Robust $\mathcal{D}$-stability of uncertain MIMO systems: LMI criteria *

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Abstract: The focal point of this paper is to provide some simple and efficient criteria to judge the $\mathcal{D}$-stability of two families of polynomials, i.e., an interval multilinear polynomial matrix family and a polytopic polynomial family. Taking advantage of the uncertain parameter information, we analyze these two classes of uncertain models and give some LMI conditions for the robust stability of the two families. Two examples illustrate the effectiveness of our results.
Key words: Interval multilinear polynomial matrix; robust $\mathcal{D}$-stability; Polytopic polynomial matrix; Linear Matrix Inequalities; Parametric uncertainty

1 Introduction

The study of robust stability problems under parameter uncertainties has been pioneered by the Russian scientist Kharitonov(1978). A rich array of useful results have been developed over the last twenty years[2]–[5],[7]–[14]. Generally speaking, by dealing directly and effectively with the real parameter uncertainties in control systems, we can identify apriori the critical subset of the uncertain parameter set over which stability will be violated. The seminal theorem of *Kharitonov*[2] points out: any real parameter interval polynomial family is Hurwitz if and only if a special subset (called Kharitonov set) is Hurwitz. To general uncertain systems, edge theorem gives a positive answer[3], which is an one-dimensional test.

Consider the unity feedback system with an interval plant and a fixed controller in forward path, its characteristic polynomial is a multilinear function of certain interval variables [9, 10, 13]. That is to say, when all but one variables are fixed, all coefficients of the polynomial are affine linear in the remaining variable. The collection of all such models is called multilinear uncertainties structure. For an MIMO system, if all relationships between any input and any output belong to corresponding polytopes (the simplest form is a line), all admissible models form a polytopic polynomial matrix family. In the past one or two decades, there is continually growing interest in the robust analysis of matrix[6, 9, 10, 11, 13]. Unlike polynomial case, the vertex result does not hold for interval matrix. In fact, there does not even exist a result similar to Edge Theorem for general matrix. Many problems still remain open until now. Work reported to date shows that a reduced dimensional test holds[13] for polytopic polynomial

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matrices. Limited by the complexity of such problems, several methods have been proposed, such as eigenvalues estimation, *Lyapunov* approach, algebraic approach and spectrum theory, etc.[6]–[9]. Usually, the algebraic approach based on Kharitonov theorem is hardly effective and convenient when used for the robust analysis of matrix directly. *Lyapunov* approach is an appealing method developed in the context of *Lyapunov* theory, which presents in the form of Linear Matrix Inequalities(LMI). However, until now there is few useful result on robust test for matrix families under parameter uncertainties.

The purpose of this paper is to address the $\mathcal{D}$-stability of interval multilinear polynomial matrix family and polytopic polynomial family. Even though they are nonlinear problems, we still can establish some efficient robust stability tests, which are usually negative for a general nonlinear case. Recent work addressed the $\mathcal{D}$-stability problem for polytope of matrices using *Lyapunov* approach[9] and spectrum theory[6]. In this paper, we adopt both of two methods to analyze two classes of uncertain MIMO models and give several LMI criteria on robust stability. Our technique not only captures the uncertain parameter information, but also is easy to use. In the end two examples demonstrate the effectiveness of our results.

2 Definitions and Notations

In this paper, we use the following standard notations and definitions.

Definition 1 Given an open convex region D in the complex plane, a scalar matrix is termed D-stable, if all its eigenvalues lie in D; and a polynomial matrix is termed D-stable, if all roots of its determinant lie in D; a matrix family is termed D-stable, if all its members are D-stable.

Definition 2 Let M be an arbitrary set, we define $convM$ as the convex hull of M, i.e., the smallest convex set which contains M.

Definition 3 For a matrix A, its right null-space $\mathcal{N}_A$ is defined as the space whose every element N_A satisfies $AN_A = 0$. For simplicity, we denote $\mathcal{N}_A$ a basis for the right null-space of A.

Definition 4 A polynomial matrix is a matrix with all of its entries being polynomials; an interval multilinear polynomial matrix is a polynomial matrix with all of its entries being multilinear dependent on some interval coefficients; a family of such matrices is called interval multilinear polynomial matrix family, such as the model in (5); a polytopic polynomial matrix is a matrix with all of its entries being polytopic polynomials; a family of such matrices is called polytopic polynomial matrix family, such as the model in (6).

Definition 5 Let $\mathcal{D} \subset \mathcal{C}$ be an open convex set of the form

$$\mathcal{D} = \left\{ s \in \mathcal{C} : \begin{bmatrix} 1 \\ s \end{bmatrix}^* B \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\},$$

where B is an 2×2 matrix and $B^* = B$. $\mathcal{D}$ is called an LMI region([8][16]).

