ON THE BARENBLATT EQUATION OF ELASTO-PLASTIC FILTRATION

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IMA Preprint Series # 817
May 1991

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Introduction

We consider the equation

$$u_t + \gamma |u_t| = \Delta u\,, \quad 0 < |\gamma| < 1 \tag{E}$$

in $Q = \{(x,t) : x \in \mathbf{R}^N, t > 0\}$. This equation arises in the theory of an elastic fluid in an elasto-plastic porous medium, under the assumption that the porous medium is irreversibly deformable, and was studied by Barenblatt and his collaborators, see [B], [BK], or the recent book [BER]. There the equation is written in the form

$$F(u_t) = \Delta u \quad \text{with} \quad F(s) = \begin{cases} (1+\gamma)s & \text{for} \quad s \geq 0 \\ (1-\gamma)s & \text{for} \quad s < 0\,. \end{cases} \tag{E'}$$

Equation (E) can also be written as

$$u_t = \begin{cases} \max\{L_1(u), L_2(u)\} & \text{if} \quad \gamma < 0 \\ \min\{L_1(u), L_2(u)\} & \text{if} \quad \gamma > 0\,, \end{cases} \tag{E''}$$

where $L_1(u) = \Delta u/(1-\gamma)$ and $L_2(u) = \Delta u/(1+\gamma)$. Written in this form it is a particular case of the so-called *parabolic Bellman equations of dynamic programming*. We refer to the paper by Evans and Lenhart [EL] for more on these equations and relevant literature.

Existence of solutions for equation (E) in $N = 1$ was shown by Kamin [K1] for the Cauchy Problem. The same author also showed [K2] that for $\gamma \neq 0$ there exists no fundamental solution, i.e. no solution taking the Dirac mass as initial data. Barenblatt and Sivashinski [BS] have constructed in the one-dimensional case a selfsimilar solution of this equation with a singularity at $x = 0$, $t = 0$. This solution has the form

$$B(x,t) = t^{-\alpha/2} f(\eta)\,, \quad \eta = x/t^{1/2}\,, \tag{0.1}$$

where $\alpha < 1$ if $\gamma < 0$, $\alpha = 1$ if $\gamma = 0$ and $\alpha > 1$ if $\gamma > 0$ and behaves asymptotically

$$f(\eta) \approx \eta^{\alpha-1} e^{-\eta^2/4} \qquad \text{as} \quad \eta \to \infty\,. \tag{0.2}$$

The solution is constructed by matching explicit parabolic cylinder fuctions. They also found that $\alpha > 1$ when $\gamma > 0$, $\alpha < 1$ if $\gamma < 0$.

Here we consider an arbitrary space dimension $N \geq 1$ and show that for every $\gamma \in (-1,1)$ there exists a unique value of $\alpha = \alpha(\gamma) > 0$ for which (E) has a solution of the form (0.1) that decays exponentially at infinity. We also show that this solution is unique up to multiplicative constant in the class of selfsimilar solutions with exponential decay. We show that when $\gamma \neq 0$ the similarity exponent $\alpha(\gamma)$ is anomalous, which means that it differs from the usual $\alpha = N$ of the heat equation. Moreover, we prove the following properties of $\alpha(\gamma)$:

(i) $\alpha(\gamma)$ is a strictly increasing and continuous function of γ,

(ii)
$$\lim_{\gamma\to -1} \alpha(\gamma) = \max\{N-2, 0\}\,, \quad \lim_{\gamma\to 1} \alpha(\gamma) = \infty\,.$$

and of course for the heat equation, $\gamma = 0$, we get $\alpha(0) = N$.

The construction of the solutions is carried out in Section 2. In doing this we also prove that for every $0 < \alpha < \alpha(\gamma)$ and every $A > 0$ there exists a unique solution of the form (0.1) which takes on initial data $u(x,0) = A\,|x|^{-\alpha}$. This solution does not have fast decay : indeed, it decreases like $O(|x|^{-\alpha})$ as $|x| \to \infty$ for every fixed $t > 0$. The investigation of the range covered by α is performed in Section 3. Section 1 covers the basic existence and uniqueness theory in several space dimensions.

For the heat equation a basic conservation law states that the mass of a solution $M(t) = \int u(x,t)\,dx$ is invariant in time. This does not happen for equation (E) when $\gamma \neq 0$. Indeed, integration of the equation with respect to x gives

$$\frac{dM}{dt} = -\gamma \int_{\mathbf{R}^N} |u_t|\,dx\,. \tag{0.3}$$

Therefore, the mass of a solution increases in time if $\gamma < 0$ and decreases if $\gamma > 0$. This implies that for the selfsimilar solution (0.1) the exponent α has to be less than N in the former case, larger than N in the latter. It then follows from (0.1) that

$$\int_{\mathbf{R}^N} B(x,t)\,dx = C\,t^{\frac{N-\alpha}{2}}\,. \tag{0.4}$$

It follows that B is a fundamental solution only if $\gamma = 0$. It is *very singular* (VSS) if $\gamma > 0$, in the sense that the mass of the initial singularity is infinite. On the other hand, B is a *mildly singular solution* (MSS) (i.e. the mass of the initial singularity is 0) if $\gamma < 0$.

Recently, Goldenfeld and his collaborators [GMOL] have used renormalization group techniques to find the dependence of α and the solution on γ near $\gamma = 0$.

In Section 4 we use these selfsimilar solutions with exponent $\alpha(\gamma)$ to obtain the asymptotic behaviour as $t \to \infty$ of the solution of the Cauchy Problem with nonnegative initial data which decay fast as $|x| \to \infty$. This answers a conjecture stated by Barenblatt in his book [B, pages 55-59; see also Foreword]. As the selfsimilar solutions with $\alpha < \alpha(\gamma)$ show the decrease condition on the initial data is important, and solutions with data decaying slowly as $|x| \to \infty$ will have different asymptotic behaviour.

Our asymptotic result proves not only convergence towards Barenblatt profiles as $t \to \infty$, but it also shows that the ratio between $u(x,t)$ and $B(x,t)$ tends to a constant value $c > 0$. In Section 5 we discuss the properties of the nonlinear functional $J : u_0 \mapsto c$, which generalizes the Mass Conservation Law valid for the Heat Equation to the values $\gamma \neq 0$.

The situation described above could reflect some general phenomenon. Indeed, selfsimilar solutions with a singularity at the origin, $x = 0$, $t = 0$, of the very singular and mild types have been recently found by Bernis, Hulshof and Vazquez [BHV] in a different model of elastoplasticity, $u_t = |u_{xx}|^{m-1}u_{xx}$. For $m \neq 1$ this is also an equation of non-divergence type where the law of conservation of mass does not hold. In this case VSS occur to $m > 1$ while MSS occur to $0 < m < 1$. In that paper the connections with the porous medium equation are explored. Very singular selfsimilar solutions were considered before in [BPT] and [KP] in the study of nonlinear heat equations of the form $u_t = \Delta u - u^p$, with $p > 1$.

1. Existence and uniqueness of a solution

We set our Cauchy problem with nonnegative data in the space $C_0(\mathbf{R}^N)$ of continuous functions which tend to zero as $|x| \to \infty$.

THEOREM 1.1. *For every $u_0 \in C_0(\mathbf{R}^N)$ such that $u_0(x) \geq 0$ there exists a unique classical solution $u(x,t) \in C([0,\infty) : C_0(\mathbf{R}^N))$ of equation (E) with $u(0) = u_0$. The solution u is twice continuously differentiable in x and once in t, and u and its derivatives are Hölder continuous functions in Q. Moreover, the maps $u_0 \mapsto u(\cdot,t)$ form a semigroup of contractions in $C_0(\mathbf{R}^N)$ (endowed with the* sup *norm).*

PROOF: *Uniqueness.* It is based on the following property of L^∞-contraction, which can be proved by the usual Maximum Principle techniques.

LEMMA 1.1. *Given two classical solutions u_1 and u_2 of equation (E) with initial data u_{01} and u_{02}, we have for every $t > 0$*

$$\sup_{x\in\mathbf{R}^N} \{u_1(x,t) - u_2(x,t)\}_+ \leq \sup_{x\in\mathbf{R}^N} \{u_{01}(x) - u_{02}(x)\}_+ . \tag{1.1}$$

In particular, the sup*-norm of the difference of two solutions decreases in time and the Maximum Principle holds: if $u_{01} \leq u_{02}$ then for every $t > 0$ we have $u_1(\cdot,t) \leq u_2(\cdot,t)$.*

This property is well-known and referred to in the literature as the *m-accretivity* of the operator $A(u) = -\Phi(\Delta u)$ ($\Phi = F^{-1}$) defined in some convenient subspace of $C_0(\mathbf{R}^N)$.

Existence. Using the Theory of Nonlinear Semigroups of Crandall and Liggett, see [C], the property of L^∞-contraction can also provide with the existence of a solution $u \in C([0,\infty) : C_0(\mathbf{R}^N))$ which solves equation (E) in some *mild* sense, see for instance [BC]. In this way we obtain a semigroup of contractions in $C_0(\mathbf{R}^N)$. The idea is to discretize the equation implicitly in time, solve the resulting sequence of elliptic problems and then pass to the limit according to the Semigroup Generation Theorem. We then have to prove that the solution is actually a classical solution of (E).

