GLOBAL ATTRACTORS AND APPROXIMATE INERTIAL MANIFOLDS FOR ABSTRACT DISSIPATIVE EQUATIONS

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Abstract. An abstract differential equation is considered, $u' + Au + F(u) = f(t)$, using assumptions appropriate for systems of reaction-diffusion equations on multi-dimensional spatial domains. A priori estimates establish the existence of absorbing balls in relevant function spaces, and consequently the existence of a global attractor is verified. It is shown that the monotonicity of A gives rise to approximate inertial manifolds for this equation. These are finite-dimensional manifolds which have exponentially attracting neighborhoods under the flow. The dimension of the manifold and the thickness of the attracting neighborhood are inversely related.

Key words. attractor, absorbing ball, finite-dimensional approximation, inertial manifold, reaction-diffusion equations

AMS(MOS) subject classifications. 34G20, 35B40, 35K55, 35K57

§**1. Introduction.** The use of partial differential equations and functional differential equations to model systems evolving in time is pervasive in applied mathematics. From a geometrical perspective these models can be viewed as infinite-dimensional dynamical systems (cf. [14], [32]), in which the underlying state space is usually taken to be an infinite-dimensional Banach or Hilbert space. This perspective is a natural generalization of its classical counterpart, where the models used are ordinary differential equations and the state spaces are finite-dimensional Euclidean spaces. An obvious hinderance in the analysis of the former type of systems is their infinite-dimensionality. However, it has been observed that some of these systems, in particular the so called dissipative systems (cf. [8], [10], [12], [14], [32]), behave asymptotically like finite-dimensional systems. In some cases they have compact global attractors of finite Hausdorff dimension (cf. [2], [14], [19], [22], [32]). In these situations one can hope to return to the finite-dimensional setting without losing any essential information contained in the model. It is for this purpose that inertial manifolds are employed. These are finite-dimensional manifolds within the state space that are invariant and exponentially attracting under the flow of the original dynamical system. Thus asymptotically the flow of the original system is the same as the flow on the inertial manifold. When both an inertial manifold and attractors exist, it is clear that the inertial manifold must contain all of the attractors in the system.

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These appealing features of inertial manifolds have precipitated many investigations (cf. [7], [8], [10], [16], [20], [21], [25], [26], [29]) into their existence for a variety of systems. Generally one considers an equation in Banach space of the form $u' + Au + F(u) = 0$, with A an unbounded linear operator and F a nonlinear Lipschitz operator. A crucial requirement for existence is that the spectrum of A have large gaps relative to the strength of the nonlinearity. This is typically referred to as the gap condition. For scalar semilinear reaction-diffusion equations it has been shown (cf. [20]) that the gap condition is satisfied if the spatial domain Ω has dimension $n = 1$ or if $n \in \{2,3\}$ and Ω has a special form (rectangles and cubes). In other cases it is unknown whether the gap condition is satisfied without severely restricting F, and in general this remains as an obstacle in the application of existence theories.

This work deals with an alternative to the construction of an inertial manifold, which eliminates the need for the gap condition. We show that within a reasonably general framework, it is always possible to construct finite-dimensional manifolds having properties similar to those of an inertial manifold. For example the property of exponential attraction is still present, but in a modified form; solutions are exponentially attracted to a neighborhood of the manifold. In addition the thickness of the neighborhood about the manifold can be made arbitrarily small at the expense of increasing the dimension of the manifold. The property of invariance is also weaker. In general we can only expect the manifold to be invariant with respect to a perturbed equation. In this way our theory is related to the theory of approximation dynamics proposed by Sell [28]. Manifolds of this type have been called approximate inertial manifolds. Their use was originally proposed in [13], and subsequently considered by other authors [17], [23], [33].

The approximate inertial manifolds constructed here are obtained from a decomposition of the steady-state problem. Thus necessarily they contain all steady-states of the problem. The ideas involved are analogous to those employed in the method of alternative problems (cf. [4], [6]). Manifolds obtained in this way were also used in [17] and [33], while in [21] a slightly different method is used. A distinguishing feature of our approach is the use of monotonicity. This property is exploited in two ways. First it is used to obtain the manifolds in question, in a way that is independent of the Lipschitz constant for F. Thus manifolds of low dimension are available. Secondly it is used to show that a neighborhood of the manifolds are exponentially attracting. By using monotonicity we are able to obtain a precise description of the rate of decay, and to bound the decaying distance by a term involving only the initial data and the decaying exponential. This is done in both the H and H^1 norms (i.e. $L^2(\Omega)$ and $H^1(\Omega)$). Our estimates for the thickness of the neighborhood are in asymptotic agreement with those of [17], [33] and with those for the first approximate inertial manifold in [21]. Manifolds having higher order asymptotic estimates are also constructed in [21], where L^∞ estimates and additional smoothness are available.

Our interest in the problem was originally motivated by the question of asymptotic behavior of systems of reaction-diffusion equations in several space variables. Our belief is that such systems provide perhaps the simplest examples to which a general theory may apply. As such we have chosen a correspondingly simple abstract framework in which to work. We consider an abstract equation of the form $u' + Au + F(u) = f(t)$, where u is a Hilbert space valued function of $t > 0$, and allow for a nonhomogeneous time-dependent forcing term. Throughout we assume an abstract dissipative condition (see (3.1)) on F. In applications this conditions amounts to a L^2 inner product condition. Similarly the monotonicity required (see (4.5)) is an L^2 condition. In applications, L^∞ estimates are only needed if one obtains F by a truncation procedure.

The outline of the paper is as follows. In section 2 we present the abstract framework employed, and recall basic facts about the abstract Cauchy problem. In section 3 we discuss regularity of solutions and show that absorbing balls (and hence attractors) exist for the problem, under the abstract condition on dissipativeness and assuming bounded forcing terms. The associated steady-state problem is considered in section 4, where we show the existence of the finite-dimensional Lipschitz manifolds to be used subsequently. In section 5 we establish some preliminary lemmas which are used in section 6 to prove our main result, Theorem 6.2. Finally in section 7 we apply the results of the previous sections to systems of reaction-diffusion equations in several variables.

§2. Notations and Assumptions. Let H be a real separable Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$. Let A denote a closed (unbounded) symmetric linear operator in H, with domain $\mathcal{D}(A) = \{u \in H : Au \in H\}$. We assume for simplicity (see remark following Theorem 2.1) that there is an orthonormal set of eigenfunctions $\{\varphi_i\}_{i\in I}$ of A, which is complete in H. Here I denotes a countable index set (for example of multi-indices). Let $\{\lambda_i\}_{i\in I}$ denote the corresponding set of eigenvalues; we have $A\varphi_i = \lambda_i\varphi_i$ for all $i \in I$. We further assume without loss of generality that $\lambda_* = \inf\{\lambda_i : i \in I\} > 0$; if this is not the case we can replace A by $A + aI$ and $F(u)$ by $F(u) - a(u)$ without qualitatively changing the hypotheses on F.

By completeness, any $u \in H$ has the representation

$$u = \sum_{i\in I} u_i\varphi_i, \qquad \text{where } u_i = (u, \varphi_i).$$

There are Sobolev subspaces $H^s \subset H$ corresponding to the fractional powers of A and the norms $|\cdot|_s$, depending on the real parameter s, which are defined by

$$|u|_s = \left\{\sum_{i\in I} \lambda_i^s u_i^2\right\}^{1/2} = |A^{s/2}u|. \tag{2.1}$$

We assume that the embedding $H^1 \subset H$ is compact. Clearly we have $H^2 = \mathcal{D}(A)$ equipped with the graph norm, and in general $A : H^{s+2} \to H^s$.

There is a dual pairing $H^s \times H^{-s} \to \mathbb{R}$ which we will have occasion to use. The notation $\langle u, v\rangle_s$ will be used to denote this pairing.

