

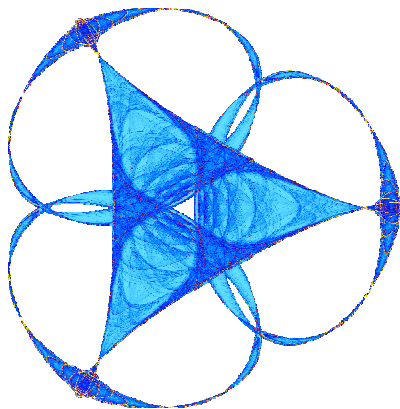
ERROR ANALYSIS FOR A MONOLITHIC DISCRETIZATION OF COUPLED
DARCY AND STOKES PROBLEMS

By

Vivette Girault, Guido Kanschat, Béatrice Rivière

IMA Preprint Series #2390

(APRIL 2012)



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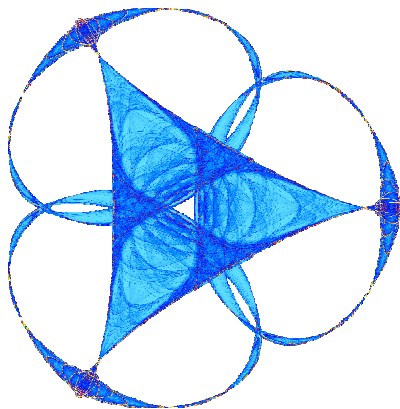
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ERROR ANALYSIS FOR A MONOLITHIC DISCRETIZATION OF COUPLED Darcy AND Stokes PROBLEMS

VIVETTE GIRAULT, GUIDO KANSCHAT, AND BÉATRICE RIVIÈRE

ABSTRACT. The coupled Stokes and Darcy equations are approximated by a strongly conservative finite element method. The discrete spaces are the divergence-conforming velocity space with matching pressure space such as the Raviart-Thomas spaces. This work proves optimal error estimate of the velocity in the L^2 norm in the domain and on the interface under weak regularity assumptions.

1. INTRODUCTION

Recently, a monolithic strongly conservative finite element method that combines the discontinuous Galerkin method for the Stokes equations with the mixed finite element method for the Darcy equations was introduced by Kanschat and Rivière [28]. Mass conservation is achieved in the \mathbf{H}^{div} sense, meaning that the divergence of the velocity is pointwise equal to zero inside the mesh elements. Energy error estimates were derived in [28] but these are not optimal in the Darcy region. In fact, standard duality arguments cannot be applied. In this article, we complete the error analysis of our scheme by deriving optimal L^2 error estimates and thus providing mathematical proof of the convergence rates observed in [28]. All estimates are done with weak regularity assumptions, such that they apply to cases where the interface is not smooth. The argument is based on a sharp estimate for the trace of the solution on the interface. As a result of our analysis, we find that the orders of approximation of the velocity in the Darcy and Stokes subdomains are balanced.

Modeling of Darcy-Stokes coupling traces back to the seminal work of Beavers and Joseph [5] and Saffman [36]. The Beavers-Joseph-Saffman model can be derived by homogenization (see Jäger & Mikelić [25] and references therein) and its well-posedness has been established by Layton et al. [29]. In the latter, a finite element method based on continuous and \mathbf{H}^{div} elements in the Stokes and Darcy subdomains, respectively, is introduced. The coupling is achieved by a mortar. Alternatively, a primal formulation has been applied in the Darcy subregion (see Discacciati & Quarteroni [15] and references therein); this formulation also allows for interface conditions involving the tangential Darcy velocity, but mass conservation is not as natural anymore. Riviere and Yotov [35] and Gatica et al [18] proposed a primal formulation in the Stokes region coupled with a dual formulation in the Darcy region, which means that different elements are used in both regions. In [17], Gatica et al. analyzed a fully mixed formulation that solves the pseudo-stress and velocity in the Stokes region and requires Lagrange multipliers. Finally, the work of Arbogast and

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Brunson [2] uses a finite element with continuity requirements changing between \mathbf{H}^1 and \mathbf{H}^{div} as needed. Our discretization is distinguished from these by

- (a) employing a conservative formulation in the whole domain
- (b) using the same \mathbf{H}^{div} conforming element pair in the Stokes and the Darcy subdomain, and
- (c) a discontinuous Galerkin method to address the resulting inconsistency in the Stokes subdomain.

Thus, it addresses the issue of mass loss reported for instance by Hanspal et al. [24] and it does not require a mortar between the subdomains. Due to the fact that the same element is used over the whole domain, it lends itself to particularly simple implementations.

The outline of the paper is as follows. Section 2 defines the model problem including the interface conditions. Across the interface, we assume continuity of normal component of velocity, balance of forces and the Beavers-Joseph-Saffman law. The numerical scheme is then defined. Instead of specifying a particular discontinuous Galerkin form for the Stokes subdomain, we make generic assumptions and show they are satisfied by the interior penalty discontinuous Galerkin methods and the local discontinuous Galerkin method. Section 3 contains the error analysis. It begins with our main result, Theorem 5, followed by the auxiliary results needed for its proof.

2. MODEL PROBLEM AND DISCRETIZATION

Let Ω be a bounded Lipschitz domain in \mathbb{R}^d , $d = 2, 3$, split into two Lipschitz subregions Ω_S and Ω_D of free and porous media flow, respectively. By Γ_S , Γ_D , and Γ_{SD} we denote the boundaries of the subdomains according to the definition

$$(1) \quad \Gamma_{SD} = \partial\Omega_S \cap \partial\Omega_D \quad \Gamma_S = \partial\Omega \cap \partial\Omega_S, \quad \Gamma_D = \partial\Omega \cap \partial\Omega_D.$$

When $d = 3$, we assume that the boundaries of Γ_S , Γ_D , and Γ_{SD} are all Lipschitz. To simplify the discussion, in the case when Ω_S is not connected, we assume that each of its connected components is adjacent to a portion of Γ_S with positive measure. The results of this work remain true in the general situation, but some proofs are more involved. The geometry of the boundary components also determines the regularity of solutions, in particular for a nondifferentiable interface. Since we are not aware of a thorough study of the expected regularity, we derive estimates under weak assumptions, leaving the regularity of the solution as a parameter.

The coupled Darcy/Stokes problem in conservative form reads

$$(2a) \quad -\nabla \cdot (2\nu\varepsilon(\mathbf{u})) + \nabla p = \mathbf{f}, \quad \text{in } \Omega_S,$$

$$(2b) \quad K^{-1}\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega_D,$$

$$(2c) \quad \nabla \cdot \mathbf{u} = g, \quad \text{in } \Omega.$$

The deformation tensor is $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$. The coefficient $\nu > 0$ is the fluid kinematic viscosity. The variable K is the ratio of the intrinsic permeability to the fluid viscosity. It is a symmetric positive definite tensor with eigenvalues $\{\lambda_i\}_{i=1,\dots,d}$, uniformly bounded above and away from zero. The intrinsic permeability is allowed to vary continuously over Ω_D . The data \mathbf{f} and g are chosen respectively in the space of vector-valued functions $\mathbf{L}^2(\Omega)$ and the scalar-valued space $L^2(\Omega)$, with the restriction (4) below. Here, we consider the most general—not necessarily physically meaningful—case of inhomogeneous right-hand sides in both equations and both subdomains. Studying the problem in this generality will allow us for instance to apply our results to dual problems as they occur in a posteriori error estimation. Whenever we want to distinguish between the solution of

the Stokes and the Darcy subproblem, we refer to $\mathbf{u}_S = \mathbf{u}|_{\Omega_S}$ and $\mathbf{u}_D = \mathbf{u}|_{\Omega_D}$ and analogue for the pressures p_S and p_D . On the interface Γ_{SD} , this notation refers to the traces taken from the respective subdomains.

To complete this problem, we need interface and boundary conditions. For any bounded domain \mathcal{O} , let \mathbf{n} denote the unit normal vector to $\partial\mathcal{O}$, exterior to \mathcal{O} ; in the case of the interface Γ_{SD} , \mathbf{n} is the unit normal vector to Γ_{SD} directed from Ω_S to Ω_D , and $\boldsymbol{\tau}^j$, $1 \leq j \leq d-1$, is an orthonormal set of tangent vectors on the tangent plane to Γ_{SD} . On the interface, we impose the Beavers-Joseph-Saffman conditions (see Beavers and Joseph [5] and Saffman [36]), and on the boundary, we assume no-slip and Neumann for simplicity:

$$\begin{aligned}
 (3a) \quad & \mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}, & \text{on } \Gamma_{SD}, \\
 (3b) \quad & p_S - 2\nu\varepsilon(\mathbf{u}_S)\mathbf{n} \cdot \mathbf{n} = p_D, & \text{on } \Gamma_{SD}, \\
 (3c) \quad & \gamma^2 K^{-\frac{1}{2}} \mathbf{u}_S \cdot \boldsymbol{\tau}^j + 2\nu\varepsilon(\mathbf{u}_S)\mathbf{n} \cdot \boldsymbol{\tau}^j = 0, & \text{on } \Gamma_{SD}, j \in [1, d-1], \\
 (3d) \quad & \mathbf{u}_S = 0, & \text{on } \Gamma_S, \\
 (3e) \quad & \mathbf{u}_D \cdot \mathbf{n} = 0, & \text{on } \Gamma_D.
 \end{aligned}$$

Here, the tensor $K^{-\frac{1}{2}}$ is obtained from K by replacing the eigenvalues λ_i by $\lambda_i^{-\frac{1}{2}}$. The function γ is a positive, smooth function on the interface bounded uniformly above and away from zero.

