

INFINITELY REPEATED GAMES WITH DISCOUNTING:  
A GENERAL THEORY

BY  
DILIP ABREU

IMA Preprint Series # 90  
August 1984

**INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS**  
**UNIVERSITY OF MINNESOTA**  
514 Vincent Hall  
206 Church Street S.E.  
Minneapolis, Minnesota 55455

- | #  | Author(s)  | Title  | #  | Author(s)                            | Title   |
|----|--|--|----|--------------------------------------|---|
| 1  | Workshop Summaries from the September 1982 workshop on Statistical Mechanics, Dynamical Systems and Turbulence |  | 25 | Scott J. Spector,                    | On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension  |
| 2  | Raphael De laLlave,  | A Simple Proof of C. Siegel's Center Theorem   | 26 | W.A. Coppel,                         | Maps on an Interval   |
| 3  | H. Simpson, S. Spector,  | On Copositive Matrices and Strong Ellipticity for Isotropic Elastic Materials  | 27 | James Kirkwood,                      | Phase Transitions in the Ising Model with Transverse Field  |
| 4  | George R. Sell,  | Vector Fields in the Vicinity of a Compact Invariant Manifold  | 28 | Luis Magalhães,                      | The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different             |
| 5  | Milan Miklavcic,   | Non-linear Stability of Asymptotic Suction   | 29 | Charles Tresser,                     | Homoclinic Orbits for Flow in $R^3$   |
| 6  | Hans Meinberger,   | A Simple System with a Continuum of Stable Inhomogeneous Steady States   | 30 | Charles Tresser,                     | About some Theorems by L.P. Sil'nikov   |
| 7  | Bau-Sen Du,  | Period 3 Bifurcation for the Logistic Mapping  | 31 | Michael Alzenmann,                   | On the Renormalized Coupling Constant and the Susceptibility in $\phi^4$ Field Theory and the Ising Model in Four Dimensions <sup>4</sup> |
| 8  | Hans Meinberger,   | Optimal Numerical Approximation of a Linear Operator   | 32 | C. Eugene Wayne,                     | The KAM Theory of Systems with Short Range Interactions I   |
| 9  | L.R. Angel, D.F. Evans, B. Minham,   | Three Component Ionic Microemulsions   | 33 | M. Slemrod, J. E. Marsden,           | Spatial Chaos in a Van der Waals Fluid due to Periodic Thermal Fluctuations   |
| 10 | D.F. Evans, D. Mitchell, S. Mukherjee, B. Minham,  | Surfactant Diffusion: New Results and Interpretations  | 34 | J. Kirkwood, C.E. Wayne,             | Percolation in Continuous Systems   |
| 11 | Lelf Arkerlyd,   | A Remark about the Final Aperiodic Regime for Maps on the Interval   | 35 | Luis Magalhães,                      | Invariant Manifolds for Functional Differential Equations Close to Ordinary Differential Equations  |
| 12 | Luis Magalhães,  | Manifolds of Global Solutions of Functional Differential Equations   | 36 | C. Eugene Wayne,                     | The KAM Theory of Systems with Short Range Interactions II  |
| 13 | Kenneth Meyer,   | Tori in Resonance  | 37 | Jean De Canniere,                    | Passive Quasi-Free States of the Noninteracting Fermi Gas   |
| 14 | C. Eugene Wayne,   | Surface Models with Nonlocal Potentials: Upper Bounds  | 38 | Elias C. Alfantis,                   | Maxwell and van der Waals Revisited   |
| 15 | K.A. Perciak-Spector,  | On Stability and Uniqueness of Fluid Flow Through a Rigid Porous Medium  | 39 | Elias C. Alfantis,                   | On the Mechanics of Modulated Structures  |
| 16 | George R. Sell,  | Smooth Linearization Near a Fixed Point  | 40 | William Ruckle,                      | The Strong $\phi$ Topology on Symmetric Sequence Spaces   |
| 17 | David Molkind,   | A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification  | 41 | Charles R. Johnson,                  | A Characterization of Borda's Rule Via Optimization   |
| 18 | Pierre Collet,   | Local $C^\infty$ Conjugacy on the Julia Set for some Holomorphic Perturbations of $z \rightarrow z^2$                  | 42 | Hans Meinberger,                     | Kazuo Kishimoto, The Spatial Homogeneity of Stable Equilibria of Some Reaction-Diffusion Systems on Convex Domains                        |
| 19 | Henry C. Simpson,  | Scott J. Spector, On the Modified Bessel Functions of the First Kind On Barrelling for a Material in Finite Elasticity | 43 | K.A. Perciak-Spector, M.O. Williams, | On Work and Constraints in Mixtures   |
| 20 | George R. Sell,  | Linearization and Global Dynamics  | 44 | H. Rosenberg, E. Toubiana,           | Some Remarks on Deformations of Minimal Surfaces  |
| 21 | P. Constantin, C. Foias,   | Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 2D Navier-Stokes Equations    | 45 | Stephan Pelikan,                     | The Duration of Transients  |
| 22 | Milan Miklavcic,   | Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator  | 46 | V. Capasso, K.L. Cooke, M. Witten,   | Random Fluctuations of the Duration of Harvest  |
| 23 | P. Collet, H. Epstein, G. Gallavotti,  | Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties                 | 47 | E. Fabes, D. Stocck,                 | The $L^p$ -Intergrability of Green's Functions and Fundamental Solutions for Elliptic and Parabolic Equations                             |
| 24 | J.E. Dunn, J. Serrin,  | On the Thermodynamics of Interstitial working  | 48 | H. Brezis,                           | Semilinear Equations in $R^N$ without conditions at infinity  |
|    |  |  | 49 | M. Slemrod,                          | Lax-Friedrichs and the Viscosity-Capillarity Criterion  |

