

**ASYMPTOTIC BEHAVIOUR FOR AN EQUATION  
OF SUPERSLOW DIFFUSION. THE CAUCHY PROBLEM**

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# ASYMPTOTIC BEHAVIOUR FOR AN EQUATION OF SUPERSLOW DIFFUSION. THE CAUCHY PROBLEM

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**Abstract.** We investigate the asymptotic behaviour as  $t \rightarrow \infty$  of the nonnegative weak solution to the Cauchy problem for the equation of superslow diffusion

$$u_t = (e^{-1/u})_{xx} \text{ for } x \in \mathbb{R}, t > 0$$

with nonnegative initial function  $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ ,  $u_0 \not\equiv 0$ . We prove that asymptotic separation of variables takes place if we make the change of variables  $v = e^{-1/u}$  and  $\eta = x/\log t$ . The precise result says that as  $t \rightarrow \infty$

$$t v(\eta \log t, t) \rightarrow \frac{1}{2}(a^2 - \eta^2)_+,$$

and the convergence is uniform in  $\eta \in \mathbb{R}$ . The constant  $a > 0$  is exactly one half of the initial energy:  $a = \frac{1}{2} \int u_0(x) dx > 0$ . This implies that  $u$  evolves for large  $t$  towards a mesa-like profile of height  $1/(\log t)$  and width  $= 2a \log t$ .

**Key words.** Nonlinear diffusion equation, asymptotic behaviour, explicit solutions

**AMS(MOS) subject classifications.** 35K55, 35K65, 34E10

**Introduction.** In this paper we investigate the asymptotic behaviour of the solution to the Cauchy problem for the equation

$$(0.1) \quad u_t = (\phi(u))_{xx} \text{ in } Q = \mathbb{R} \times (0, \infty),$$

with diffusion law of the exponential type:

$$(0.2) \quad \phi(u) = e^{-1/u} \text{ for } u > 0, \quad \phi(0) = 0,$$

and the initial condition

$$(0.3) \quad u(x, 0) = u_0(x) \text{ in } \mathbb{R}.$$

The initial function  $u_0$  satisfies

Equation (0.1), (0.2) for  $u \geq 0$  is an example of equation of *superslow diffusion*, so-called because the diffusivity  $\phi'(u) = e^{-1/u}u^{-2}$  vanishes at  $u = 0$  faster than any power of  $u$ . A large class of such equations has been studied in [F]. Existence and uniqueness of a continuous nonnegative weak solution of (0.1)–(0.3) is well-known. The solution is smooth at any point of positiveness. See a full list of references given in [K].

In order to state our main result we introduce the change of variable  $v = \phi(u)$ , i.e.,

$$(0.5) \quad v(x, t) = e^{-1/u(x,t)}.$$

Then,  $0 \leq v(x, t) < 1$  in  $Q$ , and  $v(x, t)$  solves the quasilinear equation

$$(0.6) \quad v_t = v(\log v)^2 v_{xx} \text{ in } Q.$$

The asymptotic behaviour of  $v(x, t)$  is exactly described by the following result.

**THEOREM 1.** *Under hypotheses (0.4) we have*

$$(0.7) \quad \lim_{t \rightarrow \infty} t v(\eta(\log t), t) = F_a(\eta) \equiv \frac{1}{2}(a^2 - \eta^2)_+$$

uniformly for  $\eta \in \mathbb{R}$ , where  $a$  is one half of the initial energy:

$$(0.8) \quad a = \frac{1}{2} \int u_0(x) dx > 0.$$

If we translate this result (0.7) to the function  $u(x, t)$  by means of the inverse transformation

$$(0.9) \quad u(x, t) = -\frac{1}{\log v(x, t)},$$

we get the asymptotic formula:

$$(0.10) \quad \lim_{t \rightarrow \infty} (\log t) u(\eta(\log t), t) = 1$$

uniformly in any set  $\{|\eta| \leq c\}$ , where  $c = \text{const} \in (0, a)$ , while for  $|\eta| \geq a$  we have

$$(0.11) \quad \lim_{t \rightarrow \infty} (\log t) u(\eta(\log t), t) = 0.$$

Thus, in terms of the initial variable  $u(x, t)$  we observe a mesa-like behaviour. Notice that the only parameter which appears in the formulas is the normalized length of the support of  $u$ , namely  $2a$ . This parameter is easily calculated from the law of conservation of mass  $\int u(x, t) dx = \text{const}$ , since for large  $t$  it follows from (0.10), (0.11) that  $\int u(x, t) dx \approx 2a$ .

Hence,  $\|u_0\|_1 = 2a$ . Any further information about the asymptotic spatial structure of the solution as  $t \rightarrow \infty$  is lost in the  $u$  variable.

