PLAY-THE-WINNER RULE AND INVERSE SAMPLING IN SELECTING THE BETTER OF TWO BINOMIAL POPULATIONS

by

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1. Introduction.

The problem of selecting the better of two independent binomial populations (i.e., the one with the highest probability of success p on a single trial) has been formulated in different ways (see [1] and [2] and their references). In this paper, as in [2], we consider it in the framework of ranking and selection problems. For preassigned constants P^* and Δ^*, with $\frac{1}{2} < P^* < 1$ and $0 < \Delta^* < 1$, it is required that the probability of a correct selection (CS) should be at least P^* when the true difference in the p-values (denoted by Δ) is at least Δ^*, i.e., we want a procedure R such that

$$(1.1) \qquad P\{CS|R\} \geq P^*$$

whenever $\Delta \geq \Delta^*$. It is assumed that tests can be made one at a time on either population and that the results are immediately available.

As in [2] we are again interested in comparing two different sampling rules, but in this paper we use inverse sampling (i.e., terminate when any one population has r successes) as a termination rule, while in [2] the termination rule was based on the difference in the number of successes. One of these sampling rules is the Play-the-Winner (PW) rule suggested by Robbins (see the references in [2]) in which a success generates a new trial on the same population and a failure implies that a switch is to be made to the other population. The other sampling rule is the Vector-at-a-time (VT) rule in which we take two observations at each stage, one from each population, and do not consider stopping between these two observations. Let R_I and R_I' denote the two inverse sampling procedures, based on the PW and VT sampling rules, respectively. We will show that R_I is preferable to R_I' in the limit $(\Delta^* \to 0)$. The procedures based on the absolute difference in the number of successes defined in [2] are preferable to R_I and R_I' for Δ^* sufficiently small but the reverse is true for $\Delta = 0$ (or small) with Δ^* fixed and P^* sufficiently close to one. Hence there is no result based on expected total number of trials (or on the loss defined in [2]) that is uniform in both Δ^* and P^*. Another reason for studying the inverse sampling procedures is that they can be generalized to select the best of $k > 2$ binomial population (cf. [3]), while the analogous generalizations of procedures based on the difference generally leads to difficult mathematical problems.

2. The Procedure R_I: Exact Results.

Under inverse sampling we stop when any population attains r successes and declare that the treatment associated with that population is the better treatment; the integer $r \geq 1$ is predetermined so that (1.1) is satisfied. We wish to find the probability of a correct selection $P\{CS|R_I\}$ under procedure R_I.

Let A denote the better population and B the worse one; let S_A and S_B denote the current number of successes for each, so that $r - S_A = T_A$ is the number A needs to be selected andd $r - S_B = T_B$ is the number B needs. Let $\underset{\sim}{T} = (T_A, T_B)$ and let $p > p'$ denote the single-trial success probabilities of A and B, respectively. We define probabilities $U_{m,n}$ and $V_{m,n}$ by

$$U_{m,n} = P\{CS \mid \underset{\sim}{T} = (m, n) \text{ and the next observation is on A}\}$$

$$V_{m,n} = P\{CS \mid \underset{\sim}{T} = (m, n) \text{ and the next observation is on B}\}.$$

From the PW sampling rule, we have the recursions

$$\begin{aligned} U_{m,n} &= p\, U_{m-1,n} + q\, V_{m,n} \\ V_{m,n} &= p' V_{m,n-1} + q' U_{m,n} \end{aligned} \tag{2.2}$$

with boundary conditions given by

$$U_{0,n} = 1,\ V_{m,0} = 0 \quad \text{for} \quad m,n > 0. \tag{2.3}$$

To solve (2.2) we use generating functions $U = U(x, y)$ and $V = V(x,y)$ defined by

$$U = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{m,n} x^m y^n; \quad V = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{m,n} x^m y^n. \tag{2.4}$$

