LAURICELLA - SARAN TRIPLE HYPERGEOMETRIC FUNCTIONS OF MATRIX ARGUMENTS -I

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ABSTRACT

The Lauricella - Saran triple hypergeometric functions of matrix arguments F_K, F_G and F_E were introduced by us in one of our previous studies [6]. In the present paper the Lauricella - Saran functions F_F, F_M and F_T have been introduced for the matrix arguments case and eight results have been established - two each for the functions F_F and F_M and one each for the functions F_T, F_G, F_K and F_E.

INTRODUCTION

Lauricella, in 1893, had studied the multiple hypergeometric functions. He also gave the properties of the four triple hypergeometric functions $F_A^{(3)}, F_B^{(3)}, F_C^{(3)}$ and $F_D^{(3)}$ and conjectured the existence of ten other such functions, where all these fourteen functions are complete and of the second order. Saran [4] in 1954 introduced and studied the remaining ten functions, thus completing the Lauricella's conjectured set of functions. We have

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earlier introduced the three Lauricella - Saran functions F_K, F_G and F_E of matrix arguments [6] and here we propose to introduce the functions F_F, F_M and F_T of matrix arguments and study some of the properties of these functions. All the matrices appearing in this paper are (pxp) real symmetric positive definite matrices and the meanings of all the other symbols used are the same as in the works of Mathai [2,3].

1. Preliminary Definitions

DEFINITION 1.1: The Lauricella - Saran function F_F of matrix arguments,

$$F_F = F_F(a_1, a_1, a_1, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$$

is defined as that class of functions for which the matrix - transform (M-transform) is the following:

$$M(F_F) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times |Z|^{\rho_3-(p+1)/2} F_F(a_1, a_1, a_1, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)\times dXdYdZ$$

$$= \frac{\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)}{\Gamma_p(a_1)}\frac{\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1)}\frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)}\times \frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)}$$

for $\mathrm{Re}(a_1-\rho_1-\rho_2-\rho_3, b_1-\rho_1-\rho_3, b_2-\rho_2, c_1-\rho_1, c_2-\rho_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$ (1.1)

DEFINITION 1.2:

$$F_M = F_M(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$$

$$M(F_M) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}\times |Z|^{\rho_3-(p+1)/2} F_M(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)\times dXdYdZ$$

$$= \frac{\Gamma_p(a_1-\rho_1)}{\Gamma_p(a_1)} \frac{\Gamma_p(a_2-\rho_2-\rho_3)}{\Gamma_p(a_2)} \frac{\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)} \times$$

$$\frac{\Gamma_p(c_1)}{\Gamma_p(c_1-\rho_1)} \frac{\Gamma_p(c_2)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)} \qquad (1.2)$$

for $\mathrm{Re}(a_1-\rho_1, a_2-\rho_2-\rho_3, b_1-\rho_1-\rho_3, b_2-\rho_2, c_1-\rho_1,$
$c_2-\rho_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$

DEFINITION 1.3:

$$F_T = F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; -X, -Y, -Z)$$

$$M(F_T) = \int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} \times$$
$$|Z|^{\rho_3-(p+1)/2} F_T(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_1, c_1; -X, -Y, -Z) \times$$
$$dX dY dZ$$

$$= \frac{\Gamma_p(a_1-\rho_1)}{\Gamma_p(a_1)} \frac{\Gamma_p(a_2-\rho_2-\rho_3)}{\Gamma_p(a_2)} \frac{\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1)} \frac{\Gamma_p(b_2-\rho_2)}{\Gamma_p(b_2)} \times$$

$$\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_1-\rho_1-\rho_2-\rho_3)}$$

for $\mathrm{Re}(a_1-\rho_1, a_2-\rho_2-\rho_3, b_1-\rho_1-\rho_3, b_2-\rho_2,$
$c_1-\rho_1-\rho_2-\rho_3, \rho_1, \rho_2, \rho_3) > (p-1)/2.$ (1.3)