We use the standard notation $H^s(\mathcal{O})$ for the Sobolev space of order s on a bounded domain \mathcal{O} , see Adams & Fournier, Lions & Magenes, or Nečas [1,30,31], as well as $\mathbf{H}^s(\mathcal{O})$ for spaces of vector-valued functions. When k is an integer, the norm and seminorm of $H^k(\mathcal{O})$ are defined as usual by

$$\|v\|_{H^k(\mathcal{O})} = \left(\sum_{j=0}^k \sum_{|\alpha|=j} \|\partial^\alpha v\|_{L^2(\mathcal{O})}^2 \right)^{\frac{1}{2}}, \quad |v|_{H^k(\mathcal{O})} = \left(\sum_{|\alpha|=k} \|\partial^\alpha v\|_{L^2(\mathcal{O})}^2 \right)^{\frac{1}{2}},$$

respectively. We refer to [1, 30, 31] for the extension of these definitions when s is not an integer. In order to obtain weak solutions to the set of equations (2)–(3), we introduce the spaces for the velocity part of the solution

$$\begin{aligned}
 \mathbf{H}^{\text{div}}(\Omega) &= \{ \mathbf{v} \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \mathbf{v} \in L^2(\Omega) \}, \\
 \mathbf{H}_0^{\text{div}}(\Omega) &= \{ \mathbf{v} \in \mathbf{H}^{\text{div}}(\Omega) \mid \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \},
 \end{aligned}$$

see for instance Girault & Raviart [19]. Since the domain of the Stokes operator in (2a) is $\mathbf{H}^1(\Omega_S)$ for the velocity, we have to require additional differentiability. To begin with, it is easy to check on the one hand that equations (2)–(3) are unchanged if an arbitrary constant is added to the pressure p . We choose this undetermined constant by prescribing that

$$\int_{\Omega} p = 0.$$

On the other hand, the boundary conditions (3d) and (3e) imply that

$$\int_{\Omega} \nabla \cdot \mathbf{u} = 0.$$

Therefore g must satisfy the compatibility condition

$$(4) \quad \int_{\Omega} g = 0.$$

Thus, the function spaces for our weak formulation are

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}_0^{\text{div}}(\Omega) \mid \mathbf{v}|_{\Omega_S} \in \mathbf{H}^1(\Omega_S) \text{ and } \mathbf{v}|_{\Gamma_S} = \mathbf{0} \right\},$$

$$Q = \left\{ q \in L^2(\Omega) \mid \int_{\Omega} q = 0 \right\} = L_0^2(\Omega).$$

The restriction of \mathbf{V} to Ω_S is denoted by \mathbf{V}_S . The space \mathbf{V}_D is defined in section 3.4. We also introduce the subspace of divergence-free vectors $\mathring{\mathbf{V}}$ of \mathbf{V} :

$$\mathring{\mathbf{V}} = \left\{ \mathbf{v} \in \mathbf{V} \mid \nabla \cdot \mathbf{v} = 0 \right\}.$$

The measure of a bounded set \mathcal{O} is denoted by $|\mathcal{O}|$. We use the scalar product notation and norm on Ω , boundaries, faces, and subsets of those, namely

$$(5a) \quad (\phi, \psi)_{\Omega} = \int_{\Omega} \phi \odot \psi \, d\mathbf{x}, \quad \langle \phi, \psi \rangle_{\Gamma} = \int_{\Gamma} \phi \odot \psi \, ds,$$

$$(5b) \quad \|\phi\|_{L^2(\Omega)} = \left(\int_{\Omega} |\phi|^2 \, d\mathbf{x} \right)^{\frac{1}{2}}, \quad \|\phi\|_{L^2(\Gamma)} = \left(\int_{\Gamma} |\phi|^2 \, ds \right)^{\frac{1}{2}}.$$

The pointwise multiplication operator $\phi \odot \psi$ refers to the product $\phi\psi$, the scalar product $\phi \cdot \psi$ and the double contraction $\phi : \psi$ for scalar, vector and tensor arguments, respectively. The modulus $|\phi| = \sqrt{\phi \odot \phi}$ is defined accordingly. The space Q is equipped with the L^2 norm in Ω , the space \mathbf{V} has the norm

$$\forall \mathbf{v} \in \mathbf{V}, \|\mathbf{v}\|_{\mathbf{V}} = \left(\|K^{-\frac{1}{2}} \mathbf{v}\|_{\mathbf{L}^2(\Omega_D)}^2 + \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)}^2 + \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega_S)}^2 \right)^{\frac{1}{2}},$$

and is a Hilbert space for this norm, owing to Poincaré's inequality.

2.1. Weak formulation. For $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ and $q \in Q$, we introduce the bilinear forms

$$(6) \quad \begin{aligned} a_D(\mathbf{u}, \mathbf{v}) &= (K^{-1} \mathbf{u}, \mathbf{v})_{\Omega_D}, \\ a_S(\mathbf{u}, \mathbf{v}) &= 2\nu (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\Omega_S}, \\ a_I(\mathbf{u}, \mathbf{v}) &= \sum_{j=1}^{d-1} \langle \gamma^2 K^{-\frac{1}{2}} \mathbf{u}_S \cdot \boldsymbol{\tau}^j, \mathbf{v}_S \cdot \boldsymbol{\tau}^j \rangle_{\Gamma_{SD}}, \\ a(\mathbf{u}, \mathbf{v}) &= a_D(\mathbf{u}, \mathbf{v}) + a_I(\mathbf{u}, \mathbf{v}) + a_S(\mathbf{u}, \mathbf{v}), \\ b(\mathbf{v}, q) &= -(\nabla \cdot \mathbf{v}, q)_{\Omega}. \end{aligned}$$

The weak formulation of problem (2)–(3) reads: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$(7) \quad \begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v})_{\Omega}, & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= -(g, q)_{\Omega}, & \forall q \in Q. \end{aligned}$$

Note that since both $\nabla \cdot \mathbf{u}$ and g belong to $L_0^2(\Omega)$, we can relax the zero mean value constraint on the test functions q and this last equation is valid for all q in $L^2(\Omega)$. It can be shown that the weak formulation (7) is equivalent to the boundary value problem (2)–(3); see Girault & Rivièrè [20] for an interpretation of the boundary conditions on the interface. The bilinear form $b(\cdot, \cdot)$ satisfies an inf-sup condition that easily follows from the standard inf-sup condition between $\mathbf{H}_0^1(\Omega)$ and $L_0^2(\Omega)$ (see for instance [19]): There exists a constant $\kappa > 0$ such that

$$(8) \quad \forall q \in L_0^2(\Omega), \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}} \geq \kappa \|q\|_{L^2(\Omega)}.$$

As $\mathbf{H}_0^1(\Omega) \subset \mathbf{V}$ with continuous imbedding, (8) immediately implies that there exists a constant $\beta > 0$ such that

$$(9) \quad \forall q \in L_0^2(\Omega), \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{V}}} \geq \beta \|q\|_{L^2(\Omega)}.$$

The first inf-sup condition (8) allows in particular to lift the nonzero divergence constraint:

Proposition 1. *Given g in $L_0^2(\Omega)$, there is a function \mathbf{u}_g in $\mathbf{H}_0^1(\Omega)$ such that*

$$(10) \quad \nabla \cdot \mathbf{u}_g = g \quad \|\mathbf{u}_g\|_{H^1(\Omega)} \leq \frac{1}{\kappa} \|g\|_{L^2(\Omega)}.$$

This leads to the well-posedness of (7). Indeed the auxiliary function $\overset{\circ}{\mathbf{u}} = \mathbf{u} - \mathbf{u}_g$ solves the following problem that is obviously equivalent to (7):

$$(11) \quad \begin{aligned} a(\overset{\circ}{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v})_{\Omega} - a(\mathbf{u}_g, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\ b(\overset{\circ}{\mathbf{u}}, q) &= 0, & \forall q \in Q. \end{aligned}$$

Moreover, considering that Korn's inequality holds for functions in $\mathbf{H}^1(\Omega_S)$ that vanish on Γ_S , the bilinear form $a(\cdot, \cdot)$ is elliptic and continuous on $\mathbf{V} \times \mathbf{V}$. Therefore a straightforward application of the Babuška-Brezzi theory (see for instance [19]) shows that problem (11) is well-posed, i.e. it has a unique solution that depends continuously on the data.

In the analysis that follows, we make use of an adjoint problem defined as follows: for ψ given in $L^2(\Omega_S)$, find (\mathbf{u}^*, p^*) in $\mathbf{V} \times Q$, solution of

$$(12) \quad \begin{aligned} a(\mathbf{v}, \mathbf{u}^*) + b(\mathbf{v}, p^*) &= (\mathbf{v}, \psi)_{\Omega_S}, & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}^*, q) &= 0, & \forall q \in Q. \end{aligned}$$

This problem has a unique weak solution, and since the form $a(\cdot, \cdot)$ is symmetric, this solution (\mathbf{u}^*, p^*) satisfies the interior equations (2) with $\mathbf{f} = \psi$ in Ω_S , $\mathbf{f} = \mathbf{0}$ in Ω_D , and $g = 0$, namely:

$$(13) \quad \begin{aligned} -\nabla \cdot (2\nu \varepsilon(\mathbf{u}^*)) + \nabla p^* &= \psi, & \text{in } \Omega_S, \\ K^{-1} \mathbf{u}^* + \nabla p^* &= \mathbf{0}, & \text{in } \Omega_D, \\ \nabla \cdot \mathbf{u}^* &= 0, & \text{in } \Omega, \end{aligned}$$

the interface conditions (3a)–(3c) and the boundary conditions (3d)–(3e). We make the following assumption on its regularity.

Assumption 1. *The boundaries Γ_S , Γ_D and Γ_{SD} are such that, for some real number α , $\frac{1}{2} < \alpha \leq 1$, (\mathbf{u}^*, p^*) can be estimated by*

$$(14) \quad \|\mathbf{u}_S^*\|_{H^{1+\alpha}(\Omega_S)} \leq C \|\psi\|_{L^2(\Omega_S)}, \quad \|\mathbf{u}_D^*\|_{H^\alpha(\Omega_D)} \leq C \|\psi\|_{L^2(\Omega_S)},$$

$$(15) \quad \|p_S^*\|_{H^\alpha(\Omega_S)} \leq C \|\psi\|_{L^2(\Omega_S)}, \quad \|p_D^*\|_{H^{1+\alpha}(\Omega_D)} \leq C \|\psi\|_{L^2(\Omega_S)}.$$

Note that the assumption on p_D^* follows immediately from the assumption on \mathbf{u}_D^* .

2.2. Discretization. From now on, we assume that Ω_S and Ω_D are polygons or Lipschitz polyhedra, according to the dimension. Let \mathbb{T}_h be a regular family (in the sense of Ciarlet [11]) of conforming subdivisions of $\bar{\Omega}$ into simplices, parallelograms or parallepipeds, such that the interface Γ_{SD} is the union of element faces. The case of arbitrary quadrilaterals or hexahedra is more complex because of the anisotropic behavior of the Piola transform (see Arnold, Boffi, and Falk [3]); all caveats in this reference apply to our method as well. For any element $T \in \mathbb{T}_h$, we denote by h_T its diameter, by ρ_T , the maximum diameter of

spheres inscribed in T , and by h the maximum diameter over all mesh elements. The regularity assumption on the family \mathbb{T}_h states that there exists a constant $\theta > 0$, independent of h such that

$$(16) \quad \forall T \in \mathbb{T}_h, \quad \frac{h_T}{\rho_T} \leq \theta.$$

Additionally, we introduce the notation \mathbb{T}_h^S for all cells of \mathbb{T}_h which lie in $\overline{\Omega_S}$. Denote by Γ_i^S the set of faces that are interior to Ω_S and by Γ_b^S the set of those that lie on the boundary Γ_S , furthermore let $\Gamma_h^S = \Gamma_b^S \cup \Gamma_i^S$.