Definition 6 For every $i \in \{1, \ldots, N\}$, $A_i(s)$ is an $n \times n$ polynomial matrix of the form

$$A_i(s) = A_0^i + A_1^i s + \ldots + A_l^i s^l, \tag{1}$$

and the $n \times nl$ scalar matrix $\mathcal{A}_i \triangleq (A_0^i, \ldots, A_l^i)$ is the coefficient matrix of $A_i(s)$.

Definition 7 For every $i \in \{1, \ldots, n\},\ j \in \{1, \ldots, n\}$, $\mathcal{P}_{ij}(s)$ is a polytope of polynomials , i.e.,

$$\mathcal{P}_{ij}(s) = \left\{ \sum_{k=1}^{m} \lambda_k p_{ij}^{(k)}(s) : \quad \lambda_k \geq 0,\ \sum_{k=1}^{m} \lambda_k = 1 \right\}, \tag{2}$$

where $p_{ij}^{(1)}(s), \ldots, p_{ij}^{(m)}(s)$ are fixed kth-order polynomials. Apparently, the vertex set of every $\mathcal{P}_{ij}(s)$ is

$$K_{ij}(s) = \{p_{ij}^{(k)}(s), \quad k = 1, \ldots, m\} \tag{3}$$

and the exposed edge set of every $\mathcal{P}_{ij}(s)$ is included in the following set

$$E_{ij}(s) = \left\{ \lambda p_{ij}^{(k)}(s) + (1-\lambda) p_{ij}^{(t)}(s),\ k, t = 1, \ldots, m,\ k \neq t,\ \lambda \in [0, 1] \right\}. \tag{4}$$

Definition 8 Denote $\mathcal{Q} = \left\{ (q_1, \ldots, q_m)^T :\ q_i \in [q_i^L, q_i^U] \right\}$. $a_1(q_1, \ldots, q_m), \ldots, a_N(q_1, \ldots, q_m)$ are multilinear functions of $q_1, \ldots, q_m$.

Definition 9

$$R = \begin{pmatrix} I & & & O \\ & \ddots & & \vdots \\ & & I & O \\ O & I & & \\ \vdots & & \ddots & \\ O & & & I \end{pmatrix}$$

is a $2nl \times (l+1)n$-dimensional scalar matrix and I, O are $n \times n$ unity, zero matrices respectively. As usual, $\otimes$ stands for the *Kronecker* product.

3 Preliminary results

In what follows, two kinds of uncertain models are considered:

$$\mathcal{MA}(s) = \left\{ \sum_{i=1}^{N} a_i(q_1, \ldots, q_m) A_i(s), \quad (q_1, \ldots, q_m)^T \in \mathcal{Q} \right\} \tag{5}$$

$$\mathcal{PA}(s) = \{(p_{ij}(s))_{n \times n} : p_{ij}(s) \in \mathcal{P}_{ij}(s),\ i, j = 1, \ldots, n\} \tag{6}$$

The vertex sets of $\mathcal{MA}(s)$ and $\mathcal{PA}(s)$, respectively, are

$$\begin{array}{l} K_{\mathcal{MA}}(s) = \left\{ \sum_{i=1}^{N} a_i(q_1, \ldots, q_m) A_i(s),\ q_j \in \{q_j^L, q_j^U\}, j = 1, \ldots, m. \right\} \\ K_{\mathcal{PA}}(s) = \{(p_{ij}(s))_{n \times n} : p_{ij}(s) \in K_{ij}(s),\ i, j = 1, \ldots, n\}. \end{array} \tag{7}$$

Let P_n^n be the collection of all permutations of $1, 2, \ldots, n$, and define

$$E_{\mathcal{PA}}(s) = \left\{ (p_{ij}(s))_{n \times n} : \begin{array}{l} p_{kl_k}(s) \in E_{kl_k}(s),\ (l_1, \ldots, l_n) \in P_n^n,\ k = 1, \ldots, n \\ p_{ki_k}(s) \in K_{ki_k}(s),\ i_k = 1, \ldots, l_k - 1, l_k + 1, \ldots, n \end{array} \right\} \tag{8}$$

It is easy to see that, $E_{\mathcal{PA}}(s)$ is a subset of $\mathcal{PA}(s)$ produced by taking only one entry from its exposed edge set in every row/column and all other entries from their vertex sets.

The lemma below is due to *Henrion*, et al.

Lemma 1[9] A polynomial matrix $A_i(s)$ is stable if there exists a matrix P_i solving the LMI feasibility problem

$$\mathcal{N}^*_{\mathcal{A}_i} R^*(B \otimes P_i) R \mathcal{N}_{\mathcal{A}_i} < 0, \quad P_i = P_i^* > 0.$$

Where $\mathcal{N}_{\mathcal{A}_i}$ is the right null-space of $\mathcal{A}_i$, and $*$ denotes the conjugate transpose operator.