2. Results

THEOREM 2.1:

$$F_F(a_1, a_1, a_1, b_1, b_2, b_1; c_1, c_2, c_2; -X, -Y, -Z)$$

$$= \frac{1}{\Gamma_p(a_1)\Gamma_p(b_1)} \int_{R_1>0}\int_{R_2>0} e^{-\mathrm{tr}(R_1+R_2)} |R_1|^{a_1-(p+1)/2} \times$$

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$$|R_2|^{b_1-(p+1)/2}\ {}_0F_1(\ ;c_1;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2})\times$$

$$\Phi_3(b_2;c_2;-R_1^{1/2}YR_1^{1/2},-R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2})dR_1dR_2 \quad (2.1)$$

for $\mathrm{Re}(a_1,b_1)>(p-1)/2.$

PROOF: Taking the M-transform of the right side of eq.(2.1) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we have,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$

$${}_0F_1(\ ;c_1;-R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2})\Phi_3(b_2;c_2;-R_1^{1/2}YR_1^{1/2}, \quad (2.2)$$

$$-R_2^{1/2}R_1^{1/2}ZR_1^{1/2}R_2^{1/2})dXdYdZ$$

Making use of the transformations,

$$X_1=R_2^{1/2}R_1^{1/2}XR_1^{1/2}R_2^{1/2},Y_1=R_1^{1/2}YR_1^{1/2},Z_1=R_2^{1/2}R_1^{1/2}Z\times$$

$$R_1^{1/2}R_2^{1/2};\text{ so that, } dX_1=|R_2|^{(p+1)/2}|R_1|^{(p+1)/2}dX,$$

$$dY_1=|R_1|^{(p+1)/2}dY,dZ_1=|R_2|^{(p+1)/2}|R_1|^{(p+1)/2}dZ;\text{ and,}$$

$$|X_1|=|R_2||R_1||X|,|Y_1|=|R_1||Y|,|Z_1|=|R_2||R_1||Z|;$$

in the expression (2.2) and then writing the M-transforms of the ${}_0F_1$ and Φ_3 - functions leads us to,

$$|R_1|^{-\rho_1-\rho_2-\rho_3}|R_2|^{-\rho_1-\rho_3}\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(c_2)\Gamma_p(b_2-\rho_2)}{\Gamma_p(c_1-\rho_1)\Gamma_p(b_2)}\times$$

$$\frac{\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)} \quad (2.3)$$

Substituting this expression on the right side of eq.(2.1) and then integrating out the variables R_1 and R_2 in the resulting expression by using a Gamma integral gives $M(F_F)$ as given by eq.(1.1) above.

THEOREM 2.2:

$$F_F(a_1,a_1,a_1,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)$$

$$= \frac{\Gamma_p(c_2)}{\Gamma_p(b_2)\Gamma_p(c_2-b_2)} \int_0^I |T|^{b_2-(p+1)/2} |I-T|^{c_2-b_2-(p+1)/2} \times$$

$$\left|I+T^{1/2}YT^{1/2}\right|^{-a_1} F_4[a_1,b_1;c_1,c_2-b_2;-(I+T^{1/2}YT^{1/2})^{-1/2} \times \quad (2.4)$$

$$X(I+T^{1/2}YT^{1/2})^{-1/2}, -(I+T^{1/2}YT^{1/2})^{-1/2}(I-T)^{1/2}Z\times$$

$$(I-T)^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2}]dT$$

for $\text{Re}(b_2, c_2-b_2) > (p-1)/2$.