We use the same convention as above to denote the scalar products and norms of vectors or tensors, but since we have in mind a DG method for the Stokes operator, we introduce suitable norms on the cells and faces of \mathbb{T}_h and Γ_h^S ; thus we define analogously to (5)

$$\begin{aligned} (\phi, \psi)_{\mathbb{T}_h} &= \sum_{T \in \mathbb{T}_h} \int_T \phi \odot \psi \, d\mathbf{x}, & \langle \phi, \psi \rangle_{\Gamma_\star^S} &= \sum_{F \in \Gamma_\star^S} \int_F \phi \odot \psi \, ds, \\ \|\phi\|_{\mathbb{T}_h} &= \left(\sum_{T \in \mathbb{T}_h} \int_T |\phi|^2 \, d\mathbf{x} \right)^{\frac{1}{2}}, & \|\phi\|_{\Gamma_\star^S} &= \left(\sum_{F \in \Gamma_\star^S} \int_F |\phi|^2 \, ds \right)^{\frac{1}{2}}, \end{aligned}$$

where the index \star stands for b , i , or h .

2.2.1. Discretization spaces and projections. For each integer $k \geq 0$, let \mathbb{P}_k be the space of polynomials in two or three variables of *total* degree less than or equal to k , and \mathbb{Q}_k the space of polynomials of degree less than or equal to k in *each* variable. For the discrete spaces, we use pairs of a divergence-conforming velocity space $\mathbf{V}_h \subset \mathbf{H}_0^{\text{div}}(\Omega)$ and the matching pressure space $Q_h \subset Q$ of degree k , relative to \mathbb{P}_k or \mathbb{Q}_k on simplices or parallelograms/parallepipeds, respectively. The simplest choice is the Raviart-Thomas element [32] of order k . We assume that these pairs of spaces satisfy the following key property, like in the continuous case.

Assumption 2.

$$(17) \quad \nabla \cdot \mathbf{V}_h = Q_h.$$

An immediate consequence of this assumption is that for each function $\mathbf{v} \in \mathbf{V}_h$, the condition $(\nabla \cdot \mathbf{v}, q_h)_\Omega = 0$ for all $q_h \in Q_h$ implies that $\nabla \cdot \mathbf{v} = 0$. Thus, we define the divergence-free subspace

$$(18) \quad \mathring{\mathbf{V}}_h = \{ \mathbf{v}_h \in \mathbf{V}_h \mid \nabla \cdot \mathbf{v}_h = 0 \}.$$

The Stokes operator is discretized by a DG method. We shall not describe a particular DG method (see for example [4, 14, 27, 33, 34]), but just introduce some general tools and assumptions. A unit normal vector \mathbf{n}_F is defined on each face F of Γ_h^S ; its direction can be chosen arbitrarily for each interior face, while it is outward for boundary faces. In the following, it will simply be denoted by \mathbf{n} . The jump of traces of a discontinuous function \mathbf{v} across a face F of Γ_i^S , in the direction of \mathbf{n}_F is denoted by $[[\mathbf{v}]]$. The jump on a boundary face is simply $[[\mathbf{v}]] = \mathbf{v}$. Note that by the choice of spaces, the normal components of both jumps are zero, so that the jumps only act on the tangential components. The following DG-norm is used:

$$(19) \quad \|\mathbf{v}\|_{1,h} = \left(\|\nabla \mathbf{v}\|_{\mathbb{T}_h^S}^2 + \|\sigma_h [[\mathbf{v}]]\|_{\Gamma_h^S}^2 + \sum_{j=1}^{d-1} \|\gamma K^{-\frac{1}{4}} \mathbf{v} \cdot \boldsymbol{\tau}^j\|_{\Gamma_{SD}}^2 \right)^{\frac{1}{2}},$$

where on an interior face F between two mesh cells T_1 and T_2 , we choose the weight $\sigma_h^F = \sqrt{\frac{1}{2}(\sigma_h^{F,1} + \sigma_h^{F,2})}$. The parameter $\sigma_h^{F,i}$ has the form

$$(20) \quad \sigma_h^{F,i} = \frac{\sigma}{h_{F,i}},$$

with $\sigma > 0$, independent of h , and $h_{F,i}$ the diameter of the cell T_i measured perpendicular to F . On a face F in Γ_b^S , where there is only a single cell T_1 , we choose $\sigma_h^F = \sqrt{2\sigma_h^{F,1}}$ (see e.g. [27]). In the sequel, these weight functions are denoted σ_h . The norm of \mathbf{V}_h is obtained by combining the \mathbf{H}^{div} norm on Ω and the DG-norm on Ω_S :

$$\|\mathbf{v}\|_{\mathbf{V}_h} = \left(\|K^{-\frac{1}{2}}\mathbf{v}\|_{\mathbf{L}^2(\Omega_D)}^2 + \|\nabla \cdot \mathbf{v}\|_{L^2(\Omega)}^2 + \|\mathbf{v}\|_{1,h}^2 \right)^{\frac{1}{2}}.$$

Note that this is a norm owing to Poincaré's inequality for DG methods derived by Brenner in [6, 7].

Next, we introduce an approximation operator $R_h \in \mathcal{L}(\mathbf{V} \cap \mathbf{H}^s(\Omega); \mathbf{V}_h)$, $s > 0$, satisfying the following approximation properties.

Assumption 3. *Let $k \geq 1$ be the degree of Q_h . Then, we assume the following properties of the operator R_h : First, for any $s > 0$ and $\mathbf{v} \in \mathbf{V} \cap \mathbf{H}^s(\Omega)$ there holds*

$$(21) \quad (\nabla \cdot R_h(\mathbf{v}), q_h)_\Omega = (\nabla \cdot \mathbf{v}, q_h)_\Omega \quad \forall q_h \in Q_h,$$

Then, for any cell $T \in \mathbb{T}_h$ and function $\mathbf{v} \in \mathbf{H}^s(T)$ with $0 < s \leq k + 1$ there holds

$$(22) \quad \|R_h(\mathbf{v}) - \mathbf{v}\|_{\mathbf{L}^2(T)} \leq Ch_T^s |\mathbf{v}|_{\mathbf{H}^s(T)},$$

and for $0 \leq s \leq k$, $0 \leq t \leq s$, and $\mathbf{v} \in \mathbf{H}^{s+1}(T)$, we have

$$(23) \quad \|R_h(\mathbf{v}) - \mathbf{v}\|_{\mathbf{H}^t(T)} \leq Ch_T^{s+1-t} |\mathbf{v}|_{\mathbf{H}^{s+1}(T)},$$

$$(24) \quad \|\nabla \cdot (R_h(\mathbf{v}) - \mathbf{v})\|_{L^2(T)} \leq Ch_T^s |\nabla \cdot \mathbf{v}|_{\mathbf{H}^s(T)},$$

$$(25) \quad \|R_h(\mathbf{v}) - \mathbf{v}\|_{\mathbf{L}^2(F)} \leq Ch_T^{s+\frac{1}{2}} |\mathbf{v}|_{\mathbf{H}^{s+1}(T)},$$

where F is any face of T . All constants C are independent of h , T , or F .

Note that by definition, $R_h(\mathbf{v})$ is in \mathbf{V}_h ; therefore

$$(26) \quad R_h(\mathbf{v}) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

For example, Assumption 3 is satisfied by the Raviart-Thomas interpolant R_h , see e.g. [23, 27]. For Navier-Stokes equations, the BDM pair was suggested in [12], the Raviart-Thomas pair in [13] (see also [38]).

An immediate consequence of this assumption is that for $s \leq k$ and for any $\mathbf{v} \in \mathbf{H}^{s+1}(\Omega)$ there holds

$$(27) \quad \|R_h(\mathbf{v}) - \mathbf{v}\|_{1,h} \leq Ch^s |\mathbf{v}|_{\mathbf{H}^{s+1}(\Omega)}.$$

It yields stability in the \mathbf{V}_h norm over the whole region and this is sufficient to establish the following uniform inf-sup condition: There exists a constant $\beta^* > 0$, independent of h , such that

$$(28) \quad \forall q_h \in Q_h, \quad \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\mathbf{V}_h}} \geq \beta^* \|q_h\|_{L^2(\Omega)}.$$

This follows easily from (10), (21), (24) and (27) with $s = 0$.

2.2.2. *The discontinuous Galerkin bilinear form.* We use a discrete bilinear form $a_{S,h}(\cdot, \cdot)$ based on a discontinuous Galerkin method in the Stokes subdomain and formally define for all sufficiently smooth \mathbf{u}, \mathbf{v}

$$a_h(\mathbf{u}, \mathbf{v}) = a_D(\mathbf{u}, \mathbf{v}) + a_I(\mathbf{u}, \mathbf{v}) + a_{S,h}(\mathbf{u}, \mathbf{v}).$$

Then, for solution pairs $(\mathbf{u}, p) \in \mathbf{V} \times Q$ of (7) and $(\mathbf{u}^*, p^*) \in \mathbf{V} \times Q$ of (12), and for any test function $\mathbf{v} \in \mathbf{V} + \mathbf{V}_h$, we define the primal and dual residual operators

$$(29a) \quad \text{Res}(\mathbf{u}, p; \mathbf{v}) = a_h(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - (\mathbf{f}, \mathbf{v})_\Omega,$$

$$(29b) \quad \text{Res}^*(\mathbf{v}; \mathbf{u}^*, p^*) = a_h(\mathbf{v}, \mathbf{u}^*) + b(\mathbf{v}, p^*) - (\boldsymbol{\psi}, \mathbf{v})_{\Omega_S}.$$

This definition is motivated by the next lemma.