Lemma 2[3] (Edge Theorem) Suppose $\mathcal{D} \subset \mathcal{C}$ is a simply-connected region, for any polynomial polytope Ω without degree dropping, the root set of Ω is contained in $\mathcal{D}$ if and only if the root set of all exposed edges of Ω is contained in $\mathcal{D}$.

Another lemma is on the $\mathcal{D}$-stability of the family $\mathcal{PA}(s)$,

Lemma 3[13] $\mathcal{PA}(s)$ is $\mathcal{D}$-stable if and only if $E_{\mathcal{PA}}(s)$ is $\mathcal{D}$-stable.

Proof: For all $A^0(s) \in \mathcal{PA}(s)$, let $A^0(s) = (p_{ij}(s))_{n\times n}$, where $p_{ij}(s) \in \mathcal{P}(s)$. For simplicity, we write p_{ij} for every $p_{ij}(s) \in \mathcal{P}_{ij}(s)$. In what follows, we will construct several sets in terms of $A^0(s)$. Let $\mathcal{A}_k = \{A_k(i_1, \ldots, i_n; s), \quad i_1, \ldots, i_k \in \{1, \ldots, n\}\}$ $(k = 0, \ldots, n)$, where

$$A_k(i_1,\ldots,i_n;s) = \begin{pmatrix} q_{11} & \cdots & q_{1k} & \\ \cdots & \cdots & \cdots & \\ q_{i_1 1} & \cdots & q_{i_1 k} & \\ \cdots & \cdots & \cdots & (p^0_{vt})_{n\times(n-k)} \\ q_{i_k 1} & \cdots & q_{i_k k} & \\ \cdots & \cdots & \cdots & \\ q_{n1} & \cdots & q_{nk} & \end{pmatrix} \qquad \begin{array}{c} q_{lt} \in \begin{cases} E_{i_t t}(s), & l = i_t \\ K_{i_t t}(s), & l \neq i_t \end{cases} \\ p^0_{lv} \text{ are entries of } A \\ t = 1,\ldots,k \quad l = 1,\ldots,n \\ v = k+1,\ldots,n \end{array}$$

It is easy to see that $A^0(s) = \mathcal{A}_0$ and $\mathcal{A}_k \subset \mathcal{A}_{k+1}$. In the sequel, we will prove our statement in two steps:

1) Firstly, we will show that $\mathcal{A}_n$ is $\mathcal{D}$-stable if $E_{\mathcal{PA}}(s)$ is $\mathcal{D}$-stable. By definition, we have

$$\mathcal{A}_n = \left\{ \begin{array}{l} A_n(i_1,\ldots,i_n;s) \\ = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \cdots & \cdots & \cdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \end{array} \begin{array}{l} q_{lt} \in \begin{cases} E_{i_t t}(s), & l = i_t \\ K_{i_t t}(s), & l \neq i_t \end{cases} \\ t = 1,\ldots,n \\ l = 1,\ldots,n : \end{array} \quad i_1,\ldots,i_n \in \{1,\ldots,n\} \right\}.$$

For all $A_n(i_1,\ldots,i_n;s) \in \mathcal{A}_n$, if $(i_1,\ldots,i_n) \in P^n_n$, then $A_n(i_1,\ldots,i_n;s) \in E_{\mathcal{PA}}(s)$. Otherwise, there must exist some pair i_s, i_t satisfying $i_s = i_t$. Without loss of generality, suppose $i_1 = i_2 = 1$, namely

$$A_n(i_1,\ldots,i_n;s) = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} \qquad q_{11} \in E_{11}(s) \quad q_{12} \in E_{12}(s)$$

By using Laplace formula on the first row of $A_n(i_1,\ldots,i_n;s)$, we have

$$det(A_n(i_1,\ldots,i_n;s)) = q_{11}M_{11} + q_{12}M_{12} + \sum_{i=3}^{n} q_{1i}M_{1i}$$

where M_{1i} is the algebraic complement of q_{1i}. By Lemma 2,

$$A_n(i_1,\ldots,i_n;s) \text{ is } \mathcal{D}\text{-stable} \quad \Leftrightarrow \quad \begin{array}{l} q_{11}M_{11} + q^0_{12}M_{12} + \sum_{i=3}^{n} q_{1i}M_{1i} \text{ and} \\ q^0_{11}M_{11} + q_{12}M_{12} + \sum_{i=3}^{n} q_{1i}M_{1i} \text{ are } \mathcal{D}\text{-stable.} \end{array}$$

The corresponding matrices are

$$\begin{pmatrix} q_{11}^0 & q_{12} & \cdots & q_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix} \text{ and } \begin{pmatrix} q_{11} & q_{12}^0 & \cdots & q_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix},$$

where $q_{11}^0 \in K_{11},\ q_{12}^0 \in K_{12}$. For these two uncertain matrices, if they do not belong to $E_{PA}(s)$, then there must exist at least two equal indexes. Repeat the same process to them, in the end, we have

$$E_{\mathcal{PA}}(s) \text{ is } \mathcal{D}\text{-stable} \Rightarrow \mathcal{A}_n \text{ is } \mathcal{D}\text{-stable.}$$