Given a real number $\mu > 0$, define index subsets of I by $I_\mu^- = \{i \in I : \lambda_i < \mu\}$ and $I_\mu^+ = \{i \in I : \lambda_i \geq \mu\}$. There is a corresponding decomposition of H into subspaces $H = X + Y$, where X is the linear span of the set $\{\varphi_i : i \in I_\mu^-\}$ and Y is the closure of the linear span of $\{\varphi_i : i \in I_\mu^+\}$. We assume that for any given $\mu > 0, \dim(X) < +\infty$. These subspaces are obviously orthogonal and invariant under A. Analogous decompositions $H^s = X^s + Y^s$ also result. Let $P : H^s \to X^s$ denote the projection onto X^s and set $Q = I - P$, the complementary projection. Clearly the following inequalities are valid:

$$|y| \leq \mu^{-s/2}|y|_s, \quad y \in Y^s \tag{2.2}$$

$$\lambda_*^{s/2}|x| \leq |x|_s \leq \mu^{s/2}|x|, \quad x \in X \tag{2.3}$$

When H is decomposed as above we say the decomposition is induced by μ.

Let $F : H \to H$ be a (nonlinear) operator in H, and $f : [0, +\infty) \to H$ be an H-valued function of $t \in [0, +\infty)$. We consider the initial-value problem (the Cauchy problem)

$$\frac{du}{dt} + Au + F(u) = f(t), \quad u(0) = u_0 \in H. \tag{2.4}$$

A function $u : [0, T) \to H$ is called a (classical) solution (cf. [27]) of (2.4) on $[0, T)$ if $u(t)$ satisfies: i) $u \in C([0, T); H)$, ii) $u' \in C((0, T); H)$, iii) $u(t) \in H^2$ for $t \in (0, T)$, and iv) $u(t)$ satisfies (2.4) on $(0, T)$. Assuming solutions exist on $[0, +\infty)$ our goal is to describe their asymptotic behavior in terms of a finite-dimensional system. The following hypotheses provide the proper framework for this investigation. Assume that F, f satisfy

$$|F(u_1) - F(u_2)| \leq M|u_1 - u_2|, \quad \text{for all } u_1, u_2 \in H, \tag{2.5}$$

$$|f(t_1) - f(t_2)| \leq L|t_1 - t_2|^\theta, \quad \text{for all } t_1, t_2 \in J, \tag{2.6}$$

where M, L, θ are positive constants with $0 < \theta \leq 1$, and $J \subset [0, +\infty)$ is bounded. The constant K may depend on J but we assume this dependence is only on the length of J. Thus the same constant can be used for any J with length less than a given constant. A function $f : [0, +\infty) \to H$ satisfying (2.6), where K has the dependence on J as described, will be called uniformly locally Hölder continuous on $[0, +\infty)$ with exponent θ. The collection of all such functions will be denoted $C^{0,\theta}([0, +\infty); H)$. If (2.6) holds for $J \subset [t_0, +\infty)$ we will replace $[0, +\infty)$ by $[t_0 + \infty)$ in this notation. Notice that if $\beta < \theta$ then $C^{0,\theta}([t_0, +\infty); H) \subset C^{0,\beta}([t_0, +\infty); H)$.

The following result on global existence is a combination of the results found in [27].

THEOREM 2.1. *If $F \in C(H, H)$ satisfies (2.5) and $f \in C^{0,\theta}([0, +\infty); H)$ for $0 < \theta \leq 1$, then for every $u_0 \in H$ the initial-value problem (2.4) has a unique (classical) solution*

$u \in C((0,+\infty);H^2)$ *on* $[0,+\infty)$. *In addition, for any* $\beta \in (0,1)$ *and* $0 < t_0 < T < +\infty$, *the functions* u, Au, u' *are all Hölder continuous on* $[t_0, T]$ *with exponent* $\gamma = \min(\beta, \theta)$.

Remark. What is essentially required of A, here and subsequently, is that it be the generator of an analytic semigroup, that A^{-1} be compact, and for an infinite sequence $\mu_j \to +\infty$ there are invariant subspaces X_j, Y_j with the properties attributed to X, Y above.

§3. **Regularity and A Priori Estimates.** Briefly stated, a system is called dissipative if there is an absorbing ball (cf. [14]). Let $u(t;u_0)$ denote the solution of (2.4). For $\rho > 0$ we use $B(\rho) \subset H$ and $B^1(\rho) \subset H^1$ to denote the balls $B(\rho) = \{u \in H : |u| \leq \rho\}$ and $B^1(\rho) = \{u \in H^1 : |u|_1 \leq \rho\}$. We will say (cf. [32]) that $B(\rho)$ is an absorbing ball in H if, for any $R > 0$, there is a time $T(R)$ such that $u(t;u_0) \in B(\rho)$, for all $t \geq T(R)$ and $u_0 \in B(R)$. Furthermore we will say that $B^1(\rho)$ is absorbing in H if, for any $R > 0$, there is a time $T(R)$ such that $u(t;u_0) \in B^1(\rho)$, for all $t > T(R)$ and $u_0 \in B(R)$.

We consider the following condition (3.1) on F as an abstract condition for dissipativity.

The next result shows that it implies, in the presense of bounded forcing terms, there are absorbing balls for (2.4) in H. We assume there are constants $F_0 \geq 0$ and $m_* < \lambda_*$ such that

$$(F(u),u) \geq -F_0 - m_*|u|^2, \qquad \text{for all } u \in H. \tag{3.1}$$

Lemma 3.1. *Suppose that* F *satisfies (2.5), (3.1) and that* $f \in L^\infty((0,+\infty);H)$ *satisfies (2.6). Then the solution of (2.4) belongs to* $L^\infty((0,+\infty);H)$ *and satisfies*

$$|u(t)|^2 \leq |u_0|^2 e^{-(\lambda_* - m_*)t} + C(F_0, \lambda_*, m_*, f), \qquad t \geq 0, \tag{3.2}$$

where $C(F_0,\lambda_*,m_*,f) = (\lambda_* - m_*)^{-2}[2F_0(\lambda_* - m_*) + \|f\|^2_{L^\infty((0,+\infty);H)}]$. *Thus* $B(\rho)$ *is an absorbing ball in* H *for every* $\rho > \sqrt{C(F_0,\lambda_*,m_*,f)}$.

Proof. Taking the inner product of u with (2.4) shows

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + |A^{1/2}u(t)|^2 + (F(u(t)),u(t)) = (f(t),u(t)), \qquad t > 0. \tag{3.3}$$

It follows from (2.1), (3.1) and Cauchy's inequality with $\epsilon > 0$ that

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 \leq -(\lambda_* - m_* - \epsilon)|u(t)|^2 + F_0 + \frac{1}{4\epsilon}|f(t)|^2, \qquad t > 0.$$

Choosing $\epsilon = (\lambda_* - m_*)/2$ for convenience and rearranging terms leads to

$$\frac{d}{dt}|u(t)|^2 + (\lambda_* - m_*)|u(t)|^2 \leq 2F_0 + (\lambda_* - m_*)^{-1}\|f\|^2_{L^\infty((0,+\infty);H)}, t > 0.$$

Integration of this inequality from 0 to $t > 0$ produces (3.2). □

More can be said about the Hölder continuity of a solution $u(t)$ when we know that both $|u(t)|$ and $|f(t)|$ are bounded on $(0,+\infty)$; that is when $u, f \in L^\infty((0,+\infty);H)$. In this case we can show that u is uniformly locally Hölder continuous in the sense described previously.