Lemma 2. *Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and let the solution (\mathbf{u}, p) of (2), (3) satisfy $p_D \in H^1(\Omega_D)$, $\mathbf{u} \in \mathbf{V}$, $\mathbf{u}_S \in \mathbf{H}^s(\Omega)$, and $p_S \in H^{s-1}(\Omega_S)$ for some real number $s > \frac{3}{2}$. Then we have for all $\mathbf{v} \in \mathbf{V} + \mathbf{V}_h$:*

$$(30) \quad (\mathbf{f}, \mathbf{v})_\Omega = a_h(\mathbf{u}, \mathbf{v}) - a_{S,h}(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + 2\nu \left((\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} - \langle \varepsilon(\mathbf{u})\mathbf{n}, [\mathbf{v}] \rangle_{\Gamma_h^S} \right).$$

Proof. We have

$$(\mathbf{f}, \mathbf{v})_\Omega = (-\nabla \cdot (2\nu \varepsilon(\mathbf{u})) + \nabla p, \mathbf{v})_{\Omega_S} + (K^{-1}\mathbf{u} + \nabla p, \mathbf{v})_{\Omega_D}.$$

The regularity of the functions \mathbf{f} , \mathbf{u}_D , and p_D , Green's formula, the value of \mathbf{v} on Γ_D and the orientation of the normal on Γ_{SD} easily give

$$(K^{-1}\mathbf{u} + \nabla p, \mathbf{v})_{\Omega_D} = (K^{-1}\mathbf{u}, \mathbf{v})_{\Omega_D} - (p, \nabla \cdot \mathbf{v})_{\Omega_D} - \langle p_D, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma_{SD}}.$$

In particular, as $\mathbf{v} \cdot \mathbf{n} = 0$ on Γ_D , the work of Galvis and Sarkis in [16] implies that the last duality makes sense. Next, the regularity assumptions on \mathbf{u}_S and p_S imply that the traces of both $\varepsilon(\mathbf{u}_S)$ and p_S on elements' faces belong to H^t with $t = s - \frac{3}{2} > 0$, hence are in L^2 . Then the regularity of \mathbf{f} and \mathbf{v} , Green's formula, and the value of \mathbf{v} on Γ_S yield

$$\begin{aligned} (-\nabla \cdot (2\nu \varepsilon(\mathbf{u})) + \nabla p, \mathbf{v})_{\Omega_S} &= 2\nu \left((\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} - \sum_{T \in \mathbb{T}_h^S} \langle \varepsilon(\mathbf{u})\mathbf{n}_T, \mathbf{v} \rangle_{\partial T} \right) \\ &\quad - (p, \nabla \cdot \mathbf{v})_{\Omega_S} + \langle p_S, \mathbf{v} \cdot \mathbf{n} \rangle_{\Gamma_{SD}}, \end{aligned}$$

where \mathbf{n}_T denotes the unit normal vector on ∂T , exterior to T . Again, the regularity assumption on \mathbf{u}_S imply that

$$\sum_{T \in \mathbb{T}_h^S} \langle \varepsilon(\mathbf{u})\mathbf{n}_T, \mathbf{v} \rangle_{\partial T} = \langle \varepsilon(\mathbf{u})\mathbf{n}, [\mathbf{v}] \rangle_{\Gamma_h^S} + \langle \varepsilon(\mathbf{u}_S)\mathbf{n}, \mathbf{v} \rangle_{\Gamma_{SD}}.$$

Collecting these two equalities, we obtain

$$\begin{aligned} (\mathbf{f}, \mathbf{v})_\Omega &= (K^{-1}\mathbf{u}, \mathbf{v})_{\Omega_D} - (p, \nabla \cdot \mathbf{v})_\Omega + 2\nu (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} \\ &\quad - 2\nu \langle \varepsilon(\mathbf{u})\mathbf{n}, [\mathbf{v}] \rangle_{\Gamma_h^S} + \langle -2\nu \varepsilon(\mathbf{u}_S)\mathbf{n} + (p_S - p_D)\mathbf{n}, \mathbf{v} \rangle_{\Gamma_{SD}}. \end{aligned}$$

Then the boundary conditions (3b) and (3c) on the interface Γ_{SD} give

$$\langle -2\nu \varepsilon(\mathbf{u}_S)\mathbf{n} + (p_S - p_D)\mathbf{n}, \mathbf{v} \rangle_{\Gamma_{SD}} = a_I(\mathbf{u}, \mathbf{v}),$$

and (30) follows immediately from the definition of $a_h(\cdot, \cdot)$. \square

From this lemma, we deduce immediately the following corollary.

Corollary 3. *Under the assumption of Lemma 2, the residuals can be expressed as:*

$$(31) \quad \text{Res}(\mathbf{u}, p; \mathbf{v}) = a_{S,h}(\mathbf{u}, \mathbf{v}) - 2\nu(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} + 2\nu\langle \varepsilon(\mathbf{u})\mathbf{n}, \llbracket \mathbf{v} \rrbracket \rangle_{\Gamma_h^S},$$

$$(32) \quad \text{Res}^*(\mathbf{v}; \mathbf{u}^*, p^*) = a_{S,h}(\mathbf{v}, \mathbf{u}^*) - 2\nu(\varepsilon(\mathbf{u}^*), \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} + 2\nu\langle \varepsilon(\mathbf{u}^*)\mathbf{n}, \llbracket \mathbf{v} \rrbracket \rangle_{\Gamma_h^S}.$$

Thus none of the residuals depend on p or p^* and from now on we denote them by $\text{Res}(\mathbf{u}; \mathbf{v})$ and $\text{Res}^*(\mathbf{v}; \mathbf{u}^*)$.

Now, instead of specifying $a_{S,h}(\cdot, \cdot)$, we make some generic assumptions.

Assumption 4. *The bilinear form $a_{S,h}(\mathbf{u}, \mathbf{v})$ of the DG method chosen in the Stokes sub-domain Ω_S has the following four properties:*

(a) **Stability in the DG norm:** *There exists a constant α^* independent of h , such that for any functions $\mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h$ there holds*

$$(33) \quad a_{S,h}(\mathbf{u}_h, \mathbf{u}_h) \geq \alpha^* \|\mathbf{u}_h\|_{1,h}^2.$$

(b) **Boundedness in the DG norm:** *There exists a constant M^* , independent of h , such that for any functions $\mathbf{u}, \mathbf{v} \in \mathbf{V} + \mathbf{V}_h$ there holds*

$$(34) \quad a_{S,h}(\mathbf{u}, \mathbf{v}) \leq M^* \|\mathbf{u}\|_{1,h} \|\mathbf{v}\|_{1,h}.$$

(c) **Consistency:** *With the assumptions and notation of Lemma 2 regarding the solution (\mathbf{u}, p) of (7), there exists a constant C , independent of h , such that for all $\mathbf{v} \in \mathbf{V} + \mathbf{V}_h$*

$$(35) \quad |\text{Res}(\mathbf{u}; \mathbf{v})| \leq Ch^{s-1} \|\mathbf{u}\|_{H^s(\Omega_S)} \|\sigma_h \llbracket \mathbf{v} \rrbracket\|_{\Gamma_h^S}.$$

(d) **Adjoint consistency:** *With the assumptions and notation of Lemma 2 regarding the solution (\mathbf{u}^*, p^*) of (12), there exists a constant C^* , independent of h , such that for all $\mathbf{v} \in \mathbf{V} + \mathbf{V}_h$*

$$(36) \quad |\text{Res}^*(\mathbf{v}; \mathbf{u}^*)| \leq C^* h^{s-1} \|\mathbf{u}^*\|_{H^s(\Omega_S)} \|\sigma_h \llbracket \mathbf{v} \rrbracket\|_{\Gamma_h^S}.$$

The assumptions imposed here follow those in [37], and the application to various DG methods is explained there. As an example, we reproduce the form of the interior penalty method for which (34) holds. To this end, we first introduce the lifting operator \mathcal{L} from $\mathbf{V} + \mathbf{V}_h$ into the space $\Sigma_h = \{\tau \in L^2(\Omega; \mathbb{R}^{d \times d}) \mid \forall T \in \mathbb{T}_h : \tau_{ij}|_T \in \mathcal{P}(T)\}$ of discontinuous $d \times d$ tensor functions with components in the space $\mathcal{P}(T)$ on each cell T . The space $\mathcal{P}(T)$ is a possibly mapped polynomial space on T such that $\varepsilon(v_T) \in \mathcal{P}(T)$ for all v_T in the discrete velocity space on T .

$$(\mathcal{L}\mathbf{v}, \tau)_{\mathbb{T}_h^S} = \langle \llbracket \mathbf{v} \rrbracket, \{\{\tau \mathbf{n}\}\} \rangle_{\Gamma_h^S} \quad \forall \tau \in \Sigma_h,$$

where $\{\{\tau \mathbf{n}\}\}$ denotes the pointwise mean value of $\tau \mathbf{n}$ on the two adjacent cells, or the value of $\tau \mathbf{n}$ itself on a boundary face and where $\mathbf{n}|_F = \mathbf{n}_F$. Then, the symmetric interior penalty discretization of the Stokes problem is

$$(37) \quad a_{S,h}(\mathbf{u}, \mathbf{v}) = 2\nu \left((\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} - (\mathcal{L}\mathbf{u}, \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} - (\varepsilon(\mathbf{u}), \mathcal{L}\mathbf{v})_{\mathbb{T}_h^S} + \langle \sigma_h^2 \llbracket \mathbf{u} \rrbracket, \llbracket \mathbf{v} \rrbracket \rangle_{\Gamma_h^S} \right),$$

which, restricted to the discrete space \mathbf{V}_h , is equivalent to the usual form

$$a_{S,h}(\mathbf{u}, \mathbf{v}) = 2\nu \left((\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} + \langle \sigma_h^2[\mathbf{u}], [\mathbf{v}] \rangle_{\Gamma_h^S} - \langle [\mathbf{u}], \{\{\varepsilon(\mathbf{v})\mathbf{n}\}\} \rangle_{\Gamma_h^S} - \langle \{\{\varepsilon(\mathbf{u})\mathbf{n}\}\}, [\mathbf{v}] \rangle_{\Gamma_h^S} \right).$$

Following [37], the lifting version of the LDG discretization (see [14]) is obtained as

$$(38) \quad a_{S,h}(\mathbf{u}, \mathbf{v}) = 2\nu \left((\varepsilon(\mathbf{u}) - \mathcal{L}\mathbf{u}, \varepsilon(\mathbf{v}) - \mathcal{L}\mathbf{v})_{\mathbb{T}_h^S} + \langle \sigma_h^2[\mathbf{u}], [\mathbf{v}] \rangle_{\Gamma_h^S} \right).$$

Since it is shown in [37] that the operator \mathcal{L} is bounded on $\mathbf{V} + \mathbf{V}_h$ with respect to the norm $\|\cdot\|_{\mathbf{V}_h}$, we obtain (34). By applying Korn's inequality [8] we derive also (33). In the case of the symmetric interior penalty method, the penalty parameter is assumed to be sufficiently large.