INFINITELY REPEATED GAMES WITH DISCOUNTING:  
A GENERAL THEORY

Dilip Abreu<sup>1</sup>

Department of Economics, Princeton University  
and  
Department of Economics, Harvard University

Abstract

A particularly elementary class of strategy profiles, simple strategy profiles, is shown to suffice to obtain all subgame perfect equilibrium outcome paths of infinitely repeated games with discounting. Essential to the argument are the related notions of an optimal penal code and a simple penal code. The key result is that there exists a simple penal code which is an optimal penal code.

---

<sup>1</sup> This paper is based on [1], which was also incorporated into my thesis. I wish to thank my supervisors, Hugo Sonnenschein and Bobby Willig for much help and advice. Assistance and comments from Bob Anderson, Jacques Crémer, Andreu Mas-Colell and Ennio Stacchetti proved very valuable. I enjoyed discussing this research with, and benefitted greatly from the suggestions of, among others, Ed Green, Vijay Krishna, Eric Maskin, Roger Myerson and Ariel Rubinstein. Financial support from Princeton University, the Sloan Foundation and the IMA, University of Minnesota is gratefully acknowledged. Finally I wish to record a special debt to my colleague David Pearce for his substantial help and encouragement at all stages of this research. Of course, errors remain my own.

## 1. Introduction

This paper analyzes infinitely repeated games with discounting and in particular, the structure of the pure strategy subgame perfect equilibria (see Selten [7, 8]) of such games. Progress in understanding these equilibria has been impeded by the fact that in principle they may be extremely complex. My results provide a major dimensional simplification; they show that all (subgame) perfect equilibrium outcome paths are supportable using simple strategy profiles. Such strategy profiles have a very elementary structure. Moreover, they are extremely tractable; it may be easily checked whether or not they are perfect.

The central concept of this paper is the notion of an optimal penal code. Let  $n$  be the number of players. An optimal penal code (OPC) is an  $n$ -vector of perfect strategy profiles, the  $i^{\text{th}}$  strategy profile of which yields the  $i^{\text{th}}$  player at least as low a payoff as does any other perfect equilibrium. OPC's are important because they lead to a characterization of the set of perfect equilibrium outcome paths; their existence is critical to the main results of this paper.

