

INVERSE SCATTERING ON THE LINE WITH INCOMPLETE SCATTERING DATA

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Abstract: The Schrödinger equation is considered on the line when the potential is real valued, compactly supported, and square integrable. The nonuniqueness is analyzed in the recovery of such a potential from the data consisting of the ratio of a corresponding reflection coefficient to the transmission coefficient. It is shown that there are a discrete number of potentials corresponding to the data, all those potentials are identified, their L^2 -norms are related to each other in a simple manner, and it is also shown how an additional estimate on the L^2 -norm in the data can uniquely identify the corresponding potential. The recovery is illustrated with some explicit examples.

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1. INTRODUCTION

In this paper we analyze the recovery of the potential in the Schrödinger equation on the line from a set of scattering data containing no information on bound states. Our work is motivated by the following question of Paul Sacks: Consider two potentials in the Schrödinger equation where one potential is obtained from the other by adding a bound state. Can we compare the L^2 -norms of these two potentials, and can we conclude that the potential with fewer bound states has a smaller L^2 -norm? By using (2.13) and (2.17) these questions can be answered as follows: Take a square-integrable potential and add a bound state with bound-state energy $-\kappa^2$ and any bound-state norming constant. The new potential will have a larger L^2 -norm differing from the previous L^2 -norm by the exact value of $16\kappa^3/3$. Note that such a difference is independent of the value of the norming constant used, and hence L^2 -norms of square-integrable potentials are affected only by bound-state energies and not by norming constants.

Our work is also motivated by the work of Rundell and Sacks [1], where it was shown that a bounded, real-valued, compactly-supported potential with a sufficiently small L^2 -norm is uniquely determined by the corresponding ratio of a reflection coefficient to the transmission coefficient. With the help of the results in [2], our work here quantifies the smallness of the L^2 -norm in the result of [1]. In Section 3 we present the exact least upper bound for that L^2 -norm, below which we are assured the unique determination of a real-valued, compactly-supported, square-integrable potential in terms of the ratio of a reflection coefficient to the transmission coefficient; we do not require the potential to be bounded.

Let us now establish our notation. We consider the Schrödinger equation

$$\psi''(k, x) + k^2 \psi(k, x) = V(x) \psi(k, x), \quad x \in \mathbf{R}, \quad (1.1)$$

where the potential V belongs to the Faddeev class, i.e. it is real valued, measurable, and in $L^1_1(\mathbf{R})$, the class of measurable functions on the real axis \mathbf{R} such that $\int_{-\infty}^{\infty} dx (1+|x|) |V(x)|$ is finite. The prime is used for the derivative with respect to the spatial coordinate x . The Jost solutions f_l and f_r , from the left and right, respectively, satisfy the respective boundary conditions

$$\begin{aligned} e^{-ikx} f_l(k, x) &= 1 + o(1), & e^{-ikx} f'_l(k, x) &= ik + o(1), & x &\rightarrow +\infty, \\ e^{ikx} f_r(k, x) &= 1 + o(1), & e^{ikx} f'_r(k, x) &= -ik + o(1), & x &\rightarrow -\infty, \end{aligned}$$

and the transmission coefficient T , and the reflection coefficients L and R , from the left and right, respectively, are obtained from the spatial asymptotics

$$f_l(k, x) = \frac{e^{ikx}}{T(k)} + \frac{L(k) e^{-ikx}}{T(k)} + o(1), \quad x \rightarrow -\infty,$$

$$f_r(k, x) = \frac{e^{-ikx}}{T(k)} + \frac{R(k) e^{ikx}}{T(k)} + o(1), \quad x \rightarrow +\infty.$$

A bound state of (1.1) is a square-integrable solution, and such states occur only at the k -values on \mathbf{I}^+ in the upper half complex k -plane \mathbf{C}^+ where $T(k)$ has (simple) poles. Note that $\mathbf{I}^+ := i(0, +\infty)$ denotes the positive imaginary axis. Later we will let $\overline{\mathbf{C}^+} := \mathbf{C}^+ \cup \mathbf{R}$ and $\mathbf{I}^- := i(-\infty, 0)$. The behavior at $k = 0$ tells us whether the potential in (1.1) is generic or exceptional: The generic case occurs if $T(0) = 0$ and the exceptional case occurs if $T(0) \neq 0$. For a review of scattering and bound states of (1.1), the reader is referred to [3-9] and the references therein.

Our paper is organized as follows: In Section 2 we briefly review the effect of adding a bound state to a potential and show that certain integrals of the resulting potential remains unaffected by the bound-state norming constant but affected only by the bound-state energy and in a rather simple manner. In Section 3 we analyze a consequence of the result of Section 2 in the recovery of a real-valued, compactly-supported, square-integrable potential in terms of the data $L(k)/T(k)$. We show that, corresponding to that data, there are a discrete number of potentials, and an additional estimate on the L^2 -norm in the data allows the unique identification of a potential among all. We also illustrate the recovery with some explicit examples.