While the lifting form of the methods enables us to use boundedness and ellipticity estimates in the natural norm $\|\cdot\|_{1,h}$, it is not consistent. Nevertheless, the two conditions (35) and (36) which are a refinement of [37, Proposition 8.1] hold, as the following lemma shows. Note also that while we do not require symmetry of $a_{S,h}(\cdot, \cdot)$, it implies the equivalence of (35) and (36).

Lemma 4. *The primal and dual residuals of the discrete bilinear forms in equations (37) and (38) satisfy the estimates with constants C independent of h*

$$\begin{aligned} |\text{Res}(\mathbf{u}; \mathbf{v})| &\leq Ch^{s-1} |\mathbf{u}|_{H^s(\Omega_S)} \|\sigma_h[\mathbf{v}]\|_{\Gamma_h^S} \\ |\text{Res}^*(\mathbf{u}; \mathbf{v})| &\leq Ch^{s-1} |\mathbf{u}|_{H^s(\Omega_S)} \|\sigma_h[\mathbf{v}]\|_{\Gamma_h^S}, \end{aligned}$$

under the regularity assumptions of Lemma 2.

Proof. First, we note that both forms (37) and (38) are symmetric, and thus it suffices to show only one of the inequalities. Next, we observe that the jump $[\mathbf{u}] = \mathbf{0}$ and thus $\mathcal{L}\mathbf{u} = 0$. This implies for both forms

$$a_{S,h}(\mathbf{u}, \mathbf{v}) = 2\nu \left((\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathbb{T}_h^S} - (\varepsilon(\mathbf{u}), \mathcal{L}\mathbf{v})_{\mathbb{T}_h^S} \right).$$

Hence (31) and the regularity of \mathbf{u} give

$$\text{Res}(\mathbf{u}; \mathbf{v}) = 2\nu \left(\langle \varepsilon(\mathbf{u})\mathbf{n}, [\mathbf{v}] \rangle_{\Gamma_h^S} - (\varepsilon(\mathbf{u}), \mathcal{L}\mathbf{v})_{\mathbb{T}_h^S} \right) = 2\nu \left(\langle \{\{\varepsilon(\mathbf{u})\mathbf{n}\}\}, [\mathbf{v}] \rangle_{\Gamma_h^S} - (\varepsilon(\mathbf{u}), \mathcal{L}\mathbf{v})_{\mathbb{T}_h^S} \right).$$

Since $\mathcal{L}\mathbf{v}$ is in Σ_h , we may insert the L^2 projection Π_{Σ_h} into Σ_h in the term on the right. Then, using the definition of the lifting operator, we obtain

$$\text{Res}(\mathbf{u}; \mathbf{v}) = 2\nu \left\langle \frac{1}{\sigma_h} \{\{\varepsilon(\mathbf{u}) - \Pi_{\Sigma_h} \varepsilon(\mathbf{u})\}\}, \sigma_h[\mathbf{v}] \right\rangle_{\Gamma_h^S}.$$

Applying a trace inequality on the reference element and standard approximation results, we obtain

$$|\text{Res}(\mathbf{u}; \mathbf{v})| \leq Ch^{s-1} |\mathbf{u}|_{H^s(\Omega)} \|\sigma_h[\mathbf{v}]\|_{\Gamma_h^S}.$$

□

The discrete formulation of the problem (2)–(3) then reads: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$(39) \quad \begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h)_\Omega, & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= -(g, q_h)_\Omega, & \forall q_h \in Q_h. \end{aligned}$$

Under the above assumptions, it is easy to prove that (39) has exactly one solution [28].

3. ERROR ANALYSIS

The error analysis is conducted in several steps. First, we recall the energy norm estimate from [28]. The problem with this estimate is that it is suboptimal in the Darcy subdomain Ω_D , and that it is not possible to improve it there by a simple duality argument. Nevertheless, such a duality argument is applied in the Stokes subdomain Ω_S in subsection 3.2. In conjunction with the divergence approximation of the Raviart-Thomas element, this yields a sharp estimate on the interface Γ_{SD} in subsection 3.3, which in turn serves as an input to the optimal estimate for the Darcy velocity in subsection 3.4. The outcome of these steps will be our main result:

Theorem 5. *Let Assumptions 1–4 hold. Assume further, that the solution \mathbf{u} of the weak formulation (7) is in $\mathbf{H}^s(\Omega)$ for some $s \in]\frac{3}{2}, k+1]$ with $k \geq 1$ from Assumption 3 and $\alpha \in]\frac{1}{2}, 1]$ determined by Assumption 1. Suppose g from (2c) belongs to $H^{s-1+\alpha}(\Omega)$. Then, the solution $\mathbf{u}_h \in \mathbf{V}_h$ of (39) satisfies the L^2 error estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} \leq Ch^{s+\alpha-1} (|\mathbf{u}|_{\mathbf{H}^s(\Omega)} + |g|_{H^{s-1+\alpha}(\Omega)}),$$

with a constant C independent of h and \mathbf{u} . A direct corollary is the optimal rate h^s in the case where the dual problem has full regularity ($\alpha = 1$).

Proof. The proof is an immediate consequence of Theorems 9 and 12 below. \square

In [28, Table 3], convergence rates in $L^2(\Omega)$ had already been presented. There, we obtained for a very regular solution convergence rates h^{k+1} for u_S and u_D for $k \leq 3$; for higher order elements, the corner singularities were too strong. We refer to this publication for details.

3.1. First error analysis. Using standard stability and approximation arguments, we easily establish the following variant of a result that was proved in [28]. Note that $k \geq 1$ is required, since the strong norm in the Stokes subdomain does not allow for convergence of the lowest order Raviart-Thomas element.

Proposition 6. *Let assumptions 2–4 hold. Assume further, that the solution \mathbf{u} of the weak formulation (7) is in $\mathbf{H}^s(\Omega)$ for some $s \in]\frac{3}{2}, k+1]$. Then, the solutions \mathbf{u} and \mathbf{u}_h of the coupled systems (7) and (39) have the energy error estimate*

$$(40) \quad \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}_h} \leq Ch^{s-1} |\mathbf{u}|_{\mathbf{H}^s(\Omega)},$$

with a constant C independent of h and \mathbf{u} .

Proof. We briefly recall the proof for the readers' convenience. From (7), (29a), and (39), we have:

$$(41) \quad \begin{aligned} a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_h) &= \text{Res}(\mathbf{u}; \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u} - \mathbf{u}_h, q_h) &= 0, & \forall q_h \in Q_h. \end{aligned}$$

Inserting $R_h(\mathbf{u})$, this gives

$$(42) \quad \begin{aligned} a_h(R_h(\mathbf{u}) - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p - p_h) &= \text{Res}(\mathbf{u}; \mathbf{v}_h) - a_h(\mathbf{u} - R_h(\mathbf{u}), \mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(R_h(\mathbf{u}) - \mathbf{u}_h, q_h) &= 0, & \forall q_h \in Q_h. \end{aligned}$$

Owing to (21), this last equation is due to the fact that $b(\mathbf{u} - R_h(\mathbf{u}), q_h) = 0$ for all $q_h \in Q_h$; according to (17), it implies that $\nabla \cdot (\mathbf{u}_h - R_h(\mathbf{u})) = 0$. Then (40) easily follows by choosing $\mathbf{v}_h = R_h(\mathbf{u}) - \mathbf{u}_h \in \overset{\circ}{\mathbf{V}}_h$ in (42) and using (33), (34), (35), and the approximation properties of R_h (22)–(25). \square

Note that (17) and (41) yield, if we denote by Π_{Q_h} the L^2 projection onto Q_h :

Proposition 7. *The divergence of the discrete solution \mathbf{u}_h is the L^2 projection of the divergence of \mathbf{u} onto the discrete pressure space Q_h , or $\nabla \cdot \mathbf{u}_h = \Pi_{Q_h} \nabla \cdot \mathbf{u} = \Pi_{Q_h} g$.*

This implies the following result (see for instance [10]).

Corollary 8. *Let \mathbf{u} and \mathbf{u}_h be the solutions to (7) and (39), respectively. Suppose g from (2c) belongs to $H^s(\Omega)$ with $s \in [0, k+1]$; then*

$$(43) \quad \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)} \leq Ch^s |g|_{H^s(\Omega)}.$$

3.2. Sharper error estimate in the Stokes subdomain. The energy norm in the Stokes subdomain Ω_S involves the derivatives of \mathbf{u}_S . Thus, in this subdomain we can employ a duality argument of Aubin and Nitsche to obtain the following theorem:

Theorem 9. *Let Assumptions 1–4 hold with the constant $\alpha \in [\frac{1}{2}, 1]$ determined by Assumption 1. Let $\mathbf{u} \in \mathbf{H}^s(\Omega)$ for $s \in [\frac{3}{2}, k+1]$ be the solution of (7) and let \mathbf{u}_h be the solution to (39). Then, the following error estimate holds*

$$(44) \quad \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega_S)} \leq Ch^{s-1+\alpha} |\mathbf{u}|_{H^s(\Omega)}.$$

Proof. Let (\mathbf{u}^*, p^*) be a solution of (12) with $\psi = \chi(\Omega_S)(\mathbf{u} - \mathbf{u}_h)$, where $\chi(\Omega_S)$ is the indicator function of Ω_S . It follows from (29b) with (\mathbf{u}^*, p^*) and $\mathbf{f} = \chi(\Omega_S)(\mathbf{u} - \mathbf{u}_h)$ that for all $\mathbf{v} \in \mathbf{V} + \mathbf{V}_h$

$$(\mathbf{v}, \mathbf{u} - \mathbf{u}_h)_{\Omega_S} = a_h(\mathbf{v}, \mathbf{u}^*) + b(\mathbf{v}, p^*) - \text{Res}^*(\mathbf{v}; \mathbf{u}^*).$$

On one hand, the choice $\mathbf{v} = \mathbf{u} - \mathbf{u}_h$ gives

$$(45) \quad \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega_S)}^2 = a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}^*) + b(\mathbf{u} - \mathbf{u}_h, p^*) - \text{Res}^*(\mathbf{u} - \mathbf{u}_h; \mathbf{u}^*).$$

On the other hand, the error equation (41) with $\mathbf{v}_h = R_h \mathbf{u}^* \in \mathbf{V}_h$ implies

$$a_h(\mathbf{u} - \mathbf{u}_h, R_h \mathbf{u}^*) + b(R_h \mathbf{u}^*, p - p_h) = \text{Res}(\mathbf{u}; R_h \mathbf{u}^*).$$

But $b(R_h \mathbf{u}^*, p - p_h) = 0$ because $\nabla \cdot R_h \mathbf{u}^* = 0$; this follows from the fact that $\nabla \cdot \mathbf{u}^* = 0$, Assumption 2, and property (21) of R_h . Therefore combining with (45), we obtain for all $q_h \in Q_h$