PROOF: Taking the M-transform of the right side of eq.(2.4) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times$$

$$\left|I+T^{1/2}YT^{1/2}\right|^{-a_1} F_4[a_1,b_1;c_1,c_2-b_2;-(I+T^{1/2}YT^{1/2})^{-1/2}X \quad (2.5)$$

$$\times(I+T^{1/2}YT^{1/2})^{-1/2}, -(I+T^{1/2}YT^{1/2})^{-1/2}(I-T)^{1/2}Z\times$$

$$(I-T)^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2}]dXdYdZ$$

The application of the transformations,

$$Y_1 = T^{1/2}YT^{1/2}, X_1 = (I+T^{1/2}YT^{1/2})^{-1/2}X(I+T^{1/2}YT^{1/2})^{-1/2},$$

$$Z_1 = (I+T^{1/2}YT^{1/2})^{-1/2}(I-T)^{1/2}Z(I-T)^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2},$$

i.e. $X_1 = (I+Y_1)^{-1/2} X (I+Y_1)^{-1/2}, Z_1 = (I+Y_1)^{-1/2}(I-T)^{1/2} Z \times$

$(I-T)^{1/2}(I+Y_1)^{-1/2}$; with, $dY_1 = |T|^{(p+1)/2}\, dY, dX_1 =$

$|I+Y_1|^{-(p+1)/2}\, dX, dZ_1 = |I+Y_1|^{-(p+1)/2}\, |I-T|^{(p+1)/2}\, dZ;$

and, $|Y_1| = |T||Y|, |X_1| = |I+Y_1|^{-1}|X|, |Z_1| = |I+Y_1|^{-1}|I-T||Z|;$

to the above expression and then writing the M-transform of an F_4- function and integrating out Y_1 by using a type-2 Beta integral yields,

$$|T|^{-\rho_2}\,|I-T|^{-\rho_3}\,\frac{\Gamma_p(\rho_2)\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)\Gamma_p(c_1)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(c_1-\rho_1)\Gamma_p(c_2-b_2-\rho_3)}\times \quad (2.6)$$
$$\Gamma_p(c_2-b_2)\Gamma_p(b_1-\rho_1-\rho_3)\Gamma_p(\rho_1)\Gamma_p(\rho_3)$$

Substituting this expression on the right side of eq.(2.4) and then integrating out the variable T in the resulting expression by using a type -1 Beta integral yields $M(F_F)$ as given by eq.(1.1) above.

THEOREM 2.3:

$$F_M(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)$$
$$=\frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(a_1)\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(c_1-a_1)\Gamma_p(c_2-b_2-b_1)}\times$$
$$\iint\int_0^I |U|^{b_2-(p+1)/2}\,|V|^{b_1-(p+1)/2}\,|T|^{a_1-(p+1)/2}\times$$
$$|I-T|^{c_1-a_1-(p+1)/2}\left|I+T^{1/2}XT^{1/2}\right|^{-b_1}\times$$
$$|I-U-V|^{c_2-b_2-b_1-(p+1)/2}\left|I+U^{1/2}YU^{1/2}+\right.$$

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$$(I+T^{1/2}XT^{1/2})^{-1/2}V^{1/2}ZV^{1/2}(I+T^{1/2}XT^{1/2})^{-1/2}\Big|^{-a_2} \times$$

$$dUdVdT \tag{2.7}$$

where, $T = T' > 0, 0 < T < I;\ U = U' > 0, V = V' > 0, 0 < U + V < I,$

and for $\mathrm{Re}(a_1, b_1, b_2, c_1 - a_1, c_2 - b_2 - b_1) > (p-1)/2.$

PROOF: Taking the M-transform of the right side of eq.(2.7) with respect to the variables X,Y,Z and the parameters ρ_1, ρ_2, ρ_3 respectively, we have,

$$\int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2} \times$$

$$\left|I+T^{1/2}XT^{1/2}\right|^{-b_1}\Big|I+U^{1/2}YU^{1/2}+(I+T^{1/2}XT^{1/2})^{-1/2} \times \tag{2.8}$$

$$V^{1/2}ZV^{1/2}(I+T^{1/2}XT^{1/2})^{-1/2}\Big|^{-a_2} dXdYdZ$$

Applying the transformations,

$$X_1 = T^{1/2}XT^{1/2}, Y_1 = U^{1/2}YU^{1/2}, Z_1 = (I+T^{1/2}XT^{1/2})^{-1/2}V^{1/2} \times$$

