THE CONVERGENCE FOR NODAL EXPANSION METHOD

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THE CONVERGENCE FOR NODAL EXPANSION METHOD

Huang Aixiang and Zhang Bo (*)

Abstract

In this paper, we prove the convergence of the nodal expansion method, a new numerical method for partial differential equations and provide the error estimates of approximation solution .

1. Introduction

The nodal method was presented in 1975, and extended to the higher-order approximation in 1977, by H. finneman, F. Bennewitz, M.R. Wegner at el. In the practical computations for neutron diffusion equations these methods have been shown very efficient and economical.

The theoretical foundations of the nodal expansion methods were not established until 1985, we first discussed the relationship between the nodal expansion method and the primal hybrid finite element method, and proved that the 0-order nodal expansion method was convergent. The essential idea of nodal expansion method is that the principal part and the non-principal part of the linear elliptic equations are approximated by different finite element subspace respectively. Following this idea, we present a generalized primal hybrid variational formulation. By taking different test functions and trial functions, we achieve the aim of dealing differently with the principal part, and hence, generalized primal hybrid finite element method is derived. We prove that the nodal expansion methods are special cases of generalized primal hybrid finite element methods, and thereby establish, the theoretical foundations of

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the nodal exapansion method with arbitrary order approximations are established.

2. Nodal Expansion Method and Primal Hybrid Finite Element Method

For simplicity we consider boundary value problem

$$-\Delta u + u = f \quad \text{in} \quad \Omega \tag{2.1}$$

$$u|_\Gamma = 0 \qquad \text{on} \quad \Gamma = \partial\Omega \ \text{is boundary of} \ \Omega$$

The nodal expansion method consists of the following step:

1) Let $\underline{J} = -\operatorname{grad} u$, then (2.1) can be reduced into

$$\begin{aligned} &\underline{J} + \operatorname{grad} u = 0 \\ &\operatorname{div} \underline{J} + u = f \end{aligned} \tag{2.2}$$

2) Domain Ω is decomposed into N rectangular denoted by Fh. In each element k, function u is approximated by the following polynomials

$$u(x,y) = \sum_{i,j=0}^{i+j \leq n+2} C_{ij} \; g_i \left(\frac{x-x_k}{h_1}\right) g_j \left(\frac{Y-y_k}{h_1}\right) \tag{2.3}$$

where

n	order of nodal expansion method
C_{ij}	constants to be determined
(h_1,h_2)	the lengths of sides of elements
g_i	Legendre polynomials of i degree in $[0,1]$

$$g_0(t) = 1$$

$$g_1(t) = 2t-1$$

$$g_2(t) = -6t(1-t) + 1$$

$$g_3(t) = 6t(1-t)(2t-1)$$

$$g_4(t) = 6t(1-t)(5t^2 - 5t + 1)$$

$$g_5(t) = 6t(1-t)(2t - 1)(6t^2 - 6t + 1)$$

- - - -

For example, when $n = 0$, this is called the 0 - order nodal expansion method. In this case, there exist six coefficients C_{ij} in (2.3), but considering symmetry we only select five coefficients as solution parameters which are expressed by average of u in element, and four sides of element respectively. For $n=1$ (it is called 1 - order nodal expansion method) there exist ten coefficients in (2.3), but only seven coefficients are selected as solving parameter which are expressed by average of u on the element and its sides respectively, and the moments of u .

3) In order to obtain approximating equations we employ a weighted Galerkin method

$$\begin{aligned} &\int_k w_\ell \,(\mathrm{div}\, \underline{J} + u)\,dxdy = \int_k w_\ell \, f \, dxdy \qquad \ell=1,2,\dots \ \forall\, k \in F_h \\ &\int_{\partial k} \overset{o}{w}_i \,(\underline{J} + \mathrm{grad}\, u) \cdot \underline{n} \,|_{\partial k}\, ds = 0 \qquad i=1,2,3,4 \end{aligned} \tag{2.4}$$

Therefore for example $n=0$, we select $w_\ell=1$ and

$$\overset{o}{w}_i = \begin{cases} 1 & \text{on } \partial_i k \\ 0 & \text{other sides} \end{cases} \qquad i = 1,2,3,4$$

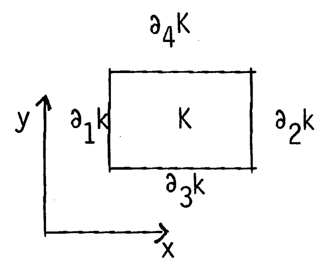
(2.4) can be rewritten

$$\int_k (\text{div } \underline{J} + u)dxdy = \int_k f \; dxdy \qquad \forall\, k \in F_h \tag{2.5}$$

$$\int_{\partial_i k} (\underline{J} + \text{grad } u) \cdot \underline{n} \,|_{\partial_i k} \; dS = 0 \qquad i=1,2,3,4$$

or n=1, we may select $w_\ell = 1$, $g_3\left(\frac{x-x_k}{h_2}\right)$, $g_3\left(\frac{y-y_k}{h_2}\right)$ and

$$\overset{o}{w}_i = \begin{cases} 1 & \text{on } \partial_i k \\ 0 & \text{other sides} \end{cases} \qquad i=1,2,3,4$$



So we have

$$\int_k (\text{div } \underline{J} + u)dxdy = \int_k f \; dxdy$$

$$\int_k g_3\left(\frac{x-x_k}{h_1}\right) (\text{div } \underline{J} + u)dxdy = \int_k g_3 \left(\frac{x-x_k}{h_1}\right) f \; dxdy$$

$$\int_K g_3\left(\frac{y-y_k}{h_2}\right) (\text{div } \underline{J} + u)dxdy = \int_k g_3 \left(\frac{y-y_k}{h_2}\right) f \; dxdy \tag{2.6}$$

$$\int_{\partial_i k} (\underline{J} + \text{grad } u) \cdot \underline{n} \,\Big|_{\partial_i k} \; ds = 0 \qquad i=1,2,3,4$$

4) Let us introduce symbols:

ϕ_k - average value of u in K

$$\phi_k = \frac{1}{|k|} \int_k u dxdy \,, \quad |K| \text{ is area of } K \tag{2.7}$$

ψ_{10}, ψ_{20}, ψ_{01}, ψ_{02} - average value of u on $\partial_i k$ respectively

$$\psi_{10} = \frac{1}{h_2} \int_{\partial_1 k} u\,d\ell \ , \ \psi_{20} = \frac{1}{h_2} \int_{\partial_2 k} u\,d\ell$$

$$\psi_{01} = \frac{1}{h_1} \int_{\partial_3 k} u\,d\ell \ , \ \psi_{02} = \frac{1}{h_1} \int_{\partial_4 k} u\,d\ell \qquad (2.8)$$

J_{10}, J_{20}, J_{01}, J_{02} - average value of flux $\underline{J}$ through $\partial_i k$ along x or y respectively.

$$J_{10} = -\frac{1}{h_2} \int_{\partial_1 k} \underline{J} \cdot \underline{n}\,d\ell \ , \quad J_{20} = \frac{1}{h_2} \int_{\partial_2 k} \underline{J} \cdot \underline{n}\,d\ell$$

$$J_{01} = \frac{-1}{h_1} \int_{\partial_3 k} \underline{J} \cdot \underline{n}\,d\ell, \qquad J_{02} = \frac{1}{h_1} \int_{\partial_4 k} \underline{J} \cdot \underline{n}\,d\ell \qquad (2.9)$$