LEMMA 3.2. *Let $F \in C(H,H)$ satisfy (2.5), (3.1) and suppose $f \in L^\infty((0,+\infty);H)$ satisfies (2.6). Then $u \in C^{0,\beta}([t_0,+\infty);H)$ for any $t_0 > 0$ and $\beta \in (0,1)$; that is u is uniformly locally Hölder continuous on $[t_0,+\infty)$.*

Proof. We modify a proof in [27] slightly. From the assumptions on A it follows that, in the linear operator norm of $\mathcal{B}(H) = \{$ bounded linear operators on $H\}$, we have $|\exp(-tA)| \leq 1$ for $t \geq 0$, and $|\exp(-tA) - \exp(-\tau A)| \leq k(\tau)\exp(-\lambda_*\tau)|t-\tau|$ for $t \geq \tau > 0$, where $k(\tau) = \tau^{-1}$ if $\tau < \lambda_*^{-1}$ and $k(\tau) = \lambda_*$ if $\tau > \lambda_*^{-1}$. From the variation of constants formula $u(t) = \exp(-tA)u_0 + v(t)$, where

$$v(t) = \int_0^t e^{-(t-s)A}[f(s) - F(u(s))]ds.$$

Since $\exp(-tA)u_0$ is Lipschitz continuous on $[t_0,+\infty)$ we need only consider $v(t)$. Let $G(s) = f(s) - F(u(s))$. We have $G \in L^\infty((0,+\infty);H)$. If $t \geq \tau \geq t_0$ then

$$v(t) - v(\tau) = \int_0^\tau \left[e^{-(t-s)A} - e^{-(\tau-s)A}\right]G(s)ds + \int_\tau^t e^{-(t-s)A}G(s)ds$$

Obviously the second integral has the desired property. We now consider the first integral. From the above estimates it follows that

$$\left|\int_0^\tau \left[e^{-(t-s)A} - e^{-(\tau-s)A}\right]G(s)ds\right| \leq 2||G||_{L^\infty((0,+\infty);H)} \int_0^\tau b(\tau-s, t-\tau)e^{-\lambda_*(\tau-s)}ds$$

where

$$b(s,h) = \begin{cases} 1, & 0 \leq s < h, \\ h/s, & h < s < \lambda_*^{-1}, \\ h\lambda_*, & s > \lambda_*^{-1}. \end{cases}$$

This assumes $h < \lambda_*^{-1}$. If $h \geq \lambda_*^{-1}$ then we set $b(s,h) = h/s$ for all $s > h$. Set $h = t - \tau$. For any $\tau \geq t_0$ and $0 < h < \lambda_*^{-1}$ we have

$$\int_0^\tau b(\tau-s,h)e^{-\lambda_*(\tau-s)}ds = \int_0^\tau b(s,h)e^{-\lambda_* s}ds \leq \int_0^{+\infty} b(s,h)e^{-\lambda_* s}ds$$
$$\leq 2h + h\int_h^{\lambda_*^{-1}} \frac{e^{-\lambda_* s}}{s}ds.$$

Using Hölder's inequality with conjugate indices $p^{-1} + q^{-1} = 1$ we find

$$\int_h^{\lambda_*^{-1}} \frac{e^{-\lambda_* s}}{s}ds \leq \left\{\int_h^{+\infty} s^{-p}ds\right\}^{1/p}\left\{\int_h^{+\infty} e^{-\lambda_* qs}ds\right\}^{1/q} \leq \left(\frac{h^{1-p}}{p-1}\right)^{1/p}\left(\frac{1}{\lambda_* q}\right)^{1/q}.$$

Choosing $p = \beta^{-1}$ we obtain

$$h\int_h^{\lambda_*^{-1}} \frac{e^{-\lambda_* s}}{s}ds \leq c(\beta,\lambda_*)h^\beta,$$

where $c(\beta,\lambda_*)$ depends only on β, λ_* and is unbounded as $\beta \to 1^-$. Clearly the same argument applies when $h \geq \lambda_*^{-1}$. Combining all estimates now yields the stated result. □

COROLLARY 3.3. *Under the assumptions of Lemma 3.1, both Au and du/dt are uniformly locally Hölder continuous on $[t_0, +\infty)$ with exponent $\gamma = \min(\beta, \theta)$, for any $t_0 > 0$ and $\beta \in (0,1)$.*

Proof. Since $u \in C^{0,\beta}([t_0, +\infty); H)$ for any $\beta \in (0,1)$ it follows that $F(u) \in C^{0,\beta}([t_0, +\infty); H) \cap L^\infty((0,+\infty); H)$. Hence $G(t) = f(t) - F(u(t))$ belongs to $C^{0,\gamma}([t_0, +\infty); H) \cap L^\infty((0, +\infty; H)$. Standard arguments (cf. [27]) now yield the conclusion. □

In addition to the implications of Hölder continuity, the condition $u \in L^\infty((0,+\infty); H)$ also implies that $A^{1/2}u \in L^\infty((t_0, +\infty); H)$ for any $t_0 > 0$. Furthermore the existence of an absorbing ball in H implies there is a ball $B^1 \subset H^1$ which is absorbing in H.

LEMMA 3.4. *Under the assumptions of Lemma 3.1 we have $A^{1/2}u \in L^\infty([t_0, +\infty); H)$, for any $t_0 > 0$; equivalently $u \in L^\infty([t_0, +\infty); H^1)$. Moreover the following a priori estimate holds:*

$$|u(t)|_1^2 \le C_1(t_0)e^{-\lambda_*(t-t_0)} + \lambda_*^{-1}\|f - F(u)\|^2_{L^\infty((0,+\infty);H)}, t \ge t_0 > 0, \tag{3.4}$$

where $C_1(t_0) = t_0^{-1}|u_0|^2 + (1+M)\|u\|^2_{L^\infty((0,+\infty);H)} + \|f\|^2_{L^\infty((0,+\infty);H)}$. Thus there is a ball $B^1 \subset H^1$ which is absorbing in H.

Proof. First we integrate (3.3) from 0 to $t_0 > 0$ to obtain

$$\begin{aligned}
\frac{1}{2}|u(t_0)|^2 + \int_0^{t_0} |A^{1/2}u(t)|^2 dt &= \frac{1}{2}|u_0|^2 + \int_0^{t_0} (f(t), u(t))dt - \int_0^{t_0} (F(u(t)), u(t))dt \\
&\le \frac{1}{2}\left(|u_0|^2 + \int_0^{t_0} |u(t)|^2 dt + \int_0^{t_0} |f(t)|^2 dt\right) + M\int_0^{t_0} |u(t)|^2 dt \\
&\le |u_0|^2 + t_0\left[(1+M)\|u\|^2_{L^\infty((0,+\infty);H)} + \|f\|^2_{L^\infty((0,+\infty);H)}\right]
\end{aligned}$$

Thus

$$\frac{1}{t_0}\int_0^{t_0} |A^{1/2}u(t)|^2 dt \le C_1(t_0),$$

where $C_1(t_0)$ is the constant described above. According to the mean value theorem for integrals, there is a $t_1 \in (0, t_0)$ such that $|A^{1/2}u(t_1)|^2 \le C_1(t_0)$.

Now since $u \in C((0,+\infty); H^2)$ and A is symmetric a straight forward argument shows

$$\frac{d}{dt}|A^{1/2}u(t)|^2 = 2(Au(t), f(t) - Au(t) - F(u(t))), \quad t \ge t_0. \tag{3.5}$$

For $u \in \mathcal{D}(A)$ we have $|Au|^2 \ge \lambda_*|A^{1/2}u|^2$; hence

$$\begin{aligned}
\frac{d}{dt}|A^{1/2}u(t)|^2 &= -2|Au(t)|^2 + 2(Au(t), f(t) - F(u(t))) \\
&\le -|Au(t)|^2 + |f(t) - F(u(t))|^2 \\
&\le -\lambda_*|A^{1/2}u(t)|^2 + \|f - F(u)\|^2_{L^\infty((0,+\infty);H)}.
\end{aligned}$$

