PROPERTIES OF POWER FUNCTIONS OF SOME TESTS CONCERNING COVARIANCE MATRICES OF MULTIVARIATE NORMAL DISTRIBUTIONS

by

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Summary.

Some properties, e.g., unbiasedness, of a class of test procedures for the one-population and the two-population problems concerning dispersion matrices of multivariate normal distributions are studied using the methods in [2] and [3].

Problem 1.

Consider $N_p(\mu, \Sigma)$. The problem is to test the hypothesis H_{01}: $\Sigma = I_p$ on the basis of a random sample $X_1, \ldots, X_{n+1}$ from $N_p(\mu, \Sigma)$, μ being unknown. $(n \geq p)$. Let

$$S = \sum_{\alpha=1}^{n+1} (X_\alpha - \bar{X})(X_\alpha - \bar{X})', \quad \text{where } \bar{X} = \frac{1}{n+1} \sum_{\alpha=1}^{n+1} X_\alpha.$$

The following results are known so far in the area under study:

(a) Let w be a region in the space of the characteristic roots $C_1, \ldots, C_p$ of S. Suppose

$$(C_1, \ldots, C_p) \in w \Rightarrow (C_1^-, \ldots, C_p^-) \in w, \quad \text{where } C_i^- \leq C_i$$

Then $\Pr(w|\Sigma)$ is a function of the characteristic roots of Σ and decreases as each of these roots increases (Anderson and Das Gupta [1]).

(b) The modified likelihood ratio test of H_{01} against H_{11}: $\Sigma \neq I_p$ having the critical region

$$|S|^{\frac{n}{2}} \operatorname{etr}(-S|2) \leq \lambda$$

is unbiased. (Suguira and Nagao, [3]).

(c) The power of the modified likelihood-ratio test monotonically increases as $|\gamma - 1|$ increases for each characteristic root γ of Σ. (Das Gupta, [2]).

(d) The likelihood-ratio test of H_{01} against H_{11} is biased. (Das Gupta, [2]).

We shall now mention some more results which can be obtained from [2] and [3].

(e) Let w_r be a region in the space of the roots of S given by

$$w_r: \quad |S|^{r/2} \operatorname{etr}(-S/2) \le \lambda$$

where $r \ge 0$. Then the power of this test monotonically increases as $|\gamma - r/n|$ increases for each root γ of Σ. In particular, w_r is an unbiased critical region for testing H_{01} against H_{12}: $Ch(\Sigma) > I_p$ if $r \le n$ and for testing H_{01} against H_{13}: $Ch(\Sigma) < I_p$ if $r \ge n$. (Notation: $Ch(\Sigma)$ denotes the $p \times p$ diagonal matrix with the characteristic roots of Σ as its diagonal elements. We write $A > B$ for two matrices $A = [a_{ij}]$, $B = [b_{ij}]$ of the same dimension if $A \neq B$ and $a_{ij} \ge b_{ij}$ for all i, j.)

The proof of the above result follows from Lemmas 2.1, 2.2, and the proof of Theorem 2.1 of Das Gupta [2] when $r > 0$, and the result (a) confirms the case when $r = 0$.

(f) It can be shown, following the method of Suguira and Nagao [3], that

$$P(w_r \,|\Sigma) - P(w_r \,|\Sigma = I_p)$$
$$\ge K(n, p) \exp(-\lambda/2) \left[1 - |\Sigma|^{-\frac{n-r}{2}}\right] \int_{\bar{w}_r} |S|^{\frac{n-1-p-r}{2}} ds$$

where $K(n, p)$ is a constant depending on n, p, and $\bar{w}_r$ is the complement of w_r.

This gives a lower bound for the power function of the test whose critical region is w_r. In particular, it follows that the test is unbiased, if

$$|\Sigma| \geq 1 \quad \text{and} \quad r \leq n$$

or, if

$$|\Sigma| \leq 1 \quad \text{and} \quad n \leq r.$$

Problem 2.

Consider $N_p(\mu_1, \Sigma_1)$ and $N_p(\mu_2, \Sigma_2)$. The problem is to test H_{02}: $\Sigma_1 = \Sigma_2$ on the basis of a random sample $X_1, \ldots, X_{n_1+1}$ from $N_p(\mu_1, \Sigma_1)$ and a random sample $Y_1, \ldots, Y_{n_2+1}$ from $N_p(\mu_2, \Sigma_2)$; μ_1, μ_2 are unknown. Let

$$S_1 = \sum_{\alpha=1}^{n_1+1} (X_\alpha - \bar{X})(X_\alpha - \bar{X})' \,, \quad \bar{X} = \frac{1}{n_1+1} \sum_{\alpha=1}^{n_1+1} X_\alpha \,,$$

$$S_2 = \sum_{\alpha=1}^{n_2+1} (Y_\alpha - \bar{Y})(Y_\alpha - \bar{Y})' \,, \quad \bar{Y} = \frac{1}{n_2+1} \sum_{\alpha=1}^{n_2+1} Y_\alpha \,.$$

The following results are known so far in the area under study:

(a) Let w be a region in the space of the characteristic roots $C_1, \ldots, C_p$ of $S_1 S_2^{-1}$ such that

$$(C_1, \ldots, C_p) \in w \Rightarrow (C_1^-, \ldots, C_p^-) \in w, \quad \text{where} \quad C_i^- \leq C_i \,.$$

Then $Pr(w|\Sigma_1, \Sigma_2)$ is a function of the characteristic roots of $\Sigma_1 \Sigma_2^{-1}$ and decreases as each of these roots increases. (Anderson and Das Gupta, [1]).