2. EFFECT OF BOUND STATES ON NORMS OF A POTENTIAL

Let $V^{[0]}$ denote a potential in the Faddeev class with no bound states. We use $V^{[N]}$ for the potential obtained from $V^{[0]}$ by adding N bound states at $k = i\kappa_j$ with the corresponding bound-state dependency constants γ_j , where we have the ordering $0 < \kappa_1 < \dots < \kappa_N$. The superscript $[j]$ refers to quantities associated with the potential $V^{[j]}$; for example, $T^{[j]}$, $R^{[j]}$, and $L^{[j]}$ denote the scattering coefficients, and $f_l^{[j]}$ and $f_r^{[j]}$ denote the left and right Jost solutions. Recall [4,9] that the dependency constants γ_j are defined as

$$\gamma_j := \frac{f_l^{[N]}(i\kappa_j, x)}{f_r^{[N]}(i\kappa_j, x)}, \quad 1 \leq j \leq N,$$

and the sign of γ_j is such that $(-1)^{N-j}\gamma_j > 0$. It is already known that

$$T^{[N]}(k) = T^{[0]}(k) \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}, \quad (2.1)$$

$$R^{[N]}(k) = (-1)^N R^{[0]}(k) \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}, \quad L^{[N]}(k) = (-1)^N L^{[0]}(k) \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}. \quad (2.2)$$

For the known facts listed in this section, we refer the reader to [4], where it is shown that bound states can be added to a potential via the Darboux transformation. We have

$$V^{[j]}(x) - V^{[j-1]}(x) = -2\mu'_j(x), \quad 1 \leq j \leq N, \quad (2.3)$$

where we have defined

$$\mu_j(x) := \frac{\chi'_j(x)}{\chi_j(x)}, \quad \chi_j(x) := f_1^{[j-1]}(i\kappa_j, x) + |\gamma_j| f_r^{[j-1]}(i\kappa_j, x). \quad (2.4)$$

It is known that $\mu_j(x)$ is continuous, strictly positive, and differentiable. In fact, as seen from (2.4) we have

$$\mu'_j(x) = V^{[j-1]}(x) + \kappa_j^2 - \mu_j(x)^2,$$

and hence from (2.3) it follows that

$$V^{[j]}(x) + V^{[j-1]}(x) = 2[\mu_j(x)^2 - \kappa_j^2]. \quad (2.5)$$

Define

$$I_{j,n}(x) := \left[V^{[j]}(x) - V^{[j-1]}(x) \right] \left[V^{[j]}(x) + V^{[j-1]}(x) \right]^n, \quad n \geq 0, \quad 1 \leq j \leq N. \quad (2.6)$$

Theorem 2.1 *Let $V^{[j]}$ be the potential obtained from $V^{[0]}$ by adding bound states of energy $-\kappa_1^2, \dots, -\kappa_j^2$, and assume that $V^{[0]}$ belongs to the Faddeev class without any bound states. We then have*

$$\int_{-\infty}^{\infty} dx I_{j,n}(x) = (-1)^{n+1} 2^{n+2} \kappa_j^{2n+1} \frac{n!}{(2n+1)!!}, \quad n \geq 0, \quad 1 \leq j \leq N, \quad (2.7)$$

where $(2n+1)!! := (1)(3)(5)\cdots(2n+1)$.

PROOF: Taking the n th power in (2.5) and expanding the result, from (2.3) and (2.5) we get

$$I_{j,n}(x) = -2^{n+1} \frac{d}{dx} \sum_{p=0}^n (-1)^{n-p} \kappa_j^{2(n-p)} \binom{n}{p} \frac{\mu_j(x)^{2p+1}}{2p+1}, \quad (2.8)$$

where $\binom{n}{p} := \frac{n!}{p!(n-p)!}$ is the binomial coefficient. It is already known that

$$\mu_j(x) = \begin{cases} \kappa_j + o(1), & x \rightarrow +\infty, \\ -\kappa_j + o(1), & x \rightarrow -\infty. \end{cases} \quad (2.9)$$

Integrating (2.6) on \mathbf{R} and using (2.9), we get

$$\int_{-\infty}^{\infty} dx I_{j,n}(x) = (-1)^{n+1} 2^{n+2} \kappa_j^{2n+1} \sum_{p=0}^n (-1)^p \binom{n}{p} \frac{1}{2p+1}. \quad (2.10)$$

Note that the summation in (2.10) can be evaluated explicitly with the help of

$$\sum_{p=0}^n (-1)^p \binom{n}{p} \frac{1}{2p+1} = \int_0^1 dx (1-x^2)^n = \frac{n!}{(2n+1)!!}, \quad n \geq 0. \quad (2.11)$$

Thus, using (2.11) in (2.10) we establish (2.7). ■

The result in (2.7) is remarkable in the sense that even though the integrand $I_{j,n}(x)$ depends on the bound-state data $\{\kappa_p, \gamma_p\}_{p=1}^j$, its integral given in (2.7) is independent of the bound-state data, except for a rather simple κ_j -dependence.