Thus integrating this differential inequality from t_1 to t_0 yields

$$\begin{aligned}|A^{1/2}u(t_0)|^2 &\leq |A^{1/2}u(t_1)|^2 e^{-\lambda_*(t_0-t_1)} + \lambda_*^{-1}\|f - F(u)\|^2_{L^\infty((0,+\infty);H)} \\ &\leq C_1(t_0) + \lambda_*^{-1}C_2\end{aligned}$$

where we set $C_2 = \|f - F(u)\|^2_\infty$ for convenience. Integrating the same differential inequality from t_0 to t then shows

$$\begin{aligned}|A^{1/2}u(t)|^2 &\leq |A^{1/2}u(t_0)|^2 e^{-\lambda_*(t-t_0)} + \lambda_*^{-1}C_2[1 - e^{-\lambda_*(t-t_0)}] \\ &\leq [C_1(t_0) + \lambda_*^{-1}C_2]e^{-\lambda_*(t-t_0)} + \lambda_*^{-1}C_2[1 - e^{-\lambda_*(t-t_0)}].\end{aligned}$$

This is equivalent to (3.4). □

COROLLARY 3.5. *Under the hypotheses of Lemma 3.1, there is a compact, connected global attractor for (2.4).*

Proof. This follows from general considerations (cf. [32]). □

Subsequently we shall consider a system equivalent to (2.4) obtained by splitting H. Let $\mu > 0$ be given and $H = X + Y$ be the decomposition of H induced by μ, as described in section 2. Applying the projections $P : H \to X$ and $Q : H \to Y$ to (2.4) produces the system

$$x' + Ax + PF(x + y) = g(t) = Pf(t), \quad x(0) = x_0 = Pu_0, \tag{3.6}$$

$$y' + Ay + QF(x + y) = h(t) = Qf(t), \quad y(0) = y_0 = Qu_0 \tag{3.7}$$

This system is equivalent to (2.4). Hence $x(t) = Pu(t)$ and $y(t) = Qu(t)$ enjoy the same properties of regularity as $u(t)$. Moreover since (3.6) is a finite-dimensional equation it follows that, under the hypotheses of Lemma 3.1, $x(t)$ is uniformly locally Hölder continuous on $[0, +\infty)$ with exponent β, for any $\beta \in (0, 1]$. In particular

$$|x(t_1) - x(t_2)| \leq K|t_1 - t_2|^\theta, \qquad \text{for all } t_1, t_2 \in J, \tag{3.8}$$

where θ is the constant appearing in (2.6), $J \subset [0, +\infty)$ and K depends only on the length of J. At times we will consider (3.7) alone, with $x(t)$ satisfying (3.8). Clearly in this case all of the previous results on global existence, uniqueness, and regularity apply to solutions $y(t)$ of (3.7).

§4. **Finite Dimensionality of Steady-States.** A general reduction principle (cf. [4], [6], [30]) is available for the steady-state problem

$$Au + F(u) = f \tag{4.1}$$

associated with (2.4). Under reasonable assumptions this problem is equivalent to a finite-dimensional problem in the following sense. If $\mu > 0$ and $H = X + Y$ is the decomposition induced by μ, with $P : H \to X$ and $Q : H \to Y$ the corresponding projections, then (4.1) has the equivalent form

$$Ax + PF(x + y) = g = Pf, \tag{4.2}$$

$$Ay + QF(x + y) = h = Qf. \tag{4.3}$$

This splitting process, and the subsequent reduction to a single equation, is known in the literature as the method of alternative problems. If (4.3) has a unique solution $y = \sigma(x, h)$ for every $x \in X, h \in Y$ then (4.1) has a solution if and only if $x \in X$ satisfies

$$Ax + PF(x + \sigma(x, h)) = g. \tag{4.4}$$

This equation (or (4.2)) is often referred to as the bifurcation equation (cf. [4], [6]). It is a finite-dimensional equation (assuming $\dim(X) < +\infty$) and can be put into the form $G(\hat{x}) = 0$ where $G : \mathbb{R}^n \to \mathbb{R}^n$. Thus the solutions of (4.1) are in a one-to-one correspondence with the solutions of $G(\hat{x}) = 0$, and therefore the solutions of (4.1) lie on a finite-dimensional manifold in H.

Equation (4.3) can be solved for $y = \sigma(x, h)$ under a variety of hypotheses. We use the following assumption which has far-reaching implications for the corresponding time-dependent problem (3.7). We assume there is a number $m \in \mathbb{R}$ such that

$$(F(u) - F(v), u - v) \geq -m|u - v|^2, \qquad \text{for all } u, v \in H. \tag{4.5}$$

Some observations should be made at this point. First, if F satisfies (2.5) then (4.5) is satisfied with $m = M$. Thus it is reasonable to assume $m \leq M$ although this is not used in any way. Also if $m < \lambda_*$ then $-m > -\lambda_*$ and $A + F(\cdot)$ is a coercive monotone operator (cf. [3], [18]). In this case there is a unique steady-state solution of (4.1), and the resulting dynamics of (2.4) are rather trivial (see remark following Lemma 5.1). Thus the range of m which is of most interest is the interval $\lambda_* \leq m \leq M$.

LEMMA 4.1. *Suppose that $F \in C(H, H)$ satisfies (2.5) and (4.5). Then for every $\mu > m$ there is a decomposition of $H = X + Y$ and a mapping $\sigma : X \times Y \to Y^1$ such that $y = \sigma(x, h)$ is the unique solution of (4.3). Moreover σ is Lipschitz continuous and satisfies*

$$|\sigma(x_1, h_1) - \sigma(x_2, h_2)| \leq (\mu - m)^{-1} \{M|x_1 - x_2| + |h_1 - h_2|\}, \tag{4.6}$$

$$|\sigma(x_1, h_1) - \sigma(x_2, h_2)|_1 \leq \mu^{-1/2}\delta^{-1} \{M|x_1 - x_2| + |h_1 - h_2|\}, \tag{4.7}$$

where δ is any number in the interval $(0, (\mu - m)/\mu)$.

Proof. Let $\mu > m$ be given, and let $H = X + Y$ be the decomposition of H induced by μ as described in section 2. Let Y^1 be the subspace of Y determined by the requirement $|y|_1 < +\infty$. Its dual is Y^{-1}. We define a nonlinear operator $T : Y^1 \to Y^{-1}$ by $Ty = Ay + QF(x+y)$, with $x \in X$ held fixed. It follows immediately from (2.1)-(2.2), (4.5) that

$$(4.8) \qquad \langle Ty_1 - Ty_2, y_1 - y_2 \rangle_1 \geq (\mu - m)|y_1 - y_2|^2, \qquad \text{for all } y_1, y_2 \in H.$$

This shows T is a monotone operator (cf. [3], [4], [18], [30]) from Y^1 to Y^{-1}. To see that it is also coercive let $0 < \delta < \mu^{-1}(\mu - m)$. Then

$$(4.9) \qquad \begin{aligned} \langle Ty_1 - Ty_2, y_1 - y_2 \rangle_1 &\geq \delta |y_1 - y_2|_1^2 + (1-\delta)|y_1 - y_2|_1^2 - m|y_1 - y_2|^2 \\ &\geq \delta |y_1 - y_2|_1^2 + [(1-\delta)\mu - m]\,|y_1 - y_2|^2 \geq \delta |y_1 - y_2|_1^2. \end{aligned}$$

Thus $Ty = h$ (i.e. (4.3)) has a unique solution for every $h \in Y$.