2) Secondly, for all $A_n(i_1,\ldots,i_n;s) \in \mathcal{A}_n$, by using Laplace formula on the n-th column of $A_n(i_1,\ldots,i_n;s)$ and by Lemma 2, we have

$$\mathcal{A}_n \text{ is } \mathcal{D}\text{-stable } \Rightarrow \mathcal{A}_{n-1} \text{ is } \mathcal{D}\text{-stable.}$$

Continuing this process, we have

$$\mathcal{A}_k \text{ is } \mathcal{D}\text{-stable } \Rightarrow \mathcal{A}_{k-1} \text{ is } \mathcal{D}\text{-stable.}$$

Since $A^0(s) = \mathcal{A}_0$, 1) and 2) imply that $A^0(s)$ is $\mathcal{D}$-stable. That is to say, if $E_{\mathcal{PA}}(s)$ is $\mathcal{D}$-stable, then, for all $A^0(s) \in \mathcal{PA}(s)$, $A^0(s)$ is $\mathcal{D}$-stable. By definition, this means that $\mathcal{PA}(s)$ is $\mathcal{D}$-stable. This completes the proof of Sufficiency.

Necessity is obvious because $E_{\mathcal{PA}}(s)$ is a subset of $\mathcal{PA}(s)$. ◇

In this paper, we assume that both $\mathcal{PA}(s)$ and $\mathcal{MA}(s)$ have fixed degrees.

4 Main results

Although overbounding is a bit conservative, it still offers a powerful tool to solve the stability problem for control systems with multilinear uncertainties.

Theorem 1 $\mathcal{MA}(s) \subset conv\{K_{\mathcal{MA}}(s)\}$.

Proof: For any $A_0(s) \in \mathcal{MA}(s)$, by induction, we will show $A_0(s) \in conv\{K_{\mathcal{MA}}(s)\}$. Denote $q = (q_1,\ldots,q_t,q_{t+1},\ldots,q_m)^T \in \mathcal{Q}$, where $q_1,\ldots,q_t$ are interval parameters and $q_{t+1},\ldots,q_m$ are fixed.

If $t = 1$, i.e., $q_1 \in [q_1^L, q_1^U]$ and $q_2,\ldots,q_m$ are are fixed. Then, in this case,

$$\begin{aligned} \mathcal{MA}(s) &= \left\{ \textstyle\sum_{i=1}^N a_i(q_1,\ldots,q_m)A_i(s), \quad q_1 \in [q_1^L, q_1^U] \right\} \\ K_{\mathcal{MA}}(s) &= \left\{ \textstyle\sum_{i=1}^N a_i(q_1,\ldots,q_m)A_i(s), \ q_1 \in \{q_1^L, q_1^U\} \right\}. \end{aligned}$$

Since $a_1(q_1,\ldots,q_m),\ldots,a_N(q_1,\ldots,q_m)$ are linear in q_1, $\sum_{i=1}^N a_i(q_1,\ldots,q_m)A_i(s)$ is also linear in q_1. Clearly, $A_0(s) \in conv\{K_{\mathcal{MA}}(s)\}$.

Assume that the claim holds for $t = k$. When $t = k+1$, we have, in this case,

$$\begin{aligned} \mathcal{MA}(s) &= \left\{ \textstyle\sum_{i=1}^N a_i(q_1,\ldots,q_m)A_i(s), \begin{array}{l} q_i \in [q_i^L, q_i^U], i = 1,\ldots,k+1 \\ q_{k+2},\ldots,q_m \text{ are fixed.} \end{array} \right\} \\ K_{\mathcal{MA}}(s) &= \left\{ \textstyle\sum_{i=1}^N a_i(q_1,\ldots,q_m)A_i(s), \begin{array}{l} q_i \in \{q_i^L, q_i^U\}, i = 1,\ldots,k+1 \\ q_{k+2},\ldots,q_m \text{ are fixed.} \end{array} \right\} \end{aligned}$$