It is readily verified that (2.2) leads to

$$\begin{aligned} (1 - px)U - qV &= pxy/(1 - y) \\ (1 - p'x)V &= q'U \end{aligned} \tag{2.5}$$

and hence, letting $D = (1 - px)(1 - p'y) - qq'$,

$$U = \frac{pxy}{1 - y} \, \frac{(1 - p'y)}{D}$$

$$V = \frac{pxy}{1 - y} \, \frac{q'}{D}.$$

Since we commence the PW sampling rule with randomization, i.e., observing each with probability ½ at the outset, it follows that

$$P\{CS \mid R_I\} = \tfrac{1}{2}(U_{r,r} + V_{r,r}), \tag{2.7}$$

or the coefficient of $x^r y^r$ in $\frac{1}{2}(U + V)$, where r is chosen to satisfy (1.1). To get an explicit expression for (2.7) we expand $1/D$ by

$$(2.8) \qquad \frac{1}{D} = \sum_{i=0}^{\infty} \frac{(qq')^i}{\{(1-px)(1-p'y)\}^{i+1}} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (px)^j (p'y)^k \sum_{i=0}^{\infty} \binom{i+j}{i}\binom{i+k}{i}(qq')^i .$$

and similarly for $(1 - p'y)/D$. Using the well-known identity (see e.g. [2]) for the incomplete beta function

$$(2.9) \qquad q^s \sum_{k=0}^{r-1} \frac{\Gamma(s+k)}{\Gamma(s)\, k!} p^k = I_q(s, r),$$

we readily find from (2.6) that

$$(2.10) \qquad P\{CS|R_I\} = p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q^j \tfrac{1}{2}\{I_{q'}(j, r) + I_{q'}(j+1, r)\},$$

where (by definition) $I_q(0, r) = 1 = 1 - I_p(r, 0)$ for $r > 0$ and any q. In a later section we derive an approximation for the exact result (2.10), which is useful for making comparisons. It will be convenient to write (2.10) in the form $\frac{1}{2}E_r\{I_{q'}(X, r) + I_{q'}(X + 1, r)\}$.

Analogous calculations give us the expected number of trials on the poorer treatment $E\{N_B|R_I\}$ as well as the expected total number of trials $E\{N|R_I\}$ needed for termination. Let

$$(2.11) \qquad \begin{aligned} R_{m,n} &= E\{N_B|\underset{\sim}{T} = (m, n) \text{ and the next observation is on } A\} \\ S_{m,n} &= E\{N_B|\underset{\sim}{T} = (m,n) \text{ and the next observation is on } B\}. \end{aligned}$$

As in (2.2) we obtain the recursions

$$(2.12) \qquad \begin{aligned} R_{m,n} &= pR_{m-1,n} + qS_{m,n} \\ S_{m,n} &= p'S_{m,n-1} + q'R_{m,n} + 1, \end{aligned}$$

with the boundary conditions

$$(2.13) \qquad R_{0,n} = S_{m,0} = 0 \quad \text{for} \quad m,n > 0.$$

The desired result is

$$(2.14) \qquad E\{N_B|R_I\} = \tfrac{1}{2}(R_{r,r} + S_{r,r}).$$

Using (2.9) and the generating functions as in (2.5) we obtain

$$(2.15) \qquad E\{N_B|R_I\} = \frac{1}{2q'} \sum_{j=0}^{\infty} I_{q'}(j+1,r)\{I_q(j,r) + I_q(j+1,r)\}$$

$$= \frac{1}{q'} \sum_{j=0}^{\infty} I_q(j+1,r)\, I_{q'}(j+1,r) + \frac{p^r}{2q'} \sum_{j=0}^{\infty} \binom{j+r-1}{j} q^j I_{q'}(j+1,r).$$

For the expected total number of trials required for termination we can either add 1 to the first equation (2.12) or interchange p with p' (and q with q') in (2.15) to obtain $E\{N_A|R_I\}$ and then add the result to (2.15).

To simplify (2.15) we assume $p > 0$ and first prove

Lemma 1: For any positive integers r, s and any $p \geq 0$

$$(2.16) \qquad \sum_{j=1}^{s} I_p(r, j) = (r + s)\, I_p(r, s) - \frac{r}{p} I_p(r + 1, s).$$

The same result holds for any real $r \geq 0$ and in the limit as $p \to 0$.