Now let $x_i \in X, h_i \in Y (i = 1,2)$ and set $u_i = x_i + y_i$, where $y_i = \sigma(x_i, h_i)$ is the unique solution of (4.3) with $x = x_i, h = h_i$. If T_i denotes the operator $T_i y = Ay + QF(x_i + y)$, then $T_i y_i = h_i$. Hence

$$(4.10) \qquad \langle A(y_1 - y_2), y_1 - y_2 \rangle_1 + (F(u_1) - F(u_2), y_1 - y_2) = (h_1 - h_2, y_1 - y_2).$$

By adding $0 = F(x_1 + y_2) - F(x_1 + y_2)$ and using (4.5) it follows that

$$(F(u_1) - F(u_2), y_1 - y_2) \geq -m|y_1 - y_2|^2 + (F(x_1 + y_2) - F(x_2 + y_2), y_1 - y_2)$$

Using this inequality in the previous identity along with (2.2), (2.5) shows

$$(\mu - m)|y_1 - y_2|^2 \leq M|x_1 - x_2|\,|y_1 - y_2| + |h_1 - h_2|\,|y_1 - y_2|.$$

This is equivalent to (4.6). To obtain (4.7) we return to (4.10) and use (4.9) to show

$$\begin{aligned} \delta |y_1 - y_2|_1^2 &\leq M|x_1 - x_2|\,|y_1 - y_2| + |h_1 - h_2|\,|y_1 - y_2| \\ &\leq \mu^{-1/2}\left\{M|x_1 - x_2| + |h_1 - h_2|\right\}|y_1 - y_2|_1. \end{aligned}$$

Inequality (4.7) now follows. □

§5. **Preliminary Estimates.** In this section we return to consideration of the time-dependent problem. We assume throughout that $F : H \to H$ satisfies (2.5) and (4.5), that $\mu > m$ has been chosen, and that $H = X + Y$ is the decomposition induced by μ. Thus $\sigma : X \times Y \to Y^1$ is well-defined according to Lemma 4.1. The first lemma is descriptive of the influence of monotinicity in the problem.

LEMMA 5.1. *Let $F \in C(H,H)$ satisfy (2.5) and (4.5), $J = [a,b) \subset [0,+\infty), x \in X$, and $h \in Y$. If $z \in C(J,Y)$ is a solution of $z' + Az + QF(x+z) = h$ on J, then for all $t_0, t \in J$*

$$|z(t) - \sigma(x,h)| \le |z(t_0) - \sigma(x,h)| e^{-(\mu-m)(t-t_0)}, \quad 0 \le t_0 \le t, \tag{5.1}$$

$$|z(t) - \sigma(x,h)|_1 \le \left(1 + \frac{M}{\mu - m}\right) |z(t_0) - \sigma(x,h)|_1 e^{-(\mu-m)(t-t_0)/2}, \quad 0 < t_0 \le t. \tag{5.2}$$

Remark. If $m < \lambda_*$ and f is independent of t then Lemma 4.1 and 5.1 show that there is a unique steady-state $\sigma(f)$ of (2.4), which is globally exponentially stable. This follows by taking $X = \{0\}$ and $Y = H$.

Proof. Define $v_o : J \to \mathbb{R}$ via the Lyapunov functional $V_o(z) = \frac{1}{2}|z - \sigma(x,h)|^2$. If z is the solution under consideration then, by definition of σ, the function $v_o(t) = V_o(z(t))$ satisfies

$$\begin{aligned} v_o'(t) &= (z(t) - \sigma(x,h), h - Az(t) - QF(x + z(t))) \\ &= -(z(t) - \sigma, A(z(t) - \sigma) + F(x + z(t)) - F(x + \sigma)) \\ &\le -(\mu - m)|z(t) - \sigma(x,h)|^2 = -2(\mu - m)v_o(t). \end{aligned}$$

Hence $v_o(t) \le v_o(t_0) \exp[-2(\mu - m)(t - t_0)]$, for $t_0 \le t$ in J. This of course is equivalent to (5.1).

Next consider $v_1(t) = V_1(z(t))$ where $V_1(z) = \frac{1}{2}|z - \sigma(x,h)|_1^2 = \frac{1}{2}|A^{1/2}(z - \sigma(x,h))|^2$ As in (3.5) we have

$$v_1'(t) = (A[z(t) - \sigma(x,h)], h - Az(t) - QF(x + z(t))), \quad 0 < t_0 < t$$

Again using the fact that $h = A\sigma + F(x + \sigma)$, as well as (2.5) and the estimate just established, we find

$$\begin{aligned} v_1'(t) &= -|A(z(t) - \sigma)|^2 + (A[z(t) - \sigma], F(x + 2(t) - F(x + \sigma)) \\ &\le -\frac{1}{2}|A(z(t) - \sigma)|^2 + \frac{1}{2}|F(x + z(t)) - F(x + \sigma)|^2 \\ &\le -\frac{1}{2}\mu|A^{1/2}(z(t) - \sigma)|^2 + \frac{1}{2}M^2|z(t_0) - \sigma|^2 e^{-2(\mu-m)(t-t_0)}. \end{aligned}$$

Thus $v_1' \le -\mu v_1 + ke(t)$, where $k = \frac{1}{2}M^2|z(t_0) - \sigma|^2$ and $e(t) = \exp\{-2(\mu - m)(t - t_0)\}$. Replacing the term $-\mu v_1$ by the upper bound $-(\mu - m)v_1$ and integrating shows that $|z(t) - \sigma|_1^2 = 2v_1(t)$ satisfies

$$\begin{aligned} |z(t) - \sigma|_1^2 &\le \left\{|z(t_0) - \sigma|_1^2 + (\mu - m)^{-1}M^2|z(t_0) - \sigma|^2\right\} e^{-(\mu-m)(t-t_0)} \\ &\le |z(t_0) - \sigma|_1^2 \left[1 + (\mu - m)^{-1}M^2\mu^{-1}\right] e^{-(\mu-m)(t-t_0)} \end{aligned}$$

Since $\mu^{-1} < (\mu - m)^{-1}$ we may take square roots to obtain (5.2). □

The above result does not directly apply to solutions of (3.7), for in this case x and h will generally depend on t. However we show that it can be used to show exponential decay of solutions by an approximation process. Before proving this we need another preparatory lemma. Recall that if $f : [0, +\infty) \to H$ satisfies (2.6) then $h = Qf$ satisfies the same condition of uniform local Hölder continuity

$$|h(t_1) - h(t_2)| \leq L|t_1 - t_2|^\theta, \qquad t_1, t_2 \in J. \tag{5.3}$$

Also we have remarked that for bounded solutions $u \in L^\infty((0, +\infty); H)$ of (2.4) the component function $x(t) = Pu(t)$ satisfies (3.8). Let $J = [a, b) \subset [0, +\infty), \bar{t} \in J, \Delta t > 0$ with $\bar{t} + \Delta t \in J$. Suppose that $u(t) = x(t) + y(t)$ is a solution of (2.4) and set $\bar{x} = x(\bar{t}), \bar{y} = y(\bar{t}), \bar{h} = h(\bar{t})$. Consider the initial-value problems

$$y' + Ay + QF(x + y) = h, \quad y(\bar{t}) = \bar{y}, \tag{5.4}$$

$$z' + Az + QF(\bar{x} + z) = \bar{h}, \quad z(\bar{t}) = \bar{y}. \tag{5.5}$$

We continue to retain the hypotheses (2.5) and (4.5) on F. Obviously $y(t) = Qu(t)$ is the solution of (5.4). By the remarks at the end of section 3, the solution $z(t)$ of (5.5) exists on J and has the same regularity properties as $y(t)$. The next lemma is essentially a result on continuous dependence.

LEMMA 5.2. *With the notations and assumptions set out above let y, z be the solutions of (5.4), (5.5) respectively. If h, x satisfy (5.3), (3.8) respectively then*

$$|y(t) - z(t)| \leq (\mu - m)^{-1} C \Delta t^\theta, \quad 0 \leq \bar{t} \leq t \leq \bar{t} + \Delta t, \tag{5.6}$$

$$|y(t) - z(t)|_1 \leq (\mu - m)^{-1/2} C \left(1 + \frac{M}{\mu - m}\right) \Delta t^\theta, \quad 0 < \bar{t} \leq t \leq \bar{t} + \Delta t, \tag{5.7}$$

where $C = L + MK$ depends on the various Hölder and Lipschitz constants.

Remark. The above estimate will be used in the next section to show that bounded solutions of (2.4) approach a neighborhood of a manifold in H^1 generated by σ. In the proof we need to know that the constant $C = L + MK$ appearing in (5.6), (5.7) is independent of $\bar{t}$. Since $h(t)$ and $x(t)$ are uniformly locally Hölder continuous on $[0, +\infty)$, this will certainly be the case if Lemma 5.2 is applied only for Δt less than a fixed constant.