The theory developed here builds on the seminal work of Aumann and Shapley [2] and Rubinstein [6] on infinitely repeated games without discounting. In particular, it borrows from their work the fundamental idea of punishing a player for not participating in the punishment of another player. In other respects, however, their techniques do not extend; these depend in an essential way upon the possibility of trading future losses one-for-one against present gains. With the introduction of discounting, intertemporal trade-offs are more subtle, and detailed shapes of punishments become important. While discounted repeated games are delicate in this respect, realism demands that they be investigated. Indeed, in

most economic applications, the assumption of a zero interest rate is inappropriate; we are typically concerned with situations in which the future is less important than the present.

Much of the analysis of this paper is couched in terms of outcome paths or punishments. An outcome path (or punishment) is an infinite stream of one-period action profiles. A strategy profile may be viewed as a rule specifying (or prescribing) an initial outcome path and punishments for any deviation from the initial outcome path, or from a previously prescribed punishment. The remarks to follow are to be understood in this context.

The assumption that payoffs are discounted is an essential element of the proof that an OPC exists. Without discounting, existence is not assured; a player may be minmaxed (i. e., forced down to his one-period individually rational payoff) for  $T$  periods for any finite  $T$ , but possibly not forever. This nonexistence is related to the fact that the Aumann-Shapley-Rubinstein strategies involve punishments which are history-dependent and whose severity depends on the pattern of previous deviations.

I show that with discounting, attention may be restricted to strategies which are history-independent, apart from the identity of the last deviant: any deviation by player  $i$  is responded to by the imposition of the same punishment or outcome path  $Q^i$ . Thus in designing OPC's we may restrict attention to simple penal codes. Such penal codes are completely defined by an  $n$ -vector of punishments  $(Q^1, \dots, Q^n)$ ;  $Q^i$  is imposed if player  $i$  deviates from any previously prescribed punishment  $Q^j$ ,  $j = 1, \dots, n$ . The main result of this paper is that there

exists a simple penal code (SPC) which is globally optimal (i.e., optimal among the class of all perfect penal codes, simple or otherwise).

This result makes it unnecessary to contemplate complex hierarchies of punishments; in no sense is there any need to "make the punishment fit the crime." In general, optimality might demand that a deviant "cooperate" in his own punishment (i.e., not play a single-period best response when he is being punished). The theorem on SPC's implies that he may be persuaded to do so simply by threatening to restart the punishment already in effect. This appears paradoxical. The appropriate resolution is that in such situations, the early stages of an optimal punishment must be more unpleasant than the remainder.

My results do not apply to mixed strategy equilibria. They depend on deviations being detected with certainty, and since randomizing devices cannot sensibly be regarded as being observable, do not apply to the mixed strategy case.<sup>1</sup> This last remark should make clear that mixed strategies lead to a significant change in the nature of the model and this presents a very attractive area for further research. I hope to return to this question in the future. It should be understood in what follows that equilibria being referred to are pure strategy equilibria.

Earlier work on repeated games with discounting (see, for instance, the classic paper by Friedman [4]) has concentrated on paths supportable by Cournot-Nash punishments, i.e., players revert to single-period Cournot-Nash equilibrium behavior forever if a deviation occurs. Cournot-Nash reversion, while subgame

---

<sup>1</sup> I would like to thank Bob Anderson for first bringing up the issue of mixed strategies and pointing out that these might lead to more severe punishments.

perfect, is not in general optimal, and will therefore only suffice to support a limited range of perfect equilibrium behavior. The theory developed here is relevant only when one is attempting to design perfect penal codes more severe than Cournot-Nash reversion. Such an exercise is critical for any study of extremal equilibria. Examples of the latter include maximal collusion in a repeated oligopolistic game or efficient contracts in repeated principal-agent problems.

I now proceed to the formal analysis. Section 2 provides notation and definitions, Section 3 establishes the properties of simple strategy profiles and Section 4 proves the main theorems. Section 5 reports on an application of the theory developed here to repeated oligopolistic games, and Section 6 concludes.

## 2. Notation and Definitions

The notation and definitions presented below are adapted from Rubinstein [6].