(b) The modified likelihood-ratio test of H_{02} against H_{12}: $\Sigma_1 \neq \Sigma_2$ having the critical region

$$|S_1|^{n_1} \quad |S_2|^{n_2} \quad |S_1 + S_2|^{-(n_1+n_2)} \leq \lambda$$

is unbiased. (Suguira and Nagao, [3]).

(c) The likelihood-ratio test of H_{02} against H_{12} is biased when $n_1 \neq n_2$ (Das Gupta, [2]).

We shall now state a few more results which can be obtained from [2] and [3].

Define

$$U = S_2^{-\frac{1}{2}} \; S_1 \; S_2^{-\frac{1}{2}}$$

and consider the class of tests based on the characteristic roots of U. Without any loss of generality, we may assume $\Sigma_1 = I_p$ and $\Sigma_2 = \Gamma = Ch(\Sigma_1 \Sigma_2^{-1})$. The density of U is given by [3]

$$K(p, n_1, n_2) \; |U|^{\frac{n_1-p-1}{2}} \; |\Gamma|^{\frac{-n_2}{2}} \; |\Gamma^{-1} + U|^{\frac{-n_1+n_2}{2}}$$

Note that the distribution of U under H_{02} is the same as the distribution of $V = \Gamma^{\frac{1}{2}} U \Gamma^{\frac{1}{2}}$ under H_{12}. Consider a test having the following acceptance region:

$$w_a: \quad |U|^a \quad |I_p + U|^{-(n_1+n_2)} \geq \lambda.$$

(d) Following the proof of Theorem 3.1 of Suguira and Nagao [3] it can be seen that

$$P(w_a \mid \Gamma = I_p) - P(w_a \mid \Gamma).$$

$$\geq \lambda^{\frac{1}{2}} K(p, n_1, n_2) \left[1 - |\Gamma|^{\frac{n_1-a}{2}}\right] \int_{w_a} |U|^{\frac{n_1-p-1-a}{2}} dU.$$

This again gives a lower bound for the power function of the test with the acceptance region w_a. In particular, this test is unbiased in the following two cases:

(i) $|\Gamma| \geq 1$, when $n_1 \leq a$

(ii) $|\Gamma| \leq 1$, when $a \leq n,$

Note that for the modified likelihood-ratio test, $a = n_1$.

To study these tests further, or, more generally, the class of tests with acceptance regions of the form $w_{a,b}$: $|U|^a \; |I + U|^{-b} \geq \lambda$, consider a family of regions given by

$$R_{a,b}: \; y^a \; (1 + y)^{-b} \geq K$$

where $y > 0$. For different values of a and b the regions $R_{a,b}$ are either intervals (endpoints may be 0 or ∞) or complements of intervals. Our first object is to study

$$\beta(\gamma) = \Pr \; [Y \in R_{a,b}]$$

where γY $(\gamma > 0)$ is distributed as the F distribution with (n_1, n_2) degrees of freedom. For this purpose it is sufficient to consider the case $0 < a < b$ when the regions $R_{a,b}$ are mainly of the form $y_1 < y < y_2$ where y_1 and y_2 are finite constants, $0 \neq y_1 < y_2$ (excluding the case $y_1 = y_2$). It is obvious that $\beta(\gamma)$ is an increasing function of γ when $R_{a,b} = (0, y_2)$ and a decreasing function when $R_{a,b} = (y_1, \infty)$. We state the following lemma which is an easy generalization of Lemma 5.1 of Das Gupta [2].

Lemma. Let Y be a random variable such that γY $(\gamma > 0)$ is distributed as the F-distribution with (n_1, n_2) degrees of freedom. Let

$$\beta(\gamma) = Pr[Y^a(1 + Y)^{-b} \geq k]$$

where $0 < a < b$. Then $\beta(\gamma) \geq \beta(1)$ for all γ lying between 1 and $\gamma_0 = bn_1/a(n_1+n_2)$. (The strict inequality holds if γ is different from 1 and γ_0 and lies within 1 and γ_0.)

(e) Following the proof of Theorem 5.1 of Das Gupta [2] and using the above lemma, it can be seen that all acceptance regions $w_{a,b}$ $(0 < a < b)$ are biased for

$$I_p < \Gamma, \text{ if } 1 < \gamma_0$$

and, for

$$\Gamma < I_p, \text{ if } \gamma_0 < 1.$$

In particular, the acceptance region w_a is biased (when $\Gamma \neq I_p$) for

$$|\Gamma| \leq 1, \text{ when } n_1 < a$$

and, for

$$1 \leq |\Gamma|, \text{ when } a < n_1.$$

The above results and the lemma hold trivially for the region $w_{a,b}$ when $0 = a < b$.

Remarks.

(1) Consider the first problem. Note that the results (a) can be applied to the regions w_r only when $r = 0$ and, in that case, result

(f) is stronger than (a) so far as the 'unbiasedness' part is concerned.

The result (e) indicates that w_r is a biased critical region, when $(\Sigma \neq I_p)$

$$|\Sigma| > 1 \quad \text{and} \quad n < r$$

or, when

$$|\Sigma| < 1 \quad \text{and} \quad r < n.$$

An interesting problem is to partition the parameter space into the regions of biasedness and unbiasedness for a given test. Another problem is to find the class of unbiased tests given a set of alternatives. None of these problems are solved even in this restricted set-up under study. However, we know that the critical regions w_r are biased if each root of Σ lie between 1 and r/n. This, along with (f), gives a partial answer.

It may be noted that the technique used by Suguira and Nagao [3], which is essentially Pitman's technique (see[2]), fails to provide results on monotonicity of power functions.

Similar remarks can also be made for the second problem.

(2) Recently it has been brought to my attention that some of the results in [2] and [3] were also obtained by Nagao and published in Journ. Hiroshima Univ., 1967, pp. 147-150.

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References

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