problems	
754	**Y. Kuang and H.L. Smith**, Slowly oscillating periodic solutions of autonomous state-dependent delay equations	
755	**Emad A. Fatemi**, A new splitting method for scaler conservation laws with stiff source terms	
756	**Hi Jun Choe**, A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities	
757	**Haitao Fan**, A vanishing viscosity approach on the dynamics of phase transitions in Van Der Waals fluids	
758	**T.A. Osborn and F.H. Molzahn**, The Wigner–Weyl transform on tori and connected graph propagator representations	
759	**Avner Friedman and Bei Hu**, A free boundary problem arising in superconductor modeling	
760	**Avner Friedman and Wenxiong Liu**, An augmented drift-diffusion model in semiconductor device	
761	**Avner Friedman and Miguel A. Herrero**, Extinction and positivity for a system of semilinear parabolic variational inequalities	
762	**David Dobson and Avner Friedman**, The time-harmonic Maxwell equations in a doubly periodic structure	
763	**Hi Jun Choe**, Interior behaviour of minimizers for certain functionals with nonstandard growth	
764	**Vincenzo M. Tortorelli and Epifanio G. Virga**, Axis-symmetric boundary-value problems for nematic liquid crystals with variable degree of orientation	
765	**Nikan B. Firoozye and Robert V. Kohn**, Geometric parameters and the relaxation of multiwell energies	
766	**Haitao Fan and Marshall Slemrod**, The Riemann problem for systems of conservation laws of mixed type	
767	**Joseph D. Fehribach**, Analysis and application of a continuation method for a self-similar coupled Stefan system	
768	**C. Foias, M.S. Jolly, I.G. Kevrekidis and E.S. Titi**, Dissipativity of numerical schemes	
769	**D.D. Joseph, T.Y.J. Liao and J.-C. Saut**, Kelvin–Helmholtz mechanism for side branching in the displacement of light with heavy fluid under gravity	

770 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
771 **B. Cockburn, F. Coquel, Ph. LeFloch and C.W. Shu**, Convergence of finite volume methods
772 **N.G. Lloyd and J.M. Pearson**, Computing centre conditions for certain cubic systems
773 **João Palhoto Matos**, Young measures and the absence of fine microstructures in the $\alpha - \beta$ quartz phase transition
774 **L.A. Peletier & W.C. Troy**, Self-similar solutions for infiltration of dopant into semiconductors
775 **H. Scott Dumas and James A. Ellison**, Nekhoroshev's theorem, ergodicity, and the motion of energetic charged particles in crystals
776 **Stathis Filippas and Robert V. Kohn**, Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.
777 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity
778 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximal smoothness of solutions to certain Euler–Lagrange equations from nonlinear elasticity
779 **Jack Carr and Robert Pego**, Self-similarity in a coarsening model in one dimension
780 **J.M. Greenberg**, The shock generation problem for a discrete gas with short range repulsive forces
781 **George R. Sell and Mario Taboada**, Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains
782 **T. Subba Rao**, Analysis of nonlinear time series (and chaos) by bispectral methods
783 **Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy**, Vortex rings of one fluid in another free fall
784 **Oscar Bruno, Avner Friedman and Fernando Reitich**, Asymptotic behavior for a coalescence problem
785 **Johannes C.C. Nitsche**, Periodic surfaces which are extremal for energy functionals containing curvature functions
786 **F. Abergel and J.L. Bona**, A mathematical theory for viscous, free-surface flows over a perturbed plane
787 **Gunduz Caginalp and Xinfu Chen**, Phase field equations in the singular limit of sharp interface problems
788 **Robert P. Gilbert and Yongzhi Xu**, An inverse problem for harmonic acoustics in stratified oceans
789 **Roger Fosdick and Eric Volkmann**, Normality and convexity of the yield surface in nonlinear plasticity
790 **H.S. Brown, I.G. Kevrekidis and M.S. Jolly**, A minimal model for spatio–temporal patterns in thin film flow
791 **Chao–Nien Chen**, On the uniqueness of solutions of some second order differential equations
792 **Xinfu Chen and Avner Friedman**, The thermistor problem for conductivity which vanishes at large temperature
793 **Xinfu Chen and Avner Friedman**, The thermistor problem with one-zero conductivity
794 **E.G. Kalnins and W. Miller, Jr.**, Separation of variables for the Dirac equation in Kerr Newman space time
795 **E. Knobloch, M.R.E. Proctor and N.O. Weiss**, Finite-dimensional description of doubly diffusive convection
796 **V.V. Pukhnachov**, Mathematical model of natural convection under low gravity
797 **M.C. Knaap**, Existence and non-existence for quasi-linear elliptic equations with the p-laplacian involving critical Sobolev exponents
798 **Stathis Filippas and Wenxiong Liu**, On the blowup of multidimensional semilinear heat equations
799 **A.M. Meirmanov**, The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution
800 **Bo Guan and Joel Spruck**, Interior gradient estimates for solutions of prescribed curvature equations of of parabolic type
801 **Hi Jun Choe**, Regularity for solutions of nonlinear variational inequalities with gradient constraints
802 **Peter Shi and Yongzhi Xu**, Quasistatic linear thermoelasticity on the unit disk
803 **Satyanad Kichenassamy and Peter J. Olver**, Existence and non-existence of solitary wave solutions to higher order model evolution equations
804 **Dening Li**, Regularity of solutions for a two-phase degenerate Stefan Problem
805 **Marek Fila, Bernhard Kawohl and Howard A. Levine**, Quenching for quasilinear equations
806 **Yoshikazu Giga, Shun'ichi Goto and Hitoshi Ishii**, Global existence of weak solutions for interface equations coupled with diffusion equations
807 **Mark J. Friedman and Eusebius J. Doedel**, Computational methods for global analysis of homoclinic and hetero clinic orbits: a case study
808 **Mark J. Friedman**, Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds
809 **Peter W. Bates and Songmu Zheng**, Inertial manifolds and inertial sets for the phase-field equations
810 **J. López Gómez, V. Márquez and N. Wolanski**, Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition
811 **Xinfu Chen and Fahuai Yi**, Regularity of the free boundary of a continuous casting problem
812 **Eden, A., Foias, C., Nicolaenko, B. and Temam, R.**, Inertial sets for dissipative evolution equations Part I: Construction and applications
813 **Jose–Francisco Rodrigues and Boris Zaltzman**, On classical solutions of the two-phase steady-state Stefan problem in strips
814 **Viorel Barbu and Srdjan Stojanovic**, Controlling the free boundary of elliptic variational inequalities on a variable domain