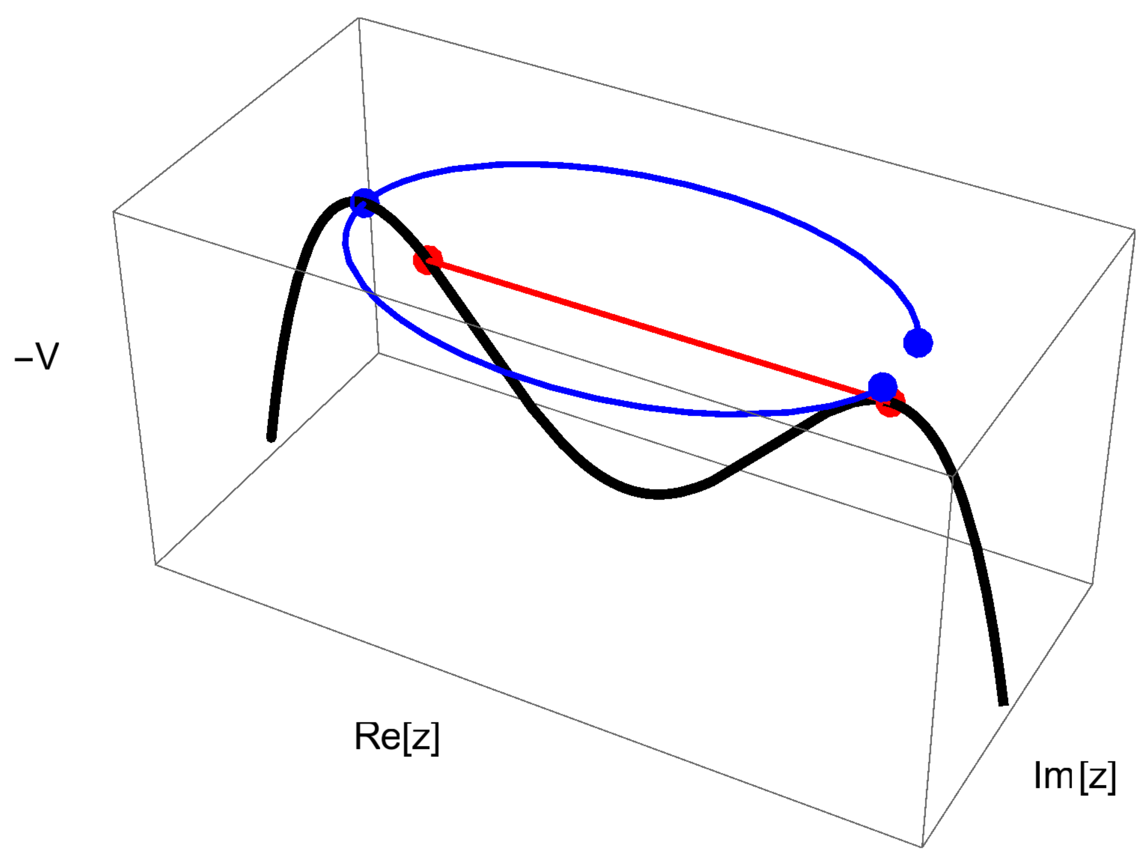
Toward Picard-Lefschetz Theory of Path Integrals and the Physics of Complex Saddles

Mithat Ünsal, North Carolina State University



with Behtash, Dunne, Sulejmanpasic, Schaefer

arXiv:1510.00978 & arxiv:1510.00978

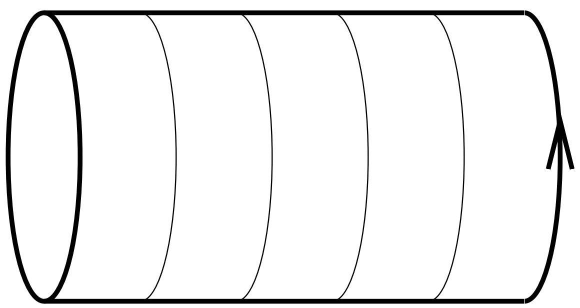
Outline

The talk is about elementary systems in QM, motivations is in QFT. The QM part of my talk should be accessible to everyone in the audience.

- Motivation from QFT, puzzles in QCD(adj) and SYM
- Necessity of complexification of path integral
- Holomorphic Newton's equations
- Semi-Classics vs. Susy algebra.
- Comments on algebraically solvable systems (QES)

Motivation: Exact results and non-trivial phases from "approximate" saddle points

Consider $SU(2)$ gauge theory on $R^3 \times S_1$



$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\,\mu\nu} - \frac{i}{g^2} \lambda^a \sigma \cdot D^{ab} \lambda^b$$

Small S_1: Calculable, and adiabatically connected to large S_1.

NP-physics can be studied using semi-classical methods,
Key elements: monopole-instantons, instantons, and bions.

Low energy fields: Holonomy b $\sim \Delta\theta$ and dual photon σ (and partners)

Increase number of flavors SYM → QCD(adj). Many remarkably similar NP-properties.

2007 to date, many people contributing

Poppitz, Shifman, Yaffe, Schaefer,
Argyres, Cherman, Anber,
Sulejmanpasic, Dunne, Dorigoni, Basar.

Zhitnitsky, Thomas
Nitta, Misumi, Sakai,
Kanazawa,
Shuryak, Zahed, ...
Ogilvie, Meisinger, Nishimura, Myers,
Voloshin,..
Bruckmann,

Latticy:
Bergner, Munster,
D'elia, Cossu,
Vairinhos

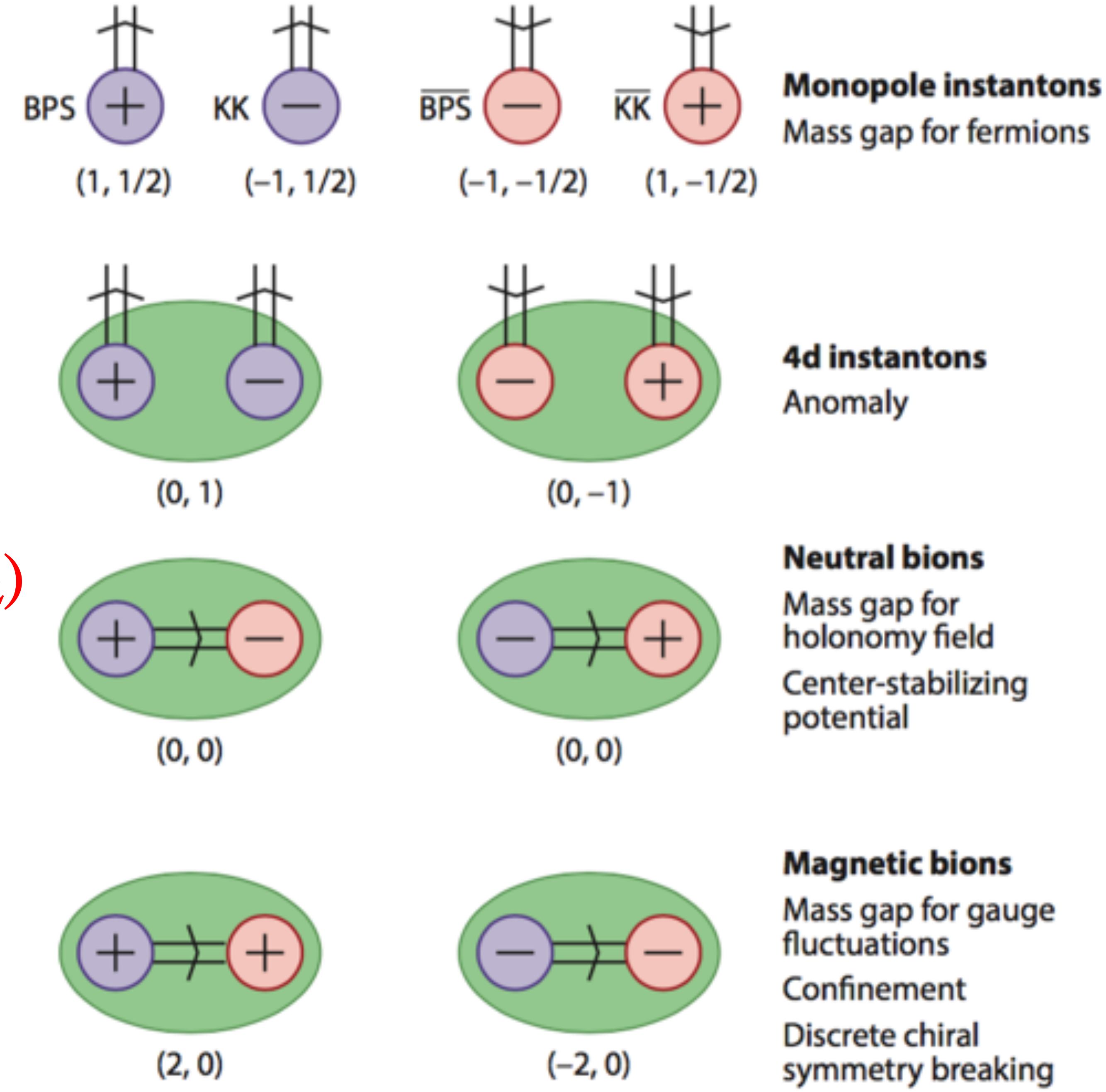
Earlier important related work:

Van Baal et. al.
Lee-Yi
Balitsky-Yung
Affleck-Harvey-Witten
Bogomolny-
Zinn-Justin,
Polyakov

Topologically non-trivial and "trivial" saddles

$$(Q_m, Q_{\rm top}) = \left(\int_{S^2} B \cdot d\Sigma, \int_{\mathbb{R}^3 \times S^1} F\tilde{F} \right)$$

Figure for SU(2)

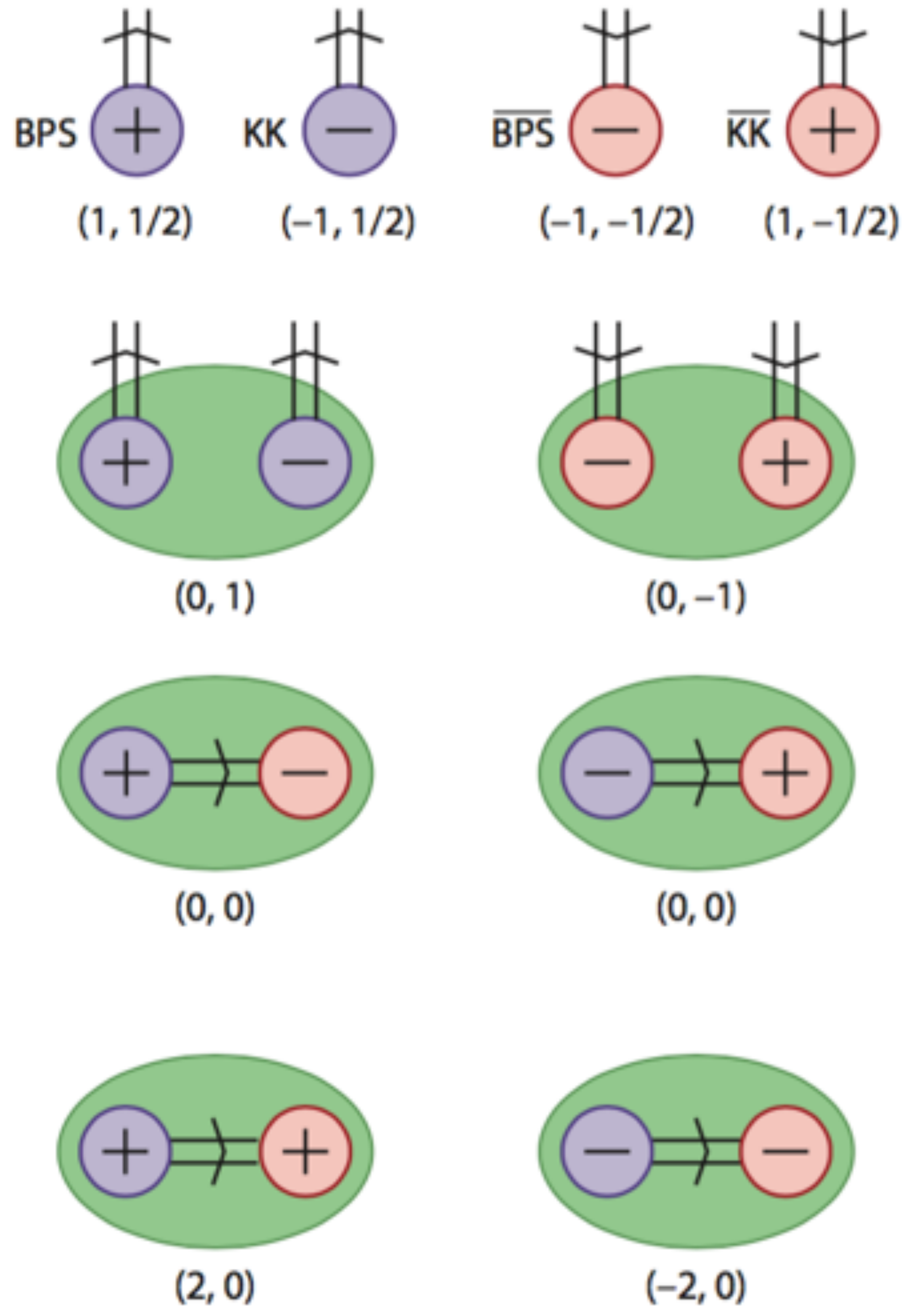


Lesson: Usual topology insufficient to classify saddles in the problem!

Topologically non-trivial and "trivial" saddles

$$(Q_m, Q_{\rm top}) = \left(\int_{S^2} B \cdot d\Sigma, \int_{\mathbb{R}^3 \times S^1} F\tilde{F} \right)$$

Operators for SU(N)



Monopole instantons
Mass gap for fermions

$$e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}} e^{-\alpha_i \cdot (b - i\sigma)} (\alpha_i \cdot \lambda)^2$$

4d instantons
Anomaly

$$e^{-\frac{8\pi^2}{g^2} + i\theta} (\lambda\lambda)^N$$

Neutral bions
Mass gap for holonomy field
Center-stabilizing potential

$$e^{-2\frac{8\pi^2}{g^2 N} + i\pi} e^{-2\alpha_i \cdot b}$$

Magnetic bions
Mass gap for gauge fluctuations
Confinement
Discrete chiral symmetry breaking

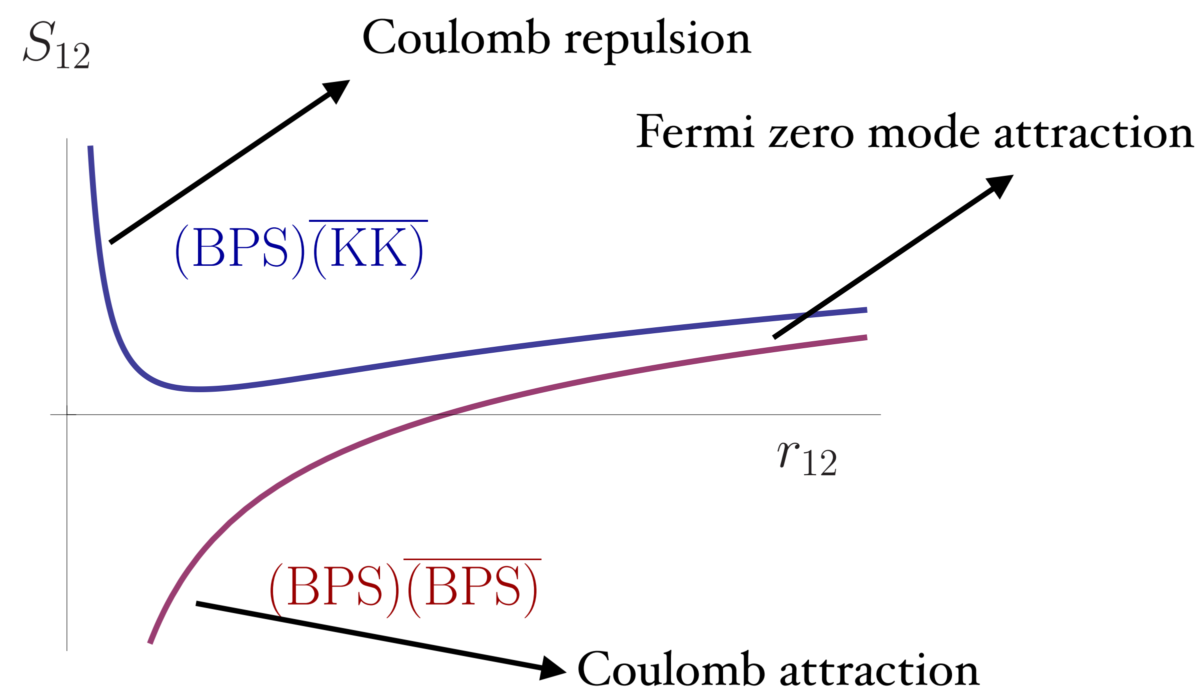
$$e^{-2\frac{8\pi^2}{g^2 N}} e^{-(\alpha_i + \alpha_{i+1}) \cdot b} e^{i(\alpha_i - \alpha_{i+1}) \cdot \sigma}$$

Lesson: Usual topology insufficient to classify saddles in the problem!

A puzzle

$$S_{12}(r) = \frac{4\pi L}{g^2 r}(q_m^1 q_m^2 - q_b^1 q_b^2) + 4\log(r)$$

$$V_{BPS,\overline{BPS}} \sim e^{-2b} e^{-2S_0} \int d^3r\, e^{-S_{12}(r)}$$



$$V(b,\sigma) \sim \frac{M_{PV}^6 L^3 e^{-2S_0}}{g^6}\left[-e^{i\pi}\cosh\left(\frac{8\pi}{g^2}(\Delta\theta - \pi)\right) - \cos(2\sigma)\right]$$

Center symmetric vacuum, via neutral bions→ Vanishing vev Polyakov loop.
Mass gap for dual photon, Debye mechanism via magnetic bions→ confinement

Looks like happy ending. But there are deep puzzles here.

How to make sense of BPS$\overline{\text{BPS}}$? In which field space does it live?

More Puzzles and Questions

Vacuum energy at confining minimum:

$$\mathcal{E} \sim \left[-e^{i\pi} e^{-2S_0} - e^{-2S_0} \right] = 0$$

How can we get a cancellation? Does tunneling not always lower the ground state energy?