Proof: Using (2.9) and the integral form for $I_p(r,j)$ with $r > 0$,

$$(2.17) \qquad \sum_{j=1}^{s} I_p(r, j) = \sum_{j=1}^{s} \frac{\Gamma(r+j)}{\Gamma(r)(j-1)!} \int_0^p t^{r-1}(1 - t)^{j-1} dt$$

$$= r \int_0^p t^{r+1} \sum_{i=0}^{s-1} \frac{\Gamma(r+i+1)}{\Gamma(r+1)i!} (1 - t)^i \frac{dt}{t^2}$$

$$= r \int_0^p I_t(r + 1, s) \frac{dt}{t^2}.$$

Integrating by parts and noting that $I_t(r+1, s)/t \to 0$ as $t \to 0$

we obtain from (2.17) the desired result (2.16). For $r = 0$ the result follows from our definition after (2.10) and for $p \to 0$ the result is easily shown.

Applying (2.16) to simplify the first part of (2.15), we obtain

$$(2.18) \qquad \sum_{j=0}^{\infty} I_q(j+1,\ r)\ I_{q'}(j+1,\ r) = p^r \sum_{j=0}^{\infty} I_{q'}(j+1,\ r) \sum_{i=j+1}^{\infty} \binom{i+r-1}{i} q^i$$

$$= p^r \sum_{i=0}^{\infty} \binom{i+r-1}{i} q^i \sum_{j=0}^{i-1} \{1 - I_{p'}(r,\ j+1)\}$$

$$= \frac{rq}{p} - p^r \sum_{i=0}^{\infty} \binom{i+r-1}{i} q^i \{(r+i)\ I_{p'}(r,\ i) - \frac{r}{p'} I_{p'}(r+1,\ i)\}$$

$$= \frac{rq}{p} - \frac{r}{p} E_{r+1}\{I_{p'}(r,\ X)\} + \frac{r}{p'} E_r\{I_{p'}(r+1,\ X)\},$$

where X has a negative binomial distribution with parameter $p > 0$ and index shown by the subscript on E and $p' > 0$. It follows that

$$(2.19) \qquad E\{N_B|R_I\} = \frac{1}{q'} [\frac{rq}{p} + \frac{r}{p'} E_r\{I_{p'}(r+1,\ X)\} - \frac{r}{p} E_{r+1}\{I_{p'}(r,\ X)\}$$

$$+ \tfrac{1}{2} E_r\{I_{q'}(X+1, r)\}].$$

If we added ones to both equations in (2.12) (or interchange p with p' in (2.19) and use lemma 2 below) and combine the result with (2.19), we find that the total number of trials N has expectation

$$(2.20) \qquad E\{N|R_I\} = (\frac{1}{q} + \frac{1}{q'})\ [\frac{rq}{p} + \frac{r}{p'} E_r\{I_{p'}(r+1,\ X)\} - \frac{r}{p} E_{r+1}\{I_{p'}(r,\ X)\}]$$

$$+ \frac{1}{2q'} - \frac{1}{2q'} E_r\{I_{p'}(r,\ X+1)\} + \frac{1}{2q} E_r\{I_{p'}(r,\ X)\}$$

It is easily shown that all four of the expectations in (2.20) approach zero as $r \to \infty$.

3. The Procedure R_I' : Exact Results.

We now seek the probability of a correct selection for the inverse sampling plan when VT sampling is used. If we consider the event that A has its rth success at the mth stage $(m \geq r)$, and B has at most $r - 1$ successes at that stage then we obtain after summing on m

$$(3.1) \qquad P\{CS|R_I'\} - \tfrac{1}{2}Q = \sum_{m=r}^{\infty} \binom{m-1}{r-1} p^r q^{m-r} \sum_{i=0}^{r-1} \binom{m}{i} (p')^i (q')^{m-i}$$

$$= p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q^j I_{q'}(j+1, r) = E_r\{I_{q'}(X+1, r)\}$$

where Q is the probability that both A and B get their r^{th} success at the same stage. To get the $P\{CS|R_I'\}$ exactly we write for Q

$$(3.2) \qquad Q = \sum_{m=r}^{\infty} \binom{m-1}{r-1}^2 (pp')^r (qq')^{m-r} = (pp')^r \sum_{j=0}^{\infty} \binom{j+r-1}{j}^2 (qq')^j$$

$$= p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q^j (p')^r \left\{ \sum_{i=j}^{\infty} \binom{i+r-1}{i} (q')^i - \sum_{i=j+1}^{\infty} \binom{i+r-1}{i} (q')^i \right\}$$

$$= E_r\{I_{q'}(X, r) - I_{q'}(X+1, r)\}$$

and hence we obtain from (3.1) and (3.2)

$$(3.3) \qquad P\{CS|R_I'\} = \tfrac{1}{2} E_r\{I_{q'}(X, r) + I_{q'}(X+1, r)\},$$

which is exactly the same as $P\{CS|R_I\}$ in (2.10). It follows that both R_I and R_I' require exactly the same integer r to satisfy (1.1).