It is interesting to compare Theorem 1 with the asymptotic behaviour of the solution to the initial-boundary value problem for equation (0.1), (0.2) in a bounded domain. In collaboration with R. Kersner we have proved in [GKV] that for the one-dimensional problem posed in  $(-l, l) \times (0, \infty)$ , where  $l > 0$  is a fixed constant, with conditions

$$(0.12) \quad u(x, 0) = u_0(x) \text{ in } (-l, l), \quad u_0 \in L^\infty, u_0 \geq 0, u_0 \not\equiv 0,$$

$$(0.13) \quad u = 0 \text{ for } x = \pm l, t > 0,$$

the following behaviour holds uniformly in  $(-l, l)$ :

$$(0.14) \quad \lim_{t \rightarrow \infty} t(\log t)^2 v(x, t) \rightarrow F_l(x).$$

Two differences appear. Firstly, the rate of decay is  $t^{-1}(\log t)^{-2}$  and not  $t^{-1}$  as in (0.7). Secondly, the particular asymptotic profile is determined by the length of domain and not by the initial function. However, it is remarkable that the family of asymptotic profiles is the same for the Cauchy and the initial-boundary value problems. This was not true for the porous medium equation  $u_t = (u^m)_{xx}$  for a fixed  $0 < m < \infty$ , see references in [K], [GV].

We also obtain a precise result on the asymptotic behaviour of interfaces of every compactly supported solution to the Cauchy problem (0.1)–(0.3).

**THEOREM 2.** *Assume that (0.4) holds and also that  $u_0$  has a compact support. Then as  $t \rightarrow \infty$*

$$s_+(t) \equiv \sup\{x \in \mathbb{R} : u(x, t) > 0\} = a \cdot \log t + O(1),$$

$$s_-(t) \equiv \inf\{x \in \mathbb{R} : u(x, t) > 0\} = -a \cdot \log t + O(1).$$

After stating our main result, let us make some comments before proceeding with the proofs. Thus, in order to understand the appearance of the asymptotic profile  $F_a(\eta)$  it is convenient to view our result in terms of the rescaled function  $\theta$  corresponding to our asymptotic formula (0.7), which is defined by

$$(0.15) \quad \theta(\eta, \tau) = (2 + t) v(\eta \log(2 + t), t),$$

where  $\tau = \log(2 + t) : [0, \infty) \rightarrow [\tau_0, \infty)$ ,  $\tau_0 = \log 2$  (the number 2 plays no special role; any number  $T > 0$  would do). Then  $\theta(\eta, \tau)$  solves the Cauchy problem

$$(0.16) \quad \theta_\tau = \mathcal{C}(\theta, \tau) \equiv \mathcal{A}(\theta) + \frac{1}{\tau}(\theta_\eta \eta - 2\theta(\log \theta)\theta_{\eta\eta}) + \frac{1}{\tau^2}\theta(\log \theta)^2\theta_{\eta\eta}$$

in  $\mathbb{R} \times (\tau_0, \infty)$ , with initial condition

$$(0.17) \quad \theta(\eta, \tau_0) = \theta_0(\eta) \equiv 2 \exp\{-1/u_0(\eta \log 2)\}.$$

The autonomous part of the operator in the right-hand side of (0.16) has the form

$$(0.18) \quad \mathcal{A}(\theta) = \theta \theta_{\eta\eta} + \theta.$$

It is easily seen that the functions  $F_a(\eta)$  given in (0.7) are precisely the radially symmetric and nonnegative weak solutions of the stationary equation  $\mathcal{A}(\theta) = 0$  which are monotone nonincreasing in  $|\eta|$ . Therefore Theorem 1 amounts to proving the convergence of the solution  $\theta(\eta, \tau)$  as  $\tau \rightarrow \infty$  to the corresponding stationary solution

$$(0.19) \quad \mathcal{A}(F) = 0 \text{ in } \mathbb{R}, \quad F \geq 0, \quad F = F(|\eta|),$$

which is uniquely determined by the total mass of the initial function, see (0.8).

It is interesting to note that the function

$$(0.20) \quad V(x, t) = t^{-1} F_a(x/\log t),$$

describing by (0.7) the asymptotic behaviour of the solution  $v(x, t)$  as  $t \rightarrow \infty$ , satisfies the nonautonomous quasilinear parabolic equation

$$(0.21) \quad v_t = (\log t)^2 v v_{xx} - \frac{x}{t \log t} v_x,$$

which looks quite different from (0.6). Thus, (0.20) is only an *approximate* self-similar solution of equation (0.6).

As for equation (0.1), (0.2), the function  $U(x, t) = (-\log V(x, t))^{-1}$ , which is an approximate self-similar solution to (0.1), (0.2), is in fact an explicit self-similar solution of the quasilinear equation

$$(0.22) \quad u_t = (\log t)^2 u^2 \left( e^{-\frac{1}{u}} \right)_{xx} - \frac{x}{t \log t} u_x.$$

We finally remark that a simple rescaling allows to extend our result to the more general equation  $u_t = (\phi(u))_{xx}$  with  $\phi(u) = a e^{-b/u}$ ,  $a, b > 0$ .