Think of gluon condensate, in old QCD literature, it was believed to be positive-definite!) (Positive operator, positive measure). In SYM, it must vanish. But SYM is QCD-like in technical sense. What is the resolution?

How can we get the exact low energy potential? (If our bions are quasi-solutions?)

The magnetic bions term: Seems like real saddle

The neutral bions: Does not look like real saddle points. → QZM, complex?

With the conventional (textbook) semi-classical approach to path integral, (See Polyakov, Zinn-Justin, Coleman, Shifman, or Witten's QFT for mathematicians, or your favorite), there is no resolution of the above puzzles and the puzzles in generic QM path integrals that I will present. (We were "lucky" in certain sense so far.)

How to do semi-classical representation of path integral adequately? Either in QFT or QM?

Could there be a way to make semi-classics exact?

A constructive (and illuminative) definition of path integral?
(Hopefully, which does not involve doing the integration.)

Complexified Path Integral & holomorphic classical mechanics

Consider complexified version of path integral in QM

$$Z = \int_\Gamma Dz\, e^{-\frac{1}{\hbar}S[z(t)]}, \quad S[z(t)] = \int dt \left(\frac{1}{2}\dot{z}^2 + V(z)\right),$$

Integration cycle, Γ (middle dimensional space in complexified field space)
Critical points: Holomorphic Newton equation

$$\frac{\delta S}{\delta z} = 0 \Rightarrow \frac{d^2 z}{dt^2} = +\frac{\partial V}{\partial z}.$$

Real and imaginary parts: $V(z) = V_r(x, y) + iV_i(x, y)$

$$\frac{d^2 x}{dt^2} = +\frac{\partial V_r}{\partial x}, \quad \frac{d^2 y}{dt^2} = -\frac{\partial V_r}{\partial y},$$

The sign is crucial!
This is not 2d classical mechanics, it is **Holomorphic classical mechanics**

Complex gradient flow and holomorphic classical mechanics

Where does the holomorphic Newton's equation come from?

Integration cycle, Γ (middle dimensional space in complexified field space).

This is indeed the standard way to apply steepest descent method to ordinary integrals. (Fedoryuk, Arnold, Pham, Berry, Howls,)

$$\frac{\partial \bar{z}(t,u)}{\partial u} = -\frac{\delta S}{\delta z} = -\Big(\frac{d^2 z}{dt^2} - \frac{\partial V}{\partial z}\Big)$$

Fixed points of the complex gradient flow (Picard-Lefschetz equations) as u $\to\infty$ are the holomorphic Newton's equations.

In the context of the complexified Chern-Simons theory, the complex gradient flow equations are given by Witten (2010).

Supersymmetric quantum mechanics (old way), and non-susy generalization

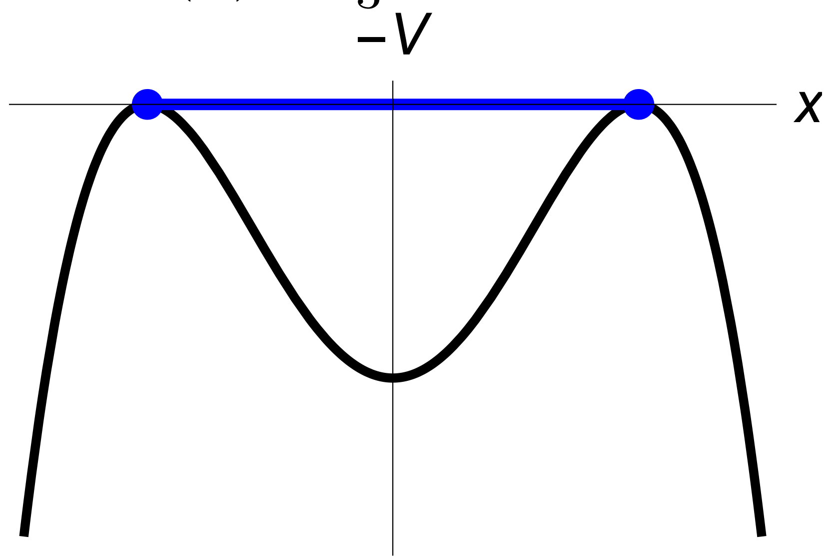
$$S = \frac{1}{g}\int dt\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}(\mathcal{W}')^2 + \left[\bar{\psi}\dot{\psi} + gp\mathcal{W}''\bar{\psi}\psi\right]\right),$$

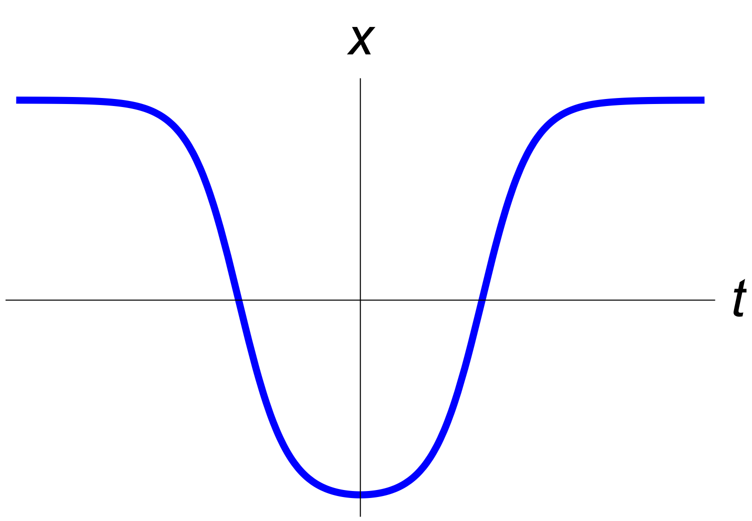
p=1 SUSY
p≠1 BY-deformation

$$S = \frac{1}{g}\int dt\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}(\mathcal{W}')^2 + \left[\bar{\psi}_i\dot{\psi}_i + g\mathcal{W}''\bar{\psi}_i\psi_i\right]\right), \qquad i = 1, \ldots, N_f$$

⇒Multiflavor generalization to mimic QCD(adj).
⇒Related to "quasi-exactly solvable" (QES) systems for integer N_f or p!

Consider $W(x) = \frac{1}{3}x^3 - x$. Bosonic potential is the double-well:





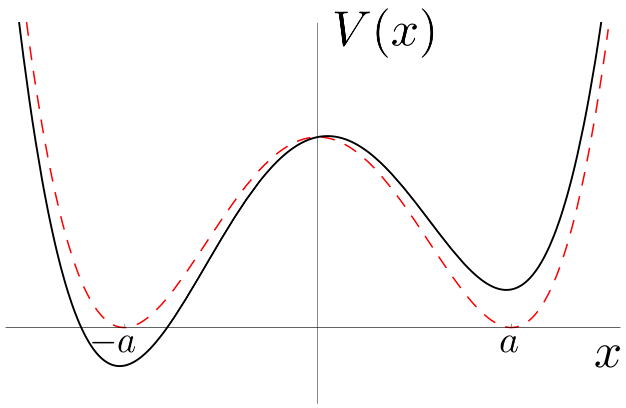
Naive IA picture.

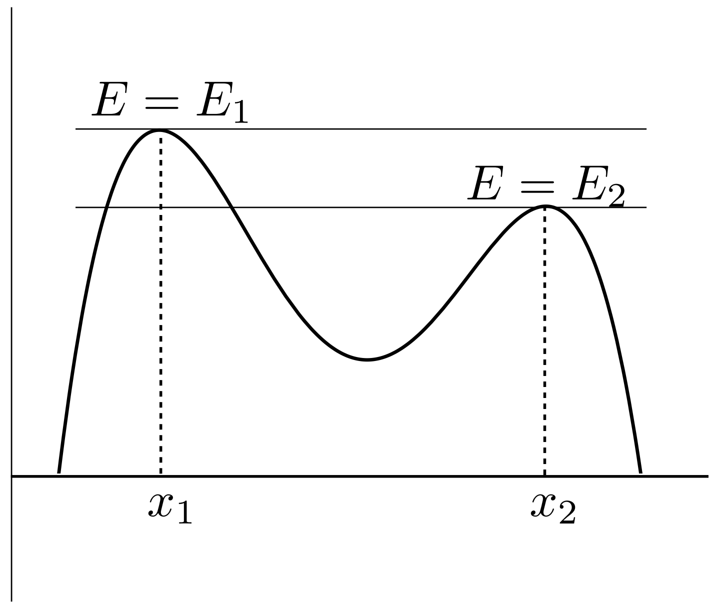
$$E_0 = \langle 0|H|0\rangle = \langle 0|\{Q,\bar{Q}\}|0\rangle \sim |\langle 0|Q|1\rangle|^2 \sim e^{-2S} > 0$$

Broken supersymmetry, positive energy. Again, positivity is puzzling!
All real saddles must contribute negative semi-definitely to ground state energy!

Supersymmetric QM and necessity of complex saddles!

Take Double-well susy QM. This system breaks susy spontaneously. (Witten, 81)
Quantize fermions and reduce the system to Bose-Fermi pair of Hamiltonians with tilted potential.





$$V_{\pm} = \frac{1}{2}(z^2 - 1)^2 \pm gz$$

Ground state energy is zero to all orders in P.T. But is known to be lifted non-perturbatively. What causes it?