For $n = 0$ and $n = 1$, respectively, from (2.7) we get

$$\int_{-\infty}^{\infty} dx [V^{[j]}(x) - V^{[j-1]}(x)] = -4\kappa_j, \quad 1 \leq j \leq N, \quad (2.12)$$

$$\int_{-\infty}^{\infty} dx [V^{[j]}(x)^2 - V^{[j-1]}(x)^2] = \frac{16}{3} \kappa_j^3, \quad 1 \leq j \leq N. \quad (2.13)$$

By summing both sides in each of (2.12) and (2.13) over j , we get the following:

Corollary 2.2 *Let $V^{[0]}$ be a potential in the Faddeev class with no bound states; add N bound states with energy $-\kappa_1^2, \dots, -\kappa_N^2$, resulting in the potential $V^{[N]}$. We then have*

$$\int_{-\infty}^{\infty} dx [V^{[N]}(x) - V^{[0]}(x)] = -4 \sum_{j=1}^N \kappa_j, \quad (2.16)$$

$$\int_{-\infty}^{\infty} dx [V^{[N]}(x)^2 - V^{[0]}(x)^2] = \frac{16}{3} \sum_{j=1}^N \kappa_j^3. \quad (2.17)$$

Let us indicate some resemblance between the result in (2.7) and the conserved quantities for an evolution equation that is exactly solvable by the inverse scattering transform [12-15]. For example, consider the time-evolution of the scattering data of (1.1) as

$T(k) \mapsto T(k)$, $L(k) \mapsto L(k)e^{-8ik^3t}$, $\kappa_j \mapsto \kappa_j$, and $\gamma_j \mapsto \gamma_j e^{-8\kappa_j^3t}$. The potential of (1.1) then evolves as $V(x) \mapsto u(x, t)$, where $u(x, t)$ satisfies the initial-value problem for the Korteweg-de Vries equation (KdV)

$$u_t - 6uu_x + u_{xxx} = 0, \quad x \in \mathbf{R}, \quad t > 0; \quad u(x, 0) = V(x).$$

It is known [12-15] that $\int_{-\infty}^{\infty} dx u(x, t)$, $\int_{-\infty}^{\infty} dx u(x, t)^2$, and an infinite number of other integrals are independent of t even though their integrands contain t explicitly. Such quantities are known as the conserved quantities for the KdV. Consider now, for example, (2.16) and (2.17), and let us time evolve the potentials $V^{[0]}(x)$ and $V^{[N]}(x)$ to obtain the corresponding solutions $u^{[0]}(x, t)$ and $u^{[N]}(x, t)$ of the KdV. Due to the fact that the bound-state energies $-\kappa_j^2$ remain unchanged during the time evolution and that the right hand sides in (2.16) and (2.17) do not contain the dependency constants γ_j , we have

$$\int_{-\infty}^{\infty} dx \left[u^{[N]}(x, t) - u^{[0]}(x, t) \right] = -4 \sum_{j=1}^N \kappa_j,$$

$$\int_{-\infty}^{\infty} dx \left[u^{[N]}(x, t)^2 - u^{[0]}(x, t)^2 \right] = \frac{16}{3} \sum_{j=1}^N \kappa_j^3.$$

Other similar conserved quantities for the KdV can be obtained with the help of (2.7).

3. RECOVERY OF THE POTENTIAL FROM $L(k)/T(k)$

In [1] the recovery of a bounded, real-valued, compactly-supported potential is considered in terms of the data $\mathcal{D}(k) := L(k)/T(k)$ known for $k \in \mathbf{R}$. In the class of such potentials corresponding to the same $\mathcal{D}(k)$, it was shown (cf. Theorem 2.3 of [1]) that there exists a positive constant C such that if V_1 and V_2 are two potentials with L^2 -norms not exceeding C then $V_1 \equiv V_2$. The uniqueness and the reconstruction were obtained by transforming the problem into an equivalent time-domain problem; however, the value of C was left unspecified. In this section, we show how the value of C can be specified.

Recently, we have analyzed [2] the recovery of the potential V of (1.1) from $\mathcal{D}(k)$ when V belongs to the Faddeev class. In this inverse problem, the construction of V is equivalent to the construction of the data $\{L(k), N, \{\kappa_j\}, \{\gamma_j\}\}$, where L is the left reflection coefficient, N is the number of bound states, the set $\{-\kappa_j^2\}_{j=1}^N$ corresponds to the bound-state energies, and the set $\{\gamma_j\}_{j=1}^N$ corresponds to the bound-state dependency constants. We have four cases to consider:

- (a) No information is available on the support of V , and the only data available is $\mathcal{D}(k)$.
- (b) In addition to $\mathcal{D}(k)$, it is known that the support of V is confined to a half line. In this case, there is no loss of generality in assuming that $V \equiv 0$ for $x < 0$.
- (c) In addition to $\mathcal{D}(k)$, it is known that the support of V is confined a finite interval. In this case, there is no loss of generality in assuming that $V \equiv 0$ for $x \notin [0, 1]$.
- (d) In addition to $\mathcal{D}(k)$ and knowledge that $V \equiv 0$ for $x \notin [0, 1]$, it is known that V is square integrable and some information related to the L^2 -norm is available. Such additional information may be in the form of a positive constant C which acts as an upper bound on the L^2 -norm.