5) By calculation it is easy to get

$$\{(g_i,g_j)\}_{\substack{i=0,1,2,3\\ j=0,1,2,3}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/5 \\ 0 & 0 & 1/5 & 0 \\ 0 & 1/5 & 0 & 6/35 \end{bmatrix}$$

$$\{(g_i',g_j')\}_{\substack{i=0,1,2,3\\ j=0,1,2,3}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 36/5 \end{bmatrix}$$

where $g'(t) = \frac{dg_i}{dt}$ and

$$(g_i,g_j) = \int_0^1 g_i(t)\ g_j(t)dt$$

Using (2.7), (2.8) and (2.9), the introduction of (2.3) with $n = 0$ into (2.4) leads to

$$(1+ \frac{12}{h_1^2} + \frac{12}{h_2^2}) \ \phi_\kappa - \frac{6}{h_1^2} (\psi_{10} + \psi_{20}) - \frac{6}{h_2^2} (\psi_{01} + \psi_{02}) = f^\kappa$$

$$\frac{1}{h_1} (\psi_{10} - \psi_{20}) + \frac{6}{h_1^2} \ (\frac{\psi_{10} + \psi_{20}}{2} - \phi_\kappa) - J_{10} = 0$$

$$-\frac{1}{h_2} (\psi_{10} - \psi_{20}) + \frac{6}{h_1^2} \ (\frac{\psi_{10} + \psi_{20}}{2} - \phi_\kappa) + J_{20} = 0$$

$$\frac{1}{h_2} (\psi_{01} - \psi_{02}) + \frac{6}{h_2^2} \ (\frac{\psi_{10} + \psi_{20}}{2} - \phi_\kappa) - J_{01} = 0$$

$$-\frac{1}{h_2} (\psi_{01} - \psi_{02}) + \frac{6}{h_2^2} \ (\frac{\psi_{01} + \psi_{02}}{2} - \phi_\kappa) + J_{02} = 0$$

where $f^\kappa = \frac{1}{|\kappa|} \int_\kappa f \, dxdy$. Replacing the first equation by the sum of the five equations we have

$$\phi_\kappa + \frac{1}{h_1} (J^\kappa_{20} - J^\kappa_{10}) + \frac{1}{h_2} (J^\kappa_{02} - J^\kappa_{01}) = f^\kappa$$

$$\frac{1}{h_1} \quad (\psi^\kappa_{10} - \psi^\kappa_{20}) + \frac{6}{h_1} \ (\ \frac{\psi_{10} + \psi_{20}}{2} \ - \ \phi_\kappa \ ) - J^k_{10} = 0$$

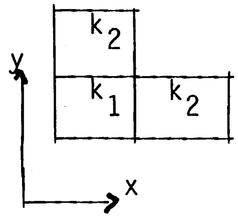
$$\frac{1}{h_1} \quad (\psi^k_{10} - \psi^\kappa_{20}) - \frac{6}{h_1} \ (\ \frac{\psi_{10} + \psi_{20}}{2} \ - \ \phi_\kappa \ ) - \ J^\kappa_{20} = 0 \qquad (2.10)$$

$$\frac{1}{h_2} \quad (\psi^k_{01} - \psi^\kappa_{02}) + \frac{6}{h_2} \ (\ \frac{\psi_{01} + \psi_{02}}{2} \ - \ \phi_\kappa \ ) - J^k_{01} = 0$$

$$\frac{1}{h_2} \quad (\psi^k_{01} - \psi^\kappa_{02}) - \frac{6}{h_2} \ (\ \frac{\psi_{10} + \psi_{20}}{2} \ - \ \phi_\kappa \ ) - \ J^\kappa_{02} = 0$$

are continuity equations

$$\psi_{10}^{k_2} = \psi_{20}^{k_1} \;,\quad \psi_{01}^{k_2} = \psi_{02}^{k_1}$$
$$J_{10}^{k_2} = J_{20}^{k_1} \;,\quad J_{01}^{k_2} = J_{02}^{k_1} \qquad (2.11)$$



So there exist nine unknowns ϕ_κ, ϕ_{10}^κ, ψ_{20}^κ, ψ_{01}^κ, ψ_{02}^κ, J_{10}^κ , J_{20}^κ , J_{01}^κ , J_{02}^κ in an element K, and nine equations (2.10) and (2.11) . The solution of (2.10) and (2.11) may be obtained by an interative method.

For n=1 we also can obtain algebraic equations of nodal expansion method of 1 - order

$$\phi_\kappa + \frac{1}{h_1}(J_{20}^k - J_{10}^k) + \frac{1}{h_2}(J_{02}^k - J_{01}^k) = f^k$$

$$\frac{1}{h_1}(\psi_{10}^k - \psi_{20}^k) + \frac{6}{h_1}\left(\frac{\psi_{10}^k + \psi_{20}^k}{2} - \phi_k\right) + \frac{35}{h_1}\psi_{11}^k - J_{10} = 0$$

$$\frac{1}{h_1}(\psi_{10}^k - \psi_{20}^k) - \frac{6}{h_1}\left(\frac{\psi_{10} + \psi_{20}}{2} - \phi_k\right) + \frac{35}{h_1}\psi_{11}^k - J_{20} = 0 \qquad (2.12)$$

$$\frac{1}{h_2}(\psi_{01}^k - \psi_{02}^k) + \frac{6}{h_2}\left(\frac{\psi_{01} + \psi_{02}}{2} - \phi_k\right) + \frac{35}{h_2}\psi_{22}^k - J_{01} = 0$$

$$\frac{1}{h_2}(\psi_{01}^k - \psi_{02}^k) - \frac{6}{h_2}\left(\frac{\psi_{01} + \psi_{02}}{2} - \phi_k\right) + \frac{35}{h_2}\psi_{22}^k - J_{02} = 0$$

$$\frac{1}{7\sqrt{5}}\left(1 + \frac{42}{h_1^2}\right)\psi_{11}^k + \frac{1}{10}(\psi_{20}^k - \psi_{10}^k) = \left(f,\ g_3\left(\frac{x-x_k}{h_1}\right)\right)/|K|$$

$$\frac{1}{7\sqrt{5}}\left(1 + \frac{42}{h_2^2}\right)\psi_{22}^k + \frac{1}{10}(\psi_{02}^k - \psi_{01}^k) = \left(f,\ g_3\left(\frac{y-y_k}{h_2}\right)\right)/|K|$$

and continuity equations (2.11).