$$ZV^{1/2}(I+T^{1/2}XT^{1/2})^{-1/2}, \text{i.e. } Z_1 = (I+X_1)^{-1/2}V^{1/2}ZV^{1/2} \times$$

$$(I+X_1)^{-1/2};\ \text{with, } dX_1 = |T|^{(p+1)/2}dX, dY_1 = |U|^{(p+1)/2}dY,$$

$$dZ_1 = \left|I+X_1\right|^{-(p+1)/2}|V|^{(p+1)/2}dZ;\ \text{and, } \left|X_1\right| = |T||X|,$$

$$\left|Y_1\right| = |U||Y|, \left|Z_1\right| = \left|I+X_1\right|^{-1}|V||Z|;$$

to the above expression and then integrating out X_1 by using a type-2 Beta integral and Y_1 and Z_1 by using a type-2 Dirichlet integral produces,

$$|T|^{-\rho_1}|U|^{-\rho_2}|V|^{-\rho_3}\frac{\Gamma_p(\rho_1)\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1-\rho_3)}\frac{\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_2)}\times \tag{2.9}$$
$$\Gamma_p(a_2-\rho_2-\rho_3)$$

Substituting this expression on the right side of eq.(2.7) an then integrating out T in the resulting expression by using a type-1 Beta integral and U and V by using atype-1 Dirichlet integral gives $M(F_M)$, as given by eq.(1.2) above.

THEOREM 2.4:

$$F_M(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_2,c_2;-X,-Y,-Z)$$
$$=\frac{1}{\Gamma_p(b_1)}\int_{R>0}e^{-tr(R)}|R|^{b_1-(p+1)/2}\,{}_1F_1(a_1;c_1;-R^{1/2}XR^{1/2})\times \tag{2.10}$$
$$\Phi_1(a_2,b_2;c_2;-Y,-R^{1/2}ZR^{1/2})dR$$

for $\mathrm{Re}(b_1)>(p-1)/2.$

PROOF: Taking the M-transform of the right side of eq.(2.10) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times \tag{2.11}$$
$${}_1F_1(a_1;c_1;-R^{1/2}XR^{1/2})\Phi_1(a_2,b_2;c_2;-Y,-R^{1/2}ZR^{1/2})dXdYdZ$$

Applying the transformations,

$$X_1=R^{1/2}XR^{1/2},\ Z_1=R^{1/2}ZR^{1/2};\ \text{with, } dX_1=|R|^{(p+1)/2}dX,$$
$$dZ_1=|R|^{(p+1)/2}dZ;\ \text{and, } |X_1|=|R||X|,|Z_1|=|R||Z|;$$

to the above expression and then using the M-transforms of the ${}_1F_1$ and Φ_1-functions, we obtain,

$$|R|^{-\rho_1-\rho_3}\frac{\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(a_1-\rho_1)}{\Gamma_p(a_1)\Gamma_p(c_1-\rho_1)}\frac{\Gamma_p(c_2)}{\Gamma_p(a_2)\Gamma_p(b_2)}\times$$
$$\frac{\Gamma_p(a_2-\rho_2-\rho_3)}{\Gamma_p(c_2-\rho_2-\rho_3)}\Gamma_p(b_2-\rho_2)\Gamma_p(\rho_2)\Gamma_p(\rho_3) \qquad (2.12)$$

Substituting this expression on the right side of eq.(2.10) and then integrating out R in the resulting expression by using a Gamma integral produces $M(F_M)$ as given by eq.(1.2) above.

THEOREM 2.5:

$$F_T(a_1,a_2,a_2,b_1,b_2,b_1;c_1,c_1,c_1;-X,-Y,-Z)$$
$$=\frac{\Gamma_p(c_1)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(c_1-a_1-a_2)}\iint|U|^{a_1-(p+1)/2}|V|^{a_2-(p+1)/2}$$
$$\times|I-U-V|^{c_1-a_1-a_2-(p+1)/2}\left|I+U^{1/2}XU^{1/2}+V^{1/2}ZV^{1/2}\right|^{-b_1} \qquad (2.13)$$
$$\times\left|I+V^{1/2}YV^{1/2}\right|^{-b_2}dUdV$$

for $\mathrm{Re}(a_1,a_2,c_1-a_1-a_2)>(p-1)/2$; where, $U>0, V>0$ and $0<U+V<I$.