In the inverted potential, there is an obvious real bounce solution, but this is not related to ground state properties.

At level E1, the classical particle will fly of to infinity, infinite action, irrelevant. So, what causes the non-zero ground state energy in bosonized description?

Graded (bosonic) formulation

Integrate out fermions exactly. Construct bosonic potentials

$$V_{\pm}(x) = \frac{1}{2}(\mathcal{W}'(x))^2 \pm \frac{pg}{2}\mathcal{W}''(x)$$

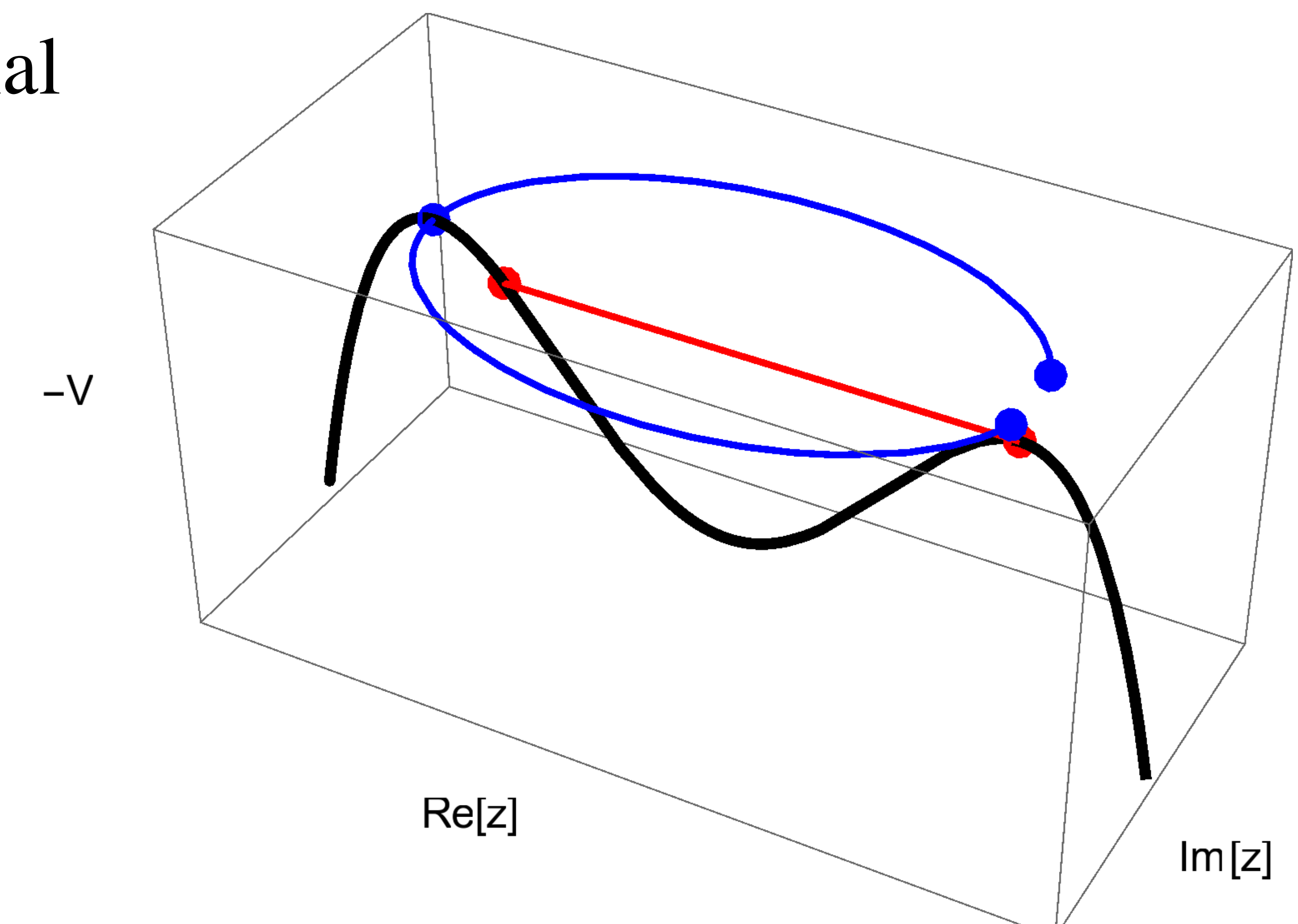
Holomorphic Newton equation with quantum modified potential

$$\frac{d^2 z}{dt^2} = \mathcal{W}'(z)\mathcal{W}''(z) + \frac{pg}{2}\mathcal{W}'''(z)$$

tilted double well potential

bounce solution

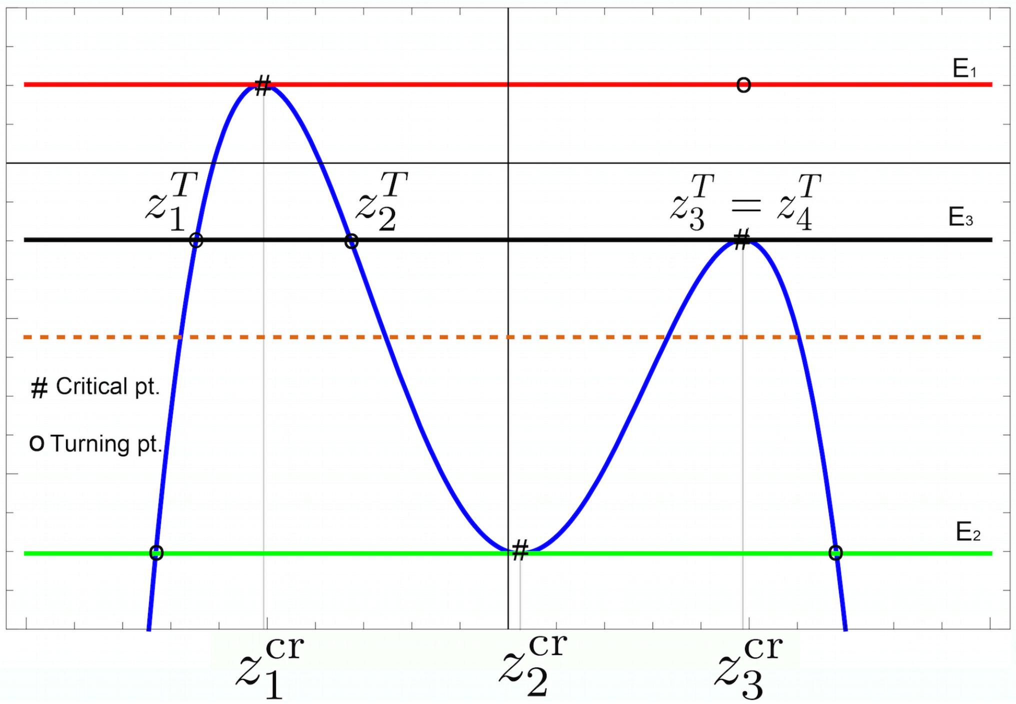
complex bions



Finding the complex bion

Use energy conservation
(with position and momenta complex)

$$\int dt = \int \frac{dz}{\sqrt{\mathcal{Q}^2}} \qquad \mathcal{Q}^2 = 2[V(z) + E]$$



This is a quartic polynomial

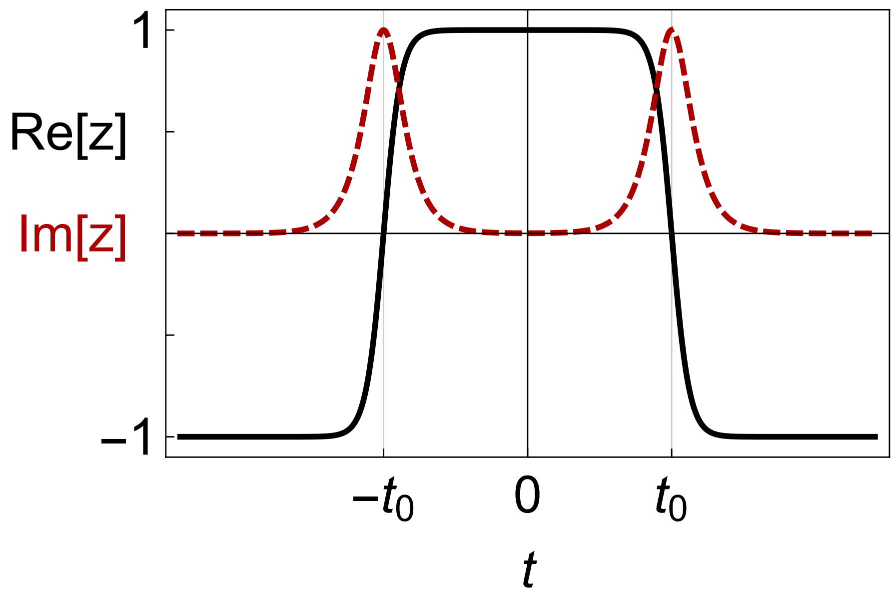
$$\mathcal{Q}^2 = \prod_k (z - z_k)$$

Solution given in terms of elliptic integrals

$$z(t) = z_T + \frac{\frac{1}{2}V'(z_T)}{\mathcal{P}(t; g_2, g_3) - \frac{1}{12}V''(z_T)}$$

Can be extend to complex turning points.
Or: Use analytic continuation in pg.