A more standard approach to the existence of a solution is the following.

(i) First, we approximate the problem by smoothing the nonlinearity F into F_ε and the initial data u_0 into $u_{0\varepsilon}$, so that a classical solution u_ε is known to exist. We can also reduce the domain to

$$Q_\varepsilon = \{x : |x| \le 1/\varepsilon\} \times (0, \infty)$$

and impose zero boundary data along the lateral boundary $\{|x| = 1/\varepsilon,\, t \ge 0\}$. Then we obtain estimates on the solution u_ε and its derivatives. These estimates will depend only on γ and on the *sup* norm of the initial data, being thus independent of ε. This allows us to pass to the limit and obtain a solution of (E), which turns out to be smooth.

(ii) *Estimates.* It is clear that for every $(x,t) \in Q_\varepsilon$ we have

$$u_\varepsilon(x,t) \le \|u_{0\varepsilon}\|_\infty \le \|u_0\|_\infty = M\,. \tag{1.2}$$

We write u instead of u_ε in the sequel for convenience. We begin with an energy estimate. We distinguish two cases: $\gamma \ge 0$ and $\gamma < 0$. If $\gamma \ge 0$ we obtain the estimate in the standard way. We multiply the equation $F(u_t) = \Delta u$ by $u(x,t)\zeta(x)^2$, where $\zeta \ge 0$ is a C_0^∞ cut-off fuction, and integrate by parts to obtain

$$\begin{aligned} &\frac{1}{2}\int u^2(x,t)\zeta^2(x)\,dx + \int_0^t \int |\nabla u(x,t)|^2\,\zeta^2(x)dx\,dt \\ &\le \frac{1}{2}\int u_0^2(x)\,dx + 2\int_0^t \int u\zeta|\nabla u \cdot \nabla\zeta|\,dx\,dt\,. \end{aligned} \tag{1.3}$$

This means that $|\nabla u|$ is uniformly bounded in L^2 on compact subsets of Q. When $\gamma < 0$ we obtain the same kind of local L^2 bound for ∇u by multiplying $F(u_t) = \Delta u$ by $f(u)\zeta^2$, where $f(u)$ is a nonnegative, bounded and decreasing function of u in the interval $I = \{0 \le u \le M\}$. Let $g(u) = \int_0^u f(s)\,ds$. We get

$$\begin{aligned} &\iint |\nabla u|^2\,(-f'(u))\zeta^2 dx\,dt + |\gamma| \iint |u_t| f(u)\zeta^2\,dx\,dt + \int g(u_0)\zeta^2\,dx \\ &\le \int g(u(x,t))\,\zeta^2(x)dx + 2\iint \nabla u \cdot \nabla\zeta\, f(u)\zeta\,dx\,dt\,. \end{aligned} \tag{1.4}$$

If we take for instance $f(u) = M - u$, we have $f'(u) = -1$ and $g(u) \le M^2/2$ in I. This gives the desired L^2 bound for ∇u. Next we obtain a bound for u_t and Δu for all $\gamma \in (-1,1)$. We multiply equation (E) by $u_t(x,t)\zeta^2(x)$ and integrate to obtain

$$\begin{aligned} &\int_\tau^t \int u_t F(u_t)\zeta^2\,dx\,dt + \frac{1}{2}\int |\nabla u(x,t)|^2\,\zeta^2(x)\,dx \\ &\le \frac{1}{2}\int |\nabla u(x,\tau)|^2\,\zeta^2(x)\,dx + 2\int_\tau^t \int |u_t|\,|\nabla u \cdot \nabla\zeta|\,\zeta\,dx\,dt\,. \end{aligned} \tag{1.5}$$

Using the fact that $sF(s) \ge (1-|\gamma|)s^2$ we obtain an estimate for u_t, and so also for Δu, in $L^2_{loc}(Q_\varepsilon)$. These two estimates imply that for every initial data $u_0 \in C_0(\mathbf{R}^N)$ the solutions u_ε will have the following uniform local bounds:

$$|\nabla u(\cdot,t)| \text{ in } L^2_{loc} \text{ for every fixed } t>0 \text{ and } \Delta u \text{ in } L^2_{loc}(Q_\varepsilon).$$

In order to obtain boundedness and compactness in Hölder spaces we write the equation for $v = \Delta u$:

$$v_t = \Delta\Phi(v)\,. \tag{1.6}$$

We have shown that for every compact subset K of Q the family $\{v_\varepsilon\}_{\varepsilon<\varepsilon_0}$ is bounded in $L^2(K)$. Note that we can write equation (1.6) in the form

$$v_t = (a_{ij}v_{x_i})_{x_j} \tag{1.7}$$

with $a_{ij} = c(v)\delta_{ij}$, $c(v) = (1+\gamma)^{-1}$ if $v>0$, $c(v) = (1-\gamma)^{-1}$ if $v<0$. Using the continuity results of Moser [M] we conclude that v_ε is uniformly bounded in some Hölder space $C^{\nu,\nu/2}$ on compact subsets of Q.

(iii) Passing to the limit to obtain a solution of (E) with initial data in C_0 is standard. #

From step (ii) of the above proof we may also conclude the following result about compactness that will be of use later in the study of the asymptotic behaviour.

LEMMA 1.2. *Every family of solutions of equation (E) which is uniformly bounded and continuous in a compact subset K of Q is uniformly bounded in $C^{2+\nu,1+\nu/2}_{x,t}(K)$.*

2. Self-similar solutions

In this section we investigate the existence and uniqueness of nonnegative self-similar solutions of the equation

$$F(u_t) = \Delta u \quad \text{in } Q = \mathbf{R}^N \times \mathbf{R}^+ \tag{2.1}$$

taking on initial data

$$u(x,0) = 0 \quad \text{for } x \neq 0\,. \tag{2.2}$$

We work in any space dimension $N \geq 1$ and the constitutive function is given by

$$F(s) = s + \gamma|s|, \qquad s \in \mathbf{R},\ |\gamma| \in (0,1). \tag{2.3}$$

We shall look for solutions of the form

$$u(x,t) = t^{-\alpha/2}f(\eta), \quad \eta = |x|/\sqrt{t},\ \alpha>0. \tag{2.4}$$

This yields upon substitution into the differential equation the following ODE for the function f:

$$f'' + \frac{N-1}{\eta}f' = \frac{1}{2}F(-(\eta f' + \alpha f)) \qquad \text{for} \quad \eta>0\,.$$

In view of the regularity of $u(x,t)$ at the origin for $t > 0$, we must have $f'(0) = 0$ and, if $x \neq 0$, we require in view of (2.2)

$$\lim_{t \downarrow 0} u(x,t) = |x|^{-\alpha} \lim_{\eta \to \infty} \eta^\alpha f(\eta) = 0.$$

Finally, because the differential equation and the conditions at the origin and infinity are not affected by a multiplicative constant, there will be no loss of generality if we set $f(0) = 1$. Thus, we shall be investigating the problem

$$\text{(I)} \quad \begin{cases} (2.5) \quad f'' + \frac{N-1}{\eta} f' = \frac{1}{2} F(-(\eta f' + \alpha f)), \quad f > 0, \qquad 0 < \eta < \infty, \\ (2.6) \quad f(0) = 1, \quad f'(0) = 0, \\ (2.7) \quad \eta^\alpha f(\eta) \to 0 \quad \text{as} \quad \eta \to \infty. \end{cases}$$

We prove the following result.

THEOREM 2.1. *For any $\gamma \in (-1,1)$ there exists a unique $\alpha(\gamma) > 0$ such that Problem (I) has a solution.*

As remarked in the Introduction this corresponds to a selfsimilar solution of equation (2.1) with a singularity at $(x = 0, t = 0)$.

Our analysis allows also to construct nonnegative and decaying selfsimilar solutions for other exponent choices.

THEOREM 2.2. *For every α such that $0 < \alpha < \alpha(\gamma)$ there is a unique positive solution of (2.5)-(2.6) satisfying*

$$\lim_{\eta \to \infty} \eta^\alpha f(\eta) = A \tag{2.8}$$

for some constant $A \in (0, \infty)$. On the contrary, for $\alpha > \alpha(\gamma)$ the solution of (2.1)-(2.2) reaches the axis $f = 0$ at a finite distance $\eta > 0$.

The solutions of (2.5), (2.6), (2.8) given by Theorem 2.2 for $0 < \alpha < \alpha(\gamma)$ do not take the initial condition (2.2). In fact, for such solutions

$$u_0(x) = C\,|x|^{-\alpha}. \tag{2.9}$$

The proof of these results will proceed via a series of Lemmas, which at the same time will provide detailed information on the behaviour of the solutions. It will be convenient to write the differential equation (2.4) as a system of first order equations and carry out a shooting technique in the phase plane, with $\alpha > 0$ as a shooting parameter. Thus we write

$$\text{(Ia)} \quad \begin{cases} f' = g \\ g' = -\frac{N-1}{\eta} g + \frac{1}{2} F(-\eta g - \alpha f), \end{cases}$$

The initial data (2.6) are now written as

$$f(0) = 1, \qquad g(0) = 0. \tag{2.10}$$

For every $\alpha \in \mathbf{R}$, (Ia) has a unique local solution $f(\eta, \alpha)$ which satisfies (2.10). We shall show that it is possible to choose α in such a way that the corresponding solution can be continued for all $\eta > 0$ and that the condition (2.7) at infinity is satisfied.