PROOF: Taking the M-transform of the right side of eq.(2.13) with respect to the variables X,Y,Z and the parameters ρ_1,ρ_2,ρ_3 respectively we obtain,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2}\times$$
$$\left|I+U^{1/2}XU^{1/2}+V^{1/2}ZV^{1/2}\right|^{-b_1}\left|I+V^{1/2}YV^{1/2}\right|^{-b_2}dXdYdZ \qquad (2.14)$$

Applying the transformations,

$$X_1=U^{1/2}XU^{1/2}, Y_1=V^{1/2}YV^{1/2}, Z_1=V^{1/2}ZV^{1/2};\ \text{with,}$$

$$dX_1 = |U|^{(p+1)/2} dX, dY_1 = |V|^{(p+1)/2} dY, dZ_1 = |V|^{(p+1)/2} dZ;$$

and, $|X_1| = |U||X|, |Y_1| = |V||Y|, |Z_1| = |V||Z|;$

to the above expression and then integrating out X_1 and Z_1 by using a type-2 Dirichlet integral and Y_1 by using a type-2 Beta integral we obtain,

$$|U|^{-\rho_1} |V|^{-\rho_2-\rho_3} \frac{\Gamma_p(\rho_1)\Gamma_p(\rho_3)\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1)\Gamma_p(b_2)} \times \quad (2.15)$$

$$\Gamma_p(\rho_2)\Gamma_p(b_2-\rho_2)$$

Substituting this expression on the right side of eq.(2.13) and then integrating out U and V in the resulting expression by using a type-1 Dirichlet integral gives $M(F_T)$ as given by eq.(1.3) above.

THEOREM 2.6 :

$$F_G(a_1, a_1, a_1, b_1, b_2, b_3; c_1, c_2, c_2; -X, -Y, -Z)$$

$$= \frac{\Gamma_p(c_1)\Gamma_p(c_2)}{\Gamma_p(b_1)\Gamma_p(b_2)\Gamma_p(b_3)\Gamma_p(c_1-b_1)\Gamma_p(c_2-b_2-b_3)} \times$$

$$\int_0^I \int_0^I \int_0^I |U|^{b_1-(p+1)/2} |S|^{b_2+b_3-(p+1)/2} |T|^{b_3-(p+1)/2} \times$$

$$|I-U|^{c_1-b_1-(p+1)/2} |I-S|^{c_2-b_2-b_3-(p+1)/2} \times$$

$$|I-T|^{b_2-(p+1)/2} \left| I + U^{1/2} X U^{1/2} + (I-T)^{1/2} S^{1/2} Y S^{1/2} \times \right.$$

$$\left. (I-T)^{1/2} + T^{1/2} S^{1/2} Z S^{1/2} T^{1/2} \right|^{-a_1} dUdSdT \quad (2.16)$$

where $0 < U < I,\ 0 < S < I,\ 0 < T < I,$ and for $\mathrm{Re}(b_1, b_2, b_3,$

$c_1 - b_1, c_2 - b_2 - b_3) > (p-1)/2.$

PROOF: Taking the M-transform of the right side of eq.(2.16) with respect to the variables X,Y,Z and the parameters ρ_1, ρ_2, ρ_3 respectively, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times \Big| I + U^{1/2}XU^{1/2} + (I-T)^{1/2}S^{1/2}YS^{1/2}(I-T)^{1/2} + \tag{2.17}$$