Complex bion solution



Re[z]: IA pair, size ln(16/pg)

Im[z]: Complex action

$$S_{\rm cb} \simeq \left(\frac{8}{3g} + p \ln\frac{16}{pg}\right) \pm i\, p\, \pi$$

Conjugate saddles do not lead to an ambiguity for p = 1.

However, $e^{i\pi}$ is a hidden topological angle (HTA) which is crucial for the ground state energy:

$$E_{gs} \sim -e^{\pm i\pi} e^{-2S_I} \sim +e^{-2S_I} > 0$$

Clash with SUSY prevented, thanks to HTA.

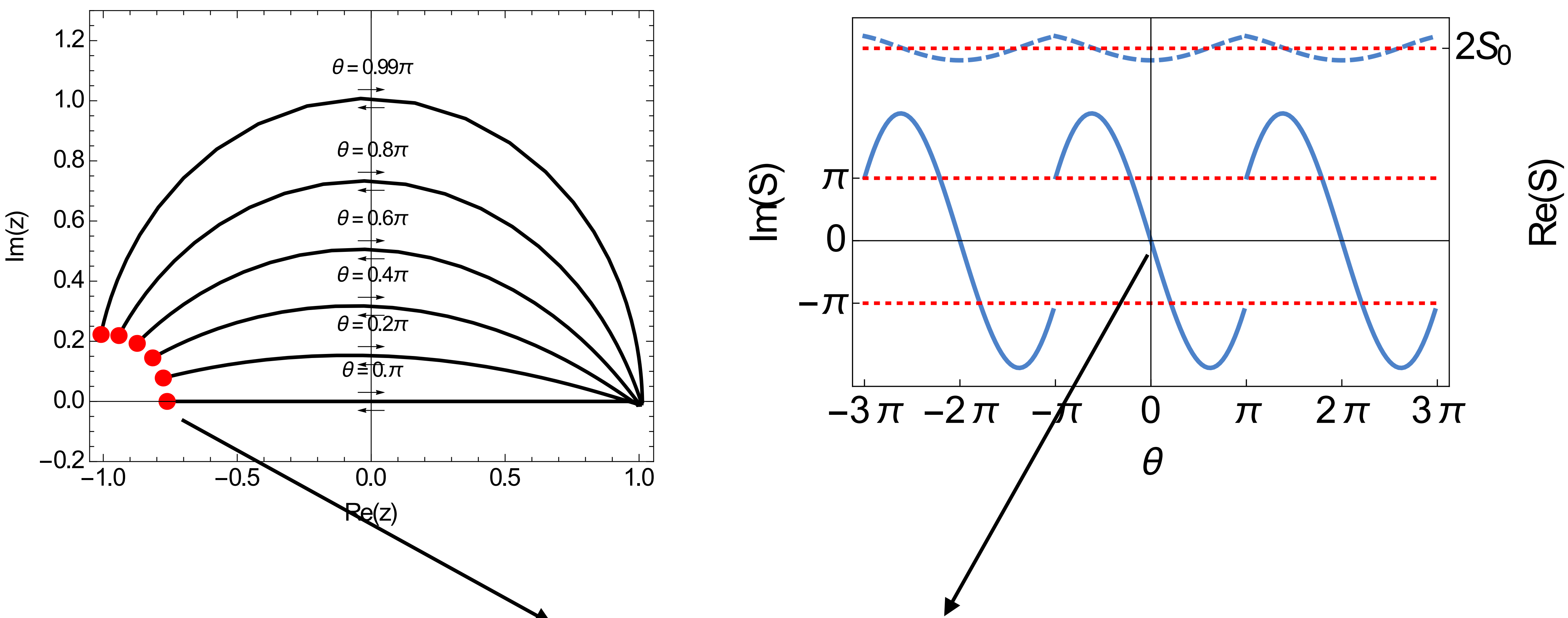
generic p : Resurgence

$p = 2, 3, \ldots$ Exact algebraic solvability (if time permits)

Complex bion by analytic continuation of real bounce

The complex bion solution can also be constructed by analytic continuation of the real bounce solution. Consider

$$V_\theta(x) = \frac{1}{2}(\mathcal{W}'(x))^2 + \frac{pe^{i\theta}g}{2}\mathcal{W}''(x)$$

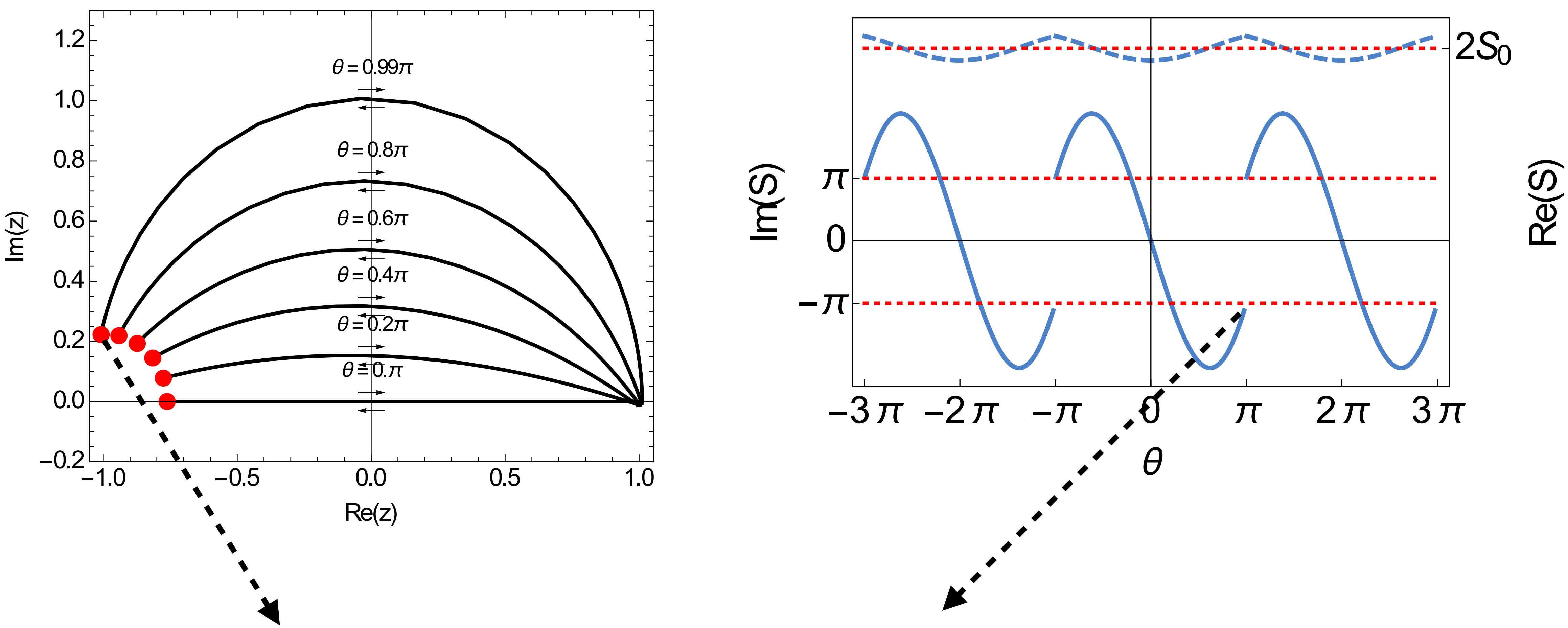


Real turning point

Complex bion by analytic continuation of real bounce

The complex bion solution can also be constructed by analytic continuation of the real bounce solution. Consider

$$V_\theta(x) = \frac{1}{2}(\mathcal{W}'(x))^2 + \frac{p e^{i\theta} g}{2}\mathcal{W}''(x)$$



One of the complex turning point

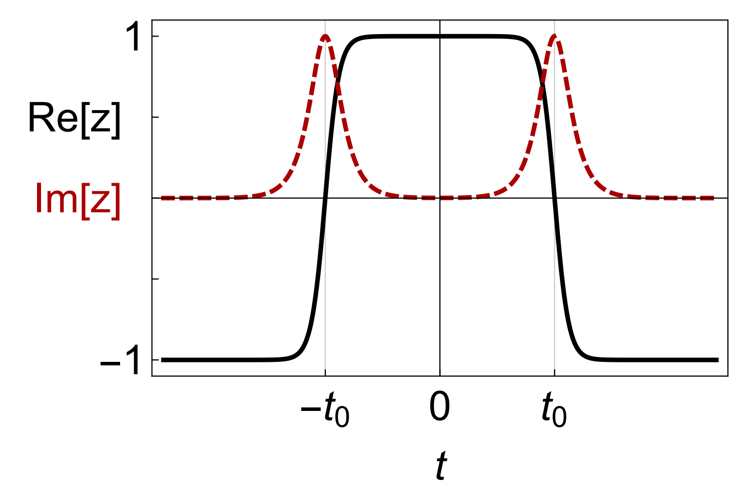
Relation to baby version of Lefshetz thimbles

Back to SUSY path integral: QZM integration over IA pairs with separation τ: (I switched p ➞ N_f)

$$I(N_f, g) = \int_{\Gamma_-^{\rm qzm}} d(m_b\tau) e^{-\left(-\frac{A}{g} e^{-m_b\tau} + N_f m_b \tau\right)}$$

Critical point of the integration is located at a complex separation

$$\tau^* = \frac{1}{2}\left[\ln\left(\frac{16}{gN_f}\right) \pm i\pi\right]$$



$$\text{Descent manifold } \Gamma_-^{\rm qzm} = \mathbb{R} \pm i\pi$$

$$I_-(N_f, g) = e^{\pm i\pi N_f} \left(\frac{g}{16}\right)^{N_f} \Gamma(N_f)$$

A CONTROVERSIAL ISSUE

Can complex, multi-valued, singular configurations contribute to a physical path integral (physical: with Hilbert space interpretation)?