Now we construct a special finite element subspace. Let $\hat{K} = \{(\xi\cdot\eta)$, $0 \leq \xi \leq 1$, $0 \leq \eta \leq 1\}$ be a reference element and a real element K can be obtained by affine transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = T_\kappa \begin{bmatrix} \xi \\ \eta \end{bmatrix} \equiv \begin{bmatrix} h, & 0 \\ 0 & h_2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \begin{bmatrix} X_k \\ Y_\kappa \end{bmatrix}$$

where (X_κ, y_κ) is coordinate of below left vertex of element K. Let us introduce shape functions in reference element $\hat{K}$

$$\begin{aligned} \hat{\Phi}_{oo}(\xi,\eta) &= 1-g_2(\xi) - g_2(\eta) = -1 + 6\xi + 6\eta - 6\xi^2 - 6\eta^2 \\ \hat{\Phi}_{10}(\xi,\eta) &= \frac{1}{2}(g_2(\xi) - g_1(\xi)) = 1 - 4\xi + 3\xi^2 \\ \hat{\Phi}_{20}(\xi,\eta) &= \frac{1}{2}(g_2(\xi) + g_1(\xi)) = -2\xi + 3\xi^2 \\ \hat{\Phi}_{01}(\xi,\eta) &= \frac{1}{2}(g_2(\eta) - g_1(\eta)) = 1 - 4\eta + 3\eta^2 \\ \hat{\Phi}_{02}(\xi,\eta) &= \frac{1}{2}(g_2(\eta) + g_1(\eta)) = -2\eta + 3\eta^2 \end{aligned} \tag{2.13}$$

and for $n=1$ we need to add shape functions

$$\hat{\Phi}_{11} = \frac{\sqrt{5}}{6} g_3(\xi) , \qquad \hat{\Phi}_{22} = \frac{\sqrt{5}}{6} g_3(\eta) \tag{2.14}$$

The shape functions in real element K are

$$\phi_{ij} = \hat{\Phi}_{ij} \circ T_\kappa^{-1} \tag{2.15}$$

On the other hand, shape functions on ∂K are

$$\lambda_{10} = \begin{cases} 1 & \text{on } \partial_1 k \\ 0 & \text{other sides} \end{cases}, \qquad \lambda_{20} = \begin{cases} -1 & \text{on } \partial_2 k \\ 0 & \text{other sides} \end{cases}$$

$$\lambda_{01} = \begin{cases} 1 & \text{on } \partial_3 k \\ 0 & \text{other sides} \end{cases}, \qquad \lambda_{02} = \begin{cases} -1 & \text{on } \partial_4 k \\ 0 & \text{other sides} \end{cases} \tag{2.16}$$

For $n=1$, the shape functions on $\partial\hat{K}$ are chosen as piecewise constants (2.16) or piecewise linear polynomials, for example

$$\hat{\lambda}_{10} = \begin{cases} 1 + g_1(\eta) & \text{on } \partial_1\hat{K} \\ 0 & \text{other sides} \end{cases}, \qquad \hat{\lambda}_{20} = \begin{cases} -1 - g_1(\eta) & \text{on } \partial_2\hat{K} \\ 0 & \text{other sides} \end{cases}$$

$$\hat{\lambda}_{01} = \begin{cases} 1 + g_1(\xi) & \text{on } \partial_3\hat{K} \\ 0 & \text{other sides} \end{cases}, \qquad \hat{\lambda}_{02} = \begin{cases} -1 - g_1(\xi) & \text{on } \partial_4\hat{k} \\ 0 & \text{other sides} \end{cases} \tag{2.17}$$

Then we can construct finite element subspaces

$$X_h = \{ v \in L^2(\Omega) \; ; \; v|_k \in P_{1k} \; , \; \forall\, k \in F_h \} \tag{2.18}$$

$$Y_h = \{ v \in L^2(\Omega) \; ; \; v|_k \in P_{ok} \; , \; \forall\, k \in F_h \} \tag{2.19}$$

$$M_h = \{ \mu \in \prod_{k \in F_h} Q_{ok} \; ; \; \mu|_{\partial k_1} + \mu|_{\partial k_2} = 0 \quad \text{on } \overline{K}_1 \cap \overline{K}_2 \, ,$$

$$k_1, k_2 \text{ are adjacent elements} \} \tag{2.20}$$

where F_h is a rectangulation of Ω, and

$$P_{1k} = \{ v|_k \in H^1(k) \; , \; v = \hat{v} \circ T_k^{-1} \; , \; \hat{v} \in \hat{P}_1 \} = \text{span } \{\phi_{oo}, \phi_{10}, \phi_{20}, \phi_{01}, \phi_{02}\}$$

$$P_{ok} = \{ v|_k \in H^1(k), \; v = \hat{v} \circ T_k^{-1} \; , \; \hat{v} \in \hat{P}_o \} = \text{span } \{\phi_{oo}\}$$

$$Q_{ok} = \text{span } \{\lambda_{10}, \lambda_{20}, \lambda_{01}, \lambda_{02}\}$$

$$\hat{P}_1 = \text{span } \{\hat{\phi}_{oo}, \hat{\phi}_{10}, \hat{\phi}_{20}, \hat{\phi}_{01}, \hat{\phi}_{02}\} \quad , \quad \hat{P}_o = \text{span } \{\hat{\phi}_{oo}\}$$

We consider Sobolev spaces

$$X = \{ v \in L^2(\Omega) \ ; \ v|_k \in H^1(k), \ \ 1 \leq K \leq N \}$$

$$M = \{ \mu \in \prod_{k=1,N} H^{-1/2}(\partial k) \ ; \ \exists\, q \in H\,(\text{div};\ \Omega)\ ,\ qn_k = \mu|_{\partial k},\ 1 \leq k \leq N \}$$

with norms respectively

$$\| v \|_X^2 = \sum_{k \in F_h} \| v \|_{1,k}^2$$

$$\| \mu \|_M = \inf \{ \| q \| ;\ \ q \in H(\text{div};\ \Omega),\ q_k = \mu \ \text{ on } \ K\ ,\ \ 1 \leq k \leq N \}$$

where

$$H(\text{div};\ \Omega) = \{ q \in (L^2(\Omega))^2 \ ; \ \text{div}\ q \in L^2(\Omega) \}$$

$$\| q \|_{H(\text{div};\ \Omega)} = (\| q \|_{0,\Omega}^2 + \| \text{div}\ q \|_{0,\Omega}^2)^{1/2}$$

It is obvious that $X_h \subset X$, $Y_h \subset L^2(\Omega)$, $M_h \subset M$.

From (2.13) and (2.14), the shape functions possess properties

$$\frac{1}{|k|} \int_k \phi_{ij} \ dx\ dy = \begin{cases} 1 & i=j=0 \\ 0 & i \neq j \end{cases}$$

$$\frac{1}{|\partial_i k|} \int_{\partial_i k} \phi_{mn}\ d\ell = \begin{cases} 1 & i=1,2;\ m \neq 0,\ n=0 \\ & i=3,4;\ m=0,\ n \neq 0 \\ 0 & \text{other} \end{cases}$$

and the following tables 1,2 :

Table 1 . $\frac{1}{|k|}(\nabla\phi_{ij}, \nabla\phi_{mn}) = \frac{1}{|k|}\int_k \nabla\phi_{ij} \cdot \nabla\phi_{mn}\, dxdy$

$(\cdot,\cdot)$ ij \ mn	0 0	10	20	01	02	11	22
00	$12(h_1^{-2}+h_2^{-2})$	$-6h_1^{-2}$	$-6h_1^{-2}$	$-6h_2^{-2}$	$-6h_2^{-2}$	0	0
10		$4h_1^{-2}$	$2h_1^{-2}$	0	0	0	0
20			$4h_1^{-2}$	0	0	0	0
01				$4h_2^{-2}$	$2h_2^{-2}$	0	0
	symmetry						
02					$4h_2^{-2}$	0	0
11						h_1^{-2}	0
22							h_2^{-2}

Table 2. $\frac{1}{|k|}(\phi_{ij},\phi_{mn}) = \frac{1}{|k|}\int_k \phi_{ij}\cdot\phi_{mn}\,dxdy$