Proceeding as in Brezis, Peletier & Terman [BPT], we introduce some notation, defining regions in the phase plane. Let

$$\begin{aligned} \mathcal{S} &= \{(f,g) : 0 \le f \le 1,\ g \le 0\} \\ \mathcal{L}_\lambda &= \{(f,g) \in \mathcal{S} : g \ge -\lambda f\} \\ \ell_\lambda &= \{(f,g) \in \mathcal{S} : g = -\lambda f\}, \end{aligned}$$

where λ is a positive number.

LEMMA 2.1. *Let (f,g) be the solution of (Ia) which starts at the point $(1,0)$. Then (f,g) enters $\mathcal{S}$ and can only leave it through the negative g-axis $\{f = 0,\ g < 0\}$.*

PROOF: If we let $\eta \to 0$ in (2.5) we obtain in view of (2.6)

$$Nf''(0) = \frac{1}{2}F(-\alpha) = \frac{(1-\gamma)\alpha}{2}.$$

Because $\alpha > 0$ this implies that $f''(0) < 0$ and hence that $g(\eta) = f'(\eta) < 0$ for η small. Since the vector field points into $\mathcal{S}$ along the positive f-axis and along the half-line $\{f = 1,\ g < 0\}$ and the origin is an equilibrium point, orbits can only leave through the negative g-axis. #

In the next lemma we prove a conditional invariance property of the sets $\mathcal{L}_\lambda$.

LEMMA 2.2. *For any $\lambda > 0$ there exists a time $\hat{\eta}_\lambda > 0$ such that if $(f(\eta_0), g(\eta_0)) \in \mathcal{L}_\lambda$ for some $\eta_0 > \eta_\lambda$ then $(f(\eta), g(\eta)) \in \mathcal{L}_\lambda$ for all $\eta \ge \eta_0$.*

PROOF: Along the top line of $\mathcal{L}_\lambda$ and along its right-hand side, the vector field points into $\mathcal{L}_\lambda$. Hence, it remains to investigate the vector field on ℓ_λ. There we have $g < 0$ and

$$\frac{dg}{df} = -\frac{N-1}{\eta} - \frac{1}{2}F(\eta - \frac{\alpha}{\lambda}), \tag{2.11}$$

(observe that for $g < 0$ we have $F(-gs) = -gF(s)$). Hence, since $f' = g < 0$, the vector field will point into $\mathcal{L}_\lambda$ if

$$-\frac{N-1}{\eta} - \frac{1}{2}F\Big(\eta - \frac{\alpha}{\lambda}\Big) < -\lambda.$$

This will be so if

$$F\Big(\eta - \frac{\alpha}{\lambda}\Big) > 2\lambda$$

that is, if

$$\eta > \hat{\eta}_\lambda = \frac{2\lambda}{1+\gamma} + \frac{\alpha}{\lambda}.$$

Thus, along ℓ_λ the vector field points into $\mathcal{L}_\lambda$ as well when $\eta > \eta_\lambda$, so $\mathcal{L}_\lambda$ is invariant when $\eta > \eta_\lambda$. #

LEMMA 2.3. *For every $\lambda > 0$ there exists an $\alpha_\lambda > 0$ such that if $0 < \alpha < \alpha_\lambda$, then the corresponding solution (f, g) will remain in $\mathcal{L}_\lambda$ for all $\eta \geq 0$.*

PROOF: Because $(f, g) = (1, 0)$ is an equilibrium solution of (Ia) when $\alpha = 0$, it follows from continuous dependence on parameters that for any given $\lambda > 0$, there exists an $\alpha_\lambda > 0$ such that (f, g) remains in $\mathcal{L}_\lambda$ for $0 \leq \eta \leq \hat{\eta}_\lambda$. Since, by Lemma 2.2, (f, g) cannot leave $\mathcal{L}_\lambda$ after $\hat{\eta}_\lambda$, it must remain in $\mathcal{L}_\lambda$ for all $\eta \geq 0$. #

Lemma 2.3 implies that if α is small enough, $f(\eta) > 0$ and $f'(\eta) < 0$ for all $\eta \geq 0$. One would then expect that $f(\eta) \to 0$ as $\eta \to \infty$. This is confirmed by the next lemma.

LEMMA 2.4. *Suppose that for some value of α, $f(\eta, \alpha) > 0$ for all $\eta \geq 0$. Then $f(\eta, \alpha) \to 0$ as $\eta \to \infty$.*

PROOF: Because $f' = g < 0$, f is decreasing. Since it is also bounded below

$$\lim_{\eta \to \infty} f(\eta) \text{ exists } = \overline{f}.$$

Suppose $\overline{f} > 0$. The equation for g says that

$$g' + \frac{N-1}{\eta} g = \frac{1}{2} F(-\eta g - \alpha f).$$

Now, it may be observed that for every $a, b > 0$ we have

$$F(a - b) \leq (1 + |\gamma|)a - (1 - |\gamma|)b.$$

Hence

$$g' + \Big(\frac{N-1}{\eta} + \frac{(1+|g|)}{2}\eta\Big) g < -\delta,$$

where $\delta = (1 - |\gamma|)\alpha\overline{f}/2 > 0$. We now deduce from an elementary computation (see also [BPT]) that

$$\lim_{\eta \to \infty} \eta g(\eta) \leq -\frac{2}{1 + |\gamma|}\delta < 0$$

so that f' is not integrable over $(0, \infty)$ and $f(\eta) \to -\infty$ as $\eta \to \infty$, a contradiction. #

Having shown in Lemma 2.4 that if f remains positive it must go to zero, we show in the next lemma that it can only do so along two directions in the phase plane: along the positive f-axis and along the negative g-axis.

LEMMA 2.5. *Suppose $f(\eta) > 0$ for all $\eta > 0$. Then*

$$\text{(2.12)} \qquad \textit{either} \quad \lim_{\eta \to \infty} \frac{g(\eta)}{f(\eta)} = 0 \quad \textit{or} \quad \lim_{\eta \to \infty} \frac{g(\eta)}{f(\eta)} = -\infty.$$

PROOF: We first show that the limit in (2.12) exists. Let

$$\text{(2.13)} \qquad \limsup_{\eta \to \infty} \frac{g(\eta)}{f(\eta)} = L^+ \quad \text{and} \quad \liminf_{\eta \to \infty} \frac{g(\eta)}{f(\eta)} = L^-$$

for some $L^+, L^- \in [-\infty, 0]$, and suppose to the contrary that $L^- < L^+$. Then there exists a sequence of points $\{\eta_k\}$ such that $\eta_k \to \infty$ as $k \to \infty$ and

$$\frac{g(\eta_k)}{f(\eta_k)} > \frac{1}{2}(L^+ + L^-) = L^* \qquad k = 1, 2, 3, \ldots.$$

Because by Lemma 2.2, $\mathcal{L}_{-L^*}$ is ultimately invariant, it follows that

$$\liminf_{\eta \to \infty} \frac{g(\eta)}{f(\eta)} \geq L^*. \tag{2.14}$$

However, by assumption $L^- < L^*$. Therefore (2.14) contradicts the second limit in (2.13).

Thus, let

$$\lim_{\eta \to \infty} \frac{g(\eta)}{f(\eta)} = L.$$

Then, because $f(\eta) > 0$ and $g(\eta) < 0$ for all $\eta > 0$, we know that $-\infty \leq L \leq 0$. Suppose that $-\infty < L < 0$. Then we deduce from the system of equations and l'Hôpital's rule that

$$L = \lim_{\eta \to \infty} \frac{g'(\eta)}{f'(\eta)} = -\lim_{\eta \to \infty} \frac{1}{2} F(\eta + \frac{\alpha}{L})$$

if the limit on the right hand side exists. And indeed it does and it is minus infinity, so that we have a contradiction. This completes the proof. #

As in [BPT] we call orbits staying in $\mathcal{S}$ for which $g/f \to -\infty$ as $\eta \to \infty$ *fast orbits* and orbits for which $g/f \to 0$ *slow orbits*. The remaining possibility for an orbit of (2.5)-(2.6) is to hit the axis $f = 0$ at a point $\eta > 0$, $\eta < \infty$ and then change sign.

It is precisely the fast orbits which will be solutions of Problem (I). Indeed, if $f(\eta) > 0$ for every $\eta > 0$ and

$$\frac{f'(\eta)}{f(\eta)} \to -\infty \quad \text{as} \quad \eta \to \infty$$

then for every $M > 0$ there exists an $\eta_M > 0$ such that

$$\frac{f'(\eta)}{f(\eta)} < -M \quad \text{if} \quad \eta > \eta_M.$$

This means that

$$f(\eta) = O(e^{-M\eta}) \quad \text{as} \quad \eta \to \infty.$$

The solution $f(\eta)$ therefore satisfies the condition (2.7) at infinity. We shall see later that slow orbits do not satisfy condition (2.7) but (2.8).