Many times rejected in the past, (usually deemed non-sense)& smoothness of the instantons is always presented as a virtue!

But the truth is that no one (*either in favor or in opposition*) had the <u>proper formalism</u> to even address this question in path integrals!

A recent paper from 2010 gives a serious deliberation on the issue (analytic continuation of Liouville theory, Harlow, Maltz, and Witten, 2011). But remains undecided, quote: "We do not have a clear rationale for why this (inclusion of multi-valeud "solutions") is allowed."

They complexify path integral once the parameters in the Lagrangian are analytically continued.

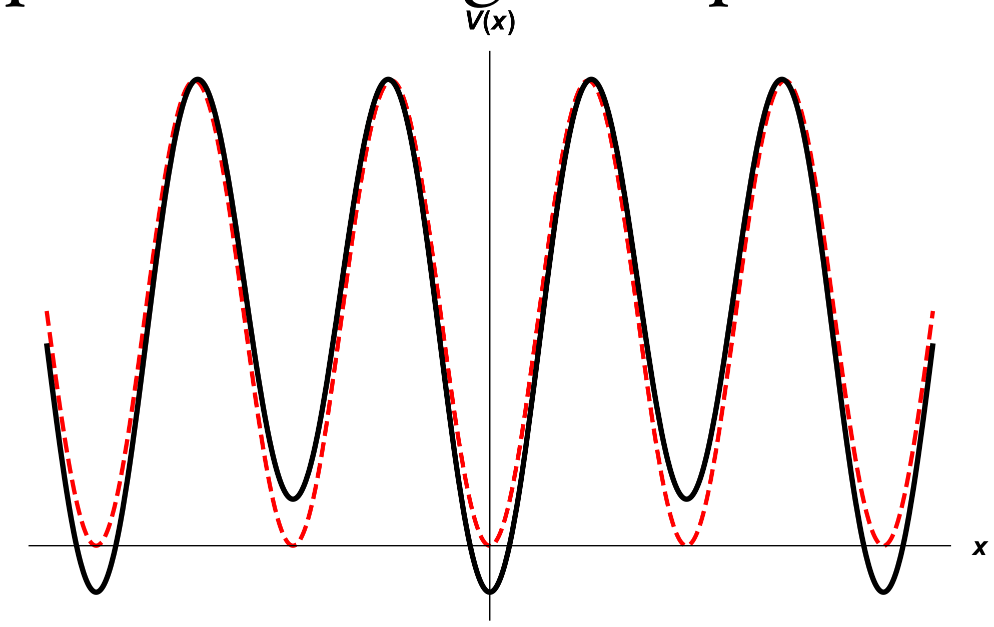
Our perspective is that in semi-classics, path integral must be complexified even when the parameters are real.

Periodic potential, some drama and surprises

Consider the superpotential W(x) = 4 cos(x/2). Then

$$V_{\pm}(x) = 2\sin^2(x/2) \pm \frac{pg}{2}\cos(x/2)$$

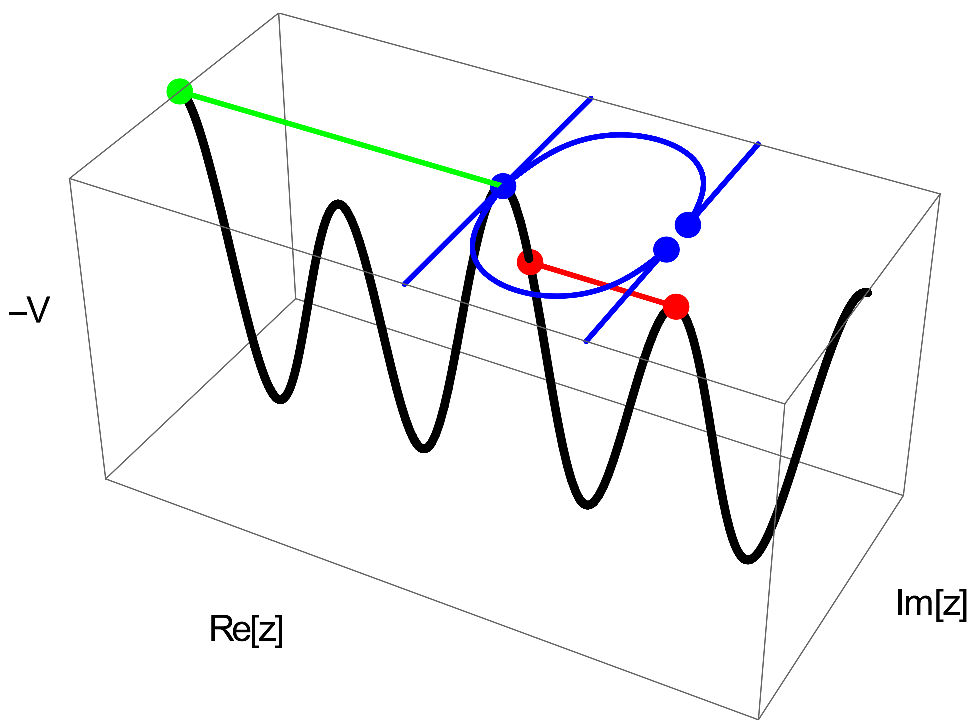
potential and graded potential



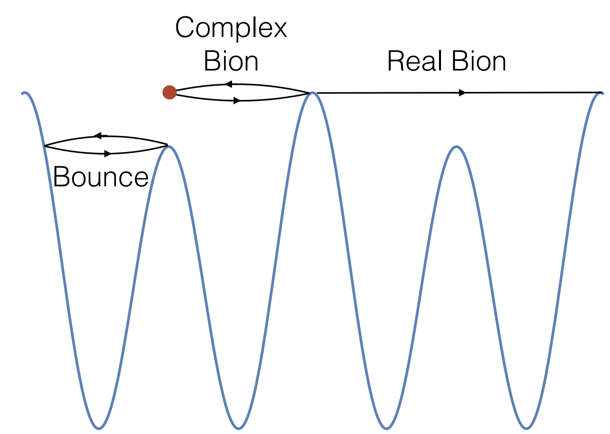
This system has Witten index zero but susy is known to be unbroken.
Two ground states, Bose-Fermi paired.

The same puzzle: If you just consider real saddles, ground state energy must be negative, in contradiction with supersymmetry algebra!

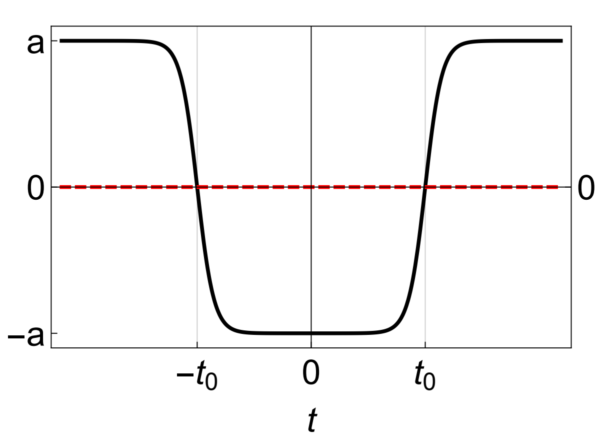
Periodic potential, some drama and surprises



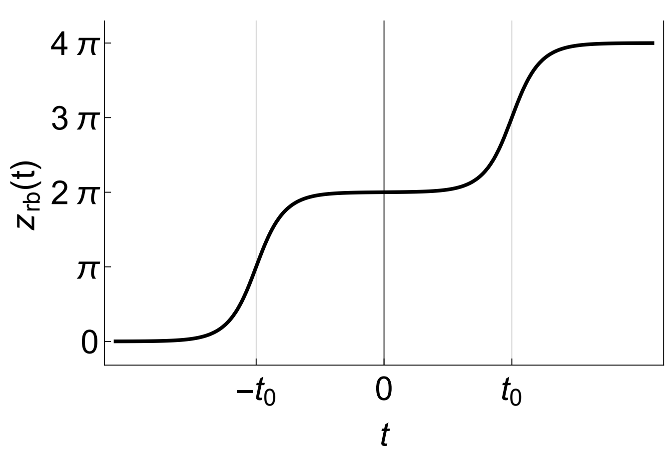
inverted graded potential
bounce solution
real bion
complex bions