For all $A_0(s) \in \mathcal{MA}(s)$, there exists an m-dimensional vector $q^0 := (q_1^0, \ldots, q_m^0)^T \in \mathcal{Q}$, satisfying

$$A_0(s) = \sum_{i=1}^{N} a_i(q_1^0, \ldots, q_m^0) A_i(s).$$

Since $a_1(q_1, \ldots, q_m), \ldots, a_m(q_1, \ldots, q_m)$ are linear in q_{k+1}, we have

$$\begin{array}{rl} a_i(q_1^0, \ldots, q_k^0, q_{k+1}^0, \ldots, q_m^0) = & \lambda_0 a_i(q_1^0, \ldots, q_k^0, q_{k+1}^L, q_{k+2}^0, \ldots, q_m^0) \\ & +(1-\lambda_0) a_i(q_1^0, \ldots, q_k^0, q_{k+1}^U, q_{k+2}^0, \ldots, q_m^0) \end{array}$$

for some $\lambda_0 \in [0,1]$. Therefore,

$$\begin{array}{rl} A_0(s) = & \sum_{i=1}^{N} \left(\lambda_0 a_i(q_1^0, \ldots, q_k^0, q_{k+1}^L, q_{k+2}^0, \ldots, q_m^0) + (1-\lambda_0) a_i(q_1^0, \ldots, q_k^0, q_{k+1}^U, q_{k+2}^0, \ldots, q_m^0)\right) A_i(s) \\ = & \lambda_0 \sum_{i=1}^{N} a_i(q_1^0, \ldots, q_k^0, q_{k+1}^L, q_{k+2}^0, \ldots, q_m^0) A_i(s) \\ & +(1-\lambda_0) \sum_{i=1}^{N} a_i(q_1^0, \ldots, q_k^0, q_{k+1}^U, q_{k+2}^0, \ldots, q_m^0) A_i(s) \end{array}$$

Since both $\sum_{i=1}^{N} a_i(q_1^0, \ldots, q_k^0, q_{k+1}^L, q_{k+2}^0, \ldots, q_m^0) A_i(s)$ and $\sum_{i=1}^{N} a_i(q_1^0, \ldots, q_k^0, q_{k+1}^U, q_{k+2}^0, \ldots, q_m^0) A_i(s)$ belong to the set

$$conv \left\{ \sum_{i=1}^{N} a_i(q_1, \ldots, q_m) A_i(s), \begin{array}{l} q_i \in \{q_i^L, q_i^U\}, i = 1, \ldots, k \\ q_{k+1}, \ldots, q_m \text{ are fixed.} \end{array} \right\},$$

we conclude

$$A_0(s) \in conv \left\{ \sum_{i=1}^{N} a_i(q_1, \ldots, q_m) A_i(s), \begin{array}{l} q_i \in \{q_i^L, q_i^U\}, i = 1, \ldots, k+1 \\ q_{k+2}, \ldots, q_m \text{ are fixed.} \end{array} \right\}.$$

That is to say, the claim holds also for $t = k+1$. Therefore, our conclusion is verified inductively.

◇

For all $B_i(s) \in K_{\mathcal{MA}}(s)$, rearrange it as $B_i(s) = B_0^i + B_1^i s + \ldots + B_l^i s^l$. Take $\mathcal{B}_i = (B_0^i, \ldots, B_l^i)$ the coefficient matrix of $B_i(s)$. It is easy to see that there exist 2^m distinct $B_i(s)$. Hence,

$$conv\{K_{\mathcal{A}}(s)\} = conv\{B_1(s), \ldots, B_{2^m}(s)\}.$$

With Lemma 1 and Theorem 1, we get an LMI condition for robust stability of interval multi-linear polynomial matrix family $\mathcal{MA}(s)$:

Theorem 2 $\mathcal{MA}(s)$ is robust $\mathcal{D}$-stable if there exist some matrices $P_i = P_i^* > 0, Q$ solving the LMI feasibility problem

$$\begin{bmatrix} R \\ \mathcal{B}_i \end{bmatrix}^* \begin{bmatrix} B \otimes P_i & Q \\ Q^* & 0 \end{bmatrix} \begin{bmatrix} R \\ \mathcal{B}_i \end{bmatrix} < 0, \quad i = 1, \ldots, 2^m. \tag{9}$$

Proof: For every $A(s) \in \mathcal{MA}(s)$, by virtue of theorem 1, there exist $\lambda_1, \ldots, \lambda_{2^m} \in [0,1]$ such that $\sum_{i=1}^{2^m} \lambda_i = 1$ and $A(s) = \sum_{i=1}^{2^m} \lambda_i B_i(s)$. Moreover, for all $i \in \{1, \ldots, 2^m\}$,

$$\begin{bmatrix} R \\ \mathcal{B}_i \end{bmatrix}^* \begin{bmatrix} B \otimes P_i & Q \\ Q^* & 0 \end{bmatrix} \begin{bmatrix} R \\ \mathcal{B}_i \end{bmatrix} < 0$$

$$\begin{array}{l} \Leftrightarrow R^* (B \otimes P_i) R + \mathcal{B}_i^* Q^* R + R^* Q \mathcal{B}_i < 0 \\ \Rightarrow \sum_{i=1}^{2^m} \lambda_i \left(R^* (B \otimes P_i) R + \mathcal{B}_i^* Q^* R + R^* Q \mathcal{B}_i\right) < 0 \\ \Leftrightarrow R^* \left(B \otimes (\sum_{i=1}^{2^m} \lambda_i P_i)\right) R + (\sum_{i=1}^{2^m} \lambda_i \mathcal{B}_i)^* Q^* R + R^* Q (\sum_{i=1}^{2^m} \lambda_i \mathcal{B}_i) < 0 \end{array}$$