The *existence* of a fast orbit follows now readily from a topological argument. Indeed, let

$$\mathcal{A} = \{\alpha > 0 : f(\eta, \alpha) > 0 \quad \text{for all } \eta \geq 0 \text{ and } \lim_{\eta \to \infty} \frac{g(\eta)}{f(\eta)} = -\infty\}.$$

$$\mathcal{A}^+ = \{\alpha > 0 : f(\eta,\alpha) > 0 \quad \text{for all } \eta \geq 0 \quad \text{and } \lim_{\eta\to\infty} \frac{g(\eta)}{f(\eta)} = 0\},$$

and

$$\mathcal{A}^- = \{\alpha > 0 : \text{there is an } \quad \eta_0 < \infty \,\text{such that}\; f(\eta_0) = 0\}.$$

These sets are disjoint, and as we have observed, they exhaust the parameter set $0 < \alpha < \infty$

$$\mathcal{A}^+ \cup \mathcal{A}^- \cup \mathcal{A} = (0,\infty)\,. \tag{2.15}$$

If we prove that both $\mathcal{A}^+$ and $\mathcal{A}^-$ are nonempty open sets, then it will follow from (2.15) and the connectedness of the half-line $(0,\infty)$ that $\mathcal{A}$ is a closed nonempty subset of $(0,\infty)$.

LEMMA 2.6. *$\mathcal{A}^+$ is a non-empty open set.*

PROOF: The result is a consequence of Lemmas 2.3-2.5 above. Thus, Lemma 2.3 says that for all sufficiently small α the orbit will live in the region $\mathcal{L}_\lambda$ for some $\lambda > 0$. For any such an orbit α necessarily belongs to $\mathcal{A}^+$ since by Lemma 2.4 f will be positive for all times and by Lemma 2.5 the limit g/f will be necessarily 0. Continuous dependence with respect to parameters also shows that if an orbit is inside $\mathcal{L}_\lambda$ and we slightly perturb α, the perturbed orbit is also in such region for the same η. Moreover, $\mathcal{L}_\lambda$ is ultimately invariant. Hence $\mathcal{A}^+$ is open.

Next we show that $\mathcal{A}^-$ is non-empty as well.

LEMMA 2.7. *There exists a number $\hat{\alpha} > 0$ such that $(\hat{\alpha},\infty) \subset \mathcal{A}^-$.*

PROOF: We use a scaling argument, writing

$$\xi = \sqrt{\alpha}\,\eta \quad \text{and} \quad z(\xi) = f(\eta)$$

Problem (2.5), (2.6) becomes in these new variables

$$z'' + \frac{N-1}{\xi}\,z' - \frac{1}{2}F\Big(-\frac{\xi}{\alpha}\,z' - z\Big) = 0, \quad z > 0, \qquad z(0) = 1,\; z'(0) = 0. \tag{2.16}$$

If we now let $\alpha \to \infty$, we formally obtain the reduced problem

$$z'' + \frac{N-1}{\xi}\,z' + \frac{1-\gamma}{2}\,z = 0, \quad z > 0, \qquad z(0) = 1,\; z'(0) = 0. \tag{2.17}$$

It is well known that the solution of this latter problem has a finite zero. Since (2.16) is a regular perturbation of (2.17) it follows that the solution of (2.16) also has a finite zero when α is large enough so that in that case $\alpha \in \mathcal{A}^-$. #

The proof that $\mathcal{A}^-$ is also an open set is again a simple consequence of the continuous dependence of the solutions with respect to parameters. As a consequence, the set $\mathcal{A}$ is not empty. This ends the proof of existence.

In order to get a more precise picture about the sets $\mathcal{A}^+$, $\mathcal{A}^-$ and $\mathcal{A}$ we introduce the change of variables

$$t = \frac{1}{4}\eta^2 \quad \text{and} \quad y(t) = \frac{-\eta f'}{f}, \tag{2.18}$$

which will turn out to be very useful in the sequel. In terms of y and t equation (2.1) becomes

$$y' = H(t, y; \alpha) \equiv -F(y - \alpha) + \frac{1}{2t}\{y^2 - (N-2)y\}. \tag{2.19}$$

The initial conditions (2.6) give $y(0) = 0$. It then follows from the equation that

$$y'(0) = \frac{2(1-\gamma)\alpha}{N}.$$

The solutions will be confined to the region $y > 0$, $t > 0$. Solutions changing sign in the (f,η)-plane correspond here to solutions blowing up in finite $t > 0$. Fast orbits correspond to solutions such that $y(t)/\sqrt{t} \to \infty$ as $t \to \infty$, while slow orbits correspond to solutions such that $y(t)/\sqrt{t} \to 0$. Then the last term in (2.16) vanishes as $t \to \infty$ and we easily see that for a slow orbit

$$y(t) \to \alpha \quad \text{as} \quad t \to \infty.$$

We will come back later to the precise behaviour of fast and slow orbits. We are now more interested in the variation of the (y,t) phase plane with the parameter α. Indeed, we see that

$$\partial H/\partial \alpha > 0.$$

This immediately implies that the solutions $y = y(t;\alpha)$ of (2.19) with $y(0) = 0$ depend monotonically on α. Combining this with the above description of slow, fast and changing sign orbits we immediately conclude that whenever $\alpha' \in \mathcal{A}^+$, $\alpha'' \in \mathcal{A}$ and $\alpha''' \in \mathcal{A}^+$ then $\alpha' < \alpha'' < \alpha'''$. In other words

LEMMA 2.8. *There exist α_1 and α_2 such that $0 < \alpha_1 \le \alpha_2 < \infty$ and*

$$\mathcal{A}^+ = (0, \alpha_1) \quad \mathcal{A} = [\alpha_1, \alpha_2] \quad \mathcal{A}^- = (\alpha_2, \infty).$$

The uniqueness part of Theorem 2.1 will consist in proving that $\alpha_1 = \alpha_2$. Before doing this we pause to consider in more detail the asymptotic behaviour of fast and slow solutions

LEMMA 2.9. *With the above notations if f corresponds to a fast orbit, then there exists a positive constant A such that*

$$f(\eta) = A\eta^{\alpha - N}\exp\Big(-\frac{1+\gamma}{4}\eta^2\Big)\cdot\{1 + O(\eta^{-2})\} \quad \text{as} \quad \eta \to \infty \tag{2.20}$$

and

$$f'(\eta) = -\frac{1+\gamma}{2}A\eta^{\alpha+1-N}\exp\Big(-\frac{1+\gamma}{4}\eta^2\Big)\cdot\{1 + O(\eta^{-2})\} \quad \text{as} \quad \eta \to \infty. \tag{2.21}$$

hence

$$y(t) = 2(1+\gamma)\,t(1+O(\frac{1}{t})). \tag{2.22}$$

PROOF: Because $g/f \to -\infty$, equation (2.5) becomes for large η

$$f'' + \frac{N-1}{\eta} f' + \frac{1+\gamma}{2}(\eta\, f' + \alpha\, f) = 0,$$

or, when we set $\eta = \frac{\xi}{\sqrt{1+\gamma}}$ and $f(\eta) = \hat{f}(\xi)$ and omit the hat again,

$$f'' + \frac{N-1}{\xi} f' + \frac{1}{2}\xi f' + \frac{\alpha}{2} f = 0.$$

We can now complete the proof following [BPT] or use standard results about Whittaker functions [AS, p. 505]. For $N = 1$ this is explicitly computed in [BS]. #

For slow solutions we have

LEMMA 2.10. *Let f correspond to a slow orbit. Then there exists a positive constant A such that*

$$f(\eta) = A\eta^{-\alpha}\{1 + O(\eta^{-2})\} \quad \text{as} \quad \eta \to \infty \tag{2.23}$$

and

$$f'(\eta) = -\alpha A\eta^{-\alpha}\{1 + O(\eta^{-2})\} \quad \text{as} \quad \eta \to \infty. \tag{2.24}$$

PROOF: We already said that equation (2.19) can be written as

$$y' = -F(y - \alpha) + \varepsilon(t),$$

where $\varepsilon(t) \to 0$ as $t \to \infty$, and that

$$y(t) \to \alpha \quad \text{as} \quad t \to \infty.$$

Thus, in fact

$$\varepsilon(t) = O(t^{-1}) \quad \text{as} \quad t \to \infty$$

and we conclude from the same argument that

$$y(t) = \alpha + O(t^{-1}) \quad \text{as} \quad t \to \infty.$$

In terms of the original variables this becomes

$$\frac{f'}{f} = -\frac{\alpha}{\eta} + O\Big(\frac{1}{\eta^3}\Big) \quad \text{as} \quad \eta \to \infty.$$

The assertion now follows upon integration. #

COROLLARY. *The slow orbits do not satisfy the initial condition (2.7). Indeed,*

$$\lim_{\eta\to\infty} \eta^\alpha f(\eta) = A > 0$$

so that the corresponding solutions of (2.1) satisfy

$$\lim_{t\to 0} u(x,t) = A|x|^{-\alpha}.$$

We proceed now with the promised uniqueness proof for fast orbits.

LEMMA 2.11. *There exists only one value of α for which Problem (I) has a solution.*

We give two proofs of this result.