**1. Preliminaries. Explicit Solution.** The weak solution to the problem (0.1)–(0.3) is a continuous nonnegative function which is smooth at any point where  $u > 0$  and has a continuous heat flux  $-(e^{-1/u})_x$  on interfaces  $\{u = 0\}$ ; see for details the list of references

in the review [K]. We also note that for the solution of nonlinear equations of the type (0.1) the law of conservation of mass holds, i.e., if the initial mass is finite

$$(1.1) \quad \int u_0(x) dx = E_0 > 0,$$

then

$$(1.2) \quad \int u(x, t) dx = E_0 \text{ for any } t > 0.$$

In view of (0.5), this implies that

$$(1.3) \quad \int_{-\infty}^{\infty} \left( -\frac{dx}{\log v(x, t)} \right) = \int_{-\infty}^{\infty} \left( -\frac{dx}{\log v_0(x)} \right) = E_0 \text{ for } t \geq 0.$$

The proof of our result is based on a careful use of a family of explicit solutions which can be found in [BMFF], [Ki]:

$$(1.4) \quad v_*(x, t; c) = \frac{1}{2t}(c^2 - w^2)_+,$$

where  $c > 0$  is a fixed arbitrary constant and the function  $w = w(x, t; c) \in [0, c]$  is determined by the equation

$$(1.5) \quad |x| = \Phi(w, c) \equiv (2 + \log(2t))w + (c - w) \log(c - w) - (c + w) \log(c + w).$$

Since the function  $\Phi(w, c)$  in the right-hand side satisfies

$$\Phi'_w = \log(2t) - \log(c^2 - w^2) \geq 0$$

for fixed  $c > 0$  and  $t \geq c^2/2$ , equation (1.5) uniquely determines the function  $w(x, t; c) \in [0, c]$  in terms of  $x \in [0, x_*(t; c))$ , where

$$(1.6) \quad x_*(t; c) = c \log t + c \log \left( \frac{e^2}{2c^2} \right),$$

which is equivalent to  $c(\log t)$  as  $t \rightarrow \infty$ . Then (1.4) is an even, continuous and nonnegative function defined for  $x \in \mathbb{R}$ ,  $t \geq c^2/2$  and satisfying

$$(1.7) \quad v_*(x, t; c) = 0 \text{ for } |x| \geq x_*(t; c),$$

$$(1.8) \quad v_*(x, t; c) > 0 \text{ for } |x| < x_*(t; c),$$

and

$$(1.9) \quad \sup_{x \in \mathbb{R}} v_*(x, t; c) = v_*(0, t; c) = \frac{c^2}{2t} < 1 \quad \text{for } t > c^2/2.$$

Going back to the variable  $u$  by means of (0.9), we get the explicit compactly supported solution

$$(1.10) \quad u_*(x, t; c) = -\frac{1}{\log v_*(x, t; c)}$$

of equation (0.1), (0.2). Indeed, one can calculate from (1.4), (1.5) that

$$\exp\{-1/u_*(x, t; c)\} = \frac{1}{t} \left\{ \frac{x_*(t) - |x|}{|\log(x_*(t) - |x|)|} \right\} (1 + o(1))$$

near the interfaces  $x = \pm x_*(t; c)$ , and hence  $\exp\{-1/u_*\} \in C^1$ , which implies the continuity of the heat flux on the interfaces. There holds [Ki]

$$(1.11) \quad \|u_*(\cdot, t; c)\|_{L^1(\mathbb{R})} = 2c \quad \text{for } t > c^2/2.$$

It is curious that at  $t_0 = c^2/2$  the function  $u_*(x, t_0; c)$  behaves near  $x = 0$  like  $|x|^{-2/3}$ , which of course is an integrable singularity, but not a  $\delta$ -function.

We begin with some simple properties of our explicit solution.

LEMMA 1. For any fixed  $c > 0$

$$(1.12) \quad v_*(x, t; c) = \frac{1}{t} F_c(\eta) + O\left(\frac{1}{t \log t}\right) \quad \text{as } t \rightarrow \infty$$

uniformly for  $\eta \in \mathbb{R}$ .

*Proof.* Using (1.5) yields

$$|\eta| = w + \frac{\log(2e^2 t/(t+2))}{\log(t+2)} w + \frac{(c-w) \log(c-w) - (c+w) \log(c+w)}{\log(t+2)}$$

for  $w \in (0, c)$ . Hence,

$$w(x, t; c) = |\eta| + O(1/\log t) \quad \text{as } t \rightarrow \infty \quad \text{in } \{|x| \leq x_*(t; c)\},$$

which by (1.4) completes the proof.  $\square$

LEMMA 2. For any fixed  $0 < c_1 < c_2$  there holds

$$(1.13) \quad \begin{aligned} v_*(x, t; c_2) &> v_*(x, t; c_1) \\ \text{for } x \in \{|x| < x_*(t; c_2)\}, t &> 2c_2^2. \end{aligned}$$

*Proof.* First we note that

$$(1.14) \quad \frac{d}{dc} x_*(t; c) > 0 \text{ for } t > 2c^2.$$

Using (1.4), we get

$$(1.15) \quad \frac{d}{dc} v_*(x, t; c) = \frac{1}{t}(c - w \cdot w'_c),$$

and (1.5) yields that  $w'_c(x, t; c)$  is well defined in  $\{|x| < x_*(t; c)\}$  for  $t > c^2/2$ . One can see that

$$(1.16) \quad w'_c = \frac{\log(c+w) - \log(c-w)}{\log(2t) - \log(c-w) - \log(c+w)} < 1$$

for  $w \in (0, c), t > 2c^2$ . This together with (1.15) implies that  $c - w \cdot w'_c > 0$  for  $w \in (0, c)$ , and hence by (1.15)

$$(1.17) \quad \frac{d}{dc} v_*(x, t; c) > 0 \text{ for } x \in \{|x| < x_*(t; c)\}, \quad t > 2c^2.$$

Using (1.14), (1.17), we get (1.13) completing the proof.  $\square$

**2. First Estimates.** Let  $u(x, t)$  be the solution of the problem (0.1)–(0.3). Assume now that  $u_0$  has a compact support in an interval  $[-b, b]$ . We begin with an upper estimate of this solution.