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega_S)}^2 &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}^* - R_h \mathbf{u}^*) + b(\mathbf{u} - \mathbf{u}_h, p^* - q_h) - \text{Res}^*(\mathbf{u} - \mathbf{u}_h; \mathbf{u}^*) \\ &\quad + \text{Res}(\mathbf{u}; R_h \mathbf{u}^*). \end{aligned}$$

This yields

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega_S)}^2 &\leq |a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}^* - R_h \mathbf{u}^*)| + |b(\mathbf{u} - \mathbf{u}_h, p^* - q_h)| \\ &\quad + |\text{Res}^*(\mathbf{u} - \mathbf{u}_h; \mathbf{u}^*)| + |\text{Res}(\mathbf{u}; R_h \mathbf{u}^*)|. \end{aligned}$$

The first term on the right-hand side is bounded as follows:

$$\begin{aligned} a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}^* - R_h(\mathbf{u}^*)) &= a_{S,h}(\mathbf{u}_S - \mathbf{u}_{Sh}, \mathbf{u}_S^* - R_h(\mathbf{u}_S^*)) \\ &\quad + a_D(\mathbf{u}_D - \mathbf{u}_{Dh}, \mathbf{u}_D^* - R_h(\mathbf{u}_D^*)) + a_I(\mathbf{u}_S - \mathbf{u}_{Sh}, \mathbf{u}_S^* - R_h(\mathbf{u}_S^*)) \\ &\leq Ch^\alpha |\mathbf{u}_S^*|_{H^{1+\alpha}(\Omega_S)} \|\mathbf{u}_S - \mathbf{u}_{Sh}\|_{1,h} + Ch^\alpha |\mathbf{u}_D^*|_{H^\alpha(\Omega_D)} \|K^{-\frac{1}{2}}(\mathbf{u}_D - \mathbf{u}_{Dh})\|_{\mathbf{L}^2(\Omega_D)} \\ &\quad + Ch^{\alpha+\frac{1}{2}} |\mathbf{u}_S^*|_{H^{1+\alpha}(\Omega_S)} \left(\sum_{j=1}^{d-1} \|\gamma K^{-\frac{1}{4}}(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \boldsymbol{\tau}^j\|_{L^2(\Gamma_{SD})} \right)^{\frac{1}{2}}, \end{aligned}$$

from the approximation properties (22), (25), and (27) of R_h . For the second term, we pick $q_h = \Pi_{Q_h} p^*$:

$$\begin{aligned} b(\mathbf{u} - \mathbf{u}_h, p^* - \Pi_{Q_h} p^*) &= -(\nabla \cdot (\mathbf{u}_S - \mathbf{u}_{Sh}), p_S^* - \Pi_{Q_h} p_S^*)_{\Omega_S} \\ &\quad - (\nabla \cdot (\mathbf{u}_D - \mathbf{u}_{Dh}), p_D^* - \Pi_{Q_h} p_D^*)_{\Omega_D}. \end{aligned}$$

From the approximation properties of Q_h , we readily derive:

$$|b(\mathbf{u} - \mathbf{u}_h, p^* - \Pi_{Q_h} p^*)| \leq Ch^\alpha \|\nabla \cdot (\mathbf{u}_S - \mathbf{u}_{Sh})\|_{L^2(\Omega)} (|p^*|_{H^\alpha(\Omega_S)} + |p^*|_{H^\alpha(\Omega_D)}).$$

The adjoint residual term $|\text{Res}^*(\mathbf{u} - \mathbf{u}_h; \mathbf{u}^*)|$ can be estimated by the adjoint consistency estimate (36):

$$|\text{Res}^*(\mathbf{u}^*; \mathbf{u} - \mathbf{u}_h)| \leq Ch^\alpha |\mathbf{u}^*|_{H^{1+\alpha}(\Omega_S)} \|\sigma_h[\mathbf{u} - \mathbf{u}_h]\|_{\Gamma_h^S}.$$

For the primal residual, in view of the regularity of \mathbf{u}^* and its boundary value on Γ_h^S , we observe that

$$\begin{aligned} |\text{Res}(\mathbf{u}; R_h(\mathbf{u}^*))| &\leq Ch^{s-1} |\mathbf{u}|_{H^s(\Omega_S)} \|\sigma_h[u^* - R_h(\mathbf{u}^*)]\|_{\Gamma_h^S} \\ &\leq Ch^{s+\alpha-1} |\mathbf{u}|_{H^s(\Omega_S)} |\mathbf{u}^*|_{H^{1+\alpha}(\Omega_S)}. \end{aligned}$$

Finally, a combination of the bounds above gives:

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega_S)} &\leq Ch^\alpha \left(\|\mathbf{u}_S - \mathbf{u}_{Sh}\|_{1,h} + \|\mathbf{u}_D - \mathbf{u}_{Dh}\|_{\mathbf{L}^2(\Omega_D)} \right. \\ &\quad \left. + h^{\frac{1}{2}} \left(\sum_{j=1}^{d-1} \|\gamma K^{-\frac{1}{4}} (\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \boldsymbol{\tau}^j\|_{L^2(\Gamma_{SD})}^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{L^2(\Omega)} + h^{s-1} |\mathbf{u}|_{H^s(\Omega_S)} \right). \end{aligned}$$

We now conclude by using (40). \square

3.3. Estimates on the interface. Since $\mathbf{u}_S - \mathbf{u}_{Sh}$ belongs to $\mathbf{H}^{\text{div}}(\Omega_S)$, its normal trace satisfies (cf. [19])

$$\|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\partial\Omega_S)} \leq C \|\mathbf{u}_S - \mathbf{u}_{Sh}\|_{\mathbf{H}^{\text{div}}(\Omega_S)}.$$

From the properties of \mathbf{V} and \mathbf{V}_h , we have

$$(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n} = 0, \quad \text{on } \Gamma_S.$$

Therefore, following the approach of Galvis and Sarkis in [16], we have

$$\|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_{SD})} \leq C \|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\partial\Omega_S)},$$

and hence

$$\|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_{SD})} \leq C \|\mathbf{u}_S - \mathbf{u}_{Sh}\|_{\mathbf{H}^{\text{div}}(\Omega_S)}.$$

On the other hand, recall the interpolation trace inequality (see for instance Brenner and Scott [9]):

$$(46) \quad \forall v \in H^1(\Omega_S), \quad \|v\|_{L^2(\partial\Omega_S)} \leq C \|v\|_{L^2(\Omega_S)}^{\frac{1}{2}} \|v\|_{H^1(\Omega_S)}^{\frac{1}{2}}.$$

For lack of regularity, it cannot be applied to \mathbf{u}_{Sh} . Nevertheless, in the appendix, we establish that

$$(47) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad \|\mathbf{v}_h\|_{\mathbf{L}^2(\Gamma_{SD})} \leq C \left(\|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega_S)} + h^{\frac{1}{2}} (\|\nabla \mathbf{v}_h\|_{\mathbb{T}_h^S}^2 + \|\sigma_h[\mathbf{v}_h]\|_{\Gamma_h^S}^2)^{\frac{1}{2}} \right. \\ \left. + \|\mathbf{v}_h\|_{\mathbf{L}^2(\Omega_S)}^{\frac{1}{2}} (\|\nabla \mathbf{v}_h\|_{\mathbb{T}_h^S}^2 + \|\sigma_h[\mathbf{v}_h]\|_{\Gamma_h^S}^2)^{\frac{1}{4}} \right),$$

with a constant C independent of h . Then we write

$$\mathbf{u}_S - \mathbf{u}_{Sh} = \mathbf{u}_S - R_h(\mathbf{u}_S) + R_h(\mathbf{u}_S) - \mathbf{u}_{Sh}.$$

We first apply (47) to $R_h(\mathbf{u}_S) - \mathbf{u}_{Sh}$:

$$\|R_h(\mathbf{u}_S) - \mathbf{u}_{Sh}\|_{L^2(\Gamma_{SD})} \leq Ch^{s-1+\frac{\alpha}{2}} |\mathbf{u}|_{H^s(\Omega)}.$$

Given the improved accuracy in the Stokes subdomain, we can now estimate the normal trace on the interface and obtain:

Lemma 10. *Let the assumptions of Theorem 9 hold. Then, the normal trace of $\mathbf{u}_S - \mathbf{u}_{Sh}$ on Γ_{SD} satisfies*

$$(48) \quad \|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{L^2(\Gamma_{SD})} \leq Ch^{s-1+\frac{\alpha}{2}} |\mathbf{u}|_{\mathbf{H}^s(\Omega)}.$$

Moreover, if g in (2c) belongs to $H^{s-1+\alpha}(\Omega)$, we have

$$(49) \quad \|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{H^{-\frac{1}{2}}(\Gamma_{SD})} \leq Ch^{s-1+\alpha} (|\mathbf{u}|_{\mathbf{H}^s(\Omega)} + |g|_{H^{s-1+\alpha}(\Omega)}).$$

We skip the proof that now follows easily from (40), (44), (43), and (47).

3.4. L^2 error in the Darcy subdomain. Now, that we have estimated the error in Ω_S and on Γ_{SD} , we are going to leave the view of a monolithic, coupled problem and consider the Darcy subproblem alone. To this end, we introduce the auxiliary function spaces $\mathbf{V}^D = \mathbf{H}_0^{\text{div}}(\Omega_D)$ as well as the corresponding Raviart-Thomas space $\mathbf{V}_h^D \subset \mathbf{V}^D$. We remark that both spaces can be extended by zero to elements in \mathbf{V} and \mathbf{V}_h , respectively. Additionally, we define Q^D and Q_h^D (resp. Q^S and Q_h^S) as the restrictions of Q and Q_h to Ω_D (resp. Ω_S). As explained in Section 2.1, we can relax the zero mean value constraint on the test functions of Q and Q_h , and hence also on the test functions of Q^S and Q_h^S , or Q^D and Q_h^D .

As a first step towards estimating the Darcy error, we consider the following

Lemma 11. *There is a function $\mathbf{w} \in \mathbf{H}^{\text{div}}(\Omega_D)$ satisfying $\nabla \cdot \mathbf{w} = 0$ in Ω_D , $\mathbf{w} \cdot \mathbf{n} = 0$ on Γ_D and $\mathbf{w} \cdot \mathbf{n} = (\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}$ on Γ_{SD} , such that*

$$(50) \quad \|\mathbf{w}\|_{\mathbf{L}^2(\Omega_D)} \leq C \|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{H^{-\frac{1}{2}}(\Gamma_{SD})},$$

$$(51) \quad \|\mathbf{w}\|_{\mathbf{H}^{\frac{1}{2}}(\Omega_D)} \leq C \|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{L^2(\Gamma_{SD})},$$

with a constant C independent of h .