First proof. Suppose Problem (II) has two solutions f_1 and f_2 corresponding to two different values of α: $\alpha_1 < \alpha_2$. Because

$$f_i''(0) = -\frac{(1-\gamma)\alpha_i}{2N}, \quad i = 1, 2,$$

it follows in view of the initial conditions ($f_i(0) = 1$, $f_i'(0) = 0$) that

$$f_1(\eta) > f_2(\eta) \quad \text{for} \quad \eta > 0 \text{ small.} \tag{2.25}$$

On the other hand, by Lemma 2.9,

$$f_1(\eta) < f_2(\eta) \quad \text{for} \quad \eta > 0 \text{ large.} \tag{2.26}$$

Thus, the graphs of f_1 and f_2 must intersect. By (2.26) it is possible to choose a constant $k > 0$ such that $kf_2 > f_1$ everywhere on $[0, \infty)$. Let

$$k_0 = \inf\{k > 0 : kf_2(\eta) > f_1(\eta) \text{ for all } \eta \geq 0\}.$$

it follows from (2.25) that $k_0 > 1$. Writing $z(\eta) = k_0 f_2(\eta)$ we have

$$z(\eta) \geq f_1(\eta) \quad \text{for all} \quad \eta \geq 0$$

and in view of (2.26) there exists a point $\eta_0 > 0$ where

$$z - f_1 = 0, \quad (z - f_1)' = 0, \quad (z - f_1)'' \geq 0.$$

If we subtract the equation for f_1 from the one for z we obtain

$$(z - f_1)'' - \frac{1}{2}F(-(\eta_0 z' + \alpha_2 z)) + \frac{1}{2}F(-(\eta_0 f_1' + \alpha_1 f_1)) = 0 \tag{2.27}$$

But

$$(\eta_0 z' + \alpha_2 z) - (\eta_0 f_1' + \alpha_1 f_1) = (\alpha_2 - \alpha_1) f_1 > 0.$$

Hence, because F is strictly increasing, the third term in (2.27) is smaller then the second one. Because the first term in nonnegative, (2.27) cannot be satisfied. This proves the uniqueness.

Second proof. We work in the (y,t)-plane. Let y_1 and y_2 be the solutions corresponding to $\alpha_1 < \alpha_2$ and let $w = y_2 - y_1$. By the monotonicity of $H(t, y, \alpha)$ with respect to α we have $w > 0$. The equation for w when t is large so that y_1 and y_2 are large is

$$w' = (1+\gamma)(\alpha_2 - \alpha_1) - (1+\gamma)w + \frac{y_1 + y_2}{2t} w - \frac{N-2}{2t} w \tag{2.28}$$

Thanks to the asymptotic behaviour of the y_i's given by Lemma 2.9 we get

$$w' \approx (1+\gamma)w + (1+\gamma)(\alpha_2 - \alpha_1) > (1+\gamma)w$$

which means that $w \geq O(e^{(1+\gamma)t})$ as $t \to \infty$. This contradicts the linear growth estimate for y_i in Lemma 2.9. #

The dependence of $\alpha(\gamma)$ on γ is investigated in the following result

THEOREM 2.3. *The exponent α of the fast orbit is a strictly increasing and continuous function of γ. Moreover, $\alpha(\gamma) > N-2$ for every $\gamma \in (-1, 1)$. Finally, $\alpha(0) = N$, so that $\alpha > N$ if and only if $\gamma > 0$.*

PROOF: (i) *Monotonicity.* Take two parameters γ_1 and γ_2 in $(-1, 1)$ with $\gamma_1 < \gamma_2$ and let α_1 and α_2 be the corresponding exponents for the fast orbit as obtained in Theorem 2.1. We want to prove that $\alpha_1 < \alpha_2$.

We argue by contradiction using the representation (2.16). Indeed, let $y_i(t)$ be the solution to $y' = H(t, y; \gamma_i, \alpha_i)$ with $y_{\alpha_i}(0) = 0$. We observe that the slope function H is monotonically increasing in α and decreasing in γ. Therefore, since $\gamma_1 < \gamma_2$, if we suppose that $\alpha_1 \geq \alpha_2$ we conclude that $y_1 > y_2$ for every $t > 0$. However, we know that the asymptotic behaviour is given by

$$y_i(t) \approx 2(1+\gamma_i)\, t\,,$$

cf. (2.22). This establishes that $\alpha_1 > \alpha_2$.

Let us go back to equation (2.19). We observe that at the points $t > 0$, where $y = \alpha$ we have

$$y' = H(t, \alpha; \alpha) = \frac{\alpha}{2t}\{\alpha - (N-2)\} \tag{2.29}$$

which means that solutions can enter the region $\{(t, x) : t > 0, y > \alpha\}$ only if $\alpha > N-2$. Therefore, a fast orbit can exist only if $\alpha > N-2$.

(ii) *continuity.* This is a consequence of uniqueness and monotonicity plus regularity that allows to take limits. Thus, given any $\gamma_0 \in (-1,1)$ we take limits in equation (2.6) as γ tends increasingly or decreasingly to γ_0. More or less standard arguments show that in both cases we obtain a solution of (2.16). Moreover, if the limit values for α are different we obtain two solutions with different α for the same $\gamma = \gamma_0$.

(iii) Finally, we observe that in the particular case $\gamma = 0$ we recover the classical heat equation and, as is well-known, the fast orbit reads

$$f(\eta) = \exp(-\frac{\eta^2}{4}) \tag{2.30}$$

with $\alpha(0) = N$. #

We note another consequence of formula (2.29): an orbit of (2.19) which enters the region $\{y > \alpha\}$ cannot go back to the values $y \le \alpha$. It follows that for our fast orbits the sign of $\alpha - y$ changes exactly once. This has an interest since the definition of the constitutive function F changes according to the sign of

$$h = \Delta f = f'' + \frac{N-1}{\eta} f'.$$

and for solutions of (2.5) with $f > 0$, $f' < 0$ we have $\text{sign}(h) = -\text{sign}(\alpha - y)$. Summing up we have

LEMMA 2.12. *For any solution of Problem (I) with $|\gamma| < 1$ the Laplacian of f has exactly one sign change, i.e. there exists $\eta_1 = \eta_1(\gamma) > 0$ such that $\Delta f < 0$ if $0 \le \eta \le \eta_1$, while $\Delta f > 0$ for $\eta_1 < \eta < \infty$.*

In view of the formula $u(x,t) = t^{-\alpha/2} f(x/\sqrt{t})$ this means that $u_t < 0$ for $|x| < \eta_1\sqrt{t}$ and $u_t > 0$ if $|x| > \eta_1\sqrt{t}$. This result will be needed in Section 3. As a curiosity, for the heat equation ($\gamma = 0$) we have $\eta_1 = \sqrt{2}$.

3. The limits as γ tends to 1 or -1

We discuss in this section the behaviour of the selfsimilar solutions constructed in Section 2 as γ approaches the extreme values 1 and -1, where the equation ceases to be parabolic. Unless stated otherwise by selfsimilar solution we will understand the unique selfsimilar solution of (E) with fast decay in x, as constructed in Section 2 and α will denote its exponent $\alpha(\gamma)$.

We begin with the limit $\gamma \to 1$.

THEOREM 3.1. *As γ converges to* 1 *we have* $\alpha \to \infty$.

PROOF: (i) We already know that as γ increases to 1 the exponent α increases. Hence, there exists the limit

$$\tilde{\alpha} = \lim_{\gamma \to 1} \alpha \tag{3.1}$$

and this limit can be either infinite or a finite number larger than N.

In order to exclude the latter possibility we take solutions for γ close to 1 so that $\alpha \approx \tilde{\alpha} < \infty$ and use the representation $y = y(t; \gamma, \alpha)$ introduced in (2.19), where we recall that $y = -\eta f'(\eta)/f(\eta)$ and $t = \eta^2/4$. As long as $0 \leq y \leq \alpha$, i.e. in an interval $0 < t \leq t_\gamma$, we have

$$y' = (1-\gamma)(\alpha - y) + \frac{y}{2t}(y - (n-2)) \tag{3.2}$$

with initial value $y(0) = 0$, while for $t \geq t_\gamma$ we have

$$y' = (1+\gamma)(\alpha - y) + \frac{y}{2t}(y - (n-2)) \tag{3.3}$$

It follows clearly from (3.3) that in the region D where

$$y > \alpha \quad \text{and} \quad t > \phi(y) = \frac{(y^2 - y(n-2))}{2(1+\gamma)(y-\alpha)} \tag{3.4}$$

the trajectories of equation (3.3) have a negative slope. Hence, if a trajectory enters such a region it is trapped inside and eventually $y(t) \to \alpha$ as $t \to \infty$. This means that the our solutions do not enter D. As a consequence t_γ has to less than $\min\{\psi(y) : \alpha < y < \infty\}$. Now, it is clear from (3.4) that this minimum depends continuously on α so that it stays bounded when α stays bounded. We conclude that if we assume that $\tilde{\alpha}$ is finite then t_γ stays bounded as $\gamma \to 1$.