Multiplying $\mathcal{N}_{\mathcal{A}}$ from the right and $\mathcal{N}_{\mathcal{A}}^*$ from the left, the inequality becomes

$$\mathcal{N}_{\mathcal{A}}^* \begin{bmatrix} R \\ \sum_{i=1}^{2^m} \lambda_i \mathcal{B}_i \end{bmatrix}^* \begin{bmatrix} B \otimes (\sum_{i=1}^{2^m} \lambda_i P_i) & Q \\ Q^* & 0 \end{bmatrix} \begin{bmatrix} R \\ \sum_{i=1}^{2^m} \lambda_i \mathcal{B}_i \end{bmatrix} \mathcal{N}_{\mathcal{A}} < 0$$

Because of $\mathcal{N}_{\mathcal{A}} = \mathcal{N}_{\sum_{i=1}^{2^m} \lambda_i \mathcal{B}_i}$, we have

$$\mathcal{N}_{\mathcal{A}}^* R^* \left(B \otimes (\sum_{i=1}^{2^m} \lambda_i P_i) \right) R \mathcal{N}_{\mathcal{A}} < 0$$

From $P_i^* = P_i > 0$, we have that $\sum_{i=1}^{2^m} \lambda_i P_i = (\sum_{i=1}^{2^m} \lambda_i P_i)^* > 0$. Now by the Lemma 1, the conclusion is obvious. $\diamond$

Remark 1 The standpoint of Theorem 2 is to transform stability problem into a positive real-like condition, and the latter can be solved using the LMI toolbox.

In regard to Lemma 1 and Lemma 3, we claim that the stability of uncertain family $\mathcal{PA}(s)$ can be inferred from whether an LMI condition holds or not for $K_{\mathcal{PA}}(s)$. This is shown by the following two theorems.

Theorem 3 $\mathcal{PA}(s)$ is $\mathcal{D}$-stable $\Leftrightarrow$ $conv(K_{\mathcal{PA}}(s))$ is $\mathcal{D}$-stable.

Proof: Sufficiency: We will show $E_{\mathcal{PA}}(s) \subset conv(K_{\mathcal{PA}}(s))$.

For all $A_1(s) \in E_{\mathcal{PA}}(s)$, by the definition of $E_{\mathcal{PA}}(s)$, there exists $(l_1, \ldots, l_n) \in P_n^n$ such that

$$A_1(s) = (p_{ij}(s))_{n\times n} : \begin{array}{l} p_{kl_k}(s) \in E_{kl_k}(s),\ k = 1, \ldots, n, \\ p_{ki_k}(s) \in K_{ki_k}(s)\ i_k = 1, \ldots, l_k - 1, l_k + 1, \ldots, n \end{array}$$

For $p_{1l_1}(s) \in E_{1l_1}(s)$, we know that there exist m real numbers $\lambda_{11}, \ldots, \lambda_{1m} \in [0, 1]$ satisfying

$$p_{1l_1}(s) = \sum_{k=1}^{m} \lambda_{1k} p_{1l_1}^{(k)} \text{ and } \sum_{k=1}^{m} \lambda_{1k} = 1.$$

Using addition of matrices, we have

$$A_1(s) = \sum_{t=1}^{m} \lambda_{1t} A_{1l_1}^{(t)}$$

where $A_{1l_1}^{(t)}$ is the matrix that all its entries coincide with $A_1(s)$ except one, which lies in the first row and the l_1-th column and equals to $p_{1l_1}^{(k)}$ for every $t \in \{1, \ldots, m\}$. Thus, for everyone of $\{A_{1l_1}^{(1)}, \ldots, A_{1l_1}^{(m)}\}$, all of its entries in the first row belong to vertex sets.

For every $A_{1l_1}^{(t)}$ $(t = 1, \ldots, m)$, applying the same process to p_{2l_2}, we can find m uncertain matrix families, and for every matrix which belongs to one of those families, all its entries in the first and second rows belong to the corresponding vertex sets. Continuing this procedure, we will get $A_1(s) \in conv(K_{\mathcal{PA}}(s))$. Therefore, $E_{\mathcal{PA}}(s) \subset conv(K_{\mathcal{PA}}(s))$. Then,

$$\begin{array}{rcl} conv(K_{\mathcal{PA}}(s)) \text{ is } \mathcal{D}\text{-stable} & \Rightarrow & E_{\mathcal{PA}}(s) \text{ is } \mathcal{D}\text{-stable} \\ (\text{ by Lemma 3}) & \Leftrightarrow & \mathcal{PA}(s) \text{ is } \mathcal{D}\text{-stable.} \end{array}$$