Proof. Let $\varphi \in H^1(\Omega_D)/\mathbb{R}$ be the unique variational solution to the problem

$$(52) \quad -\Delta\varphi = 0, \quad \text{in } \Omega_D,$$

$$(53) \quad \frac{\partial\varphi}{\partial n} = 0, \quad \text{on } \Gamma_D,$$

$$(54) \quad \frac{\partial\varphi}{\partial n} = (\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}, \quad \text{on } \Gamma_{SD},$$

where exceptionally the normal derivative of φ is defined with the normal vector pointing inside Ω_D , so as to coincide with \mathbf{n} on Γ_{SD} . This problem has a unique solution $\varphi \in H^1(\Omega_D)/\mathbb{R}$ because, as

$$\forall q_h \in Q_h^S, (\nabla \cdot (\mathbf{u}_S - \mathbf{u}_{Sh}), q_h)_{\Omega_S} = 0,$$

we have in particular,

$$(\nabla \cdot (\mathbf{u}_S - \mathbf{u}_{Sh}), 1)_{\Omega_S} = 0.$$

Then the boundary conditions on \mathbf{V} and \mathbf{V}_h imply that

$$\int_{\Gamma_{SD}} (\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n} = 0.$$

Thus

$$|\varphi|_{H^1(\Omega_D)} \leq C \|\nabla \varphi \cdot \mathbf{n}\|_{H^{-\frac{1}{2}}(\partial\Omega_D)} \leq C \|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{H^{-\frac{1}{2}}(\Gamma_{SD})}.$$

Moreover, with boundary data in $L^2(\partial\Omega_D)$, we have $\varphi \in H^{\frac{3}{2}}(\Omega_D)$ and we have the following bound [26]

$$|\varphi|_{H^{\frac{3}{2}}(\Omega_D)} \leq C \|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{L^2(\Gamma_{SD})}.$$

To conclude, it suffices to take $\mathbf{w} = \nabla \varphi$. \square

With this lemma and the estimate for the trace in the previous subsection, we are ready to prove the main result for the Darcy subregion.

Theorem 12. *Let Assumptions 1–4 hold with the constant $\alpha \in]\frac{1}{2}, 1]$ determined by Assumption 1. Let $\mathbf{u} \in \mathbf{H}^s(\Omega)$ for $s \in]\frac{3}{2}, k+1]$ be the solution of (7) and let \mathbf{u}_h be the solution to (39). Moreover, let $g \in H^{s-1+\alpha}(\Omega)$ in (2c). Then we have the estimate for the error in the Darcy subdomain:*

$$(55) \quad \|\mathbf{u}_D - \mathbf{u}_{Dh}\|_{L^2(\Omega_D)} \leq Ch^s (|\mathbf{u}|_{H^{s-1+\alpha}(\Omega)} + |g|_{H^{s-1+\alpha}(\Omega)}).$$

Proof. We observe that \mathbf{u}_D solves the problem

$$(56) \quad a_D(\mathbf{u}_D, \mathbf{v}) + b(\mathbf{v}, p_D) = (\mathbf{f}, \mathbf{v})_{\Omega_D}, \quad \forall \mathbf{v} \in \mathbf{V}^D,$$

$$(57) \quad b(\mathbf{u}_D, q) = -(g, q)_{\Omega_D}, \quad \forall q \in Q^D,$$

$$(58) \quad \mathbf{u}_D \cdot \mathbf{n} = 0, \quad \text{on } \Gamma_D,$$

$$(59) \quad \mathbf{u}_D \cdot \mathbf{n} = \mathbf{u}_S \cdot \mathbf{n}, \quad \text{on } \Gamma_{SD}.$$

Similarly, \mathbf{u}_{Dh} solves

$$(60) \quad a_D(\mathbf{u}_{Dh}, \mathbf{v}) + b(\mathbf{v}, p_{Dh}) = (\mathbf{f}, \mathbf{v})_{\Omega_D}, \quad \forall \mathbf{v} \in \mathbf{V}_h^D,$$

$$(61) \quad b(\mathbf{u}_{Dh}, q) = -(g, q)_{\Omega_D}, \quad \forall q \in Q_h^D,$$

$$(62) \quad \mathbf{u}_{Dh} \cdot \mathbf{n} = 0, \quad \text{on } \Gamma_D,$$

$$(63) \quad \mathbf{u}_{Dh} \cdot \mathbf{n} = \mathbf{u}_{Sh} \cdot \mathbf{n}, \quad \text{on } \Gamma_{SD}.$$

Subtracting (56) to (60) and adding and subtracting the interpolant $R_h(\mathbf{u}_D - \mathbf{w})$, where \mathbf{w} is defined in Lemma 11, yields the error equation for all $\mathbf{v}_h \in \mathbf{V}_h^D$:

$$\begin{aligned} a_D(\mathbf{u}_{Dh} - R_h(\mathbf{u}_D) + R_h(\mathbf{w}), \mathbf{v}_h) + a_D(R_h(\mathbf{u}_D) - \mathbf{u}_D, \mathbf{v}_h) \\ - a_D(R_h(\mathbf{w}), \mathbf{v}_h) + b(\mathbf{v}_h, p_{Dh} - p_D) = 0. \end{aligned}$$

We choose $\mathbf{v}_h = \mathbf{u}_{Dh} - R_h(\mathbf{u}_D) + R_h(\mathbf{w})$ and we note that $\mathbf{v}_h \in \mathbf{V}_h^D$. Indeed, we have

$$(\mathbf{u}_{Dh} - \mathbf{u}_D + \mathbf{w}) \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega_D,$$

and with (26), this implies

$$\mathbf{v}_h \cdot \mathbf{n} = R_h(\mathbf{u}_{Dh} - \mathbf{u}_D + \mathbf{w}) \cdot \mathbf{n} = 0.$$

From (21), we have

$$\nabla \cdot \mathbf{v}_h = \nabla \cdot (\mathbf{u}_{Dh} - R_h(\mathbf{u}_D)) + \nabla \cdot R_h(\mathbf{w}) = 0.$$

This implies $b(\mathbf{v}_h, p_{Dh} - p_D) = 0$. We then have, with a constant C that depends on the permeability tensor of the porous media only:

$$\|\mathbf{u}_{Dh} - R_h(\mathbf{u}_D) + R_h(\mathbf{w})\|_{L^2(\Omega_D)} \leq C(\|R_h(\mathbf{u}_D) - \mathbf{u}_D\|_{L^2(\Omega_D)} + \|R_h(\mathbf{w})\|_{L^2(\Omega_D)}).$$

Using the triangle inequality and property (22), we have

$$\begin{aligned} \|R_h(\mathbf{w})\|_{L^2(\Omega_D)} &\leq \|R_h(\mathbf{w}) - \mathbf{w}\|_{L^2(\Omega_D)} + \|\mathbf{w}\|_{L^2(\Omega_D)} \\ &\leq Ch^{\frac{1}{2}}\|\mathbf{w}\|_{H^{\frac{1}{2}}(\Omega_D)} + \|\mathbf{w}\|_{L^2(\Omega_D)}. \end{aligned}$$

From (50) and (51), we obtain

$$\|R_h(\mathbf{w})\|_{L^2(\Omega_D)} \leq Ch^{\frac{1}{2}}\|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{L^2(\Gamma_{SD})} + C\|(\mathbf{u}_S - \mathbf{u}_{Sh}) \cdot \mathbf{n}\|_{H^{-\frac{1}{2}}(\Gamma_{SD})}.$$

We can then conclude with Lemma 10 and with property (22) of the Raviart-Thomas interpolant. \square

APPENDIX

This section is devoted to the proof of the discrete interpolation trace inequality (47) with a constant C independent of h . More generally, the result is valid for any space of piecewise polynomial functions with degree in a fixed range, say

$$X_h = \{v_h \in L^2(\Omega_S) \mid \forall T \in \mathbb{T}_h^S, v_h|_T \in \mathbb{P}_k\},$$

where k may vary from one element to the next, but stays within a fixed range, such as $k_m \leq k \leq k_M$. To simplify, we introduce the broken seminorm on X_h :

$$|v_h|_{1,h} = \left(\|\nabla v_h\|_{\mathbb{T}_h^S}^2 + \|\sigma_h[v_h]\|_{\Gamma_h^S}^2 \right)^{\frac{1}{2}},$$

and we shall prove that for all $v_h \in X_h$

$$\|v_h\|_{L^2(\Gamma_{SD})} \leq C \left(\|v_h\|_{L^2(\Omega_S)} + h^{\frac{1}{2}}|v_h|_{1,h} + \|v_h\|_{L^2(\Omega_S)}^{\frac{1}{2}}|v_h|_{1,h}^{\frac{1}{2}} \right).$$

Our argument is based on an $H^1(\Omega_S)$ regularization of v_h , following ideas of [6, 21, 22].

Let \mathcal{P}_h denote the set of vertices of $\mathbb{T}_h^S \cap \bar{\Omega}_S$. We associate with each vertex \mathbf{a} of \mathcal{P}_h an element $T_{\mathbf{a}}$ of \mathbb{T}_h^S that has vertex \mathbf{a} (repetitions are allowed), and set

$$(64) \quad E_h(v_h)(\mathbf{a}) = v_h|_{T_{\mathbf{a}}}(\mathbf{a}),$$

$$(65) \quad \forall \mathbf{x} \in \bar{\Omega}_S, E_h(v_h)(\mathbf{x}) = \sum_{\mathbf{a} \in \mathcal{P}_h} E_h(v_h)(\mathbf{a})\varphi_{\mathbf{a}}(\mathbf{x}),$$

where $\varphi_{\mathbf{a}}$ is the standard, globally continuous basis function, that is \mathbb{P}_1 in each T , and takes the value 1 at \mathbf{a} and zero at all other vertices of \mathcal{P}_h . As $E_h(v_h)$ belongs to $H^1(\Omega_S)$, it satisfies (46) with Γ_{SD} instead of $\partial\Omega_S$:

$$\|E_h(v_h)\|_{L^2(\Gamma_{SD})} \leq C\|E_h(v_h)\|_{L^2(\Omega_S)}^{\frac{1}{2}}\|E_h(v_h)\|_{H^1(\Omega_S)}^{\frac{1}{2}},$$

with a constant C that only depends on Ω_S and Γ_{SD} . By inserting $E_h(v_h)$ and expanding the H^1 norm, this becomes

$$(66) \quad \begin{aligned} \|v_h\|_{L^2(\Gamma_{SD})} &\leq \|v_h - E_h(v_h)\|_{L^2(\Gamma_{SD})} \\ &+ C \left(\|E_h(v_h)\|_{L^2(\Omega_S)} + \|E_h(v_h)\|_{L^2(\Omega_S)}^{\frac{1}{2}}\|\nabla E_h(v_h)\|_{L^2(\Omega_S)}^{\frac{1}{2}} \right). \end{aligned}$$

We must evaluate the terms in the right-hand side of (66). To estimate $\nabla E_h(v_h)$, we use the following auxiliary lemma; to be specific, we treat the case $d = 3$. The two-dimensional case is similar.