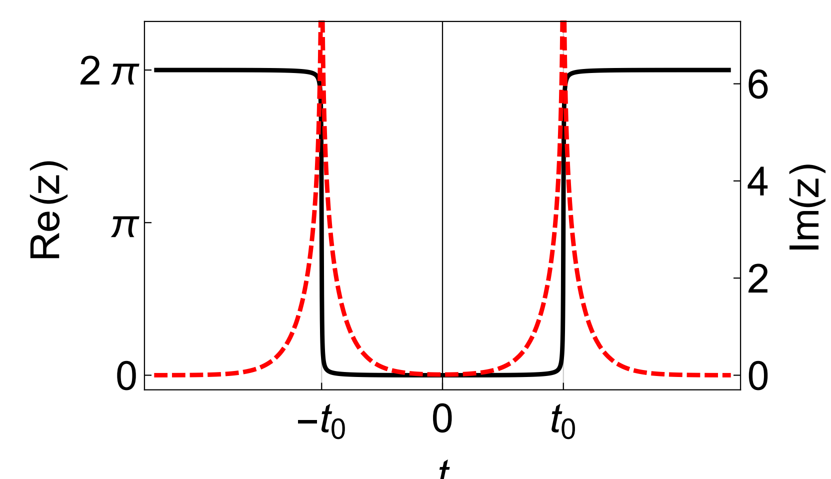
Exact bounce



Real bion



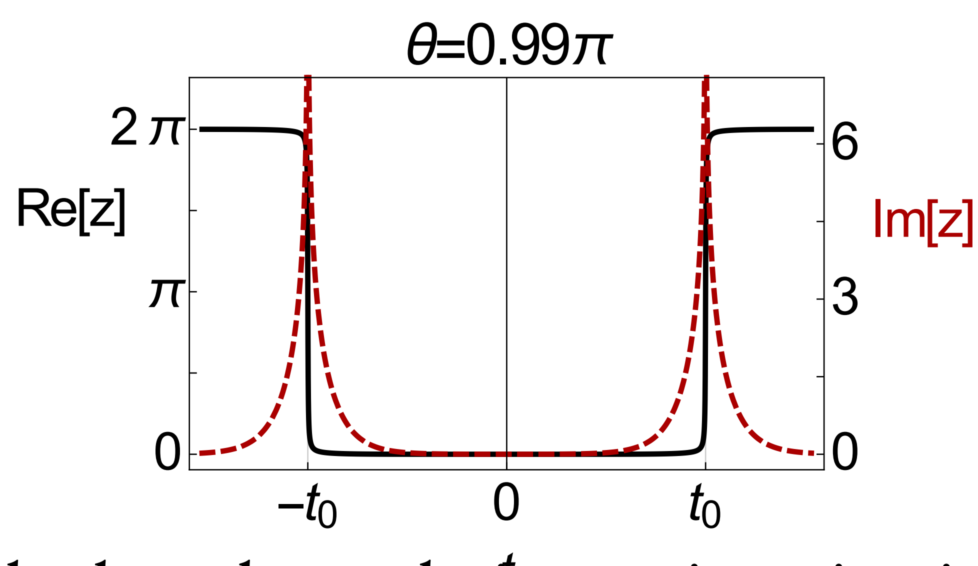
Exact complex bion



More on complex bion solution

The action is finite, and its real part is same as smooth real bion action.
Despite singular behavior of solution!

$$S_{\rm cb} \simeq \left(\frac{16}{g} + p \ln \frac{32}{pg} + \ldots \right) \pm i\, p\, \pi$$



Singularity smoothed out by analytic continuation in θ.
The solution is multi-valued, singular, complex.

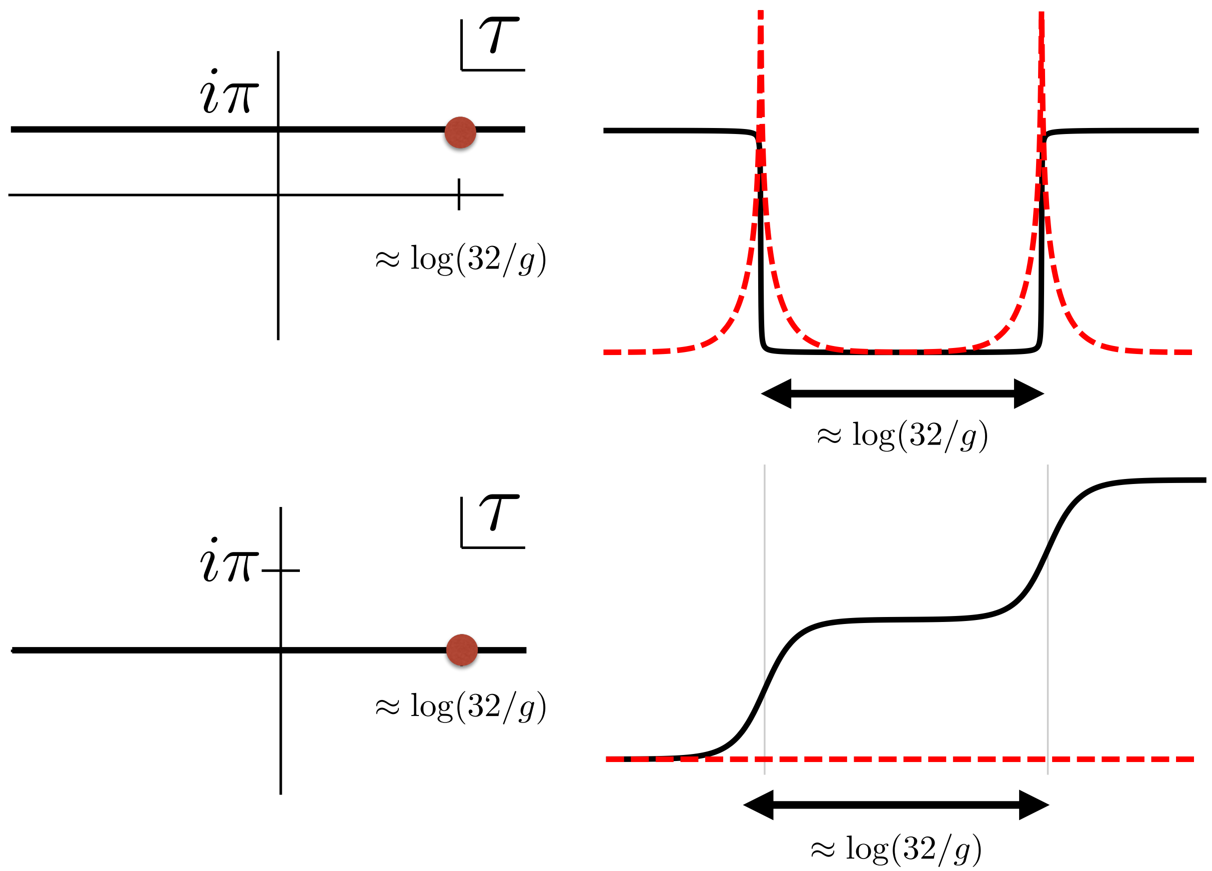
If you look any standard textbook or discussion, you will see that these are all big "sins".

From current point of view, this is the natural realization of semi-classics.

Relation to baby version of Lefshetz thimbles

Relation to instanton in the original description:

QZM integration over IA pairs with separation τ, shown are critical cycles.



Evidently, the complexification of the QZM direction is sufficient to obtain various important aspects of the exact solutions.
(Their form and Hidden topological angle)!

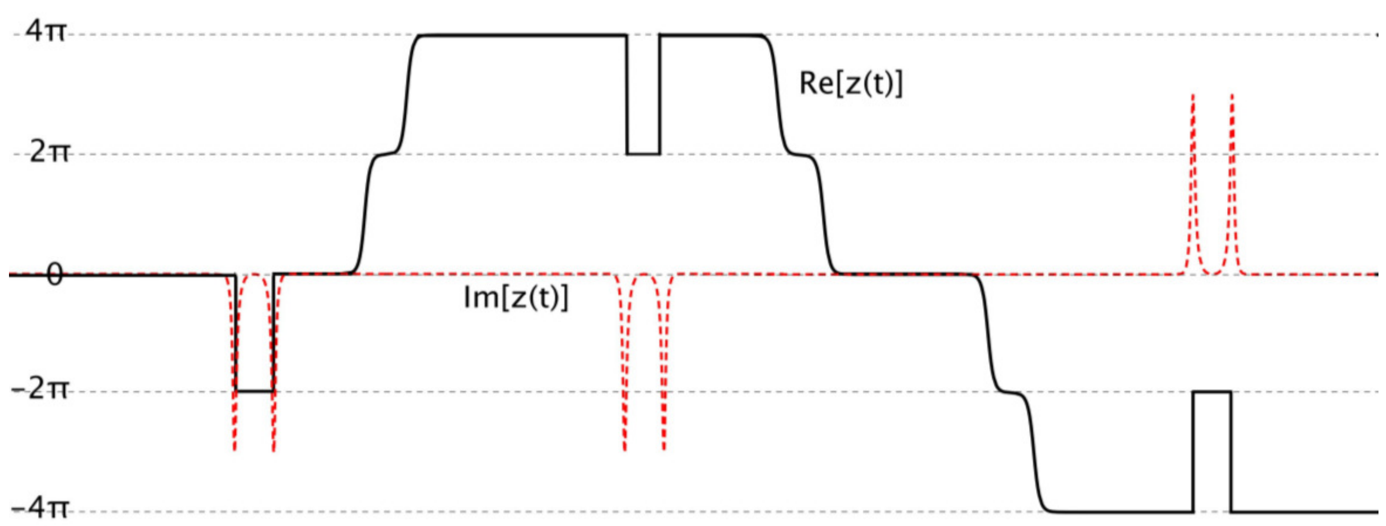
This makes me believe that magnetic bion and neutral bion in SYM/QCD are actually exact saddles.