$(\cdot,\cdot)$ ij \ mn	0 0	10	20	01	02	11	22
00	$\frac{5}{7}$	$-\frac{1}{10}$	$-\frac{1}{10}$	$-\frac{1}{10}$	$-\frac{1}{10}$	0	0
10		$\frac{2}{15}$	$-\frac{1}{30}$	0	0	$-\frac{1}{12\sqrt{5}}$	0
20			$\frac{2}{15}$	0	0	$\frac{1}{12\sqrt{5}}$	0
01				$\frac{2}{15}$	$-\frac{1}{30}$	0	$-\frac{1}{12\sqrt{5}}$
	symmetry						
02					$\frac{2}{15}$	0	$\frac{1}{12\sqrt{5}}$
11						$\frac{1}{42}$	0
22							$\frac{1}{42}$

Now let us define interpolation operators Π_{1h} and Π_{2h}:

$$\Pi_{1h} : X \to X_h\ ,\ \Pi_{1h}\,u = \phi_k\,\phi_{oo} + \psi_{10}\,\phi_{10} + \psi_{20}\,\phi_{20} + \psi_{01}\,\phi_{01} + \psi_{02}\,\phi_{02} \tag{2.21}$$

$$\Pi_{2h} : X \to Y_h\ ,\ \Pi_{2h}\,u = \phi_k\,\phi_{oo}$$

and likewise, for $\underline{J} \in H(\mathrm{div};\Omega)$, $\underline{J} \cdot \underline{n}\,|_{\partial K} = \mu$

$$\Pi_{oh} : M \to M_h,\ \Pi_{oh}\,\mu = J_{10}\,\lambda_{10} + J_{20}\,\lambda_{20} + J_{01}\,\lambda_{01} + J_{02}\,\lambda_{02} \tag{2.22}$$

where ϕ_k, $\psi_{10},\dots \psi_{02}$, $J_{10},\dots J_{02}$ are defined by (2.7) - (2.9) . It is obvious that

$$\frac{1}{|k|}\int_k \Pi_{1h}\, u\, dxdy = \phi_k = \frac{1}{|k|}\int_k u\, dxdy$$

$$\frac{1}{|\partial_i k|}\int_{\partial_i k} \Pi_{1h}\, u\, d\ell = \begin{cases} \psi_{io} & (i=1,2) \\ \psi_{oi-2} & (i=3,4) \end{cases} = \frac{1}{|\partial_i k|}\int_{\partial_i k} u d\ell$$

$$\frac{1}{|\partial_i k|}\int_{\partial_i k} \Pi_{oh}\, u\, d\ell = \begin{cases} J_{io} & (i=1,2) \\ J_{oi-2} & (i=3,4) \end{cases} = \frac{1}{|\partial_i k|}\int_{\partial_i k} u d\ell$$

i.e. Π_{1h}, Π_{2h} and Π_{oh} satisfy

$$\int_k (\Pi_{1h}\, u-u) dxdy=0 \quad , \quad \int_k (\Pi_{2h}\, u-u)\, dxdy=0 \tag{2.23}$$

$$\int_{\partial_i k} (\Pi_{1h}\, u-u) d\ell=0 \tag{2.24}$$

$$\int_{\partial_i k} (\Pi_{oh}\, \mu-\mu) d\ell=0 \tag{2.25}$$

<u>Remark</u> For $n=1$ we consider the interpolation function

$$\hat{\Pi}_{1_h} u = \phi_\kappa \phi_{oo} + \psi_{10} \phi_{10} + \psi_{10} \phi_{20} + \psi_{01} \phi_{01} + \psi_{02} \phi_{02} + \psi_{11} \phi_{11} + \psi_{22} \phi_{22}$$

It is obvious $\phi_\kappa, \psi_{10}, \psi_{20}, \psi_{01}, \psi_{02}$ can be defined by (2.7), (2.8) and

$$\psi_1 = \frac{h_1^2}{|k|} \int_k \nabla u \cdot \nabla \phi_{11} \, dxdy \quad , \quad \psi_{22} = \frac{h_2^2}{|k|} \int_k \nabla u \cdot \nabla \phi_{22} \, dxdy$$

By Table 1 , we have

$$\psi_{11} = \frac{h_1^2}{|k|} \int_k \hat{\nabla}(\Pi_{1h} u) \nabla \phi_{11} \, dxdy \ , \quad \psi_{22} = \frac{h_2}{|k|} \int_k \nabla(\Pi_{1h} u) \nabla\phi_2 \, dxdy$$

i.e. $\hat{\Pi}_{1h}$ satisfies

$$\int_k \nabla(\Pi_{1h} u-u) \nabla\phi_{ii} \, dxdy = 0 \qquad i=1,2$$

In addition, $\hat{\Pi}_{1h}$ also satisfies (2.23) - (2.25).

Now we consider the primal hybrid variational formulation associated with boundary value problem (2.1)

$$\begin{aligned} &\text{find } (u,\lambda) \in X \times M \text{ such that} \\ &a(u,v) + b(\lambda,v) = f(v) && \forall\, v \in X \\ &b(\mu,u) = 0 && \forall\, \mu \in M \end{aligned} \qquad (2.26)$$

where $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are bilinear forms on $X \times X$ and $X \times M$ respectively defined by:

$$a(u,v) = \sum_{k \subset F_a} \int_k \nabla u \, \nabla v \, dxdy + \int_\Omega uv \, dxdy$$

$$b(\lambda,v) = \sum_{k \subset F_h} \int_{\partial\kappa} \lambda\, v\, d\ell$$

and the linear form

$$f(v) = \int_{\Omega} fv\ dxdy$$

the approximation problem associated with (2.26) is as follows

$$\begin{aligned} &\text{find } (u_h,\ \lambda_h) \in X_h \times M_h \quad \text{such that} \\ &a(u_h,\ v_h) + b(\lambda_h,\ v_h) = f(v_h) \qquad \forall\ v_h \in X_h \\ &b(\mu_h,\ u_h) = 0 \qquad \forall\ \mu_h \in M_h \end{aligned} \tag{2.27}$$

Using Tables 1,2, the introduction of (2.21), (2.22) into (2.27) leads to the finite element algebraic equations

$$\begin{aligned}
&\phi_k + \frac{1}{h_1}(J_{20}^k - J_{10}^k) + \frac{1}{h_2}(J_{02}^k - J_{01}^k) = f_{00}^k + f_{10}^k + f_{20}^k + f_{01}^k + f_{02}^2 \\
&\frac{1}{h_1}\left(1+\frac{h_1^2}{12}\right)(\psi_{10}^k - \psi_{20}^k) + \frac{6}{h_1}\left(1+\frac{h_1^2}{60}\right)\left(\frac{\psi_{10}^k + \psi_{20}^k}{2} - \phi_\kappa\right) - J_{10}^k = h_1 f_{10}^k \\
&\frac{1}{h_1}\left(1+\frac{h_1^2}{12}\right)(\psi_{10}^k - \psi_{20}^k) - \frac{6}{h_1}\left(1+\frac{h_1^2}{60}\right)\left(\frac{\psi_{10}^k + \psi_{20}^k}{2} - \phi_\kappa\right) - J_{20}^k = h_1 f_{20}^k \\
&\frac{1}{h_2}\left(1+\frac{h_2^2}{12}\right)(\psi_{01}^k - \psi_{02}^k) + \frac{6}{h_2}\left(1+\frac{h_1^2}{60}\right)\left(\frac{\psi_{01} + \psi_{02}}{2} - \phi_\kappa\right) - J_{01}^k = h_2 f_{01}^k \\
&\frac{1}{h_2}\left(1+\frac{h_2^2}{12}\right)(\psi_{01}^k - \psi_{02}^k) - \frac{6}{h_2}\left(1+\frac{h_1^2}{60}\right)\left(\frac{\psi_{01} + \psi_{02}}{2} - \phi_\kappa\right) - J_{02}^k = h_2 f_{02}^k
\end{aligned} \tag{2.28}$$

and

$$\psi_{10}^{k_2} = \psi_{20}^{k_1}, \quad \psi_{01}^{k_2} = \psi_{02}^{k_1} \tag{2.29}$$

$$J_{10}^{k_2} = J_{20}^{k_1}, \quad J_{01}^{k_2} = J_{02}^{k_1}$$

where $\quad f_{ij}^{k} = \frac{1}{|k|} \int_k f \; \phi_{ij} \; dxdy = \int_{\hat{k}} f \; \hat{\phi}_{ij} \; d\xi \, d\eta$