Let us consider the construction of V or equivalently of $\{L(k), N, \{\kappa_j\}, \{\gamma_j\}\}$ in each of these four cases. For the analysis in the first three cases we refer the reader to [2] and give a brief summary below. Our results show that in case (c), given $\mathcal{D}(k)$ for $k \in \mathbf{R}$, we are able to determine all the corresponding potentials, there are a discrete number of such potentials, the L^2 -norm of each such potential is readily evaluated with the help of (2.17), and appropriate additional information on the L^2 -norm enables us to further restrict the set of potentials corresponding to $\mathcal{D}(k)$. We also explain how the constant C in Theorem 2.3 of [1] arises: that constant allows us to identify the potential with the smallest L^2 -norm among all those corresponding to the same $\mathcal{D}(k)$. By analyzing the inverse problem stated in (d), we show how to determine the precise values of C that can be used in [1].

Case (a): Recovery of V from \mathcal{D} with no Support Information

If no information other than $\mathcal{D}(k)$ is available, we have the following:

- (a.i) If $\mathcal{D}(k)$ is bounded at $k = 0$, then there is no restriction on N and hence $N \in \{0, 1, 2, \dots\}$. Note that this case corresponds to the exceptional case for (1.1).
- (a.ii) If $\mathcal{D}(k)$ is unbounded at $k = 0$, then $\lim_{k \rightarrow 0} [2ik \mathcal{D}(k)]$ is either a positive constant or a negative constant. Thus, either $\mathcal{D}(k) \rightarrow -\infty$ or $\mathcal{D}(k) \rightarrow +\infty$ as $k \rightarrow 0$ on \mathbf{I}^+ . In the former case N must be even, i.e. $N \in \{0, 2, 4, \dots\}$; in the latter case N must be odd, i.e. $N \in \{1, 3, 5, \dots\}$. Note that both these correspond to the generic case for (1.1).
- (a.iii) For each N -value resulting from (i) or (ii), given $\mathcal{D}(k)$ there corresponds a $2N$ -parameter family of potentials where the parameter set is $\{\kappa_j, \gamma_j\}_{j=1}^N$. There are no restrictions on the κ_j other than $0 < \kappa_1 < \dots < \kappa_N$. There are no restrictions on the γ_j other than $(-1)^{N-j} \gamma_j > 0$.

From the data $\mathcal{D}(k)$ known for $k \in \mathbf{R}$, one uniquely constructs

$$T^{[0]}(k) = \exp \left(\frac{-1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{\log(1 + |\mathcal{D}(s)|^2)}{s - k - i0^+} \right), \quad k \in \overline{\mathbf{C}^+}. \quad (3.1)$$

Then, with the help of (2.1) and (2.2), it is seen that the set $\{\mathcal{D}(k), N, \{\kappa_j\}\}$ leads to the left reflection coefficient given by

$$L(k) = \mathcal{D}(k) T^{[0]}(k) \prod_{j=1}^N \frac{k + i\kappa_j}{k - i\kappa_j}, \quad k \in \mathbf{R}. \quad (3.2)$$

Note that $T^{[0]}(k)$ appearing in (3.1) and (3.2) corresponds to the transmission coefficient for the potential $V^{[0]}$, which is obtained by removing all the N bound states from V . The left and right reflection coefficients, $L^{[0]}(k)$ and $R^{[0]}(k)$, respectively, corresponding to $V^{[0]}$ are uniquely determined only in the generic case as

$$L^{[0]}(k) = (-1)^N \mathcal{D}(k) T^{[0]}(k), \quad R^{[0]}(k) = (-1)^{N-1} \mathcal{D}(-k) T^{[0]}(k), \quad k \in \mathbf{R}, \quad (3.3)$$

because only in the generic case $(-1)^N$ is uniquely determined from $\mathcal{D}(k)$. In the exceptional case, the value of $(-1)^N$ cannot be determined from $\mathcal{D}(k)$ and hence there are two choices for $V^{[0]}$, which we denote by $V_1^{[0]}$ and $V_2^{[0]}$, respectively, with the corresponding scattering coefficients determined in terms of $\mathcal{D}(k)$ and $T^{[0]}(k)$ in (3.1) as follows:

$$\begin{aligned} T_1^{[0]}(k) &= T^{[0]}(k), & L_1^{[0]}(k) &= \mathcal{D}(k) T^{[0]}(k), & R_1^{[0]}(k) &= -\mathcal{D}(-k) T^{[0]}(k), \\ T_2^{[0]}(k) &= T^{[0]}(k), & L_2^{[0]}(k) &= -\mathcal{D}(k) T^{[0]}(k), & R_2^{[0]}(k) &= \mathcal{D}(-k) T^{[0]}(k). \end{aligned}$$

For the comparison of two potentials with the same transmission coefficient but with reflection coefficients differing in sign, the reader is referred to [9,10,11]. As the next proposition shows, even though $V_1^{[0]} \not\equiv V_2^{[0]}$, some of their characteristic features are related.

Proposition 3.1 *Let $V_1^{[0]}$ and $V_2^{[0]}$ be two exceptional potentials in the Faddeev class with no bound states, and assume that $T_1^{[0]} \equiv T_2^{[0]}$, $L_1^{[0]} \equiv -L_2^{[0]}$, and $R_1^{[0]} \equiv -R_2^{[0]}$, i.e. their reflection coefficients differ in sign and their transmission coefficients are the same for all $k \in \mathbf{R}$. Then we have the following:*

- (i) $\int_{-\infty}^{\infty} dx \left[V_2^{[0]}(x) - V_1^{[0]}(x) \right] \left[V_2^{[0]}(x) + V_1^{[0]}(x) \right]^n = 0, \quad n \geq 0.$
- (ii) $V_1^{[0]}$ vanishes on a half line if and only if $V_2^{[0]}$ vanishes on the same half line. Consequently, $V_1^{[0]}$ vanishes outside some interval if and only if $V_2^{[0]}$ vanishes outside that interval.