Since the probability of selecting B (or the complement of that in (3.3)) is obtained by interchanging p with p' and q with q', it follows from the above derivation of (3.3) that the expected member of stages (or trials on the poorer treatment) is

$$(3.4) \qquad E\{N_B|R_I'\} = p^r \sum_{j=0}^{\infty} (j+r) \binom{j+r-1}{j} q^j \tfrac{1}{2}\{I_{q'}(j, r) + I_{q'}(j+1, r)\}$$

$$+ (p')^r \sum_{j=0}^{\infty} (j+r) \binom{j+r-1}{j} (q')^j \tfrac{1}{2}\{I_q(j,r) + I_q(j+1,r)\}.$$

To write (3.4) in a more convenient form we first prove a useful identity

Lemma 2: For any positive integers r, s and any p, p'

$$(3.5)\qquad (p')^s \sum_{j=0}^{\infty} \binom{j+s-1}{j}(q')^j\, I_q(j+1, r) = p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q^j I_{p'}(s, j)$$

Proof: Using (2.9) the right side of (3.5) becomes

$$(3.6)\qquad p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q^j (p')^s \left[\sum_{i=0}^{j-1} \binom{i+s-1}{i}(q')^i\right] = (p')^s \sum_{i=0}^{\infty} \binom{i+s-1}{i}(q')^i p^r \sum_{j=i+1}^{\infty} \binom{j+r-1}{j} q^j$$

and, using (2.9) again, this is the left side of (3.5).

With the help of this lemma we can rewrite the second line of (3.4) and obtain

$$(3.7)\qquad E\{N_B | R_I'\} = \frac{r}{2p} E_{r+1}\{I_{q'}(X, r) + I_{q'}(X+1, r)\}$$

$$+ \frac{r}{2p'} E_r\{I_{p'}(r+1, X) + I_{p'}(r+1, X+1)\}.$$

The expected total number of trials $E\{N | R_I'\}$ is simply twice that given in (3.7).

4. Approximations.

Having obtained these exact results we now proceed to obtain approximations to them that will make the comparisons easier. We first state a useful identity. Let $j^{(s)}$ denote $j(j-1)\ldots(j-s+1)$; then for any integer $s \geq 0$

$$(4.1)\qquad p^r \sum_{j=0}^{\infty} j^{(s)} \binom{j+r-1}{j} q^j = r^{(s)} \left(\frac{q}{p}\right)^s.$$

The proof is trivial and is omitted. In particular it follows from $s = 1$ and 2 that the mean and variance of this negative binomial distribution are

$$(4.2) \qquad E(X) = \frac{rq}{p}, \qquad \sigma^2(X) = \frac{rq}{p^2};$$

we assume throughout that p and p' are both positive.

Consider the first sum in (2.10), without the coefficient $\frac{1}{2}$,

$$(4.3) \qquad S_1 = p^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} q^j I_{q'}(j, r) = E_r\{I_{q'}(X, r)\}.$$

Let Y_p and $Y_{p'}$ be two independent negative-binomial chance variables with index r and single-trial success probabilities p and p', respectively. In the limit $(r \to \infty)$ both Y_p and Y_p' (and hence also $Y_p - Y_{p'}$) tend to normally distributed random variables since they can be regarded as the sum of r independent geometrically-distributed random variables. Hence, using (4.2) and letting $\Delta = p - p'$, we can write (exactly in the first 2 steps)

$$(4.4) \qquad S_1 = P\{Y_p - Y_{p'} \leq 0\}$$

$$= P\left\{\frac{Y_p - Y_{p'} - r(\frac{q}{p} - \frac{q'}{p'})}{\sqrt{r(\frac{q}{p^2} + \frac{q'}{(p')^2})}} \leq \frac{\sqrt{r}\,\Delta}{\sqrt{q(p')^2 + q'p^2}}\right\}$$

$$\sim \Phi\left(\Delta\sqrt{\frac{r}{D}}\right) + \mathcal{O}\left(\frac{1}{r}\right),$$