Proof. Let $w(t) = y(t) - z(t)$ and define $v_o(t) = \frac{1}{2}|w(t)|^2$. Then for $t \in (\bar{t}, \bar{t} + \Delta t)$

$$\begin{aligned} v_o'(t) &= (w, h - \bar{h} - Aw - F(x + y) + F(\bar{x} + z)) \\ &\leq -(\mu - m)|w|^2 + |h - \bar{h}|\,|w| - (w, F(x + z) - F(\bar{x} + z)) \\ &\leq -(\mu - m)|w|^2 + \frac{1}{2}\eta^{-1}\left\{|h(t) - \bar{h}| + M|x(t) - \bar{x}|\right\}^2 + \frac{1}{2}\eta|w|^2 \end{aligned}$$

where $\eta > 0$ is arbitrary. With the choice $\eta = \mu - m$, and using (5.3), (3.8) we obtain

$$v_o'(t) \leq -(\mu - m)v_o(t) + [2(\mu - m)]^{-1}[L + MK]^2 \Delta t^{2\theta}, \quad \bar{t} \leq t \leq \bar{t} + \Delta t.$$

Since $v_o(\bar{t}) = 0$ integration of this inequality yields

$$v_o(t) \leq \frac{1}{2}\left[\frac{\Delta t^\theta (L + MK)}{\mu - m}\right]^2 \left(1 - e^{-(\mu - m)\Delta t}\right), \quad \bar{t} \leq t \leq \bar{t} + \Delta t.$$

Inequality (5.6) now follows.

For the H^1-norm estimate we consider $v_1(t) = \frac{1}{2}|A^{1/2}(y(t) - z(t))|^2$ and assume $t \geq \bar{t} > 0$. Differentiating and using (5.6) gives

$$\begin{aligned} v_1'(t) &= (Aw, h - \bar{h} - Aw - Q[F(x + y) - F(\bar{x} + z)]) \\ &\leq -\frac{1}{2}|Aw|^2 + \frac{1}{2}\left\{|h - \bar{h}| + |F(x + y) - F(\bar{x} + z)|\right\}^2 \\ &\leq -\frac{1}{2}\mu|A^{1/2}w|^2 + \frac{1}{2}C^2\left[1 + \frac{M}{\mu - m}\right]^2 \Delta t^{2\theta}. \end{aligned}$$

Since $\mu > \mu - m$ this shows $v_1' \leq -(\mu - m)v_1 + k$, where $k = \frac{1}{2}C^2\left[1 + M(\mu - m)^{-1}\right]^2 \Delta t^{2\theta}$. But $v_1(\bar{t}) = 0$ so that integration shows

$$|A^{1/2}(y(t) - z(t))|^2 \leq (\mu - m)^{-1}C^2\left[1 + \frac{M}{\mu - m}\right]^2 \Delta t^{2\theta}\left(1 - e^{-(\mu - m)(t - \bar{t})}\right),$$

for $0 < \bar{t} \leq t \leq \bar{t} + \Delta t$. Inequality (5.7) now follows. □

§**6. Exponential Decay to Approximate Inertial Forms.** In this section we prove our main result on the exponential attracting property of certains sets. In particular these sets are neighborhoods of a finite dimensional manifold $\mathcal{M}_\mu(t) = \{v \in H : v = x(t) + \sigma(x(t), Qf(t))\}$ generated by the map σ, which is described in section 4. Essentially we argue that the actual trajectory can be uniformly approximated by a series of trajectory segments. These segments are shown to possess the exponential decay property. Since the actual trajectory must stay close by, it also has this property. We first prove an auxilary result.

LEMMA 6.1. *Let $\theta \in (0, 1)$ and $\varphi(\sigma) = \sigma^\theta/(1 - \exp(-\sigma)), \sigma > 0$. Then $k(\theta) = \inf\{\varphi(\sigma) : \sigma > 0\}$ is attained at a unique value $\sigma_0 = \sigma_0(\theta)$. In addition $k(\theta) = \sigma_0^{\theta-1}(\theta + \sigma_0)$ and $k(\theta) \to 1^+$ as $\theta \to 1^-$.*

Proof. Clearly

$$\varphi'(\sigma) = \frac{\sigma^{\theta-1}\psi(\sigma)}{(1 - e^{-\sigma})^2}, \text{ where } \psi(\sigma) = \theta - (\theta + \sigma)e^{-\sigma}.$$

Examining $\psi(\sigma)$ we find that for each $\theta \in (0,1)$ there is a unique $\sigma_0 = \sigma_0(\theta) > 0$ such that $\psi(\sigma) < 0$ for $0 < \sigma < \sigma_0$, and $\psi(\sigma) > 0$ for $\sigma > \sigma_0$. Hence $k(\theta) = \varphi(\sigma_0)$. Since $\theta = (\theta + \sigma_0)\exp(-\sigma_0)$ implicitly defines σ_0, $\varphi(\sigma_0) = \sigma_0^{\theta-1}(\theta + \sigma_0)$.

Finally we show $k(\theta) \to 1^+$ as $\theta \to 1^-$. Again using the implicit definition $\theta = (\theta + \sigma_0)\exp(-\sigma_0)$ we find $\theta = \delta(\sigma_0)$, where $\delta(\sigma) = \sigma/(\exp\sigma - 1)$. That is $\sigma_0 = \delta^{-1}(\theta)$. It is easy to see that $\delta(0+) = 1, \delta'(0+) = -1/2$, and $\delta'(\sigma) < 0$ for all $\sigma > 0$. Clearly then $\theta + \sigma_0 \to 1^+$ as $\theta \to 1^-$. Consider the function $(\theta - 1)\log\sigma_0 = [\delta(\sigma_0) - 1]\log\sigma_0$. Since $\sigma^{-1}[\delta(\sigma) - 1] \to -1/2$ as $\sigma \to 0^+$, we have $(\theta - 1)\log\sigma_0 = \sigma_0^{-1}[\delta(\sigma_0) - 1]\sigma_0\log\sigma_0 \to 0^+$ as $\sigma \to 0^+$. Hence $\sigma_0^{\theta-1} \to 1^+$ as $\theta \to 1^-$.

THEOREM 6.2. *Let $F \in C(H,H)$ satisfy (2.5), (3.1), (4.5) and $f \in L^\infty((0,+\infty);H)$ satisfy (2.6). Then for any $\mu > m$ there is a decomposition $H = X + Y$ and a map $\sigma : X \times Y \to Y^1$ such that for any solution $u(t) = x(t) + y(t)$ of (2.4) we have*

$$(6.1)\quad |y(t) - \sigma(x(t),h(t))| \le C_1 k(\theta)(\mu - m)^{-(1+\theta)} + d_0 e^{-(\mu-m)(t-t_0)}, t \ge t_0 \ge 0$$

$$(6.2)\quad |y(t) - \sigma(x(t),h(t))|_1 \le C_2 k(\theta)(\mu - m)^{-(\frac{1}{2}+\theta)} + BD_0 e^{-(\mu-m)(t-t_0)/2}, t \ge t_0 > 0,$$

where $h(t) = Qf(t)$ is the projection of $f(t)$ onto Y, $C_1 = 2(L + MK)$ depends on the various Hölder and Lipschitz constants (cf. (2.5), (3.7), (5.3)), $B = 1 + M/(\mu - m), C_2 = C_1 B(B + \delta^{-1})$ with $\delta \in (0, (\mu - m)/\mu)$ arbitrary, $d_0 = |y(t_0) - \sigma(x(t_0), h(t_0))|$, $D_0 = |(y(t_0) - \sigma(x(t_0), h(t_0))|_1$ and $k(\theta)$ is a constant depending only on θ with the property that $k(\theta) \to 1^+$ as $\theta \to 1^-$.

Remark. Observe that $B \to 1^+$ as $\mu \to +\infty$, and if we choose $\alpha > 1$ and set $\delta = \alpha^{-1}(\mu - m)/\mu$ then $\delta^{-1} \to \alpha$ as $\mu \to +\infty$.