D1  $G = (\{S_i\}_{i=1}^n; \{\pi_i\}_{i=1}^n)$  denotes a one-shot simultaneous game.

$N = \{1, \dots, n\}$  is the set of players.  $S_i$  is a pure strategy set for player  $i$ ;  $\pi_i : S \rightarrow \mathbb{R}$  is his payoff function.  $S_i$  is assumed to contain at least two elements. Elements of  $S_i$  are denoted  $q_i$  and are referred to as actions.

$S \equiv S_1 \times S_2 \times \dots \times S_n$ ,  $q \equiv (q_1, q_2, \dots, q_n)$  and  $\pi \equiv (\pi_1, \pi_2, \dots, \pi_n)$ . Note that whenever player-subscripted symbols are used, the corresponding unsubscripted symbol refers to a Cartesian product or a vector, depending on the context. Remark:  $G$  is assumed to be simultaneous in order to abstract from problems of perfection in the one-shot game.

D2  $G^\infty(\delta)$  denotes the supergame with discounting obtained by repeating  $G$  infinitely often, and evaluating payoffs in terms of the discount factor  $\delta \in (0, 1)$ .

$\sigma_i$  denotes a pure strategy for player  $i$ . It is a sequence of functions  $\sigma_i(1), \sigma_i(2), \dots, \sigma_i(t), \dots$ , one for each period  $t$ ; The function for period  $t$  determines player  $i$ 's action at  $t$  as a function of the actions of all players in all previous periods. Formally,  $\sigma_i(1) \in S_i$  and for  $t = 2, 3, \dots$ ,  $\sigma_i(t) : S^{t-1} \rightarrow S_i$ .  $\Sigma_i$  denotes player  $i$ 's strategy set.  $\Sigma \equiv \Sigma_1 \times \Sigma_2 \times \dots \times \Sigma_n$  is the set of strategy profiles.

A stream of action profiles  $\{q(t)\}_{t=1}^\infty$  is referred to as an outcome path or punishment and is denoted by  $Q$ .  $\Omega = S^\infty$  is the set of outcome paths. Any strategy profile  $\sigma \in \Sigma$  generates an outcome path denoted  $Q(\sigma) = \{q(\sigma)(t)\}_{t=1}^\infty$ , and defined inductively below:

$$q(\sigma)(1) = \sigma(1)$$

$$q(\sigma)(t) = \sigma(t)(q(\sigma)(1), \dots, q(\sigma)(t-1)) .$$

$V_i : \Omega \rightarrow \mathbb{R}$  defines the  $i^{\text{th}}$  player's payoff from an outcome path  $Q = \{q(t)\}_{t=1}^\infty \in \Omega$ .

$V_i(Q) = \sum_{t=1}^\infty \delta^t \pi_i(q(t))$ .  $\tilde{V}_i : \Sigma \rightarrow \mathbb{R}$  is the  $i^{\text{th}}$  player's payoff function.

$\tilde{V}_i(\sigma) = V_i(Q(\sigma))$ . Note that I am discounting to the beginning of period 1, and that

I assume that period  $t$  payoffs are received at the end of period  $t$ . I also assume that all players have the same discount factor; this assumption is made for notational convenience and plays no role in the analysis.

D3  $\sigma \in \Sigma$  is a Nash equilibrium if for all  $i$ ,  $\tilde{V}_i(\sigma) \geq \tilde{V}_i(\sigma'_i, \sigma_{-i})$  for all  $\sigma'_i \in \Sigma_i$ ,

where  $\sigma \equiv (\sigma_i, \sigma_{-i})$ .

Note that "-i" is used consistently to denote a profile with a missing  $i^{\text{th}}$  element.



D4 Subgame Perfect Equilibrium. A t-period action profile

$(q(1), \dots, q(t)) \in S^t$  is denoted  $H(t)$  and is called a t-period history. For any

$\sigma \in \Sigma$ ,  $t = 1, 2, \dots$  and any history  $H(t) = (q(1), \dots, q(t)) \in S^t$ ,  $\sigma|_{H(t)} \in \Sigma$

denotes the strategy profile induced by  $\sigma$  after the history  $H(t)$ .