LEMMA 3. There exist constants  $c_1 > 0$  and  $t_1 > c_1^2/2$  such that

$$(2.1) \quad v(x, t) \leq v_*(x, t_1 + t; c_1) \quad \text{in } \mathbb{R} \times (0, \infty).$$

*Proof.* By the comparison theorem [K] we obtain that (2.1) will be valid if

$$(2.2) \quad v(x, 0) \leq v_*(x, t_1; c_1) \quad \text{in } \mathbb{R}.$$

Using properties (1.6)–(1.9), we have that (2.2) holds if

$$(2.3) \quad t_1 > c_1^2/2,$$

$$(2.4) \quad \sup_{x \in \mathbb{R}} v_*(x, t_1; c_1) \equiv \frac{c_1^2}{2t_1} > \sup_{x \in \mathbb{R}} v(x, 0) = M_1 \in (0, 1)$$

and

$$(2.5) \quad \begin{aligned} x_*(t_1; c_1) &= c_1 \log \left( \frac{e^2 t_1}{2c_1^2} \right) \gg \\ l_1 &= \sup\{|x| : x \in \text{supp } v(x, 0)\}. \end{aligned}$$

Choose  $t_1$  as follows:

$$t_1 = \frac{1}{4} c_1^2 \left( 1 + \frac{1}{M_1} \right).$$

Then (2.3), (2.4) hold, and (2.5) implies the inequality

$$c_1 \log \left( \frac{e^2}{8} \left( 1 + \frac{1}{M_1} \right) \right) \gg l_1,$$

which is valid for any  $c_1 > 0$  large enough.  $\square$

Our next estimate is a lower bound.

LEMMA 4. *There exist constants  $c_2 > 0$  and  $t_2 > c_2^2/2$  such that*

$$(2.6) \quad v(x, t) \geq v_*(x, t; c_2) \quad \text{in} \quad \mathbb{R} \times (t_2, \infty).$$

*Proof.* By well known properties of the weak solution  $u(x, t)$  [K] there exists  $t_2 \geq 0$  such that  $u(0, t_2) > 0$  and  $u(x, t_2) \in C(\mathbb{R})$ . Choose arbitrary small  $c_2 > 0$ . Then from (1.6)–(1.9) one can see that inequality

$$v(x, t_2) \geq v_*(x, t_2; c_2) \quad \text{in} \quad \mathbb{R}$$

holds, and hence by the comparison theorem [K] estimate (2.6) is valid.  $\square$

If we now perform the change of variables (0.15), then from Lemmas 1, 3 and 4 and properties (1.6)–(1.9) of the explicit solution we get the following weak form of the asymptotic behaviour, which in particular determines the rate of stabilization to 0 of  $u(x, t)$ .

LEMMA 5. *If  $u_0$  satisfies (0.4) and has a compact support, then there exist  $\tau_* > 0$  and constants  $0 < c_- < c_+$  such that*

$$(2.7) \quad F(\eta; c_-) \leq \theta(\eta, \tau) \leq F(\eta; c_+) \quad \text{in} \quad \mathbb{R} \times (\tau_*, \infty).$$

As a consequence of these estimates we can also control the growth of the support of the solution  $u(x, t)$  as  $t \rightarrow \infty$ .

COROLLARY. There exist  $t_* > 0$  and  $0 < C_- < C_+$  such that for  $t \geq t_*$

$$(2.8) \quad \left\{ |x| < C_- \log \left( \frac{e^2 t}{2C_-^2} \right) \right\} \subseteq \text{supp } u(\cdot, t) \subseteq \left\{ |x| < C_+ \log \left( \frac{e^2 t}{2C_+^2} \right) \right\}$$

and

$$(2.9) \quad - \left\{ \log \frac{C_-^2}{2t} \right\}^{-1} \leq \sup_{x \in \mathbb{R}} u(x, t) \leq - \left\{ \log \frac{C_+^2}{2t} \right\}^{-1}.$$

**3. A Sharp Estimate.** We establish here a sharp lower bound.

LEMMA 6. *There holds*

$$(3.1) \quad \lim_{t \rightarrow \infty} t \sup_{x \in \mathbb{R}} v(x, t) \geq \frac{a^2}{2},$$

where  $a$  is given by (0.8).

*Proof. Step 1.* Assume also that  $u_0$  has a compact support. By well-known properties of the weak solution to the Cauchy problem (0.1)–(0.3) there exists  $t = t_1$  such that the support

$$(3.2) \quad \text{supp } v(x, t_1) = (l_-, l_+)$$

is a connected interval and  $0 \in (l_-, l_+)$ . By Aleksandrov's Reflection Principle (see [GNN]) the solution  $v(x, t)$  is a monotone function with respect to  $x$  in  $(-\infty, -b) \cup (b, \infty)$  for any fixed  $t \geq t_1$ .