where $\Phi(x)$ is the standard normal distribution and $D = q(p')^2 + q'p^2$. The second sum in (2.10) is the same as in (4.4) except that the equality sign is dropped; hence we can disregard the second sum if we also drop the coefficient $\frac{1}{2}$ and the desired approximation is given by (4.4). Both Δ and D contain values of p and p' that are generally unknown to the experimenter. The most conservative choice of r arises from what is called the least favorable (LF) configuration, i.e., we wish to minimize (4.4) subject to the condition that $\Delta \geq \Delta^*$. First we set $\Delta = \Delta^*$ in (4.4) and then maximize

$$(4.5) \qquad D^* = (1 - p)(p - \Delta^*)^2 + (1 - p + \Delta^*)p^2$$

for p in the interval $(\Delta^*, 1)$. An elementary calculation shows that the value of p that maximizes (4.5) is

$$(4.6) \qquad p_0 = \frac{1}{6}\left(2 + 3\Delta^* + \sqrt{4 + 3(\Delta^*)^2}\right) = \frac{2}{3} + \frac{\Delta^*}{2} + \mathcal{O}\{(\Delta^*)^2\}.$$

Hence disregarding an error of order $(\Delta^*)^2$, the LF configuration is to take p and p' centered at $2/3$ with difference Δ^*. The maximum value of D^* is then $8/27 + \mathcal{O}\{(\Delta^*)^2\}$ and, disregarding an error of order $(\Delta^*)^2$, we find that for small Δ^* a lower bound to $P\{CS|R_I\}$ is given by

$$(4.7) \qquad \text{Min } P\{CS|R_I\} \sim \Phi\left(\Delta \sqrt{\frac{27}{8} r}\right).$$

We solve for r by putting $\Delta = \Delta^*$ in the right side of (4.6) and setting the result equal to P^*. If we let $\lambda = \lambda(P^*)$ denote the normal P^*-percentage point, i.e., the solution of $\Phi(\lambda) = P^*$, then we obtain

$$(4.8) \qquad r = \frac{8}{27}\left(\frac{\lambda}{\Delta^*}\right)^2 .$$

Table 1 gives some typical values of r calculated from (4.8).

In the same spirit as above we can find normal approximations to (2.18) and (2.19) and hence to (2.20). For (2.18), which we denote by T_1, we obtain a symmetric result in p and p'

$$(4.9) \qquad T_1 \sim \frac{rq}{p} - \frac{r}{p}\{1 - \Phi(y)\} + \frac{r}{p'}\{1 - \Phi(y)\}$$

$$= \frac{rq}{p}\Phi(y) + \frac{rq'}{p'}\{1 - \Phi(y)\},$$

where $y = \Delta\sqrt{r/D}$ is the same argument as in (4.4) and the error is again (1/r). Hence using (4.8)

$$(4.10)\qquad E\{N_B|R_I\} \sim \frac{r}{q'}\ \{\frac{q}{p}\ \Phi(y) + \frac{q'}{p'}[1 - \Phi(y)]\}$$

$$\sim \frac{8}{27q'}\left(\frac{\lambda}{\Delta^*}\right)^2 \{\frac{q}{p}\ \Phi\left(\frac{\lambda\Delta}{\Delta^*}\sqrt{\frac{27}{8D}}\right) + \frac{q'}{p'}\ [1 - \Phi\left(\frac{\lambda\Delta}{\Delta^*}\sqrt{\frac{27}{8D}}\right)]\}.$$

For the expected total number of trials $E\{N|R_I\}$ we multiply the result in (4.10) by $(1 + q'/q)$ as indicated by (2.20); for $\Delta^* \to 0$ the result is simply $r(q + q')/pq'$, where r is given by (4.8).

For the procedure R_I' we obtain from (3.7)

$$(4.11)\qquad E\{N_B|R_I'\} \sim \frac{r}{p}\ \Phi(y) + \frac{r}{p'}\ \{1 - \Phi(y)\},$$

where y is again as above. Since $q < q'$, we find on comparing (4.10) and (4.11) that for large r

$$(4.12)\qquad E\{N_B|R_I\} < E\{N_B|R_I'\}$$

i.e., for large values of r the procedure R_I with the PW sampling rule is always preferable to procedure R_I' which uses the VT sampling rule.