$\sigma|_{H(t)} \equiv \sigma|_{H(t)}^{(1)}, \dots, \sigma|_{H(t)}^{(r)}, \dots$  is defined as follows:

$$\left(\sigma|_{H(t)}\right)_i^{(r)}(q'(1), \dots, q'(r-1)) = \sigma_i^{(t+r)}(q(1), \dots, q(t), q'(1), \dots, q'(r-1)),$$

$$\text{all } q'(1), \dots, q'(r-1) \in S^{r-1} \quad \text{and} \quad r = 1, 2, \dots$$

$\sigma \in \Sigma$  is a subgame perfect equilibrium iff  $\sigma$  is a Nash equilibrium and for

all  $t = 1, 2, \dots$  and all histories  $H(t) \in S^t$ ,  $\sigma|_{H(t)}$  is a Nash equilibrium.

(Subgame) perfect equilibrium will henceforth be referred to as perfect equilibrium.

The following abbreviations are also used: N.E. = Nash equilibrium, P.E. = perfect equilibrium, SSP = simple strategy profile, OPC = optimal penal code, and SPC = simple penal code.

### 3. Simple Strategy Profiles

This section analyzes a particularly simple class of strategy profiles, i.e., simple strategy profiles. Before defining these I provide a brief account of the relationship among strategy profiles, deviations and outcome paths.

Recall that a strategy profile  $\sigma$  is a sequence of functions  $\sigma(1), \dots, \sigma(t), \dots$ , where  $\sigma(t) : S^{t-1} \rightarrow S$ .  $\sigma(t)$  maps from  $(t-1)$ -period histories to an action profile for period  $t$ . An outcome path  $Q = \{q(t)\}_{t=1}^{\infty}$ , on the other hand, is a sequence of action profiles. With a minimum of formality, I discuss here how strategy profiles may be described in terms of outcome paths and deviations from

previously specified outcome paths. In general this might be unnecessarily cumbersome; for simple strategy profiles (SSP's), however, this is the most natural way to proceed.

Implicit in any strategy profile is a notion of initial and subsequent deviations from behavior specified by  $\sigma$ . In the absence of deviations, conformity with  $\sigma$  results in the infinite sequence of action profiles given by the outcome path  $Q(\sigma)$ . Any particular deviation (different players might deviate, at distinct times and to varying extents) from  $Q(\sigma)$  leads into a particular subgame;  $\sigma$  induces a strategy profile, and therefore an outcome path on this subgame. A subsequent deviation from this secondary outcome path is possible; this leads to another subgame, another induced strategy profile and another outcome path which becomes the new standard with respect to which conformity with  $\sigma$  is determined. And so on, for higher order deviations.

Thus a strategy profile may be thought of as a rule specifying:

- (1) an initial outcome path  $Q^0 = Q(\sigma) \in \Omega$ ;
- (2) an outcome path  $Q^r \in \Omega$  after any particular deviation from an ongoing outcome path  $Q^{r-1} \in \Omega$ .

In the above definition,  $r$  refers to deviations and not players. A noninitial outcome path may be thought of as a "punishment." It is imposed for deviating from an ongoing outcome path. The latter may be the initial path or itself a punishment.

Observe that  $Q^r$  may depend not only on  $Q^{r-1}$  and the particular deviation from it, but more generally on the entire history of previously prescribed outcome paths or punishments and deviations from them. Indeed, the punishments

which the Aumann-Shapley-Rubinstein strategies employ are history-dependent in a nontrivial way. Punishments are tailored to "fit the crime," and the argument that such strategies are perfect depends in an essential way on this construction.

To summarize, an arbitrary strategy profile may involve in its specification an infinity of punishments and a complicated rule prescribing which punishment is imposed for any particular deviation from the initial path or an on-going punishment.