Necessity: The relationship between $\mathcal{PA}(s)$ and $conv(K_{\mathcal{PA}}(s))$ can be easily established, thereby Necessity is proved. For all $A_1(s) \in conv(K_{\mathcal{PA}}(s))$, there exist n^2 numbers $\lambda_{ij} \in [0,1]$ $\sum_{i,j=1}^n \lambda_{ij} = 1$ and n^2 matrices $F_{ij}(s) \in K_{\mathcal{PA}(s)}$ such that

$$A_1(s) = \sum_{i,j=1}^n \lambda_{ij} F_{ij}(s).$$

Denote $F_{ij}(s) = \left(f_{hl}^{ij}(s)\right)_{n\times n}$ with $f_{hl}^{ij}(s) \in K_{hl}(s)$ for all $h,l \in \{1,\ldots n\}$. By addition of matrices,

$$A_1(s) = \left(\sum_{i,j=1}^n \lambda_{ij} f_{hl}^{ij}(s)\right)_{n\times n}$$

For every $h \in \{1,\ldots,n\}, l \in \{1,\ldots,n\}$, $\sum_{i,j=1}^n \lambda_{ij} f_{hl}^{ij}(s)$ still belongs to $\mathcal{P}_{hl}(s)$ whenever $\lambda_{ij} \in [0,1]$, $\sum_{i,j=1}^n \lambda_{ij} = 1$ and $f_{hl}^{ij}(s) \in K_{hl}(s)$. Thus $conv(K_{\mathcal{PA}}(s)) \subset \mathcal{PA}(s)$. This completes the proof. $\diamond$

For all $A(s) \in \mathcal{PA}(s)$, we have $A(s) = (p_{ij}(s))_{n\times n}$, where $p_{ij}(s) \in \mathcal{P}_{ij}(s)$ with degree l. Rewriting it as

$$A(s) = A_0 + A_1 s + \ldots + A_l s^l.$$

Denote $\mathcal{A} \triangleq (A_0,\ldots,A_l)$, then $\mathcal{A}$ is an $n \times nl$ scalar matrix. By Theorem 3, we have

Theorem 4 $\mathcal{PA}(s)$ is robust $\mathcal{D}$-stable if there exist some matrices $P_{\mathcal{A}} = P_{\mathcal{A}}^* > 0, Q$ solving the LMI feasibility problem

$$\begin{bmatrix} R \\ \mathcal{A} \end{bmatrix}^* \begin{bmatrix} B \otimes P_{\mathcal{A}} & Q \\ Q^* & 0 \end{bmatrix} \begin{bmatrix} R \\ \mathcal{A} \end{bmatrix} < 0, \quad \mathcal{A} \in K_{\mathcal{PA}}(s). \tag{10}$$

Proof: By Theorem 3, this problem is equivalent to the $\mathcal{D}$-stability of $conv(K_{\mathcal{PA}}(s))$. For the latter, for all $A(s) \in conv(K_{\mathcal{PA}}(s))$, there exist n^2 numbers $\lambda_{ij} \in [0,1]$, $\sum_{i,j=1}^n \lambda_{ij} = 1$ and n^2 matrices $F_{ij}(s) \in K_{\mathcal{PA}(s)}$ such that

$$A(s) = \sum_{i,j=1}^n \lambda_{ij} F_{ij}(s).$$

Now by a similar argument as in the proof of Theorem 2, we get the desired result. $\diamond$

5 Illustrative Examples

In this section, we give examples to illustrate the utility of our main results. Example 1 is considered in the context of robust stability of interval multilinear polynomials with respect to left half plane.

Example 1 (n=1, N=2, l=3, m=3) Let $A_1(s), A_2(s)$ be two given polynomials

$$A_1(s) = s^3 + 2.64s^2 + 1.82s + 0.37 \tag{11}$$
$$A_2(s) = s^3 + 5.57s^2 + 9.04s + 3.85 \tag{12}$$

And the uncertain model is $\mathcal{A}(s) = \{a_1(q_1,q_2,q_3)A_1(s) + a_2(q_1,a_2,q_3)A_2(s)\}$, where $q_1 \in [1,2]$, $q_2 \in [3,3.8]$, $q_3 \in [0.5,0.8]$ and $a_1(q_1,q_2,q_3) = 0.6q_1 + 0.1q_2 - q_3 + 0.1q_1q_2$, $a_2(q_1,q_2,q_3) = -0.6q_1 -$