Proof. Let $\mu > m$ be given, and let $H = X + Y$ be the decomposition induced by μ as described in section 2. Let $\sigma : H \times Y \to Y^1$ be the map given by Lemma 4.1. Let $t_0 \ge 0$, choose $\Delta t > 0$ and set $h(t) = Qf(t)$. Define sequences $\{t_n\}, \{x_n\}, \{h_n\}, \{d_n\}$ by setting $t_n = t_0 + n\Delta t, x_n = x(t_n) = Pu(t_n), h_n = h(t_n)$, and $d_n = |y(t_n) - \sigma(x_n, h_n)|$ where $y(t) = Qu(t)$. We first obtain estimates on the sequence $\{d_n\}$. For all $n \ge 0$ let $z_n : [t_n, t_{n+1}] \to H$ be the solution of the initial-value problem

$$(6.3)\qquad z_n' + Az_n + QF(x_n + z_n) = h_n, \quad z_n(t_n) = y_n.$$

It follows immediately form Lemma 5.1 that

$$(6.4)\qquad |z_n(t) - \sigma(x_n, h_n)| \le d_n e^{-(\mu-m)(t-t_n)}, \quad t_n \le t \le t_{n+1}$$

For convenience set $a = \mu - m, b = \exp(-a\Delta t), C = L + MK$. Then using (4.6), (5.6), (6.4) we find

$$\begin{aligned} d_{n+1} &\le |y(t_{n+1}) - z(t_{n+1})| + |z(t_{n+1}) - \sigma(x_n, h_n)| + |\sigma(x_{n+1}, h_{n+1}) - \sigma(x_n, h_n)| \\ &\le Ca^{-1}\Delta t^\theta + d_n b + a^{-1}\{M|x_{n+1} - x_n| + |h_{n+1} - h_n|\} \\ &\le 2Ca^{-1}\Delta t^\theta + d_n b \end{aligned}$$

Observe that C is independent of n by the uniform local Hölder continuity of $x(t)$ and $h(t)$. Thus the sequence $\{d_n\}$ satisfies $d_{n+1} \leq bd_n + \eta$, where $0 < b < 1$ and $\eta = 2Ca^{-1}\Delta t^\theta$. Iterating this inequality yields $d_n \leq \eta(1-b)^{-1} + b^n d_0$ for $n \geq 1$.

We now use this discrete estimate to obtain the continuous estimate (6.1). On the interval $[t_n, t_{n+1}]$ the inequalities (5.6), (6.4) and the discrete estimate above yield

$$\begin{aligned}|y(t) - \sigma(x_n, h_n)| &\leq |y(t) - z_n(t)| + |z_n(t) - \sigma(x_n, h_n)| \\ &\leq Ca^{-1}\Delta t^\theta + \left[\eta(1-b)^{-1} + b^n d_0\right] e^{-a(t-t_n)}.\end{aligned}$$

Let $C_1(a, \Delta t) = C\left[1 + 2b(1-b)^{-1}\right]$. Since $b^n = \exp(-a(t_n - t_0))$ we have

$$|y(t) - \sigma(x_n, h_n)| \leq C_1(a, \Delta t)a^{-1}\Delta t^\theta + d_0 e^{-a(t-t_0)}, t_n \leq t \leq t_{n+1}.$$

Now let $t \in (0, +\infty)$ be given, and determine n so that $t_n \leq t \leq t_{n+1}$. From the Lipschitz property of σ and (3.7), (5.3) we have

$$|\sigma(x_n, h_n) - \sigma(x(t), h(t))| \leq Ca^{-1}\Delta t^\theta, \quad t > 0.$$

Therefore with $t_0 \geq 0$ we have

$$|y(t) - \sigma(x(t), h(t))| \leq C_2(a, \Delta t)a^{-1}\Delta t^\theta + d_0 e^{-a(t-t_0)}, \quad t \geq t_0 \geq 0,$$

where $C_2(a, \Delta t) = C_1(a, \Delta t) + C = 2C(1-b)^{-1}$. It remains to examine the dependence of $\Delta t^\theta(1-b)^{-1}$ on Δt.

We have $\Delta t^\theta(1-b)^{-1} = \Delta t^\theta/(1 - \exp(-a\Delta t)) = a^{-\theta}\varphi(a\Delta t)$, where $\varphi(\sigma) = \sigma^\theta/(1 - \exp(-\sigma)), \sigma > 0$. Therefore, according to Lemma 6.1, the term $\Delta t^\theta(1-b)^{-1}$ is smallest when $\Delta t = a^{-1}\sigma_0$. For this value of Δt we obtain $\Delta t^\theta(1-b)^{-1} = a^{-\theta}\sigma_0^{\theta-1}(\theta + \sigma_0)$. Estimate (6.1) now follows with $k(\theta) = \sigma_0^{\theta-1}(\theta + \sigma_0)$.

For estimate (6.2) we assume $t_0 > 0$ and set $d_n = |y(t_n) - \sigma(x_n, h_n)|_1^2$. Again for convenience we use $a = \mu - m$, and now $b = \exp(-a\Delta t/2)$. Set $B = 1 + M/a$. From (5.2) we have

$$|z(t) - \sigma(x_n, h_n)|_1 \leq Bd_n e^{-a(t-t_n)/2}, t_n \leq t \leq t_{n+1}. \tag{6.5}$$

Thus in the same way as before, by employing (4.7), (5.7), and (6.5) we find

$$\begin{aligned}d_{n+1} &\leq CBa^{-1/2}\Delta t^\theta + d_n b + a^{-1/2}\delta^{-1}C\Delta t^\theta \\ &\leq C(B + \delta^{-1})a^{-1/2}\Delta t^\theta + d_n b\end{aligned}$$

Iterating this inequality shows $d_n \leq \eta(1-b)^{-1} + b^n d_0$, where now $\eta = C(B+\delta^{-1})a^{-1/2}\Delta t^\theta$. Thus

$$\begin{aligned}|y(t) - \sigma(x_n, h_n)|_1 &\leq CBa^{-1/2}\Delta t^\theta + B\left[\eta(1-b)^{-1} + b^n d_0\right] e^{-a(t-t_n)/2} \\ &\leq CBa^{-1/2}\Delta t^\theta \left[1 + \frac{b(B + \delta^{-1})}{1-b}\right] + Bd_0 e^{-a(t-t_0)/2}, t_n \leq t \leq t_{n+1}.\end{aligned}$$

If $t \geq t_0$ is given and n is such that $t_n \leq t \leq t_{n+1}$, then by (4.7)

$$|y(t) - \sigma(x(t), h(t))|_1 \leq Ca^{-1/2}\Delta t^\theta (B + \delta^{-1}) \left[1 + \frac{Bb}{1-b}\right] + Bd_0 e^{-a(t-t_0)/2}, t \geq t_0 > 0.$$

Since $1 + Bb/(1-b) = (1 + a^{-1}bM)/(1-b) \leq (1 + a^{-1}M)/(1-b)$ the above inequality implies

$$|y(t) - \sigma(x(t), h(t))|_1 \leq Ca^{-1/2}B(B + \delta^{-1})\Delta t^\theta (1-b)^{-1} + Bd_0 e^{-a(t-t_0)/2}, t \geq t_0 > 0.$$

Now we have $\Delta t^\theta (1-b)^{-1} = (2/a)^\theta \varphi(a\Delta t/2)$, where again φ is the function discussed in Lemma 6.1. Thus $\Delta t^\theta (1-b)^{-1}$ is smallest when $\Delta t = 2a^{-1}\sigma_0$. With this choice $\Delta t^\theta (1-b)^{-1} = (2/a)^\theta k(\theta)$. Inequality (6.2) now follows and the proof is complete. □

COROLLARY 6.3. *Under the hypotheses of Theorem 6.2, the solution $u(t)$ of (2.4) approaches a neighborhood of the manifold $\mathcal{M}_\mu(t) = \{v \in H : v = x(t) + \sigma(x(t), Qf(t))\}$. In particular*

$$\text{dist}(u(t), \mathcal{M}_\mu(t)) \leq \text{dist}(u_0, \mathcal{M}_\mu(0))e^{-(\mu-m)t} + C(\mu - m)^{-(1+\theta)}, \quad t \geq 0, \tag{6.6}$$

where $C = 2(L + MK)k(\theta)$ is independent of μ, m, u_0, and t. An analogous inequality holds when $M_\mu(t)$ is considered as a subset of H^1.