An SSP, on the other hand, is completely defined by an  $(n+1)$ -vector of outcome paths  $(Q^0, Q^1, \dots, Q^n)$  and a very simple rule.  $Q^0$  is the initial path.  $Q^i$ ,  $i \in N$  is a player-specific punishment. Any deviation by player  $i$  alone from any ongoing prescribed path (the initial path or one of the  $n$  punishments) is responded to by imposing  $Q^i$ . Simultaneous deviations are ignored.<sup>2</sup>

D5 Let  $Q^i \in \Omega$ ,  $i = 0, 1, \dots, n$ . The simple strategy profile  $\sigma(Q^0, Q^1, \dots, Q^n)$  specifies:

- (1) play  $Q^0$  until some player deviates singly from  $Q^0$ ;
- (2) for any  $j \in N$ , play  $Q^j$  if the  $j^{\text{th}}$  player deviates singly from  $Q^i$ ,  $i = 0, 1, \dots, n$ , where  $Q^i$  is an ongoing previously specified path; continue with  $Q^i$  if no deviations occur or if two or more players deviate simultaneously.

Note for later use that  $\tilde{V}(\sigma(Q^0, Q^1, \dots, Q^n)) = V(Q^0)$ .

---

<sup>2</sup> Since the equilibrium notions employed in this paper are noncooperative, it is sufficient to restrict attention to the deterrence of uncoordinated (among players) deviations: punishments designed for groups of simultaneously defecting players are irrelevant.

The reader who finds D5 to his taste may wish to skip the more formal definition D5\*.

D5\* Let  $Q^i = \{q^i(t)\}_{t=1}^{\infty} \in \Omega$ ,  $i = 0, 1, \dots, n$ . The simple strategy profile  $\sigma(Q^0, Q^1, \dots, Q^n)$  is defined below.

For notational convenience let  $\sigma \equiv \sigma(Q^0, Q^1, \dots, Q^n)$ ;  $\mathbb{Z}_+$  denotes the positive integers.

$$\sigma(1) = q^0(1).$$

Let  $\alpha(1) = 0$  and  $\beta(1) = 1$ . For  $t = 2, 3, \dots$ ,  $\sigma(t) : S^{t-1} \rightarrow S$  is defined in terms of functions  $\alpha(t) : S^{t-1} \rightarrow \{0, 1, \dots, n\}$ ,  $\beta(t) : S^{t-1} \rightarrow \mathbb{Z}_+$  and a correspondence  $\gamma(t) : S^{t-1} \rightrightarrows N$ . Let  $H(t-1) = (q(1), \dots, q(t-1)) \in S^{t-1}$ .  $\sigma(t)$ ,  $\alpha(t)$ ,  $\beta(t)$  and  $\gamma(t)$  are specified inductively as follows:

$$\gamma(t)(H(t-1)) = \{j : q_j(t-1) \neq \sigma_j(t-1)(q(1), \dots, q(t-2))\}$$

$$\alpha(t)(H(t-1)) = \begin{cases} \gamma(t)(H(t-1)) & \text{if } \gamma(t)(H(t-1)) \text{ is a singleton} \\ \alpha(t-1)(q(1), \dots, q(t-2)) & \text{otherwise} \end{cases}$$

$$\beta(t)(H(t-1)) = \begin{cases} 1 & \text{if } \gamma(t)(H(t-1)) \text{ is a singleton} \\ \beta(t-1)(q(1), \dots, q(t-2)) + 1 & \text{otherwise} \end{cases}.$$

$$\sigma(t)(H(t-1)) = q^{\alpha(t)(H(t-1))}(\beta(t)(H(t-1)))$$

Given our interest in perfect equilibria, the "simplicity" of a strategy profile must be judged in terms of how easy it is to verify whether it is perfect. Perfection is in general a difficult criterion to apply because

- (i) we need to check that the strategy profiles induced by all histories are Nash equilibria;

(ii) it is a nontrivial exercise to determine whether any given strategy profile is an N. E. In particular, for any player, both one-shot deviations and all conceivable patterns of successive deviations must not yield a higher payoff. Recall that a one-shot deviation from  $\sigma \in \Sigma$  involves an initial deviation from  $Q(\sigma)$ , followed by conformity with  $\sigma$  thereafter. While a one-shot deviation might not be lucrative, it is possible that a more complicated, possibly infinite sequence of deviations could be.