In view of (2.13)

$$\hat{\phi}_{00} + \hat{\phi}_{10} + \hat{\phi}_{20} + \hat{\phi}_{01} + \hat{\phi}_{02} \equiv 1 \tag{2.30}$$

So

$$\sum f_{ij}^{k} = f^{k}$$

Comparing between the nodal expansion algebraic equations (2.10) and primal hybrid finite element algebraic equations (2.28) we discover that if terms of higher - order (h_1, h_2) in (2.28) are neglected then (2.28) is equivalent to (2.10).

For $n = 1$ we define $\hat{X}_h = \text{span}\{\phi_{00}, \phi_{10}, \phi_{20}, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{22}\}$. The approximation problem for (2.26) can be expressed as

find $(u_h, \lambda_h) \in \hat{X}_h \times u_h$ such that

$$a(u_h, v_h) + b(\lambda_h, v_h) = \langle f, v_h \rangle \qquad \forall\, v_h \in \hat{X}_h \tag{2.31}$$

$$b(u_h, u_h) = 0 \qquad \forall\, \mu_h \in \mu_h$$

Using (2.30) and tables 1,2 we can obtain algebraic equations from (2.31):

$$\phi_k + \frac{1}{h_1}(J_{20}^k - j_{10}^k) + \frac{1}{h_2}(J_{02}^k - J_{01}^k) = f^k$$

$$\frac{1}{h_1}(\psi_{10}^k - \psi_{20}^k) + \frac{6}{h_1}\left(\frac{\psi_{10}^k + \psi_{20}^k}{2} - \phi_k\right) + \frac{35}{h_1}\psi_{11}^k + \frac{h_1}{30}(4\psi_{10}^k - \psi_{20}^k - 3\phi_k)$$

$$-\left(\frac{35}{h_1} + \frac{h_1}{12\sqrt{5}}\right)\psi_{11}^k - J_{10}^k = h_1 f_{10}$$

$$\frac{1}{h_1}(\psi_{10}^k - \psi_{20}^k) - \frac{6}{h_1}\left(\frac{\psi_{10}^k + \psi_{20}^k}{2} - \phi_k\right) + \frac{35}{h_1}\psi_{11}^k - \frac{h_1}{30}(4\psi_{20} - \psi_{10} - 3\phi_k)$$

$$-\left(\frac{35}{h_1} + \frac{h_1^k}{12\sqrt{5}}\right)\psi_{11}^k - J_{20}^k = -h_1 f_{20}$$

$$\frac{1}{h_2}(\psi_{01}^k - \psi_{02}^k) + \frac{6}{h_2}\left(\frac{\psi_{01}^k + \psi_{02}^k}{2} - \phi_k\right) + \frac{35}{h_2}\psi_{22}^k + \frac{h_2}{30}(4\psi_{01}^k - \psi_{02}^k - 3\phi_k)$$

$$-\left(\frac{35}{h_2} + \frac{h_2}{12\sqrt{5}}\right)\psi_{22}^k - J_{01} = h_2 f_{01}^k$$

$$\frac{1}{h_2}(\psi_{01}^k - \psi_{02}^k) - \frac{6}{h_2}\left(\frac{\psi_{01}^k + \psi_{02}^k}{2} - \phi_k\right) + \frac{35}{h_2}\psi_{22}^k - \frac{h_2}{30}(4\psi_{02}^k - \psi_{01}^k \quad 3\phi_k)$$

$$-\left(\frac{35}{h_2} + \frac{h_2}{12\sqrt{5}}\right)\psi_{22}^k - J_{02}^k = -h_2 f_{02}^k$$

$$\left(\frac{1}{42} + \frac{1}{h_1^2}\right)\psi_{11}^k + \frac{1}{12\sqrt{5}}(\psi_{20}^k - \psi_{10}^k) = (f,\ \phi_{11}) \,/\, |k|$$

$$\left(\frac{1}{42} + \frac{1}{h_2^2}\right)\psi_{22}^k + \frac{1}{12\sqrt{5}}(\psi_{02}^k - \psi_{01}^k) = (f,\ \Phi_{22}) \,/\, |k|$$

and (2.29) .

Therefore we need to establish a new variational formulation corresponding nodal expansion method. We will discuss it in next section and it is called generalized primal hybrid variational formulation.

3. Variational Formulation for the Nodal Expansion Method

Let us introduce the bilinear form $X \times X \to R$:

$$\tilde{a}(u,v) = \sum_{k \in F_h} \int_k (\nabla u \ \nabla v + u \ \Pi_{2h} \ v) \, dxdy \qquad \forall \ u \in X \ , \ v \in X \qquad (3.1)$$

and the linear form $X \to R$

$$\tilde{f}(v) = \sum_{k \in F_h} \int_k f \ \Pi_{2h} \ v \, dxdy \qquad \forall \ v \in X \qquad (3.2)$$

Then the variational formulation associated with nodal expansion method of 0-order can be expressed as the following

find $(\tilde{u}, \tilde{\lambda}) \in X \times M$ such that

$$\tilde{a}(\tilde{u},v) + b(\lambda,v) = \tilde{f}(v) \qquad \forall \ v \in X \qquad (3.3)$$

$$b(\mu,\tilde{u}) = 0 \qquad \forall \ \mu \in M$$

By virtue of Sobolev's theorem of equivalent norm and Schwartz inequality we obtain

$$C_2 \ \|\psi\|_{1,k}^2 + \frac{1}{|k|^2} \left(\int_k \psi dxdy \right)^2 \leq c \ \|\psi\|_{1,k}^2 \qquad \forall \psi \in H^1(k) \qquad (3.4)$$

in consequence

$$|\tilde{a}(u,v)| \leq c \; \|u\|_{1,\Omega} \; \|v\|_{1,\Omega} \qquad \forall \; u,v \in H^1(\Omega)$$

$$|\tilde{a}(u,u)| \geq c \; \|u\|^2_{1,\Omega} \qquad \forall \; u \in H^1(\Omega)$$

In addition, it is well known that bilinear $b(\cdot,\cdot)$ and X, M satisfy the Babuśka-Brezzi condition. Furthermore, applying (3.4) we have

$$|\tilde{f}(v)| \leq \|f\|_{0,\Omega} \; \|v\|_X \qquad \forall \; v \in X$$

i.e. $\tilde{f}(v) \in X^*$ (dual space to X). According to Brezzi's theory there exists a unique solution $(\tilde{u},\tilde{\lambda})$ of (3.3) and

$$\|\tilde{u}\|_X + \|\tilde{\lambda}\|_M \leq c \; \|\tilde{f}\|_{X}^{*} \tag{3.5}$$