*Step 2.* Fix now an arbitrary small  $\varepsilon > 0$ . We replace  $v(x, t_1)$  by an approximation  $\tilde{v}_\varepsilon(x)$  such that

- i)  $\tilde{v}_\varepsilon(x) \leq v(x, t_1)$  in  $\mathbb{R}$  and  $\tilde{v}_\varepsilon(x) \equiv v(x, t_1)$  in  $(l_- + \varepsilon, l_+ - \varepsilon)$ ,
- ii)  $\int (u(x, t_1) - \tilde{v}_\varepsilon(x)) dx \leq 2\varepsilon$ , and
- iii)  $\left| \frac{d}{dx} \tilde{v}_\varepsilon(x) \right| \geq 1$  near the endpoints of its support.

*Construction.* Consider the right-hand side  $x > 0$ . It is clear that we can choose  $l_1 \in (l_+ - \varepsilon/2, l_+)$  such that

$$\int_{l_1}^{l_+} u(x, t_1) dx < \varepsilon/2.$$

To the left of  $l_1$  we draw the line  $y(x) = M(l_1 - x)$ . This line intersects the graph of  $v(x, t_1)$  for the first time in a point  $l_2 < l_1$ . If  $M > 1$  is large enough, we have  $l_2 > l_+ - \varepsilon$  and

$$\int_{l_2}^{l_+} u(x, t_1) dx < \varepsilon.$$

For such an  $M$  we define

$$\begin{aligned} \tilde{v}_\varepsilon(x) &= v(x, t_1) & \text{if } 0 \leq x \leq l_2, \\ \tilde{v}_\varepsilon(x) &= y(x) & \text{if } l_2 \leq x \leq l_1, \\ \tilde{v}_\varepsilon(x) &= 0 & \text{if } x \geq l_1. \end{aligned}$$

The same construction holds for the left-hand side  $x < 0$ .

*Step 3.* Denote  $v_\varepsilon(x, t)$  the weak solution of the Cauchy problem in  $\mathbb{R} \times (t_1, \infty)$  to the equation (0.6) with the initial function  $v_\varepsilon(x, t_1) = \tilde{v}_\varepsilon(x)$  in  $\mathbb{R}$ . Let

$$c_\varepsilon = \frac{1}{2} \int \tilde{u}_\varepsilon(x) dx \equiv \frac{1}{2} \int u_\varepsilon(x, t) dx \text{ for every } t \geq t_1,$$

so that  $a - \varepsilon \leq c_\varepsilon < a$ . Since by construction  $\tilde{v}_\varepsilon(x) \leq v(x, t_1)$  in  $\mathbb{R}$ , from the comparison theorem [K] we have  $v_\varepsilon(x, t) \leq v(x, t)$  in  $\mathbb{R} \times (t_1, \infty)$ .

We consider now the family of explicit solutions  $\{v_*(x - x_0, t + T; c_\varepsilon), x_0 \in [-b, b], T > 0\}$  having the same mass  $c_\varepsilon$  as  $u_\varepsilon(x, t)$ . For a fixed  $t \geq t_1$  we denote by  $N(t; x_0, T)$  the number of sign changes in  $\mathbb{R}$  of the difference  $w(x, t; x_0, T) \equiv v_\varepsilon(x, t) - v_*(x - x_0, t + T; c_\varepsilon)$  or, which is the same, the number of intersections in  $\mathbb{R}$  of the functions  $v_\varepsilon(x, t)$  and  $v_*(x - x_0, t + T; c_\varepsilon)$ . Then by a well-known property, see e.g. [A], [GP1], [M], [Sat] and references therein, we conclude that  $N(t; x_0, T)$  does not increase with time and, in particular,

$$(3.3) \quad N(t; x_0, T) \leq N(t_1; x_0, T) \quad \text{for } t > t_1.$$

Notice that by known properties of regularity of the weak solution at a point where it is positive [K], and by using results of [A], [KP], we may conclude that for  $t \geq t_1$  every zero of the difference in the domain of positiveness of both solutions considered is an isolated point. Since, by the properties of the explicit solutions given in Section 1, we have for an arbitrary fixed  $x_0 \in [-b, b]$

$$v_*(x - x_0, t_1 + T; c_\varepsilon) \approx \frac{c_\varepsilon^2}{2(t_1 + T)} \text{ as } T \rightarrow \infty$$

uniformly in any compact in  $x \in \mathbb{R}$ , by using the property iii) of the function  $v_\varepsilon(x, t_1)$ , we have that for every  $x_0 \in [-b, b]$  and  $T$  large enough

$$(3.4) \quad N(t_1; x_0, T) = 2.$$

This together with (3.3) yields the inequality

$$(3.5) \quad N(t; x_0, T) \leq 2 \text{ for } t \geq t_1.$$

Fix arbitrary  $x_0 \in [-b, b]$  and  $T = T_0$  large enough. We now prove that for  $t > t_1$

$$(3.6) \quad \sup_{x \in \mathbb{R}} v_\varepsilon(x, t) \geq \sup_{x \in \mathbb{R}} v_*(x - x_0, t + T_0; c_\varepsilon).$$

Assume for a moment that this is true. Then

$$\sup_{x \in \mathbb{R}} v_*(x - x_0, t + T_0; c_\varepsilon) \equiv v_*(0, t + T_0; c_\varepsilon) = \frac{c_\varepsilon^2}{2(t + T_0)},$$

and (3.6) implies that

$$\liminf_{t \rightarrow \infty} t \sup_{x \in \mathbb{R}} v(x, t) \geq \liminf_{t \rightarrow \infty} t \sup_{x \in \mathbb{R}} v_\varepsilon(x, t) \geq \frac{c_\varepsilon^2}{2}.$$

Since  $\varepsilon > 0$  is an arbitrary, we obtain the desired result (3.1). Hence, we have to prove (3.6).