Remarks:

(1) If the forcing term f is independent of t we may take $\theta = 1$ and the manifold $\mathcal{M}_\mu$ independent of t.

(2) The choice of μ, and hence the decomposition of H, is only restricted by the requirement $\mu > m$. Allowing μ to increase shows that we have essentially $\text{dist}(u(t), \mathcal{M}_\mu(t)) = \mathcal{O}((\mu - m)^{-(1+\theta)})$ as $\mu \to +\infty$.

(3) The result of Theorem 6.2 can be paraphrased by saying that solutions of (2.4) have the form $u(t) = x(t) + y(t)$ where

$$\begin{cases} x' + Ax + PF(x + \sigma(x, h) + \epsilon) = g(t) \\ \\ y = \sigma(x, h) + \epsilon. \end{cases} \tag{6.7}$$

In these equations $\epsilon(t) = y(t) - \sigma(x(t), h(t))$ is bounded according to the estimates

$$|\epsilon(t)| \leq |\epsilon(t_0)|e^{-(\mu-m)(t-t_0)} + C_1(\theta)(\mu - m)^{-(1+\theta)}, \quad t \geq t_0 \geq 0,$$
$$|\epsilon(t)|_1 \leq B|\epsilon(t_0)|_1 e^{-(\mu-m)(t-t_0)/2} + C_2(\theta)(\mu - m)^{-(\frac{1}{2}+\theta)}, \quad t \geq t_0 > 0$$

where the constants involved are described in Theorem 6.2.

§7. Systems of Reaction-Diffusion Equations. Let $\Omega \subset \mathbb{R}^n$ be an open bounded region with smooth boundary. We consider the system (cf. [2], [31], [32]; see also [5], [9], [15], [16], [24])

$$(7.1) \qquad u_t - D\Delta u + F(u) = f, \quad (t,x) \in \mathbb{R}^+ \times \Omega,$$

$$(7.2) \qquad u(t,x) = 0, \quad (t,x) \in \mathbb{R}^+ \times \partial\Omega,$$

$$(7.3) \qquad u(0,x) = u_0(x), \quad x \in \Omega,$$

where $u \in \mathbb{R}^k$ is a column vector, $D = \text{diag}\,(d_1, \cdots, d_k)$ is a $(k \times k)$ diagonal matrix with $d_i > 0$ for $1 \le i \le k, F : \mathbb{R}^k \to \mathbb{R}^k$ is Lipschitz continuous, and $f : \mathbb{R}^+ \times \Omega \to \mathbb{R}^k$. This system falls into the abstract framework of the previous sections with $H = L^2(\Omega, \mathbb{R}^k)$ and $A = -D\Delta$. The domain of A is $H^2 = H^2(\Omega, \mathbb{R}^k) \cap H_0^1(\Omega, \mathbb{R}^k)$, and $H^1 = H_0^1(\Omega, \mathbb{R}^k)$. These are the standard Hilbert-Sobolev spaces (cf. [1]).

In order to apply the results of the previous sections we make the following assumptions. Let $F : \mathbb{R}^k \to \mathbb{R}^k$ satisfy

$$(7.4) \qquad |F(u) - F(v)| \le M|u - v|, \quad \text{for all } u, v \in \mathbb{R}^k, \text{ and}$$

$$(7.5) \qquad u^T F(u) \ge 0, \text{ for all } u \in \mathbb{R}^k \text{ with } |u| \ge r,$$

where $r > 0$ is a given constant. Here u^T denotes the transpose of u. The Lipschitz property (7.4) clearly implies

$$\left\{ \int_\Omega |F(u) - F(v)|^2 dx \right\}^{1/2} \le M \left\{ \int_\Omega |u - v|^2 dx \right\}^{1/2}, \quad u, v \in L^2(\Omega; \mathbb{R}^k),$$

which in (2.5) in the present context. The inequality (7.5) implies that the function $u \to u^T F(u)$ is bounded below on $\mathbb{R}^k$. Thus

$$\int_\Omega u^T F(u) dx \ge \min_{u \in \mathbb{R}^k} u^T F(u) \cdot \text{ meas } (\Omega), \qquad u \in L^2(\Omega),$$

which is (3.1) with $m_* = 0$ and F_0 being the negative of the right-hand side above (assuming it is negative; otherwise $F_0 = 0$). From (7.4) it follows that

$$(7.6) \qquad m = -\inf \left\{ |u - v|^{-2} (u - v)^T [F(u) - F(v)] : u, v \in \mathbb{R}^k, u \ne v \right\}.$$

is well-defined. From this definition it immediately follows that

$$(7.7) \qquad \int_\Omega (u - v)^T [F(u) - F(v)] dx \ge -m \int_\Omega |u - v|^2 dx$$

which corresponds to (4.5).

Let $f : (0, +\infty) \times \Omega \to \mathbb{R}^k$ be measurable and satisfy

$$\sup_{t>0} \int_\Omega |f(t,x)|^2 dx < +\infty. \tag{7.8}$$

$$\left\{ \int_\Omega |f(t_1,x) - f(t_2,x)|^2 dx \right\}^{1/2} \le L|t_1 - t_2|^\theta, \qquad \text{for all } t_1, t_2 \in J, \tag{7.9}$$

where L depends only on the length of J and $\theta \in (0,1]$. Clearly (7.9) corresponds to (2.6), while (7.8) states that as a $L^2(\Omega, \mathbb{R}^k)$-valued function, $t \to f(t,\cdot), f$ is bounded. These conditions would certainly be satisfied if $|f(t,x)| \le f_0(x)$, where $f_0 \in L^2(\Omega; \mathbb{R}^k)$, and $|f(t_1,x) - f(t_2,x)| \le c(x)|t_1 - t_2|^\theta$, for all $x \in \Omega$ and $t_1, t_2 \in J$, where $c \in L^2(\Omega; \mathbb{R}^k)$.

THEOREM 7.1. *Let $F \in C(\mathbb{R}^k, \mathbb{R}^k)$ satisfy (7.4), (7.5) and f satisfy (7.8), (7.9). Then for every $u_0 \in L^2(\Omega, \mathbb{R}^k)$ there is a unique solution u of (7.1)-(7.3) defiend for all $t > 0$. Moreover (7.1)-(7.3) has a compact connected global attractor, and all orbits are exponentially attracted to a neighborhood of a finite-dimensional manifold. The thickness of the neighborhood can be made arbitrarily small by increasing the dimension of the manifold.*

Remark. Theorem 7.1 remains valid if we replace (7.2) with a Neuman boundary condition and (7.5) by

$$u^T F(u) \ge \epsilon |u|^2, \qquad \text{for all } u \in \mathbb{R}^k \text{ with } |u| \ge r, \tag{7.10}$$

where $\epsilon, r > 0$ are given constants with ϵ arbitrarily small. In this case we replace $-D\triangle$ by $-D\triangle + \frac{\epsilon}{2}$ and $F(u)$ by $\hat{F}(u) = F(u) - \frac{\epsilon}{2}u$. Since $u \to u^T F(u) - \frac{1}{2}\epsilon|u|^2$ is bounded below on $\mathbb{R}^k$, (3.1) is again valid with $\hat{F}$ replacing F. Notice that (7.5) is not suficient in this case as the following example shows. Let $k = 1$, $F(u) = -u^{-1}$ for $|u| \ge 1$, and f be identically zero. If $u_0 = 1$, the constant function, then $u = (2t+1)^{1/2}$ is the solution of (7.1)-(7.3), with (7.2) replaced by the Neuman boundary conditions.

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