Lemma 13. *For $\ell = 1$ or $\ell = -1$, let \hat{T}_ℓ denote the two adjacent reference elements with vertices $\hat{\mathbf{a}}_0 = (0, 0, 0)$, $\hat{\mathbf{a}}_2 = (0, 1, 0)$, $\hat{\mathbf{a}}_3 = (0, 0, 1)$, and $\hat{\mathbf{a}}_\ell = (\ell, 0, 0)$, let \hat{F} denote their common face and let \hat{v} be such that it is a polynomial of degree k_ℓ in \hat{T}_ℓ . There exists a constant \hat{C} , only depending on k_ℓ and the reference face \hat{F} such that*

$$(67) \quad |\hat{v}|_{\hat{T}_1}(\hat{\mathbf{a}}_0) - \hat{v}|_{\hat{T}_{-1}}(\hat{\mathbf{a}}_0)| \leq \hat{C} \|\hat{v}\|_{\hat{F}} \|L^2(\hat{F})\|.$$

Proof. Let $\hat{w} = (\hat{v}|_{\hat{T}_1} - \hat{v}|_{\hat{T}_{-1}})|_{\hat{F}}$. Then

$$|\hat{w}(\hat{\mathbf{a}}_0)| \leq \|\hat{w}\|_{L^\infty(\hat{F})} = \|[\hat{v}]_{\hat{F}}\|_{L^\infty(\hat{F})},$$

and (67) follows from the fact that $[\hat{v}]_{\hat{F}}$ is a polynomial of degree $\max(k_\ell)$ and all norms are equivalent in finite dimensional spaces. \square

Lemma 14. *There exists a constant C , independent of h , such that*

$$(68) \quad \forall v_h \in X_h, \|\nabla E_h(v_h)\|_{L^2(\Omega_S)} \leq C |v_h|_{1,h}.$$

Proof. Let T be an element of \mathbb{T}_h^S with vertices \mathbf{a}_i , $1 \leq i \leq d+1$. We have

$$\|\nabla E_h(v_h)\|_{L^2(T)}^2 = \left\| \sum_{i=1}^{d+1} E_h(v_h)(\mathbf{a}_i) \nabla \varphi_{\mathbf{a}_i} \right\|_{L^2(T)}^2.$$

But

$$\forall \mathbf{x} \in T, \sum_{i=1}^{d+1} \varphi_{\mathbf{a}_i}(\mathbf{x}) = 1,$$

hence

$$\forall \mathbf{x} \in T, \sum_{i=1}^{d+1} \nabla \varphi_{\mathbf{a}_i}(\mathbf{x}) = \mathbf{0},$$

and we can write

$$(69) \quad \begin{aligned} \|\nabla E_h(v_h)\|_{L^2(T)}^2 &= \left\| \sum_{i=1}^{d+1} (E_h(v_h)(\mathbf{a}_i) - v_h(\mathbf{a}_i)) \nabla \varphi_{\mathbf{a}_i} \right\|_{L^2(T)}^2 \\ &\leq \left(\sum_{i=1}^{d+1} |v_h|_{T_{\mathbf{a}_i}}(\mathbf{a}_i) - v_h|_T(\mathbf{a}_1)| \|\nabla \varphi_{\mathbf{a}_i}\|_{L^2(T)} \right)^2. \end{aligned}$$

Given a vertex \mathbf{a}_i , there exists a sequence of two-by-two adjacent elements T_1, T_2, \dots, T_j that share this vertex, and such that $T_1 = T_{\mathbf{a}_i}$, $T_j = T$. Therefore

$$(70) \quad v_h|_{T_{\mathbf{a}_i}}(\mathbf{a}_i) - v_h|_T(\mathbf{a}_1) = \sum_{n=1}^{j-1} (v_h|_{T_n} - v_h|_{T_{n+1}})(\mathbf{a}_i) + v_h|_T(\mathbf{a}_i) - v_h|_T(\mathbf{a}_1).$$

On one hand, Lemma 13 implies that

$$|(v_h|_{T_n} - v_h|_{T_{n+1}})(\mathbf{a}_i)| \leq \hat{C} \left(\frac{|\hat{F}|}{|F_n|} \right)^{\frac{1}{2}} \| [v_h]_{F_n} \|_{L^2(F_n)},$$

where F_n is the face shared by T_n and T_{n+1} . On the other hand, if $\mathbf{a}_i \neq \mathbf{a}_1$, by passing to the reference element \hat{T} , and using equivalence of norms, we have

$$|v_h(\mathbf{a}_i) - v_h(\mathbf{a}_1)| \leq h_{\hat{T}} \|\hat{\nabla} \hat{v}\|_{L^\infty(\hat{T})} \leq \hat{C} \|\hat{\nabla} \hat{v}\|_{L^2(\hat{T})}.$$

Thus

$$|v_h(\mathbf{a}_i) - v_h(\mathbf{a}_1)| \leq \hat{C} h_T \left(\frac{|\hat{T}|}{|T|} \right)^{\frac{1}{2}} \|\nabla v_h\|_{L^2(T)}.$$

By substituting these two inequalities into (70) and evaluating the basis functions in the reference element, we obtain

$$\|\nabla E_h(v_h)\|_{L^2(T)}^2 \leq \hat{C} \left[\sum_{i=1}^{d+1} \left(\sum_{n=1}^{j-1} \frac{1}{|F_n|} \|[v_h]_{F_n}\|_{L^2(F_n)}^2 + \frac{h_T^2}{|T|} \|\nabla v_h\|_{L^2(T)}^2 \right) \frac{|T|}{\rho_T^2} \|\hat{\nabla} \hat{\varphi}_i\|_{L^2(\hat{T})}^2 \right],$$

where \hat{C} takes into account the maximum number of edges or faces that meet at the vertices of \mathcal{P}_h ; this number is bounded by a constant independent of h since the family of triangulations is regular. We recover (68) by multiplying and dividing the first term of this sum with $(\sigma_h^F)^2$ and using again the regularity of the family \mathbb{T}_h^S . \square

To estimate $E_h(v_h)$, it is convenient to break it into $E_h(v_h) - v_h$ and v_h .

Lemma 15. *There exists a constant C , independent of h , such that*

$$(71) \quad \forall v_h \in X_h, \quad \|E_h(v_h) - v_h\|_{L^2(\Omega_S)} \leq Ch |v_h|_{1,h}.$$

Proof. We briefly sketch the proof. Let T be an element of \mathbb{T}_h^S with vertices \mathbf{a}_i , $1 \leq i \leq d+1$. We have

$$\|E_h(v_h) - v_h\|_{L^2(T)}^2 = \int_T \left(\sum_{i=1}^{d+1} (E_h(v_h)(\mathbf{a}_i) - v_h(\mathbf{x})) \varphi_{\mathbf{a}_i}(\mathbf{x}) \right)^2 d\mathbf{x},$$

an expression similar to (69) with \mathbf{x} instead of \mathbf{a}_1 and the basis function instead of its gradient. As \mathbf{x} and \mathbf{a}_1 are both points in T , the argument of Lemma 14 yields

$$\|E_h(v_h) - v_h\|_{L^2(T)}^2 \leq \hat{C} \left[\sum_{i=1}^{d+1} \left(\sum_{n=1}^{j-1} \frac{1}{|F_n|} \|[v_h]_{F_n}\|_{L^2(F_n)}^2 + \frac{h_T^2}{|T|} \|\nabla v_h\|_{L^2(T)}^2 \right) |T| \|\hat{\varphi}_i\|_{L^2(\hat{T})}^2 \right],$$

whence (71). \square

It remains to evaluate $v_h - E_h(v_h)$ on Γ_{SD} .

Lemma 16. *There exists a constant C , independent of h , such that*

$$(72) \quad \forall v_h \in X_h, \quad \|E_h(v_h) - v_h\|_{L^2(\Gamma_{SD})} \leq Ch^{\frac{1}{2}} |v_h|_{1,h}.$$

Proof. Again, we briefly sketch the proof. Let F be a face of $\mathbb{T}_h^S \cap \Gamma_{SD}$ with vertices \mathbf{a}_i , $1 \leq i \leq d$. Since

$$\forall \mathbf{x} \in F, \quad \sum_{i=1}^d \varphi_{\mathbf{a}_i}(\mathbf{x}) = 1,$$

we can write

$$\|E_h(v_h) - v_h\|_{L^2(F)}^2 = \int_F \left(\sum_{i=1}^d (E_h(v_h)(\mathbf{a}_i) - v_h(\mathbf{x})) \varphi_{\mathbf{a}_i}(\mathbf{x}) \right)^2 ds(\mathbf{x}).$$

Therefore, the argument of Lemma 14 yields

$$\|E_h(v_h) - v_h\|_{L^2(F)}^2 \leq \hat{C} \left[\sum_{i=1}^d \left(\sum_{n=1}^{j-1} \frac{1}{|F_n|} \|[v_h]_{F_n}\|_{L^2(F_n)}^2 + \frac{h_T^2}{|T|} \|\nabla v_h\|_{L^2(T)}^2 \right) |F| \|\hat{\varphi}_i\|_{L^2(\hat{F})}^2 \right],$$

and (72) follows easily from this. \square

By substituting (68), (71), and (72) into (66), we immediately derive the discrete interpolation trace inequality.

Theorem 17. *If the family of triangulations \mathbb{T}_h^S is regular, there exists a constant C , independent of h , such that*

$$(73) \quad \forall v_h \in X_h, \quad \|v_h\|_{L^2(\Gamma_{SD})} \leq C \left(\|v_h\|_{L^2(\Omega_S)} + h^{\frac{1}{2}} |v_h|_{1,h} + \|v_h\|_{L^2(\Omega_S)}^{\frac{1}{2}} |v_h|_{1,h}^{\frac{1}{2}} \right).$$

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