(iii) If $V_1^{[0]}$ vanishes on \mathbf{R}^- and is continuous on the interval $(0, \delta)$ for some $\delta > 0$, then $V_1^{[0]}(0^+) = -V_2^{[0]}(0^+)$. Similarly, if $V_1^{[0]}$ vanishes on \mathbf{R}^+ and is continuous on the interval $(-\delta, 0)$ for some $\delta > 0$, then $V_1^{[0]}(0^-) = -V_2^{[0]}(0^-)$.

PROOF: From (2.24) and (2.25) of [11] we have

$$V_2^{[0]}(x) = V_1^{[0]}(x) - 2\rho_1'(x) = 2\rho_1(x)^2 - V_1^{[0]}(x), \quad (3.4)$$

where

$$\rho_1(x) := \frac{f_{1l}^{[0]'}(0, x)}{f_{1l}^{[0]}(0, x)} = \frac{f_{1r}^{[0]'}(0, x)}{f_{1r}^{[0]}(0, x)},$$

with $f_{1l}^{[0]}(k, x)$ and $f_{1r}^{[0]}(k, x)$ being the left and right Jost solutions for the potential $V_1^{[0]}$. It is known that $f_{1l}^{[0]}(0, x)$ is continuous and strictly positive and that $\rho_1(x) = o(1/x)$ as $x \rightarrow \pm\infty$. Hence, with the help of (3.4) we get [cf. (2.8)]

$$\left[V_2^{[0]}(x) - V_1^{[0]}(x) \right] \left[V_2^{[0]}(x) + V_1^{[0]}(x) \right]^n = -\frac{2^{n+1}}{2n+1} [\rho_1(x)^{2n+1}]', \quad n \geq 0,$$

and integrating both sides over \mathbf{R} we obtain (i). To prove (ii), notice that there is no loss of generality in choosing the half line as \mathbf{R}^- . If $V_1^{[0]} \equiv 0$ for $x < 0$, then $f_{1r}^{[0]}(0, x) = 1$ for $x \leq 0$, and hence $\rho_1(x) = 0$ for $x \leq 0$. Thus, from (3.4) it follows that $V_2^{[0]} \equiv 0$ for $x < 0$ as well. Conversely, it follows that $V_1^{[0]} \equiv 0$ on \mathbf{R}^- whenever $V_2^{[0]} \equiv 0$ there; thus, we have proved (ii). From (3.4) we see that the first statement in (iii) holds whenever $\rho_1(0^+) = 0$, which is the case due to the continuity of $\rho_1(x)$ at $x = 0$ and $\rho_1(x) = 0$ for $x \leq 0$, which is satisfied when $V_1^{[0]} \equiv 0$ for $x < 0$. The second statement in (iii) is obtained in a similar manner. ■

Note that Proposition 3.1(i) holds even when n is a noninteger. By letting $n = 0$ and $n = 1$ there and using the fact that a potential in the Faddeev class is integrable, we obtain

$$\int_{-\infty}^{\infty} dx V_1^{[0]}(x) = \int_{-\infty}^{\infty} dx V_2^{[0]}(x), \quad \int_{-\infty}^{\infty} dx \left[V_1^{[0]}(x)^2 - V_2^{[0]}(x)^2 \right] = 0. \quad (3.5)$$

For smooth potentials, we refer the reader to (2.101) of [10] for results similar to (3.5) and their generalizations.

Case (b): Recovery of V from \mathcal{D} with Half-line Support

If $\mathcal{D}(k)$ is given for $k \in \mathbf{R}$ and if it is also known that $V \equiv 0$ for $x < 0$, then, in addition to all the results given in case (a), in particular, in addition to (a.i) and (a.ii), we have the following improvements:

- (b.iii) $\mathcal{D}(k)$ has a unique analytic extension to $k \in \mathbf{C}^+$ and such an extension is uniquely determined by our data $\mathcal{D}(k)$ known for $k \in \mathbf{R}$. The value of N must satisfy $N \leq Z + 1$, where Z denotes the number of zeros of $\mathcal{D}(k)$ on \mathbf{I}^+ . In fact, from the proof of Proposition 3.1 in [2] it follows that if $\mathcal{D}(k)$ has multiple zeros on \mathbf{I}^+ , then Z is actually the number of distinct zeros of odd multiplicity, without counting the multiplicities.
- (b.iv) For each N -value resulting from restrictions (a.i), (a.ii), and (b.iii), given $\mathcal{D}(k)$ for $k \in \mathbf{R}$, there corresponds an N -parameter family of potentials where the parameter set is $\{\kappa_j\}_{j=1}^N$. The κ_j satisfy the restrictions $0 < \kappa_1 < \cdots < \kappa_N$ and $(-1)^{N-j}\mathcal{D}(i\kappa_j) > 0$ for $j = 1, \dots, N$. The latter restriction confines the κ_j to subintervals whose endpoints are uniquely determined by the zeros of $\mathcal{D}(k)$ on \mathbf{I}^+ . The dependency constants γ_j are uniquely determined as $\gamma_j = \mathcal{D}(i\kappa_j)$ and hence they are not free parameters. The left reflection coefficient $L(k)$ given in (3.2) becomes meromorphic in \mathbf{C}^+ with simple poles at $k = i\kappa_j$ for $j = 1, \dots, N$. Thus, (3.2) now holds for $k \in \overline{\mathbf{C}^+}$ and the left reflection coefficient $L^{[0]}(k)$ given in (3.3) becomes analytic in \mathbf{C}^+ .