Lemma 1 Let $f^k = \int_k \int_k f dxdy \; / \; |k|$, $\forall \; k \in F_h$. Then $\forall \; \psi \in X$, we have

$$\Big| \sum_{k \in F_h} \int_k (f-f^k)\psi dxdy \Big| \leq \begin{cases} ch^2 \; \|f\|_X \; \|\psi\|_X & \text{for } f \in X \\ ch \; \|f\|_{0,\Omega} \; \|\psi\|_X & \text{for } f \in L^2(\Omega) \end{cases} \tag{3.6}$$

Proof For every $K \in F_h$, we have

$$\Big| \int_k (f-f^k)\psi dxdy \Big| = \Big| \int_k (f-f^k)(\psi-\psi^k) dxdy \Big| \leq \|f-f^k\|_{0,k} \; \|\psi-\psi^k\|_{0,k}$$

$$\text{by [6]} \leq \begin{cases} ch^2 \|f\|_{1,k} \|\psi\|_{1,k} & \text{for } f \in X \\ ch \|f\|_{0,k} \|\psi\|_{1,k} & \text{for } f \in L^2(\Omega) \end{cases} \quad \forall \psi \in X$$

applying the Cauchy inequality the lemma is yielded.

<u>Theorem 1</u> Suppose (u,λ) is a solution of primal variational problem (2.26) and $(\tilde{u},\tilde{\lambda})$ is a solution of (3.3). Then

$$\|u-u\|_X + \|\lambda-\tilde{\lambda}\|_M \leq \begin{cases} ch^2 (\|f\|_X + \|f\|_{0,\Omega}) & \text{for } f \in X \\ ch \|f\|_{0,\Omega} & \text{for } f \in L^2(\Omega) \end{cases} \tag{3.7}$$

<u>Proof</u> By subtracting the (3.3) from (2.26) we obtain

$$\sum_{k\in F_h} \int_k \nabla(u-\tilde{u}) \nabla v dxdy + \sum_{k\in F_h} \int_k [(u-\tilde{u})v + \tilde{u}(v - \Pi_{2h} v)]dxdy$$

$$+ b(\lambda-\tilde{\lambda}, v) = \sum_{k\in F_h} \int_k f(v - \Pi_{2h} v)dxdy \qquad \forall v \in X$$

$$b(\mu, u-u) = 0$$

let $w = u-\tilde{u}$, $\sigma = \lambda-\tilde{\lambda}$ above relation can be rewritten as

$$\begin{aligned} &a(w,v) + b(\sigma,v) = G(v) && \forall v \in X \\ &b(\mu, w) = 0 && \forall \mu \in M \end{aligned} \tag{3.8}$$

where

$$G(v) = \sum_{k\in F_h} \int_k (f-\tilde{u})(v - \Pi_{2h}v)dxdy$$

By lemma 1

$$|G(v)| \leq \begin{cases} ch^2 \, \|f-\tilde{u}\|_X \, \|v\|_X & \text{for} \quad f \in X \\ \\ ch \, \|f-\tilde{u}\|_{0\Omega} \, \|v\|_X & \text{for} \quad f \in L^2(\Omega) \end{cases}$$

Therefore, by (3.5)

$$\|G\|_X^* \leq \begin{cases} ch^2(\|f\|_X + \|f\|_{0,\Omega}) & \text{for} \quad f \in X \\ \\ ch \, \|f\|_{0,\Omega} & \text{for} \quad f \in L^2(\Omega) \end{cases}$$

Applying Brezzi's theory, problem (3.8) has a unique solution (w,σ) and

$$\|w\|_X + \|\sigma\|_M \leq c \, \|G\|_X^* \quad \leq \begin{cases} Ch^2 \, (\|f\|_X + \|f\|_{0,\Omega}) \\ \\ Ch \, \|f\|_{0,\Omega} \end{cases}$$

that is (3.7). The proof is completed.

<u>Remark</u> If $f \in L'(\Omega)$, (3.6) and (3.7) are still valid when $\|f\|_{0,\Omega}$ replaced by $\|f\|_X^*$.

4. Nodal Expansion Approximation

Assume X_h, Y_h, M_h are finite element subspaces defined by (2.18) - (2.20) respectively. Approximation problem associated with (3.3) is the following

find $(\tilde{u}_h, \tilde{\lambda}_h) \in X_h \times M_h$ such that

$$\tilde{a}\,(\tilde{u}_h,v) + b(\tilde{\lambda}_h,v) = \tilde{f}(v) \qquad \forall \; v \in X_h \tag{4.1}$$

$$b(\mu,\tilde{u}_h) = 0 \qquad \forall \; \mu \in M_h$$

Using tables 1,2 it is easy to calculate (4.1). The resulting algebraic equations are equal to nodal expansion algebraic equations (2.10),(2.11). In other words, nodal expansion method is also a nonstandard finite element method. Its corresponding variational formulation and finite element subspaces are just expressed by (3.3) and X_h, M_h .

In order to establish error estimates for the nodal expansion approximation solution we need following two lemmas

Lemma 2 For $u \in X$, we have

$$\tilde{a}(u-\Pi_{1h}u,\ v) = 0 \qquad \forall\ v \in X_h \tag{4.2}$$

Proof

$$\tilde{a}(u-\Pi_{1h}u,\ v) = \sum_{k\in F_h} \left(\int_k \nabla(u-\Pi_{1h}u) \cdot \nabla v\,dxdy + \int_k (u-\Pi_{1h}u)\Pi_{2h}v\,dxdy \right)$$

From $\Pi_{2h}v$ = constant and (2.23), we have

$$\int_k (u-\Pi_{1h}u)\ \Pi_{2h}\ v\,dxdy = (\Pi_{2h}\ v) \int_k (u-\Pi_{1h}u)dxdy = 0$$

On the other hand

$$\int_k \nabla(u-\Pi_{1h}u) \cdot \nabla v\,dxdy = \int_{\partial\kappa} (u-\Pi_{1h}u)\ \nabla v \cdot \underline{n}\ d\ell - \int_k (u-\Pi_{1h}u)\nabla^2 v\ dxdy$$

it is easy to show $\forall\ v \in X_h$, $\nabla^2 v\ |_k$ = constant. Therefore

$$\int_k (u-\Pi_{1h}u)\nabla^2 v\ dxdy = (\nabla^2 v) \int_k (u-\Pi_{1h}u)dxdy = 0 \qquad \forall\ v \in X_h$$

In addition

$$\nabla v \cdot \underline{n} = \frac{\partial v}{\partial n} = (-1)^i \frac{\partial v}{\partial x} \qquad \text{on} \quad \partial_i k (i=1,2)$$

$\forall\ v \in X_h$, $\frac{\partial v}{\partial x}$ is a linear polynomial of x on $\partial_1 \kappa$, $\partial_2 \kappa$. So

$$\int_{\partial_i k} (u - \Pi_{1h} u)\ \nabla v \cdot \underline{n}\ d\ell = 0 \qquad i=1,2$$

Likewise

$$\int_{\partial_i k} (u - \Pi_{1h} u) \nabla v \cdot \underline{n}\ d\ell = 0 \qquad i=3,4$$

The proof is completed.