$$T^{1/2}S^{1/2}ZS^{1/2}T^{1/2}\Big|^{-a_1} dXdYdZ$$

On applying the transformations,

$X_1 = U^{1/2}XU^{1/2}, Y_1 = (I-T)^{1/2}S^{1/2}YS^{1/2}(I-T)^{1/2}, Z_1 =$
$T^{1/2}S^{1/2}ZS^{1/2}T^{1/2}$; with, $dX_1 = |U|^{(p+1)/2} dX, dY_1 = |I-T|^{(p+1)/2} \times$
$|S|^{(p+1)/2} dY, dZ_1 = |T|^{(p+1)/2} |S|^{(p+1)/2} dZ$; and $|X_1| = |U||X|$,
$|Y_1| = |I-T||S||Y|, |Z_1| = |T||S||Z|$;

to the above expression and then integrating out the variables X_1, Y_1, Z_1 by using a type-2 Dirichlet integral we obtain,

$$|U|^{-\rho_1} |I-T|^{-\rho_2} |S|^{-\rho_2-\rho_3} |T|^{-\rho_3} \frac{\Gamma_p(\rho_1)\Gamma_p(\rho_2)\Gamma_p(\rho_3)}{\Gamma_p(a_1)} \times \tag{2.18}$$

$$\Gamma_p(a_1-\rho_1-\rho_2-\rho_3)$$

Substituting this expression on the right side of eq.(2.16) and then integrating out the variables U,S and T in the resulting expression by using a type-1 Beta integral produces $M(F_G)$ as given by eq.(1.2) of the author's paper [6].

THEOREM 2.7 :

$$F_K(a_1, a_2, a_2, b_1, b_2, b_1; c_1, c_2, c_3; -X, -Y, -Z)$$

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$$= \frac{\Gamma_p(c_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(a_1)\Gamma_p(a_2)\Gamma_p(b_2)\Gamma_p(c_1-a_1)\Gamma_p(c_2-b_2)\Gamma_p(c_3-a_2)} \times$$

$$\int_0^I\int_0^I\int_0^I |U|^{a_1-(p+1)/2}|V|^{a_2-(p+1)/2}|T|^{b_2-(p+1)/2} \times$$

$$|I-U|^{c_1-a_1-(p+1)/2}|I-V|^{c_3-a_2-(p+1)/2} \times$$

$$|I-T|^{c_2-b_2-(p+1)/2}\left|I+T^{1/2}YT^{1/2}\right|^{-a_2}\left|I+U^{1/2}XU^{1/2}\right.$$

$$\left.+(I+T^{1/2}YT^{1/2})^{-1/2}V^{1/2}ZV^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2}\right|^{-b_1} \times$$

$$dUdVdT$$

for $\operatorname{Re}(c_1-a_1, c_2-b_2, c_3-a_2, a_1, a_2, b_2) > (p-1)/2.$ (2.19)

PROOF: Taking the M-transform of the right side of eq.(2.19) with respect to the variables X,Y,Z and the parameters ρ_1, ρ_2, ρ_3 respectively, we get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0}|X|^{\rho_1-(p+1)/2}|Y|^{\rho_2-(p+1)/2}|Z|^{\rho_3-(p+1)/2} \times$$

$$\left|I+T^{1/2}YT^{1/2}\right|^{-a_2}\left|I+U^{1/2}XU^{1/2}+(I+T^{1/2}YT^{1/2})^{-1/2}V^{1/2} \times\right. \quad (2.20)$$

$$\left.ZV^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2}\right|^{-b_1} dXdYdZ$$

Using the transformations,

$$X_1 = U^{1/2}XU^{1/2}, Y_1 = T^{1/2}YT^{1/2}, Z_1 = (I+T^{1/2}YT^{1/2})^{-1/2}V^{1/2} \times$$

$$ZV^{1/2}(I+T^{1/2}YT^{1/2})^{-1/2}; \text{ i.e. } Z_1 = (I+Y_1)^{-1/2}V^{1/2}ZV^{1/2} \times$$

$$(I+Y_1)^{-1/2}; \text{ with, } dX_1 = |U|^{(p+1)/2}dX, dY_1 = |T|^{(p+1)/2}dY,$$