Case (c): Recovery of V from \mathcal{D} with Compact Support

If $\mathcal{D}(k)$ for $k \in \mathbf{R}$ is given and if it is also known that $V \equiv 0$ for $x \notin [0, 1]$, then, in addition to all the results in cases (a) and (b), in particular, in addition to (a.i) and (a.ii), we have the following improvements:

- (c.iii) The quantity $k\mathcal{D}(k)$ has a unique analytic extension to the entire complex plane, and such an extension is uniquely determined by our data $\mathcal{D}(k)$ known for $k \in \mathbf{R}$. Moreover, as in (b.iii) the value of N must satisfy $N \leq Z + 1$, where Z is the number of zeros of $\mathcal{D}(k)$ on \mathbf{I}^+ having odd multiplicities, without counting the multiplicities.
- (c.iv) For each N -value resulting from restrictions (a.i), (a.ii), and (c.iii), given $\mathcal{D}(k)$ for $k \in \mathbf{R}$, there correspond a discrete number of potentials where the discrete parameter set is $\{\kappa_j\}_{j=1}^N$. The set $\{\kappa_j\}_{j=1}^N$ must be a subset of $\{\beta_m\}$ and satisfy the additional restrictions $0 < \kappa_1 < \cdots < \kappa_N$ and $(-1)^{N-j}\mathcal{D}(i\kappa_j) > 0$ for $j = 1, \dots, N$. Here, each $k = -i\beta_m$ corresponds to a zero of $1/T^{[0]}(k)$ on \mathbf{I}^- , where $T^{[0]}(k)$ is the quantity in (3.1), and $k/T^{[0]}(k)$ is now entire on \mathbf{C} and uniquely constructed via (3.1) from our data $\mathcal{D}(k)$ known for $k \in \mathbf{R}$. The values $k = -i\beta_m$ correspond to the (real) resonances of $V^{[0]}$. For an answer to the question whether the set $\{\beta_m\}$ is a finite set or an infinite set, we refer the reader to [16]. Informally speaking, if $V^{[0]} \in C_0^\infty[0, 1]$ and the order of the zero of $V^{[0]}$ at $x = 0$ or at $x = 1$ is infinite, then the set $\{\beta_m\}$ may be infinite;

otherwise, it is a finite set.

Case (d): Recovery of V from \mathcal{D} with Compact Support and L^2 -norm

Let us assume that $\mathcal{D}(k)$ is given for $k \in \mathbf{R}$ and it is known that $V \equiv 0$ for $x \notin [0, 1]$, $V \in L^2[0, 1]$, and $\|V\| \leq C$, where we denote the L^2 -norm of V as $\|V\| := \sqrt{\int_{-\infty}^{\infty} dx V(x)^2}$. We will determine the precise values of C that assure a unique or nonunique determination of V from \mathcal{D} .

As outlined in case (a) below (3.3), given $\mathcal{D}(k)$ for $k \in \mathbf{R}$, we are able to uniquely determine $V^{[0]}$ when \mathcal{D} is singular at $k = 0$, and we determine two distinct potentials $V_1^{[0]}$ and $V_2^{[0]}$ if \mathcal{D} is finite at $k = 0$. In the latter case, we know from (3.5) that $\|V_1^{[0]}\| = \|V_2^{[0]}\|$. Thus, $\mathcal{D}(k)$ uniquely determines the L^2 -norm of $V^{[0]}$, even though there are two distinct choices for $V^{[0]}$ in the exceptional case. Let us denote that unique value by $\|V^{[0]}\|$.

As seen from (c.iv), for each allowed integer N , $\mathcal{D}(k)$ uniquely [2] determines a discrete number of ordered sets $\{\kappa_j\}_{j=1}^N$ with the ordering $0 < \kappa_1 < \dots < \kappa_N$ related to the bound states of $V^{[N]}$. Let us define

$$C_0 := \|V^{[0]}\|; \quad C_N := \left[\|V^{[0]}\|^2 + \frac{16}{3} \sum_{j=1}^N \kappa_j^3 \right]^{1/2}, \quad 1 \leq j \leq N.$$

Thus, for each N , C_N consists of a sequence of values. Clearly, C_0 consists of a single number. By listing all the elements in C_N for all allowed N -values, we obtain a discrete set of ordered positive numbers, which we denote by $\{C_N\}$. This set is a subset of $\{\beta_m\}$, as indicated in (c.iv). The smallest number in the ordered set $\{C_N\}$ is strictly less than the next larger number due to the fact that each set $\{\kappa_j\}_{j=1}^N$ with the largest allowable N consists of distinct positive elements. This allows us to determine the value of C in the inequality $\|V\| \leq C$ in order to determine a unique potential V corresponding to our data \mathcal{D} . By choosing C as greater than or equal to the smallest number in the set $\{C_N\}$ but strictly less than the next larger element, we will uniquely determine the potential V . Next we illustrate this determination with some explicit examples.