Lemma 3 The bilinear form $b(\cdot,\cdot)$ and the finite element subspaces X_h, M_h defined by (2.18), (2.20) satisfy the discrete Babuška-Brezzi condition

$$\sup_{0 \neq v_h \in X_h} \frac{b(\lambda_h, v_h)}{\| v_h \|_X} \geq \gamma \| \lambda_h \|_M \qquad \forall\ \lambda_h \in M_h \tag{4.3}$$

where constant γ is independent of h.

Proof In order to prove (4.3) it is sufficient to prove[5] that the interpolation operators have following proportion

$$\| \Pi_{1h} v \|_X \leq c\ \| v \|_X \tag{4.4}$$

$$b(v - \Pi_{1h} v,\ \mu_h) = 0 \qquad \forall\ \mu_h \in M_h \tag{4.5}$$

In view of the construction of the subspace M_h and interpolation proportion (2.23), (2.24) it is easy to get (4.5).

On the other hand, applying tables 1,2, we have

$$|\Pi_{1h}\, v|^2_{1,k} = \int_k \Big(\sum_{ij} \phi_{ijx} \sum_{mn} \phi_{mnx} + \sum_{ij} \phi_{ijy} \sum_{mn} \phi_{mny} \Big) \psi^k_{ij}\, \psi^k_{mn}\, dxdy$$

$$= \int_k \sum_{ij} \sum_{mn} \nabla\phi_{ij}\, \nabla\phi_{mn}\, \psi_{ij}\, \psi_{mn}\, dxdy$$

$$= 2|k| \Big[6 \Big(\frac{1}{h_1^2} + \frac{1}{h^2}\Big) \phi_k^2 + \frac{2}{h_1^2} (\psi_{10}^2 + \psi_{20}^2) + \frac{2}{h_2^2} (\psi_{01}^2 + \psi_{02}^2) -$$

$$- \frac{6}{h_1^2} (\phi_k\, \psi_{10} + \phi_k\, \psi_{20}) - \frac{6}{h_2^2} (\phi_k\, \psi_{01} + \phi_k\, \psi_{02}) + \frac{2}{h_1^2}\, \psi_{10}\, \psi_{20} + \frac{2}{h_2^2}\, \psi_{01}\, \psi_{02} \Big]$$

$$= \frac{4}{h_1^2} |k| \, [(\psi_{10} - \psi_{20})^2 + 3(\psi_{10} - \phi_k)(\psi_{20} - \phi_k)] +$$

$$+ \frac{4}{h^2} |k| \, [(\psi_{01} - \psi_{02})^2 + 3(\psi_{01} - \phi_k)(\psi_{02} - \phi_k)]$$

where $\psi_{00} \equiv \phi_k$, $\psi_{ij} \equiv \psi^k_{ij}$. Likewise

$$\|\Pi_{1h} v\|^2_{0,k} = \int_k \sum_{ij} \sum_{mn} \phi_{ij}\, \phi_{mn}\, \psi_{ij}\, \psi_{mn}\, dxdy$$

$$= |k|\, \phi_k^2 + \frac{1}{15} |k| \, \{2(\psi_{10} - \psi_{20})^2 + 3\, (\psi_{10} - \phi_k)(\psi_{20} - \phi_k)] +$$

$$+ \frac{1}{15} |k| \, [2(\psi_{01} - \psi_{02})^2 + 3\, (\psi_{01} - \phi_k)(\psi_{02} - \phi_k)]$$

If $v|_k \in D(\bar{k})$ then we have

$$(\psi_{10} - \psi_{20})^2 = \frac{1}{h_2^2} \left(\int_k \frac{\partial v}{\partial x} dxdy \right)^2 \leq \frac{|k|}{h_2^2} \int_k \left(\frac{\partial v}{\partial x} \right)^2 dx\, dy$$

$$(\psi_{01} - \psi_{02})^2 \leq \frac{|k|}{h_1^2} \int_k \left(\frac{\partial v}{\partial y}\right)^2 dxdy$$

$$(\psi_{10}-\phi_k)(\psi_{20}-\phi_k) = \frac{1}{|k|^2} \int_k (v(x_k,y)-v(x,y))dxdy \cdot \int_k v(x_k+h_1,y)-v(x,y))dxdy$$

$$= \frac{1}{|k|^2} \int_k \left(\int_x^{x_k} \frac{\partial v}{\partial t} dt\right)dxdy \cdot \int_k \left(\int_x^{x_k+h_1} \frac{\partial v}{\partial t} dt\right)dxdy$$

$$\leq \frac{1}{|k|^2} \frac{|k|h_1}{2} \int_k \left(\int_{x_k}^{x_k+h_1} \left(\frac{\partial v}{\partial t}\right)^2 dt\right)dxdy$$

$$= \frac{h_1}{2h_2} \int_k \left(\frac{\partial v}{\partial x}\right)^2 dxdy$$

$$(\psi_{01} - \phi_k)(\psi_{02} - \phi_k) \leq \frac{h_2}{2h_1} \int_k \left(\frac{\partial v}{\partial y}\right)^2 dxdy$$

$\forall\, v|_k \in H^1(K)$, due to the density of $D(\bar{k})$ in $H^1(k)$ the above inequalities are also valid. In view of the regularity assumption of partition F_h we have

$$|\, \Pi_{1h} v\, |_{1,K}^2 \leq c\, |v|_{1,K}^2 \qquad \forall\, v \in H^1(K)$$

$$\|\, \Pi_{1h} v\, \|_{0,k}^2 \leq \|v\|_{0,k}^2 + ch^2\, |v|_{1,k}^2 \qquad \forall\, v \in H^1(K)$$

Consequently, we obtain (4.3). This proof completed.

Concerning the bilinear form $\tilde{a}(.,.)$ it is clear to yield

$$|\tilde{a}(u_h,v_h)| \leq M \|u_h\|_X \|v_h\|_X \qquad \forall\, u_h,v_h \in X_h \tag{4.6}$$

$$|\tilde{a}(u_h,u_h)| \geq \alpha \|u_h\|_X^2 \qquad \forall\, u_h \in X_h \tag{4.7}$$

<u>Theorem 2</u> There exist a unique solution $(\tilde{u}_h,\tilde{\lambda}_h)$ for problem (4.1) and the following inequality is valid

$$\|\tilde{u}-\tilde{u}_h\|_X + \|\tilde{\lambda}-\tilde{\lambda}_h\|_M \leq c \left\{ \inf_{v_h\in X_h} \|\tilde{u}-v_h\|_X + \right.$$

$$\left. + \inf_{\mu_h\in M_h} \|\tilde{\lambda}-\mu_h\|_M + \inf_{\mu\in M_h} \sup_{v_h\in X_h} b(\tilde{\lambda}-\mu_h,\, v_h) \,/\, \|v_h\|_X \right\} \tag{4.8}$$

<u>Proof</u> firstly, we have

$$\|\tilde{u}-\tilde{u}_h\|_X \leq \|\tilde{u}-\Pi_{1h}\tilde{u}\|_X + \|\Pi_{1h}\tilde{u}-\tilde{u}_h\|_X \tag{4.9}$$

Secondly, by virtue of lemma 2 and (4.6), (4.7)

$$\alpha\|\tilde{u}-\Pi_{1h}\tilde{u}\|_X^2 \leq \tilde{a}(\tilde{u}-\Pi_{1h}\,\tilde{u},\, \tilde{u}-\Pi_{1h}\,\tilde{u}) = \tilde{a}(\tilde{u}-\Pi_{1h}\tilde{u},\, \tilde{u}-v_h)$$

$$\|\tilde{u}-\Pi_{1h}\,\tilde{u}\|_X^2 \leq c \inf_{v_h\in X_h} \|\tilde{u}-v_h\|_X \tag{4.10}$$