However, we can obtain a bound from below for t_γ by studying equation (3.2). Indeed, it is easy to show that the line $y = ct$ is supersolution of (3.2) in the interval $0 \leq t \leq a$ if

$$c > (1-\alpha)\alpha + \frac{c}{2}(ca + 2 - N),$$

for example if

$$c = \frac{4(1-\gamma)\alpha}{N} \quad \text{and} \quad a = \frac{N}{2c}. \tag{3.5}$$

Both c and a depend on γ, but as $\gamma \to 1$ then c converges to 0 while a tends to ∞. Since our solution will satisfy

$$y(t) \leq ct \quad \text{for} \quad 0 \leq t \leq a.$$

We conclude that $t_\gamma \geq a$, hence t_γ cannot be bounded. All possibilities for $\tilde{\alpha}$ being finite being excluded, it follows that $\tilde{\alpha} = \infty$. #

The situation for $\gamma \to -1$ is as follows

THEOREM 3.2. *For $N \geq 2$ we have*

$$\lim_{\gamma \to -1} \alpha(\gamma) = N - 2\,, \tag{3.7}$$

while in one space dimension the limit is 0.

PROOF: Since α decreases as γ decreases to -1 and $\alpha > 0$ there exists the limit $\lim_{\gamma \to -1} \alpha = \alpha_1 \geq 0$. Also $\alpha_1 \geq N - 2$, $\alpha \geq 0$.

Let us investigate what happens for $y \geq \alpha$. Equation (3.3) becomes in the limit $\gamma = -1$

$$y' = \frac{1}{2t}(y^2 - (N-2)y)\,. \tag{3.8}$$

For $N = 1$ or 2 any solution of (3.8) which takes values in the region $y > 0$ necessarily blows up in finite time. This can checked by explicit integration. The same is true in dimensions $N \geq 3$ for the solutions which enter the region where $y > N - 2$.

Let us see what happens if $N = 1$ or 2 and $\alpha_1 > 0$. Then, for $\gamma \approx -1$ we have equation (3.2) as long as $0 < y \leq \alpha$. The slope at 0 is $y'(0) = (2(1-\gamma)\alpha)/N$, which tends to $4\alpha_1/N > 0$ as $\gamma \to -1$. Moreover, it is easy to see that in a neighbourhood of the origin, $0 \leq t \leq a$, a function of the form $Y = ct$ is a subsolution of (3.2) if $0 < c < c_1$, and a and c_1 can be chosen independent of $\gamma \approx -1$, hence we may pass to the limit and find a nontrivial solution of the equation

$$y' = 2(\alpha_1 - y) + \frac{y^2 - (N-2)}{2t}, \quad 0 < y < \alpha_1 \tag{3.9}$$

which should be continued for $y \geq \alpha_1$ with a solution of (3.8). As we said, such a solution would blow up in finite time, $t = t_*$. By continuity with respect to the parameters, the solution for $\alpha \approx \alpha_1$, $\gamma \approx -1$ would attain a very high value of y for $t \approx t_*$. Given the form (3.3) of the equation in that region, such a solution would necessarily blow up, a contradiction. This proves that $\alpha_1 = 0$ for $N = 1$ or 2. We remark that the same argument implies that the solution $y = y(t;\gamma)$ has to tend to 0 everywhere in $\mathbf{R}^+$ as $\gamma \to -1$

Assume now that $N \geq 3$ and $\alpha_1 > N - 2$. A similar analysis of equation (3.2) in the region $0 \leq y \leq \alpha_1$ allows to conclude that, in the limit $\alpha \to \alpha_1$ we get a positive solution of (3.9) in some interval $0 < t < t_1$ with $y(t_1) = \alpha_1$. Thsi is continued for $t > t_1$ with a solution of (3.8), which necessarily blows up since $\alpha_1 > N - 2$, and the blow-up argument is extended to $g \approx -1$, $\alpha \approx \alpha_1$ by continuity, thus arriving at a contradiction. Hence $\alpha_1 = N - 2$. #

4. Asymptotic behaviour

In this section we consider a solution $u(x,t)$ of the Cauchy Problem for equation (E) with initial data

$$u(x,0) = u_0(x)\,, \tag{4.1}$$

which are assumed to be continuous, nonnegative, $u_0 \not\equiv 0$, and such that

$$u_0(x) = O(e^{-Ax^2}) \quad \text{as} \quad |x| \to \infty \tag{4.2}$$

for some $A > 0$. We want to show that the large-time behaviour of u is precisely described by a certain multiple of the selfsimilar solution with fast decay $B(x,t)$ constructed in Section 2. Actually, we prove

THEOREM 4.1. *Under the above assumptions there exists a constant $c > 0$ such that*

$$t^{\alpha/2}|u(x,t) - cB(x,t)| \to 0 \quad \text{as} \quad t \to \infty \tag{4.3}$$

uniformly on sets of the form $\{|x| \le Ct^{\frac{1}{2}}\}$, $C > 0$. Here α is the time-decay exponent of B, defined in (0.1).

The proof will be organized into a series of partial results. We begin with the following information on the change of the profile of B with time.

LEMMA 4.1. *For every t and $\tau > 0$ we have*

$$\lim_{x\to\infty} \frac{B(x,t+\tau)}{B(x,t)} = \infty. \tag{4.4}$$

PROOF: We have $B(x,t) = t^{-\alpha/2}f(\eta)$ with $\eta = xt^{-1/2}$ and f such that

$$f(\eta) \approx \eta^{\alpha-1}e^{-\eta^2/4} \quad \text{as} \quad \eta \to \infty\,.$$

Hence

$$\frac{B(x,t+\tau)}{B(x,t)} \approx \left(\frac{t}{t+\tau}\right)^{\alpha-\frac{1}{2}} \exp\left(\frac{x^2\tau}{4t(t+\tau)}\right). \quad \#$$

COROLLARY 4.1. *For every t and $\tau > 0$ there exists $c = c(t,\tau)$ such that*

$$B(x,t) \le cB(x,t+\tau) \quad \textit{for every} \quad x \in \mathbf{R}^N \tag{4.5}$$

and $c(t,\tau) \to 1$ as $\tau \to 0$. $\quad \#$

More precise information about the change of the selfsimilar profiles with time is given by the following consequence of Lemma 2.12

LEMMA 4.2. *There exists $R_1 = R_1(t,\tau) > 0$ such that $B(x,t+\tau) > B(x,t)$ for $|x| > R_1$ and $B(x,t+\tau) < B(x,t)$ for $0 < |x| < R_1$. Moreover, there exists $\eta_1 > 0$ such that as $\tau \to 0$*

$$R_1(t,\tau) \to \eta_1\sqrt{t}. \tag{4.6}$$

We now proceed with the study of the evolution of a solution u under the assumptions of Theorem 4.1. The basic idea is to adapt a solution cB as closely as possible to our solution u.

LEMMA 4.3. *For every time delay $\tau > 1/(4A)$ there exists a constant $c > 0$ (depending on u and τ) such that*

$$u(x,t) \le cB(x,t+\tau) \tag{4.7}$$

for every $x \in \mathbf{R}^N$ and every $t \ge 0$. If u_0 has compact support then τ can be arbitrarily small.

PROOF: We compare the functions $u(x,t)$ and $cB(x,t+\tau)$, $c > 0$. If $\tau > 1/(4A)$ and c is large enough the latter will be larger than the former in $\mathbf{R}^N$ at $t = 0$, hence the conclusion by the Maximum Principle. #

We can now define for t and $\tau > 0$ the optimal upper bound

$$C(t,\tau) = \inf\{c > 0 : u(x,t) \le cB(x,t+\tau) \text{ for } x \in \mathbf{R}^N\}. \tag{4.8}$$

As a consequence of Lemma 4.3 $C(t,\tau)$ is finite for $\tau > 1/(4A)$. It is a simple consequence of the Maximum Principle that $C(t,\tau)$ is a decreasing function of t for fixed τ. Therefore, there exists the limit

$$C(\infty,\tau) = \lim_{t\to\infty} C(t,\tau) \tag{4.9}$$

and this limit is nonnegative. To continue we also need an estimate for u from below.

LEMMA 4.4. *For every $t_1 > 0$ there exists $c(t_1) > 0$ such that*

$$u(x,t) \ge c(t_1)B(x,t) \tag{4.10}$$

if $x \in \mathbf{R}^N$ and $t \ge t_1$.

PROOF: Without loss of generality (the equation is translation-invariant) we may assume that $u_0(0) > 0$ so that by continuity there exist ε, τ and $\delta > 0$ such that $u(x,t) \ge \delta$ for all $|x| \le \varepsilon$, $0 \le t \le \tau$. We now consider the domain $S_\varepsilon = \{(x,t) : |x| \ge \varepsilon, 0 < t < t_1\}$

with $0 < t_1 \le \tau$ and compare u and cB in this domain. It is easily checked that $u \ge cB$ on the parabolic boundary of S_ε if $c = c_1$ is small enough. Therefore, $u \ge c_1 B$ in S_ε, and consequently

$$u(x, t_1) \ge c_1 B(x, t_1) \quad \text{for} \quad |x| \ge \varepsilon .$$

On the other hand, we may easily find a constant $c_2 > 0$ such that the inequality $u(x, t_1) \ge c_2 B(x, t_1)$ holds in the ball $|x| < \varepsilon$, since the ball is bounded and $u(x, t_1)$ is positive there. Taking $c(t_1) = \min\{c_1, c_2\}$, the result follows in the case $t_1 \le \tau$.