As our scattering data let us use $\mathcal{D}(k) = \frac{-\epsilon e^{ik} \sin \sqrt{k^2 + \epsilon}}{2ik \sqrt{k^2 + \epsilon}}$, where ϵ is a positive parameter. In fact, one corresponding potential is the square well of depth ϵ supported on the interval $[0, 1]$. For each value of ϵ , let us obtain all the potentials corresponding to $\mathcal{D}(k)$ with support confined to $[0, 1]$ and specify their L^2 -norms. We have $\lim_{k \rightarrow 0} [2ik \mathcal{D}(k)] = -\sqrt{\epsilon} \sin \sqrt{\epsilon}$, and hence the exceptional case occurs when $\sqrt{\epsilon}/\pi$ is a positive integer and the

generic case occurs otherwise. The zeros of $\mathcal{D}(k)$ on \mathbf{I}^+ occur when $\sin \sqrt{k^2 + \epsilon} = 0$, and hence these are all simple zeros occurring at $k = i\sqrt{\epsilon - (j-1)^2\pi^2}$ for $j = 1, \dots, Z$, with Z being equal to $\lfloor \sqrt{\epsilon}/\pi \rfloor$, i.e. the greatest integer less than or equal to $\sqrt{\epsilon}/\pi$. As $k \rightarrow \infty$ on \mathbf{I}^+ , we have $\mathcal{D}(k) \rightarrow 0^+$. As $k \rightarrow 0$ on \mathbf{I}^+ , we get $(-1)^Z \mathcal{D}(k) \rightarrow 0^+$ in the exceptional case, and $(-1)^Z \mathcal{D}(k) \rightarrow +\infty$ in the generic case. Define

$$\frac{1}{\tau(k)} := e^{ik} \left[\cos \sqrt{k^2 + \epsilon} + \frac{2k^2 + \epsilon}{2ik\sqrt{k^2 + \epsilon}} \sin \sqrt{k^2 + \epsilon} \right].$$

Note that $\tau(k)$ corresponds to the transmission coefficient of the square-well potential of depth ϵ supported on $[0, 1]$. It is known that $1/\tau(k)$ has exactly $Z + 1$ (simple) zeros on \mathbf{I}^+ , which we denote by ξ_j with the ordering $0 < \xi_1 < \dots < \xi_{Z+1}$. The quantity in (3.1) is obtained as

$$\frac{1}{T^{[0]}(k)} = \frac{1}{\tau(k)} \prod_{j=1}^{Z+1} \frac{k + i\xi_j}{k - i\xi_j}. \quad (3.6)$$

Example 3.2 When $\epsilon = 5$, we are in the generic case and $Z = 0$. Hence, $N \leq 1$, but $\mathcal{D}(k) \rightarrow +\infty$ on \mathbf{I}^+ indicates that N must be odd. Thus, $N = 1$ is the only allowed value. In this case, $1/T^{[0]}(k)$ given in (3.6) has two zeros on \mathbf{I}^- at $k = -i\beta_j$ with $\beta_1 = 1.5433\bar{4}$ and $\beta_2 = 1.585\bar{7}$. We use an overline to indicate roundoff. In (3.6) we have $\xi_1 = \beta_2$. Corresponding to $\mathcal{D}(k)$ we have two potentials $V_1^{[1]}$ and $V_2^{[1]}$, having bound states at $k = i\beta_1$ and $k = i\beta_2$, respectively. Note that $V_2^{[1]}$ is the square well of depth ϵ . We have $\|V_1^{[1]}\| = 4.8312\bar{6}$ and $\|V_2^{[1]}\| = 5$. Thus, knowledge of any C satisfying $\|V_1^{[1]}\| \leq C < \|V_2^{[1]}\|$ helps us to identify $V_1^{[1]}$ or $V_2^{[1]}$ as the unique potential corresponding to $\mathcal{D}(k)$. The left reflection coefficients $L_1^{[1]}$ and $L_2^{[1]}$ corresponding to $V_1^{[1]}$ and $V_2^{[1]}$, respectively, are obtained from (3.2) as

$$L_j^{[1]}(k) = \mathcal{D}(k) T^{[0]}(k) \frac{k + i\beta_j}{k - i\beta_j}, \quad j = 1, 2.$$