$$dZ_1 = \left|I+Y_1\right|^{-(p+1)/2} |V|^{(p+1)/2}\, dZ; \text{ and, } \left|X_1\right| = |U||X|,$$

$$\left|Y_1\right| = |T||Y|, \left|Z_1\right| = \left|I+Y_1\right|^{-1} |V||Z|;$$

in the above expression and then integrating out X_1 and Z_1 by using a type-2 Dirichlet integral and Y_1 by using a type-2 Beta integral yields,

$$|U|^{-\rho_1} |T|^{-\rho_2} |V|^{-\rho_3} \frac{\Gamma_p(\rho_1)\Gamma_p(\rho_3)\Gamma_p(b_1-\rho_1-\rho_3)}{\Gamma_p(b_1)} \times$$
$$\frac{\Gamma_p(\rho_2)}{\Gamma_p(a_2-\rho_3)} \Gamma_p(a_2-\rho_2-\rho_3) \tag{2.21}$$

Substituting this expression on the right side of eq.(2.19) and then integrating out U,V,T in the resulting expression by using a type-1 Beta integral generates $M(F_K)$ as given by eq.(1.1) of the author's paper [6].

THEOREM 2.8:

$$F_E(a_1,a_1,a_1,b_1,b_2,b_2;c_1,c_2,c_3;-X,-Y,-Z)$$
$$= \frac{1}{\Gamma_p(a_1)} \int_{U>0} e^{-tr(U)} |U|^{a_1-(p+1)/2}\, {}_1F_1(b_1;c_1;-U^{1/2}XU^{1/2}) \times$$
$$\Psi_2(b_2;c_2,c_3;-U^{1/2}YU^{1/2},-U^{1/2}ZU^{1/2})dU \tag{2.22}$$

for $\text{Re}(a_1) > (p-1)/2$.

PROOF: This result was stated in eq.(1.20) of the author's paper [6]. To prove it, we take the M-transform of the right side of eq.(2.22) with respect to the variables X,Y,Z and the parameters ρ_1, ρ_2, ρ_3 respectively to get,

$$\int_{X>0}\int_{Y>0}\int_{Z>0} |X|^{\rho_1-(p+1)/2} |Y|^{\rho_2-(p+1)/2} |Z|^{\rho_3-(p+1)/2} \times$$
$${}_1F_1(b_1;c_1;-U^{1/2}XU^{1/2})\Psi_2(b_2;c_2,c_3;-U^{1/2}YU^{1/2},-U^{1/2}ZU^{1/2}) \tag{2.23}$$
$$\times dXdYdZ$$

On using the transformations,

$$X_1 = U^{1/2} X U^{1/2}, Y_1 = U^{1/2} Y U^{1/2}, Z_1 = U^{1/2} Z U^{1/2}; \text{ with, } dX_1 = |U|^{(p+1)/2} dX, dY_1 = |U|^{(p+1)/2} dY, dZ_1 = |U|^{(p+1)/2} dZ; \text{ and, } |X_1| = |U||X|, |Y_1| = |U||Y|, |Z_1| = |U||Z|;$$

in the last expression and then writing the M-transforms of the ${}_1F_1$ and Ψ_2 functions yields,

$$|U|^{-\rho_1-\rho_2-\rho_3} \frac{\Gamma_p(b_1-\rho_1)\Gamma_p(c_1)\Gamma_p(\rho_1)\Gamma_p(c_2)\Gamma_p(c_3)}{\Gamma_p(b_1)\Gamma_p(c_1-\rho_1)\Gamma_p(b_2)\Gamma_p(c_2-\rho_2)} \times \frac{\Gamma_p(b_2-\rho_2-\rho_3)}{\Gamma_p(c_3-\rho_3)} \Gamma_p(\rho_2)\Gamma_p(\rho_3) \tag{2.24}$$

Substituting this expression on the right side of eq.(2.22) and then integrating out U in the resulting expression by using a Gamma integral produces $M(F_E)$ as given by eq.(1.3) of the author's paper [6].

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