Note finally that now that the inequality is established for $t_1 = \tau$, the Maximum Principle implies that it will hold for $t_1 > \tau$ with the same constant. #

Again, by the Maximum Principle the family $\{c(t)\}$ is monotone nondecreasing in t, hence there exists the limit

$$c(\infty) = \lim_{t \to \infty} c(t) .$$

After all this preparatory work we may now proceed with the core of the proof. We introduce the rescaling operator $\mathcal{T}_k$, with parameter $k > 0$, which acts on solutions of (1.1) according to the formula

$$(\mathcal{T}_k u)(x, t) = k^\alpha u(kx, k^2 t) \tag{4.11}$$

to produce a new solution of (1.1). Moreover B is invariant under such a transformation. Let us abbreviate $\mathcal{T}_k u = u_k$. Given a solution $u(x, t)$ under the assumptions of Theorem 4.1 we consider the family $\{u_k(x, t)\}$ for $k \ge 1$. By known compactness of the solutions of (1.1) we may extract from every sequence in this family a convergent subsequence (on compact subsets of Q) and the limit

$$U(x, t) = \lim_{j \to \infty} u_{k_j}(x, t)$$

is again a solution of (1.1). Note that

$$\mathcal{T}_k(B(x, t + \tau)) = B(x, t + \frac{\tau}{k^2}) . \tag{4.12}$$

We will prove below that $U = cB$ for some $c > 0$. The main technical difficulty will arise from the "tails of the solutions", namely from the behaviour of the solutions for large x, which makes comparison tricky.

LEMMA 4.5. *Let U be the limit of a subsequence $\{u_{k_j}\}$ as above. Then for every $\tau > 0$*

$$c(\infty) B(x, t) \le U(x, t) \le C(\infty, \tau) B(x, t) \quad \text{in} \quad Q . \tag{4.13}$$

PROOF: We have

$$\begin{aligned} u_k(x, t) &= k^\alpha u(kx, k^2 t) \\ &\le C(k^2 t, \tau) k^\alpha B(kx, k^2 t + \tau) \\ &= C(k^2 t, \tau) B(x, t + (\tau / k^2)) . \end{aligned}$$

Now let $k = k_j \to \infty$ to obtain the upper bound. The lower bound is obtained in a similar way. #

We now consider the set $\mathcal{S}$ of all sequence limits $\{U\}$ of subsequences $\{u_{k_j}\}$ and introduce the following optimal constant

$$C_* = \inf\{c > 0 : \text{ there exists } U \in \mathcal{S} \text{ such that } U(x,t) \le cB(x,t) \text{ in } Q\}. \tag{4.14}$$

By Lemma 4.5, $c(\infty) \le C_* \le C(\infty, \tau)$ and hence C_* is a positive and finite constant. The proof of Theorem 4.1 will consist in showing that C_* is precisely the constant for which the asymptotic formula (4.3) is satisfied.

A basic ingredient in the subsequent proof is the following approximation result.

LEMMA 4.6. *Let $U \in \mathcal{S}$ be a sequence limit such that*

$$U(x,t) \le CB(x,t+\sigma) \quad in \quad Q \tag{4.15}$$

for some C and $\sigma > 0$. Then for every $\varepsilon > 0$ there exist $\tau > \sigma$ and $t_1 > 0$ such that

$$u(x,t) \le (C+\varepsilon)B(x,t+\tau) \quad for \quad t \ge t_1. \tag{4.16}$$

PROOF: Given $\varepsilon > 0$, bt Corollary 4.1 there exists $\tau_1 = \tau_1(\varepsilon) > 0$ small enough such that

$$B(x,1+\sigma) \le \frac{C+\varepsilon}{C+\frac{\varepsilon}{2}} B(x,1+\sigma+\tau) \tag{4.17}$$

for $0 < \tau < \tau_1$. On the other hand, we know from Lemma 4.3 that there exist τ_2 and C_0 such that for every $x \in \mathbf{R}^N$ and $t \ge 1$

$$u(x,t) \le C_0 B(x,t+\tau_2). \tag{4.18}$$

Besides, given $s > 0$ there exists $R = R(s)$ such that for $|x| > R$ we have for every $r \in (0,1)$

$$C_0 B(x,1+r) \le CB(x,1+q) \quad \text{with} \quad q+1 = (1+r)(1+s). \tag{4.19}$$

even if $C < C_0$. Therefore, for every $|x| > R(s)$ and every $k \ge 1$ we have from (4.18) and (4.19)

$$u_k(x,1) \le C_0 B(x, 1+\frac{\tau_2}{k^2}) \le CB(x,1+\tau_3) \tag{4.20}$$

with $\tau_3 = (1+\tau_2 k^{-2})(1+s) - 1$, so that $\tau_3 > s$ and tends to s as $k \to \infty$. Let us fix $s = \sigma$ and assume that $k \ge k_0$ is so large that $\tau_3 < \sigma + \tau_1$.

Let us now take a small $\delta > 0$ and work in the remaining ball $\{|x| \le R\}$. We can find $k_1 \ge k_0$ sufficiently large (depending on R and δ) so that

$$|u_{k_1}(x,1) - U(x,1)| \le \delta \quad \text{if} \quad |x| \le R.$$

Hence, by the assumption (4.15), putting $\delta = \frac{\varepsilon}{2}\min\{B(x,1) : |x| \le R\}$, we get

$$u_{k_1}(x,1) \le \delta + U(x,1) \le \delta + CB(x,1) \le (C + \frac{\varepsilon}{2})B(x, 1+\sigma).$$

Using (4.17) we arrive at

$$u_{k_1}(x,1) \le (C+\varepsilon)B(x, 1+\tau_3) \tag{4.21}$$

if $|x| \le R$. Combining this with (4.20) we conclude that the inequality is true in $\mathbf{R}^N$. Undoing the rescaling we get for every $x \in \mathbf{R}^N$

$$u(x,t_1) \le (C+\varepsilon)B(x, t_1 + \tau),$$

where $t_1 = k_1^2$ and $\tau = k_1^2\tau_3$. By the Maximum Principle the comparison extends to $t > t_1$. #

An important consequence follows from Lemma 4.6.

COROLLARY 4.2. *Let U be any sequence limit, $U \in \mathcal{S}$. Then for every $(x,t) \in Q$ we have*

$$U(x,t) \le C_* B(x,t). \tag{4.22}$$

PROOF: By definition of C_*, given $\varepsilon > 0$ there exists a sequence $\{u_{k_j}\}$ with limit $\hat{U}$ such that

$$\hat{U}(x,t) \le (C_* + \varepsilon)B(x,t) \tag{4.23}$$

for every $(x,t) \in Q$. Therefore, by Corollary 4.1 if $\sigma > 0$ is small enough we have for $x \in \mathbf{R}^N$ and $t > 1$

$$\hat{U}(x,t) \le (C_* + 2\varepsilon)B(x, t+\sigma). \tag{4.24}$$

Apply now Lemma 4.6 with $C = C_* + 2\varepsilon$ to obtain for $t > t_1$

$$u(x,t) \le (C_* + 3\varepsilon)B(x, t+\tau).$$

Observe that this means that for *any* $k \ge 1$ we have

$$u_k(x,t) \le (C_* + 3\varepsilon)B(x, t + \tau k^{-2}). \tag{4.25}$$

if $t \ge t_1/k^2$. This means that any limit U taken along any subsequence will satisfy

$$U(x,t) \le (C_* + 3\varepsilon)B(x,t). \tag{4.26}$$

Since ε is arbitrary the conclusion follows. #

The second ingredient consists in showing that the upper bounds either correspond to exact equality or otherwise they can be improved as time goes on.

LEMMA 4.7. *Let v be a nonnegative solution of (1.1) and assume that for some constant C_0 and some time $t_0 > 0$ we have*

$$v(x,t_0) \leq C_0 B(x,t_0) \quad \text{and} \quad v(x,t_0) \not\equiv C_0 B(x,t_0)\,. \tag{4.27}$$

Then for every $t_1 > t_0$ there exist $\tau > \sigma$ and $\delta > 0$ such that

$$v(x,t) \leq (C_0 - \delta) B(x,t+\tau) \quad \text{if} \quad x \in \mathbf{R}^N, t \geq t_1\,. \tag{4.28}$$

PROOF: We begin with the observation that under the hypotheses (4.27) we have $u(x,t) \leq C_0 B(x,t)$ for every $t \geq t_0$ thanks to the Maximum Principle in its classical form. Moreover, the Strong form of the Principle implies that $v(x,t)$ will actually be strictly less than $C_0 B(x,t)$ at every (x,t) such that $t > t_0$.