Example 3.3 When $\epsilon = \pi^2$, we are in the exceptional case and $Z = 0$. Hence, both $N = 0$ and $N = 1$ are allowed. In this case $1/T^{[0]}(k)$ given in (4.1) has only one zero on \mathbf{I}^- at $k = -i\beta_1$ with $\beta_1 = 2.52258\bar{8}$. In (4.1) we have $\xi_1 = \beta_1$. Corresponding to $\mathcal{D}(k)$ we have two potentials $V^{[0]}$ and $V^{[1]}$, the former with no bound states and the latter with one bound state at $k = i\beta_1$. Note that $V^{[1]}$ is the square well of depth ϵ . We have $\|V^{[0]}\| = 3.3853\bar{7}$ and $\|V^{[1]}\| = \pi^2$. Thus, knowledge of any C satisfying $\|V^{[0]}\| \leq C < \|V^{[1]}\|$ helps us to identify either $V^{[0]}$ or $V^{[1]}$ as the unique potential corresponding to $\mathcal{D}(k)$. The left reflection

coefficients $L^{[0]}$ and $L^{[1]}$ corresponding to $V^{[0]}$ and $V^{[1]}$, respectively, are obtained from (3.2) as

$$L^{[0]}(k) = \mathcal{D}(k) T^{[0]}(k), \quad L^{[1]}(k) = \mathcal{D}(k) T^{[0]}(k) \frac{k + i\beta_1}{k - i\beta_1}.$$

Example 3.4 When $\epsilon = 20$, we are in the generic case and $Z = 1$. Hence, $N \leq 2$, but $\mathcal{D}(k) \rightarrow -\infty$ on \mathbf{I}^+ indicates that N must be even. Thus, $N = 0$ and $N = 2$ are the only possibilities. In this case $1/T^{[0]}(k)$ given in (4.1) has two zeros on \mathbf{I}^- at $k = -i\beta_1$ with $\beta_1 = \xi_1 = 1.9302\bar{1}$ and $k = -i\beta_2$ with $\beta_2 = \xi_2 = 3.9255\bar{6}$, where ξ_1 and ξ_2 are the quantities in (3.6). When $N = 2$, the only potential $V^{[2]}$ corresponding to $\mathcal{D}(k)$ is the square well of depth ϵ with support $[0, 1]$. When $N = 0$, the corresponding potential $V^{[0]}$ is uniquely determined from $\mathcal{D}(k)$ and its left reflection coefficient $L^{[0]}(k)$ is obtained from (3.2) as

$$L^{[0]}(k) = \mathcal{D}(k) T^{[0]}(k) \frac{(k - i\beta_1)(k - i\beta_2)}{(k + i\beta_1)(k + i\beta_2)}.$$

In this case we have $\|V^{[0]}\| = 6.2463\bar{5}$ and $\|V^{[2]}\| = 20$. Thus, an appropriate specification of the upper limit on the L^2 -norm of the potential allows the unique identification of $V^{[0]}$ or $V^{[2]}$ from $\mathcal{D}(k)$.

Example 3.5 When $\epsilon = 130$, the allowed values for N are 0, 2, and 4. In this case $1/T^{[0]}(k)$ given in (4.1) has six zeros on \mathbf{I}^- at $k = -i\beta_j$ with $\beta_1 = 4.8729\bar{5}$, $\beta_2 = 8.2260\bar{7}$, $\beta_3 = 8.3286\bar{5}$, $\beta_4 = 10.087\bar{9}$, $\beta_5 = 10.740\bar{7}$, $\beta_6 = 11.08\bar{5}$. For $N = 0$, the only potential corresponding to $\mathcal{D}(k)$ has norm $\|V^{[0]}\| = 23.96\bar{8}$. For $N = 2$, there are five potentials corresponding to $\mathcal{D}(k)$ with norms $\|V_1^{[2]}\| = 64.50\bar{9}$, $\|V_2^{[2]}\| = 65.366\bar{8}$, $\|V_3^{[2]}\| = 91.956\bar{6}$, $\|V_4^{[2]}\| = 115.38\bar{7}$, $\|V_5^{[2]}\| = 120.19\bar{7}$, where $V_1^{[2]}$ has bound states $\{-\beta_1^2, -\beta_2^2\}$, $V_2^{[2]}$ has $\{-\beta_1^2, -\beta_3^2\}$, $V_3^{[2]}$ has $\{-\beta_1^2, -\beta_6^2\}$, $V_4^{[2]}$ has $\{-\beta_4^2, -\beta_6^2\}$, and $V_5^{[2]}$ has $\{-\beta_5^2, -\beta_6^2\}$. For $N = 4$, there are four potentials corresponding to $\mathcal{D}(k)$ with norms $\|V_1^{[4]}\| = 130$, $\|V_2^{[4]}\| = 130.43\bar{2}$, $\|V_3^{[4]}\| = 134.28\bar{7}$, $\|V_4^{[4]}\| = 134.70\bar{5}$, where $V_1^{[4]}$ has bound states $\{-\beta_1^2, -\beta_2^2, -\beta_4^2, -\beta_6^2\}$, $V_2^{[4]}$ has $\{-\beta_1^2, -\beta_3^2, -\beta_4^2, -\beta_6^2\}$, $V_3^{[4]}$ has $\{-\beta_1^2, -\beta_2^2, -\beta_5^2, -\beta_6^2\}$, $V_4^{[4]}$ has $\{-\beta_1^2, -\beta_3^2, -\beta_5^2, -\beta_6^2\}$. Thus, some appropriate knowledge on the L^2 -norm of the potential allows us to pick a unique potential among all these 16 potentials corresponding to the same $\mathcal{D}(k)$. Note that $V_1^{[4]}$ is the square well of depth ϵ .

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