*Step 4.* Suppose (3.6) is not valid and

$$t_* = \sup\{\tau > 0 : (3.6) \text{ holds for all } t \in [t_1, t_1 + \tau]\} < \infty.$$

Let  $x_* \in [-b, b]$  be a point of maximum of the function  $v_\varepsilon(x, t_*)$  and hence by a definition of  $t_*$  we have

$$(3.7) \quad v_\varepsilon(x_*, t_*) = \frac{c_\varepsilon^2}{2(t_* + T_0)}.$$

Consider the explicit solution  $v_*(x - x_*, t + T_0; c_\varepsilon)$ . By construction we have

$$(3.8) \quad \begin{aligned} w(x, t_*; x_*, T_0) &\equiv v_\varepsilon(x, t_*) - v_*(x - x_*, t_* + T_0; c_\varepsilon) = 0 \quad \text{for } x = x_*, \\ w_x(x, t_*; x_*, T_0) &= 0 \quad \text{for } x = x_*. \end{aligned}$$

Suppose first that  $x = x_*$  is the *point of tangency* of the functions  $v_\varepsilon(x, t_*)$  and  $v_*(x - x_*, t_* + T_0; c_\varepsilon)$ , i.e., the difference  $w(x, t_*; x_*, T_0)$  satisfying (3.8) does not change sign in a small neighborhood of the point  $x = x_*$ . Since these have the same masses we get

$$(3.9) \quad N(t_*; x_*, T_0) \geq 1.$$

Indeed, if (3.9) is not valid and  $N(t_*; x_*, T_0) = 0$ , then by the Strong Maximum Principle [Fr] it follows that, since  $v_\varepsilon \not\equiv v_*$ , for arbitrarily small  $\delta > 0$  either  $v_\varepsilon(x, t_* + \delta) < v_*(x -$

$x_*, t_* + T_0 + \delta; c_\varepsilon)$  or  $v_\varepsilon(x, t_* + \delta) > v_*(x - x_*, t_* + T_0 + \delta; c_\varepsilon)$  in the domain of positiveness of both functions, contradicting the equality of masses. Hence, there exists at least one point of intersection, i.e., a point  $x_1$  where the difference  $w$  changes sign, and  $x_1 \neq x_*$ . Assume without loss of generality that the difference  $w(x, t_*; x_*, T_0) \leq 0$  in a small neighborhood  $I_r = (x_* - r, x_* + r)$  of the point  $x = x_*$  with  $r \ll |x_1 - x_*|$ . Then by using the continuous dependence of the function  $v_*(x - x_*, t_* + T_0; c_\varepsilon)$  with respect to a small perturbation of the value of  $T_0$ , we obtain that for any small  $\delta > 0$  there exist at least two points of sign change for perturbed difference  $w(x, t_*; x_*, T_0 + \delta)$  in  $I_r$ , one to the left of  $x = x_*$  and to the right, and also an intersection point which lies not far from  $x_1$ , and anyway is outside  $I_r$ . Therefore, for small  $\delta > 0$  we have

$$(3.10) \quad N(t_*; x_*, T_0 + \delta) \geq 3.$$

This leads to a contradiction with (3.5) for  $t = t_*$ ,  $x_0 = x_* \in [-b, b]$  and  $T = T_0 + \delta$ .

Now, if the maximum  $x = x_*$  is an *inflection point* for the difference  $w(x, t_*; x_*, T)$  satisfying (3.8), namely that it changes sign in any neighborhood of the point  $x = x_*$ , then we use the idea given in [GP1], [GP2]. Assume without loss of generality that  $w(x, t_*; x_*, T_0) > 0$  in a small left-hand neighborhood of  $x = x_*$  and  $w(x, t_*; x_*, T_0) < 0$  in a small right-hand one. Then it is easily seen that for any  $\lambda > 0$  small enough there holds

$$(3.11) \quad N(t_*; x_* - \lambda, T_0) \geq 3$$

(see a similar detailed analysis in [GP1], [GP2]) contradicting (3.5) for  $t = t_*$ ,  $x_0 = x_* - \lambda$  and  $T = T_0$  and completing the proof of Lemma 6 in the case of compactly supported data.

*Step 5.* If  $u_0$  is not compactly supported, the proof is made by approximation from below with compactly supported functions.  $\square$

**4. Semiconvexity.** For the proof of Theorem 1 we need also the following lower estimate of the second derivative of the solution.

LEMMA 7. *Let  $u(x, t)$  be a solution of (0.1)–(0.3). Then for every  $\varepsilon > 0$  there exists  $T_\varepsilon > 0$  such that*

$$(4.1) \quad v_{xx} \geq -\frac{1 + \varepsilon}{t(\log t)^2}$$

in the domain  $\{(x, t) : x \in \mathbb{R}, t \geq T_\varepsilon\}$ .