SUSY-QM vacuum: Dilute bion gas

Ground state: Dilute gas of complex and real bions

$$E_{gs} \sim -e^{-S_{\rm cb}} - e^{-S_{\rm rb}} = -e^{\pm i\pi} e^{-2S_{\rm rb}} - e^{-2S_{\rm rb}} = 0$$

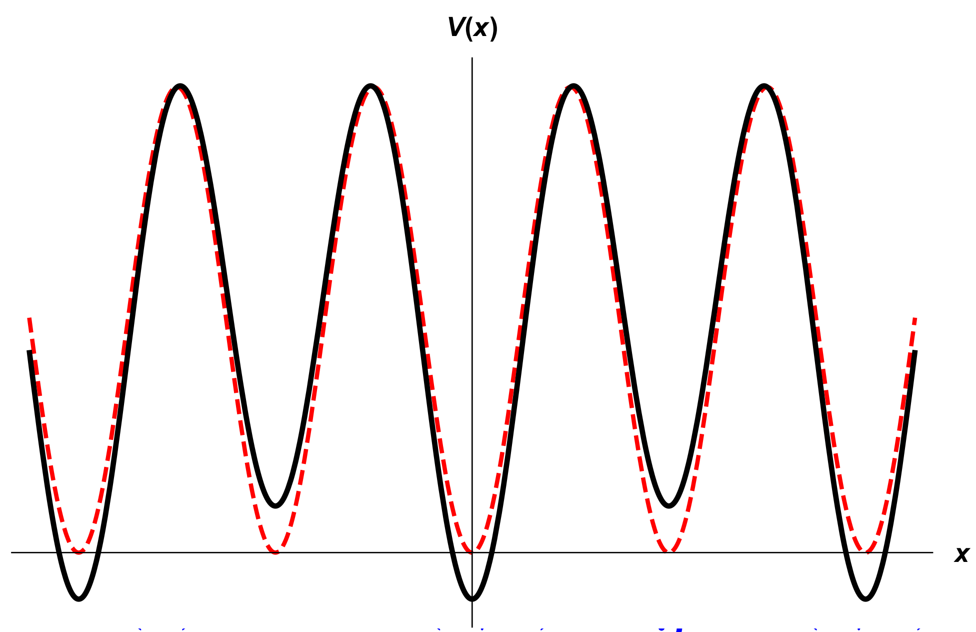
Supersymmetry consistent with semi-classics thanks to multivalued complex saddle.

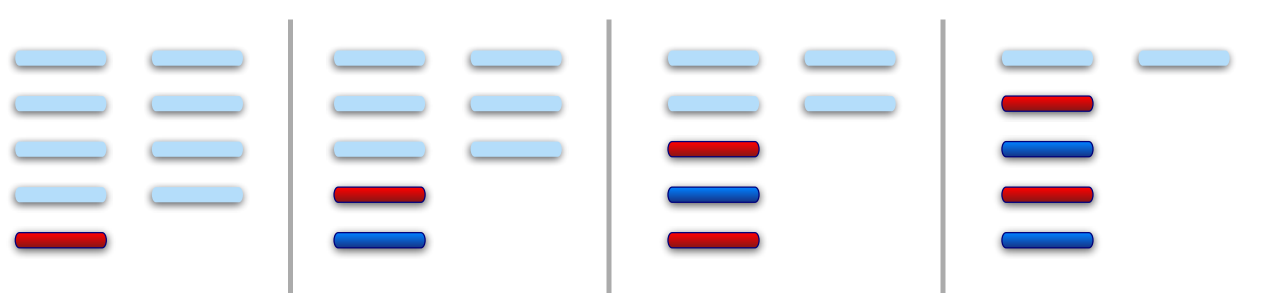


QES-systems

Can Kozcaz, Yuya Tanizaki,
Tin Sulejmanpasic, MU, 2016

Turbiner and Shuryak independently mentioned me the following puzzle (observation):
In QES systems, certain energy eigenlevels can algebraically be obtained and there are no exp[-A/g] effects.
But there are (in suitable cases) obvious real saddles (instantons). Why there are no NP contributions? Eg.





p=1 p=2 p=3 p=4

SUSY QES-systems

QES-systems

Can Kozcaz, Yuya Tanizaki,
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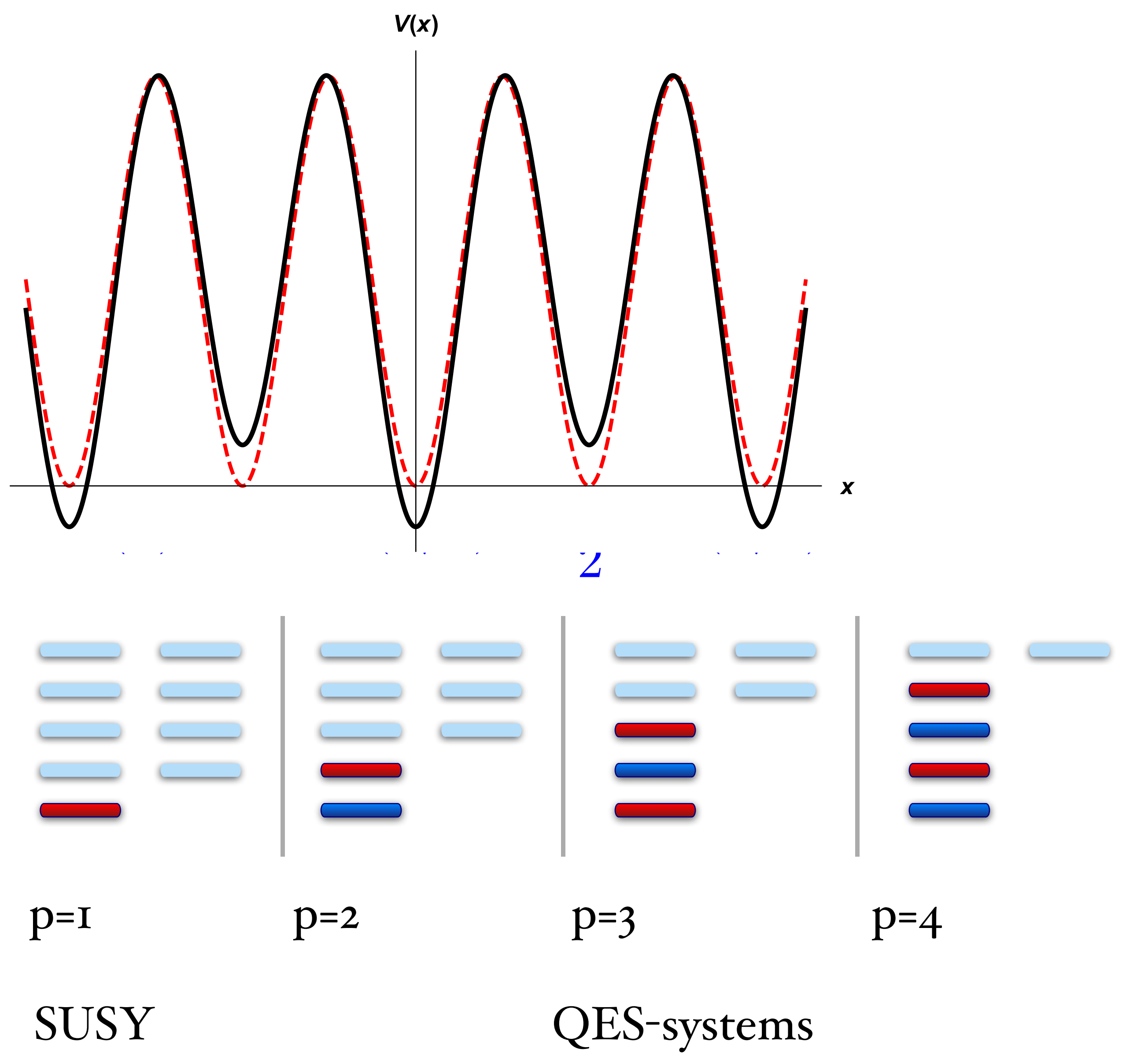

Well, you can guess the resolution now:

⇒Red states: NP contributions from real and complex saddles are present, and cancel each other out, exactly as in susy theory.

⇒Blue states: NP contributions from real and complex saddles are present, and do add up. as in dynamically broken susy theories.

⇒Remarkable amount of structure:
Convergent/divergent perturbation theory.
Exact results from Bender-Wu method.
Complex/real saddles
HTA (quantized in units of **π**) for integer p.
Unquantized HTA.

Relation to QFT. (SYM⇒QCD(adj))?

Stay tuned.

Conclusion/Prospects

Many field theories and quantum mechanical models require complex saddles and phases (hidden topological angles) from non-BPS saddle points.

These appear from solutions of complexified path integral. We found exact solutions of holomorphic Newton equations, corresponding to saddle points of complexified path integral. These solutions are potentially singular.

Imaginary part of S is either un-ambiguous (quantized), and related to HTA, or ambiguous and related to resurgence.

Remarkable amount of similarity between SUSY-QM → QES-QM
SYM → QCD(adj)

Accident?