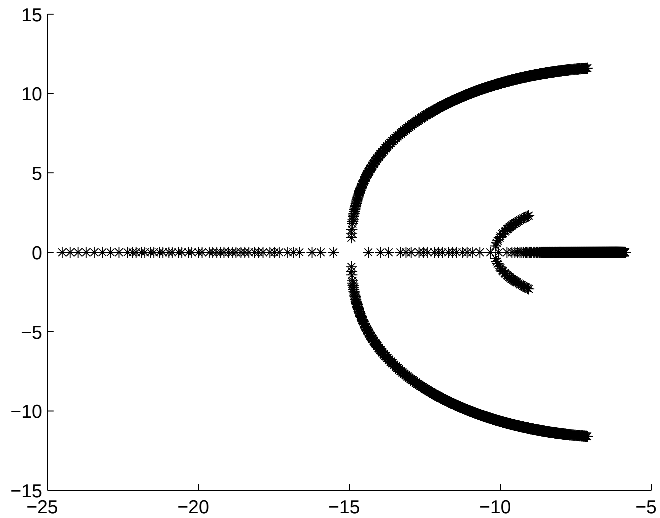

Figure 1: Example 1

$0.1q_2 + q_3 + 1 - 0.01q_2q_3$. For *Hurwitz* stable, the 2×2 Hermite matrix B corresponds to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Applying Theorem 2 to this problem, it suffices to solve 16 linear matrix inequalities. Using the LMI Toolbox in *Matlab*, then it is easy to check that the corresponding LMI problem is feasible. Thus, we conclude that the whole polynomials family is robust *Hurwitz* stable.

Example 2 (n=3, N=2, l=3, m=3) Consider the third order uncertain model below

$$\mathcal{A}(s) = \{a_1(q_1, q_2, q_3)A_1(s) + a_2(q_1, a_2, q_3)A_2(s)\}$$
$$A_1(s) = \frac{1}{10}$$
$$\begin{pmatrix} 15s^3 + 2.5s^2 + 12s - 1 & 13s^3 + 12s^2 + 0.22s - 1.1 & 2.7s^3 - 7.3s^2 - 7.9s - 3.7 \\ 10s^2 - 7.4s + 2.8 & 23s^3 + 4.7s^2 - 1.3s + 3 & 11s^3 - 16s^2 + 15s + 7.6 \\ 12s^2 - 1.1s - 4.7 & 2.8s^2 + 0.99s - 0.5 & 25s^3 - 2.2s^2 - 3.8s + 0.035 \end{pmatrix}$$
$$A_2(s) = \frac{1}{10}$$
$$\begin{pmatrix} 20s^3 - 13s^2 - 18s - 0.96 & -3.7s^3 + 13s^2 + 0.078s - 6.7 & -20s^3 - 4.5s^2 + 9.4s - 0.27 \\ -9.8s^2 + 14s - 4.5 & 19s^3 + 0.35s^2 - 7.5s + 3.4 & -11s^3 - 5.5s^2 - 20s + 3.8 \\ 6s^2 + 17s + 7.1 & -1.3s^2 + 1.1s + 1.5 & 10s^3 - 1.9s^2 - 10s - 5.1 \end{pmatrix}$$

with $q_1 \in [1, 1.2]$, $q_2 \in [2.1, 2.4]$, $q_3 \in [1.5, 1.8]$ and $a_1(q_1, q_2, q_3) = q_1 - q_2 + q_3 + 0.1q_1q_2$, $a_2(q_1, q_2, q_3) = -q_1 + q_2 - q_3 + 1 - 0.01q_2q_3$. In this Example, the quadratic stability region is the unity circle, thus the associated matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Solving the corresponding LMI in Theorem 2, it is easy to show that the uncertain model is robust *Schur* stable.

Remark 2 Our results can also be verified by the plots of root loci of the whole polynomials family in following figures. From the plots of root loci, we can see that our LMI criteria are not very conservative, and can provide correct, effective information on robust stability of uncertain systems.

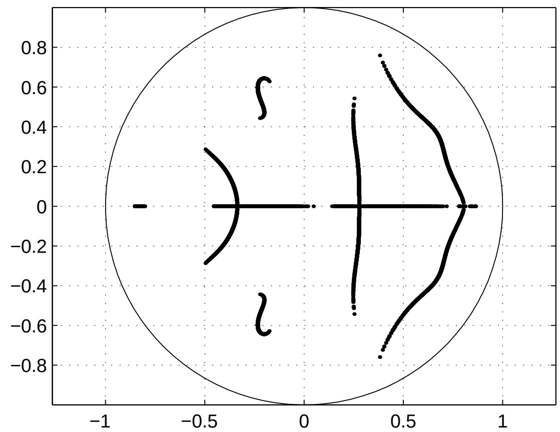

Figure 2: Example 2

6 Conclusion

In this paper, we have dealt with the performance robustness of interval multilinear polynomial matrix families and polytopic polynomial matrix families. Some computationally tractable and nonconservative sufficient conditions for these two classes of system models have been obtained.

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