Fix $t_1 > t_0$ and take $L_1 > 2\eta_1\sqrt{t_1}$ with η_1 as in Lemma 4.2. Thanks to the above remark there exists a $\delta_1 > 0$ such that

$$v(x,t_1) \leq (C_0 - \delta_1) B(x,t_1) \quad \text{for} \quad |x| \leq L_1\,. \tag{4.29}$$

By continuity we can choose a $\tau > 0$ so small that

$$v(x,t) \leq (C_0 - (\delta_1/2)) B(x,t_1+\tau) \quad \text{for} \quad |x| \leq L_1\,. \tag{4.30}$$

Fix τ small enough. By our choice of L_1 we have for some $\delta_2 > 0$ small enough

$$v(x,t_1) \leq C_0 B(x,t_1) < (C_0 - \delta_2)\, B(x,t_1+\tau) \quad \text{for} \quad |x| > L_1\,. \tag{4.31}$$

Putting $\delta = \min\{\delta_1/2, \delta_2\}$, we can conclude from (4.30) and (4.31) that

$$v(x,t_1) \leq (C_0 - \delta) B(x,t_1+\tau) \qquad \text{in} \quad \mathbf{R}^N\,,$$

and the Maximum Principle allows us to extend the inequality to $t > t_1$. #

The last step in the proof consists in showing that $\mathcal{S}$ reduces to a point.

LEMMA 4.8. *For every sequence limit U we have $U(x,t) \equiv C_* B(x,t)$.*

PROOF: Consider a sequence limit $U \in \mathcal{S}$. By Corollary 4.2 we have $U(x,t) \leq C_* B(x,t)$. If $U \neq C_* B$ then Lemma 4.7 applies to show that there exist ε and $\sigma > 0$ such that

$$U(x,t) \leq (C_* - \varepsilon) B(x,t+\sigma) \tag{4.32}$$

for every $t \geq 1$. Then by Lemma 4.6 we obtain a time t_1 and a delay $\tau > \sigma$ such that for $t > t_1$

$$u(x,t) \leq (C_* - \frac{\varepsilon}{2}) B(x,t+\tau) \tag{4.33}$$

and this gives a contradiction, since if (4.35) were true it would imply that any subsequence limit would satisfy

$$U(x,t) \le (C_* - \frac{\varepsilon}{2})B(x,t)\,, \tag{4.34}$$

against the very definition of C_*. #

Once we have proved that every subsequence limit coincides with $C_*B(x,t)$, the whole family $\{u_k\}$ must necessarily converge towards C_*B (uniformly on compact subsets of Q). This is precisely the meaning of formula (4.3). The proof of Theorem 4.1 is complete.

REMARK: A careful inspection of the proof shows that in fact the limit constants $C(\infty,\tau)$ (for $\tau > 1/(4A)$) and $c(\infty)$ coincide with C_*. Therefore, for every $\varepsilon > 0$ there exists $t_1 > 0$ such that

$$(C_* - \varepsilon)B(x,t) \le u(x,t) \le (C_* + \varepsilon)B(x,t+\tau) \tag{4.35}$$

for every $x \in \mathbf{R}^N$ and $t \ge t_1$.

5. Discussion of the asymptotic result. A conservation law

Theorem 4.1 states that for large times any nonnegative solution of equation (E) having the right decay rate on its initial data will look like the Barenblatt solution $B(x,t)$. However, the result says more, namely that there is a precise constant c such that

$$\lim_{t\to\infty} t^{\alpha(\gamma)/2}|u(x,t) - cB(x,t)| = 0$$

uniformly in $x \in \mathbf{R}^N$. This well-defined asymptotic constant allows us to define a functional $J : X \mapsto \mathbf{R}^+$ in the space of initial data

$$X = \{f \in C(\mathbf{R}^N) :\ f \ge 0 \text{ and } f(x) = O(e^{-A|x|^2}) \ \text{ for some } A > 0\}$$

by means of the formula

$$J(u_0) = c\,, \tag{5.1}$$

where c is the constant given by Theorem 4.1. It is well known that in the heat equation case, $\gamma = 0$, the constant c is just the mass of the solution

$$c = \int u_0(x)\,dx\,. \tag{5.2}$$

(We normalize the fundamental solution to have unit mass). In this way the asymptotic constant is given by a conserved quantity expressed in simple terms as a function of the initial data. Now, we do not know of any conserved quantity expressed in closed form for equation (E) when $\gamma \neq 0$. Such fortune is not to be expected since the equation represents a kind of constrained dynamics with nonholonomic constraints.

Moreover, *the functional J expresses a conservation law for equation (E)*, since obviously we have the same asymptotic constant for all the family of data corresponding to different stages of the evolution of a solution. To be specific, if u is a solution with data $u_0 \in X$ and we put $u(t) = u(\cdot, t)$ then

$$J(u(t)) = J(u_0) \quad \text{for every} \quad t > 0\,.$$

It follows easily from the Maximum Principle that J is an ordered operator, i.e. $f \geq g$, with $f, g \in X$ implies $J(f) \geq J(g)$. It is also clear that

$$J(\lambda f) = \lambda J(f) \quad \text{for every} \quad \lambda > 0\,. \tag{5.3}$$

We would like to know as much as possible about the functional J. For instance, J is a linear operator when for $\gamma = 0$, but it is not for $\gamma \neq 0$. Let us prove this important fact that reflects the essential nonlinearity of the equation.

THEOREM 5.1. *$J : X \to \mathbf{R}_+$ is a sublinear operator if $\gamma < 0$, while it is superlinear if $\gamma > 0$. J is not a linear operator whenever $\gamma \neq 0$.*

PROOF: Assume that $\gamma > 0$, u_1 and u_2 are two solutions of the equation with initial data f anf g respectively, and let $u = u_1 + u_2$. Using the fact that $|u_{1,t} + u_{2,t}| \leq |u_{1,t}| + |u_{2,t}|$ we get

$$u_t + \gamma |u_t| \leq \Delta u\,. \tag{5.4}$$

Therefore, $u_1 + u_2$ is just a subsolution of the equation, and the solution with initial data $f+g$ will have an asymptotic constant not less than the sum of the constants corresponding to f and g. In other words,

$$J(f+g) \geq J(f) + J(g) \quad \text{if} \quad \gamma > 0\,. \tag{5.5}$$

A similar argument proves that

$$J(f+g) \leq J(f) + J(g) \quad \text{if} \quad \gamma < 0\,. \tag{5.6}$$

Let us now construct an example to show the strict nonlinearity of the functional when $\gamma \neq 0$. For simplicity we work in one space dimension though the argument is easily generalized. Let us also call $\varepsilon = \alpha(\gamma) - 1$, the anomaly. We take as basic initial function

$$f(x) = 1 \quad \text{if} \quad |x| \leq 1, \quad f(x) = 0 \quad \text{otherwise}\,. \tag{5.7}$$

Let u be the corresponding solution and $c = J(f)$. Consider now the scaled function $g(x) = f(x/2)$. The corresponding solution v is then related to u by

$$v(x,t) = u(\frac{x}{2}, \frac{t}{4})\,. \tag{5.8}$$

It follows from the asymptotic formula and (5.8) that

$$J(g) = \lim_{t \to \infty} t^{(1+\varepsilon)/2} v(0,t) = 2^{1+\varepsilon} \lim_{t \to \infty} t^{(1+\varepsilon)/2} u(0,t) = 2^{1+\varepsilon} J(f). \tag{5.9}$$

Now, it is clear tfrom the special form of f and g that $g(x) = f(x-1) + f(x+1) = f_1(x) + f_2(x)$ and also that $J(f_1) = J(f_2) = J(f)$. We thus obtain

$$J(f_1 + f_2) = 2^{\varepsilon}(J(f_1) + J(f_2)). \tag{5.10}$$

Since $\varepsilon > 0$ for $\gamma > 0$ and $\varepsilon < 0$ for $\gamma < 0$, the correct inequality always follows. In fact, the careful reader will have observed that the initial functions we use are not continuous. But, since a slight modification of the functions and the scaling parameters permits to construct a close example with continuous functions, and since this only encumbers the computation, we have preferred to omit such unessential details. #

An interesting question is that of efficiently computing $J(f)$. We have already used the fact that for any given data u_0 the quantity $J(u_0)$ can be obtained as

$$J(u_0) = \lim_{t \to \infty} t^{\alpha(\gamma)/2} u(x,t) \tag{5.11}$$

for any fixed $x > 0$, if we normalize the Barenblatt solution $B(x,t)$ so that $B(0,1) = 1$. We can also use the integral formula

$$J(u_0) = \frac{1}{c_1} \lim_{t \to \infty} \int |x|^{\varepsilon} u(x,t)\, dx\,, \tag{5.12}$$

with $\varepsilon = \alpha(\gamma) - N$, which generalizes (5.2). Observe that the integral expression in the second member is invariant in time for the Barenblatt solution. c_1 is precisely its value for B.

ACKNOWLEDGEMENTS

This paper was completed in the Institute for Mathematics and its Applications, Univ. of Minnesota, during long-term stays of the authors to participate in the Special Year in Phase Transitions and Free Boundaries. We are also grateful to the Univ. of Montréal and Leiden for support during previous stages of the work. This research is a part of the project Filtration and Nonlinear Diffusion, funded by Grant SC1-0019-C(TT) of the European Community.

The authors would like to thank G. I. Barenblatt for his interest and comm~~m~~ents, which originated Chapter 5 of the paper. In particular, he drew our attention to the significance

of the asymptotic constant as an invariant of the dynamics. N. Goldenfeld suggested the use of formula (5.12).

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