*Proof.* By approximation we may assume that  $u_0$  is continuous, bounded and positive in  $\mathbb{R}$ . Then  $v(x, t)$  is a classical solution of equation (0.6). Differentiating it twice with respect to  $x$  we obtain the equation satisfied by  $z = v_{xx}$ :

$$(4.2) \quad \begin{aligned} z_t &= v(\log v)^2 z_{xx} + 2[(\log v)^2 + 2 \log v] v_x z_x + \\ &\frac{2}{v} (\log v + 1) v_x^2 z + ((\log v)^2 + 2 \log v) z^2. \end{aligned}$$

Following the method of [AB] we want to try an explicit subsolution for this equation. If we try  $z = -1/\varphi(t)$ , with  $\varphi > 0$ , we easily check that a sufficient condition is that

$$(4.3) \quad \varphi'(t) \leq \inf_{x \in \mathbb{R}} ((\log v(x, t))^2 + 2 \log v(x, t)).$$

Now, for large  $t > 0$  from Lemma 5 we have  $v \leq \text{const}/t$ , hence  $(\log v)^2 + 2 \log v \geq (\log t)^2(1 - \frac{\varepsilon}{4})$  for  $\varepsilon$  small if  $t$  is large enough. Therefore, an admissible choice is

$$(4.4) \quad \varphi(t) = (1 - \frac{\varepsilon}{2})(t - T)(\log t)^2$$

if  $t > T$  for some large  $T$ . Since with this choice  $z$  will be a subsolution of equation (4.2) in  $D = \{(x, t) : x \in \mathbb{R}, t > T\}$  with  $z(x, T) = -\infty$ , we conclude from the Maximum Principle that

$$v_{xx} \geq z \text{ in } D,$$

hence the conclusion in the limit.  $\square$

*Remark.* The sharpness of estimate (4.1) is checked by looking at the explicit solution  $v_*(x, t; c)$ , for which we have the estimates

$$(i) \quad (v_*)_{xx} \geq -\frac{1}{t(\log t)^2} + O\left(\frac{1}{t(\log t)^3}\right) \quad \text{as } t \rightarrow \infty,$$

$$(ii) \quad (v_*)_{xx}(0, t; c) = -\frac{1}{t \log^2(2t/c^2)} \quad \text{for } t > c^2/2.$$

**5. Proof of Theorem 2.** We now prove, under the additional assumption that  $u_0$  has a compact support, explicit estimates of the support of the solution

$$\text{supp } u(x, t) = (s_-(t), s_+(t)),$$

which is a connected interval for large  $t$ , say  $t > t_1$ .

We take the function  $U(x, t) = u_*(x - d_+, t + T; a)$ , where  $a$  is one half of the energy of  $u_0$ ,  $T > a^2/2$  and  $d_+ = s_+(t_1) + x_*(T; a)$ , so that the support of  $U(x, 0)$ ,  $(S_-(0), S_+(0))$ , lies to the right of the support of  $u_0$ . Then, by the shifting comparison principle [V] (see similar results for quasilinear heat equation with source proved by intersection comparison in [SGKM, p. 248]) we have a comparison of the interfaces of  $u$  and  $U$ , i.e., for  $t > t_1$

$$s_+(t) \leq S_+(t) = d_+ + x_*(t + T; a),$$

$$s_-(t) \leq S_-(t) = d_+ - x_*(t + T; a).$$

A similar argument by shifting to the left gives

$$\begin{aligned} s_+(t) &\geq -d_- + x_*(t+T; a), \\ s_-(t) &\geq -d_- - x_*(t+T; a), \end{aligned}$$

where  $d_- = s_-(t_1) - x_*(T; a)$ . In view of the formula for  $x_*(t+T; a)$  we have

$$\begin{aligned} s_+(t) &= a \log t + O(1), \\ s_-(t) &= -a \log t + O(1), \end{aligned}$$

which completes the proof.  $\square$

**6. Proof of Theorem 1.** Consider the Cauchy problem (0.16), (0.17) for the quasilinear parabolic equation which is a perturbation of the autonomous (“stationary”) equation

$$(6.1) \quad \theta_\tau = \mathcal{A}(\theta).$$

By Lemma 5 the evolution trajectory  $\{\theta(\cdot, \tau), \tau > \tau_*\}$  is uniformly bounded, and hence, by a general regularity result [DB1], [DB2], it is compact in  $C_0(\mathbb{R})$ . We now prove that  $\omega$ -limit set, given by the formula

$$\begin{aligned} \omega(\theta_0) &= \{f \in C_0(\mathbb{R}) : f \geq 0 \text{ and there exists } \{\tau_j\} \rightarrow \infty \text{ such that} \\ &\quad \theta(\cdot, \tau_j) \rightarrow f(\cdot) \text{ as } j \rightarrow \infty \text{ uniformly in } \mathbb{R}\}, \end{aligned}$$

is precisely

$$(6.2) \quad \omega(\theta_0) = \{F_a(\cdot)\},$$

which indeed yields (0.7).