By (2.19) the left member of (4.12) is close to $(rq/pq') + 1/2q'$ for large r and by (3.7) the right member of (4.12) is close to r/p for large r. Hence we can approximate the value of r above which (4.12) holds by the solution of

$$(4.13)\qquad \frac{1}{q'}\ (\tfrac{1}{2} + \frac{rq}{p}) = \frac{r}{p},$$

i.e., by $p/2\Delta$.

In comparing the results for procedures R_I and R_I' with the procedures R_S and R_S' (see [2]) based on the absolute difference in the number of successes, we note that the latter procedures, for which $E\{N\}$ is proportional to $(\Delta\Delta^*)^{-1}$, are preferable to the former, where $E\{N\}$ is proportional to $(\Delta^*)^{-2}$, when Δ^* is sufficiently small

However the reverse holds for fixed Δ^* if Δ is small (or zero) and P^* is sufficiently close to one. Hence there is no result based on expected total number of trials (or on the loss defined in [2]) that is uniform in both Δ^* and P^*.

TABLE 1

Values of r Needed Under Procedures R_I and R_I' for Given Values of P^* and Δ^*

Δ^*	$P^* = .90$	$P^* = .95$	$P^* = .99$
.1	38	49	69
.2	10	13	18
.3	5	6	8
.4	3	4	6

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APPENDIX

As an alternative procedure R_I^* we could consider waiting for a fixed number r of failures(instead of r successes). Under the PW sampling rule the number of failures for different populations differ by at most one. Hence it will be convenient to wait for r failures from each of the populations. Since the results for procedure R_I^* are so similar to and comparable with those for R_I we chose to include them here as an appendix.

The first thing to be noted about R_I^* is that by considering as a new chance variable Y_p the total number of observations until r failures(for one population), we can treat the populations separately and we do not need the recursion formula approach. For any such Y_p, using (2.9),

$$(A\ 1) \qquad P\{Y_p \leq y\} = q^r \sum_{m=r}^{y} \binom{m-1}{r-1} p^{m-r}$$

$$= q^r \sum_{j=0}^{y-r} \binom{j+r-1}{j} p^j = I_q(r, y-r+1),$$

the mean $E\{Y_p\} = r/q$, and the variance $\sigma^2(Y_p) = rp/q^2$. Hence the exact probability of a correct selection (CS) is

$$(A\ 2) \qquad P\{CS|R_I^*\} = P\{Y_p > Y_{p'}\} + \tfrac{1}{2}P\{Y_p = Y_{p'}\}$$

$$= q^r \sum_{j=0}^{\infty} \binom{j+r-1}{j} p^j\{I_{q'}(r,j) + \tfrac{1}{2}[I_{q'}(r,j+1) - I_{q'}(r,j)]\}$$

$$= \tfrac{1}{2}E_r'\{I_{q'}(r, X) + I_{q'}(r, X + 1)\},$$

where $p > p'$ are the two single-trial success probabilities as before and E_r' differs from E_r, following (2.10), only in that p and q are interchanged.

The exact value for the total number of observations required by R_I^* is simply

(A 3) $$E\{N|R_I^*\} = E\{Y_p + Y_{p'}\} = r(\frac{1}{q} + \frac{1}{q'}).$$

To determine r we minimize the right side of (A 2) subject to the condition that $\Delta = p - p' = q' - q \geq \Delta^*$, set the result equal to P^*, and solve for r. It is clear from (A 2) that for a minimum we set q' equal to its lowest value $q + \Delta^*$. The minimization in q is treated asymptotically $(r \to \infty)$ in direct analogy with the method used in (4.4), obtaining

(A 4) $$P\{CS|R_I^*\} \sim E_r' \{I_{q'}(r, X)\} \sim \Phi(\Delta\sqrt{\frac{r}{D}})$$

where D and Φ are as above after (4.4). As in (4.5) we maximize D obtaining the same solution as in (4.6) for the minimizing value of p. Setting the right side of (A 4) equal to P^*, we obtain the same solution for r as in (4.8). Hence in comparing the asymptotic result in (4.10) (setting $y = \infty$ for Δ^* small) with (A 3) above, we find that for small Δ^* (which implies a large r) procedure R_I is preferable when

(A 5) $$\frac{q}{p} < 1 \quad \text{or} \quad p > \tfrac{1}{2}$$

and R_I^* is preferable when $p < \frac{1}{2}$.