On the other hand, from (4.1) and (3.3), we have

$$\tilde{a}(\tilde{u}-\tilde{u}_h,\, v_h) + b(\tilde{\lambda}-\tilde{\lambda}_h,\, v_h) = 0 \qquad \forall\, v_h \subset X_h \tag{4.11}$$

$$b(\mu_h,\, \tilde{u}-\tilde{u}_h) = 0 \tag{4.12}$$

By lemma 2 and (4.11)

$$\|\Pi_{1h}\,\tilde{u}-u_h\|_X^2 \leq c\,\tilde{a}\,(\Pi_{1h}\,\tilde{u}-\tilde{u}_h\,,\ \Pi_{1h}\,\tilde{u}-\tilde{u}_h)$$

$$= C\tilde{a}\,(\tilde{u}-\tilde{u}_h\,,\ \Pi_{1h}\,\tilde{u}-\tilde{u}_h) = cb(\tilde{\lambda}-\tilde{\lambda}_h,\ \Pi_{1h}\,\tilde{u}-\tilde{u}_h)$$

$$= Cb(\tilde{\lambda}-\mu_h,\ \Pi_{1h}\,\tilde{u}-\tilde{u}_h) + cb(\mu_h - \tilde{\lambda}_h\,,\ \Pi_{1h}\,\tilde{u}-\tilde{u}_h) \tag{4.13}$$

but in view of (4.12)

$$b(\mu_h,\ \tilde{u}_h - \Pi_{1h}\,\tilde{u}) = b(\mu_h,\ \tilde{u} - \Pi_{1h}\,\tilde{u}) + b(\mu_h,\ \tilde{u}_h - \tilde{u})$$

$$= b(\mu_h,\ \tilde{u} - \Pi_{1h}\tilde{u})$$

However, from (2.20) and (2.24)

$$b(\mu_h,\ \tilde{u} - \Pi_{1h}\tilde{u}) = 0 \qquad\qquad \forall\ \mu_h \in M_h \tag{4.14}$$

Therefore

$$b(\mu_h,\ \tilde{u}_h - \Pi_{1h}\tilde{u}) = 0 \tag{4.15}$$

By (4.13)

$$\|\Pi_{1h}\,\tilde{u} - \tilde{u}_h\|_X \leq cb(\tilde{\lambda}-\mu_h,\ \Pi_{1h}\,\tilde{u}-\tilde{u}_h)\ /\ \|\Pi_{1h}\,\tilde{u}-\tilde{u}_h\|_X$$

$$\leq c \inf_{\mu_h\in M_h}\ \sup_{v_h\in X_h}\ b(\tilde{\lambda}-\mu_h,\ v_h)\ /\ \|v_h\|_X \tag{4.16}$$

Combining (4.9),(4.10) and (4.16) we have

$$\|\tilde{u}-\tilde{u}_h\|_X \leq c\ \{\ \inf\ \|\tilde{u}-v_h\| + \inf_{\mu_h\in M_h}\ \sup_{v_h\in X_h}\ b(\tilde{\lambda}-\mu_h,\ v_h)\ /\ \|v_h\|_X\ \} \tag{4.17}$$

Furthermore, lemma 3 yields that

$$\alpha \|\tilde{\lambda}_h - u_h\|_M \ \|v_h\|_X \leq b(\tilde{\lambda}_h - \mu_h, v_h)$$

but

$$b(\tilde{\lambda}_h - \mu_h, v_h) = \tilde{a}(\tilde{u}-\tilde{u}_h, v_h) + b(\tilde{\lambda}-\mu_h, v_h)$$

so

$$\alpha \|\tilde{\lambda}_h - \mu_h\|_M \leq \|\tilde{u}-\tilde{u}_h\|_X + \|\tilde{\lambda}-\mu_h\|_M$$

$$\|\tilde{\lambda}-\tilde{\lambda}_h\|_M \leq \|\tilde{\lambda}-\mu_h\|_M + \|\tilde{\lambda}_h - \mu_h\|_M \quad .$$

$$\leq c_1 \|\tilde{u}-\tilde{u}_h\|_X + c_2 \|\tilde{\lambda}-\mu_h\|_M$$

i.e.

$$\|\tilde{\lambda}-\tilde{\lambda}_h\|_M \leq c \ \{\|\tilde{u}-\tilde{u}_h\|_X + \inf_{\mu_h \in M_h} \|\tilde{\lambda}-\mu_h\|_M \} \tag{4.18}$$

Combining (4.17) and (4.18) we can be obtain (4.8). The proof is completed.

It is easy to prove that $\{X_h, M_h\}$ satisfy the condition of lemma 9 in [8]. Hence for $\psi \in H^1(\Omega)$

$$\inf_{\mu_h \in M_h} \ \sup_{v_h \in X_h} \ b(\psi-\mu_h, v_h) \ / \ \|v_h\|_X \leq ch\|\psi\|_{1,\Omega} \tag{4.19}$$

On the other hand, by interpolation theory in Sobolev space

$$\inf_{v_h \in X_h} \|\tilde{u}-v_h\|_X \leq \inf_{v_h \in s_h} \|\tilde{u}-v_h\|_X \leq ch \ \|\tilde{u}\|_{2,\Omega} \tag{4.20}$$

where

$$S_h = \{v_h \in C^{\circ}(\Omega);\ v_h|_k \in P_1(K),\ v_h|_\Gamma = 0\} \subset X_h$$

From (4.18), (4.19) and (4.20) we have

<u>Theorem 3</u> $\forall f \subset L^2(\Omega)$, if problem (3.3) has a solution $(\tilde{u},\tilde{\lambda}) \in (X \cap H^2(\Omega)) \times M$, then

$$\|\tilde{u}-\tilde{u}_h\|_X + \|\tilde{\lambda}-\tilde{\lambda}_h\|_M \leq ch\ \|\tilde{u}\|_{2,\Omega} \tag{4.21}$$

From theorems 2 and 3 , we can yield

<u>Theorem 4</u> Suppose (u,λ) is a weak solution of boundary value problem (2.1), $(\tilde{u}_h, \tilde{\lambda}_h)$ is a solution of nodal expansion approximation problem (4.1). If the solution $(\tilde{u},\tilde{\lambda})$ of problem (3.3) satisfies $\tilde{u} \in H^2(\Omega)$ for $f \in L^2(\Omega)$, then

$$\|u-\tilde{u}_h\|_X + \|\lambda-\tilde{\lambda}_h\|_M \leq c_h \|f\|_{0,\Omega} \tag{4.22}$$

<u>Proof</u> Using the triangle inequality

$$\|u-\tilde{u}_h\|_X + \|\tilde{\lambda}-\tilde{\lambda}_h\|_M \leq \|u-\tilde{u}\|_X + \|\lambda-\tilde{\lambda}\|_M + \|\tilde{u}-u_h^{\sim}\|_X + \|\tilde{\lambda}-\tilde{\lambda}_h\|_M$$

By virtue of (3.7) and (4.21) the inequality (4.22) is obtained, theorem is proved.

<u>Remark</u> Using Nitsch's trick it is easy to get

$$\|\tilde{u}-\tilde{u}_h\|_{0,\Omega} \leq ch^2\ \|f\|_{0,\Omega}$$

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