Choose an arbitrary  $f \in \omega(\theta_0)$ , so that there exists a sequence  $\tau_j \rightarrow \infty$  such that

$$(6.3) \quad \theta(\cdot, \tau_j) \rightarrow f(\cdot) \text{ as } j \rightarrow \infty \text{ uniformly in } \mathbb{R}.$$

Applying Aleksandrov’s Reflection Principle and passing to the limit  $\tau \rightarrow \infty$  we have  $f = f(|\eta|)$  and  $f$  does not increase in  $|\eta|$  (see a detailed argument in [KV], Section 5).

We now prove that  $f(a) = 0$ . Suppose for a contradiction that  $f(a) > 0$  and hence by continuity

$$(6.4) \quad \text{mes supp } f > 2a.$$

Using the conservation law (1.3) for the rescaled function  $\theta$  with  $\tau = \tau_j$  yields

$$(6.5) \quad \int_{-\infty}^{+\infty} \left[ 1 - \frac{\log \theta(\eta, \tau_j)}{\log \tau_j} \right]^{-1} d\eta \equiv E_0 = 2a$$

for  $j = 1, 2, \dots$ . It follows from (6.3) that for given small  $\varepsilon > 0$  there exists  $j_\varepsilon > 0$  such that

$$(6.6) \quad \theta(\cdot, \tau_j) \geq (f(\cdot) - \varepsilon)_+ \text{ in } \mathbb{R} \text{ for any } j > j_\varepsilon.$$

Therefore (6.4), (6.6) imply that for any  $\varepsilon > 0$  small enough

$$(6.7) \quad \text{mes supp } \theta(\cdot, \tau_j) \geq \text{mes supp}(f(\cdot) - \varepsilon)_+ > 2a$$

for  $j > j_\varepsilon$ . Combining (6.5)–(6.7) yields the estimate

$$(6.8) \quad \int_{-\infty}^{+\infty} \left[ 1 - \frac{\log \theta(\eta, \tau_j)}{\log \tau_j} \right]^{-1} d\eta \geq \int_{-\infty}^{+\infty} \left[ 1 - \frac{\log((f(\eta) - \varepsilon)_+)}{\log \tau_j} \right]^{-1} d\eta \\ \rightarrow \text{mes supp}(f(\cdot) - \varepsilon)_+ > 2a \text{ as } j \rightarrow \infty,$$

contradicting the conservation law (6.5).

Thus,  $f(a) = 0$  and  $\text{mes supp } f \leq 2a$ . Using Lemma 6 yields

$$(6.9) \quad f(0) \geq \frac{a^2}{2}.$$

Rewriting estimate (4.1) for the function  $\theta(\eta, \tau)$ , integrating this inequality twice and passing to the limit  $\tau = \tau_j \rightarrow \infty$ , we obtain

$$(6.10) \quad f(\eta) \geq \frac{1}{2}(f(0) - \eta^2)_+.$$

By using (4.1), we may also conclude that  $f_{\eta\eta} \geq -1$  a.e. Since  $F_{\eta\eta}(\eta; a) \equiv -1$  in  $[0, a]$ , from (6.9) and (6.10) we have that the difference  $z(\eta) = f(\eta) - F_a(\eta)$  satisfies  $z \geq 0$ ,  $z_{\eta\eta} \geq 0$  a.e. in  $[0, a]$ , and since  $z(a) = 0$ , one can see that  $z_\eta(a) \leq 0$ . Assume for contradiction that  $z \not\equiv 0$  and hence  $z_{\eta\eta} > 0$  in a set of nonzero measure in  $[0, a]$ . Then integrating the inequality  $z_{\eta\eta} \geq 0$  over  $(0, a)$  yields

$$z_\eta(0) < z_\eta(a) \leq 0,$$

contradicting the symmetry condition at the origin. This completes the proof.  $\square$

**7. Final Remarks.** It is interesting to consider our equation (0.1), (0.2) as some kind of limit of the porous medium equation (PME)  $u_t = (u^m)_{xx}$  as  $m \rightarrow \infty$ . Thus, the PME admits a family of explicit self-similar solutions (the Barenblatt solutions) which decay in time according to

$$(7.1) \quad u_m(x, t) \leq a_m \|u_0\|_1^{\frac{2}{m+1}} t^{-\frac{1}{m+1}}, a_m > 0,$$

while their support is confined by the interfaces  $\pm s_m(t)$ ,

$$(7.2) \quad s_m(t) = c_m \|u_0\|_1^{\frac{m-1}{m+1}} t^{\frac{1}{m+1}}, c_m > 0.$$

Moreover, these estimates are true for every nonnegative solution while initial data are compactly supported [V]. Put now  $v = u^m$  (in analogy to (0.5)) and let  $m \rightarrow \infty$  to obtain a formal expression for the upper bound in the limit

$$(7.3) \quad v_\infty(x, t) \leq a_\infty \|u_0\|_1^2 t^{-1},$$

which agrees with Theorem 1. Agreement with Theorem 2 for the interfaces necessitates replacing  $t^{1/(m+1)}$  in the limit by  $\log t$  (and not by 1), then obtaining

$$(7.4) \quad s_\infty(t) = c_\infty \|u_0\|_1 \log t.$$

On the other hand, a rigorous limit  $m \rightarrow \infty$  in the equation  $u_t = (\phi_m(u))_{xx}$  leads to the so called “mesa problem” studied by several authors, cf. [CF], [EHKO], [FH], [Sac].

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