In terms of the perfection criterion, SSP's are very simple indeed. We need consider only  $(n+1)$  induced strategy profiles. Furthermore, each of these may be verified to be an N. E. by checking one-shot deviations alone. This is the content of Theorems 1 and 3 below. They together establish that the SSP  $\sigma(Q^0, Q^1, \dots, Q^n)$  is perfect iff no one-shot deviation by any player  $j \in N$  from  $Q^i$ ,  $i = 0, 1, \dots, n$  yields him a higher payoff, given that he and all other players will conform with  $Q^j$  after the deviation.

The proofs of these results are elementary, but notationally burdensome. The reader might well find the following verbal argument compelling. Consider  $\sigma(Q^0, Q^1, \dots, Q^n)$  and suppose that no one-shot deviations by any player from any of the  $(n+1)$  paths is lucrative. The way SSP's are constructed, any deviation by player  $j \in N$  results in  $Q^j$  being imposed. Suppose that  $Q^j$  has just been imposed. Since  $Q^j$  is restarted after any deviation by  $j$  and since one-shot deviations do not yield a higher payoff, no finite sequence of deviations by  $j$  from  $Q^j$  will be lucrative. However, given discounting, if an infinite sequence of deviations is lucrative, then a large enough finite sequence will also be. Thus, under our hypothesis,

player  $j$  cannot do better than conform with  $Q^j$  once it is imposed. But now it is clear that player  $j$  will conform with  $Q^i$ ,  $i = 0, 1, \dots, n$ , since a deviation from  $Q^i$  results in  $Q^j$  being imposed, from which, we have just argued, no further deviation is profitable. Finally note that the induced strategy profile after any history is an SSP  $\sigma(\hat{Q}^i, Q^1, \dots, Q^n)$ , where  $\hat{Q}^i$  is a truncation of one of the  $(n+1)$  paths  $Q^i$ . (By a truncation  $\hat{Q}^i$ , I mean a path  $\{q^i(t+s)\}_{t=1}^{\infty}$  for some  $s = 0, 1, 2, \dots$ .) Clearly if deviations from an entire path are not lucrative, deviations from any truncation of it cannot be.

If the reader finds the prospect of struggling with the paragraph above more inviting than the notation below, he should subsequently absorb D7, D8 and the statement of Theorem 4, and then proceed to the next section.

Theorem 1. Let  $Q^i \in \Omega$ ,  $i = 0, 1, \dots, n$ .  $\sigma(Q^0, Q^1, \dots, Q^n)$  is a P.E. iff for all  $i = 0, 1, \dots, n$ ,  $\sigma(Q^i, Q^1, \dots, Q^n)$  is an N.E.

Proof.

"if"

For any  $\tau \in \Sigma$ , let  $h(\tau, s) = (q(\tau)(1), \dots, q(\tau)(s))$ ,  $s = 1, 2, \dots$ . Adopt the convention  $\tau|_{h(\tau, 0)} = \tau$ . Observe that if  $\tau$  is an N.E. then  $\tau|_{h(\tau, s)}$  is an N.E.  $s = 0, 1, 2, \dots$ , i.e., an N.E. strategy profile induces an N.E. profile on all subgames which lie along the equilibrium path. For notational convenience denote  $\sigma(Q^i, Q^1, \dots, Q^n)$  by  $\sigma^i$ . It follows from the definition of  $\sigma^0$  that for any history  $H(t) \in S^t$ ,

$$\sigma^0|_{H(t)} = \sigma^i|_{h(\sigma^i, s)} \text{ for some } i = 0, 1, \dots, n \text{ and some } s = 0, 1, 2, \dots$$

"only if"

For  $i \in N$ , consider  $q_i^* \in S_i$  such that  $q_i^* \neq q_i^0(1)$ . Let  $q^* \equiv (q_i^*, q_{-i}^0(1))$ .

Finally, observe that  $\sigma^0 \Big|_{q^*} = \sigma^i$ .

Q. E. D.

The next three definitions provide a convenient notation for strategies which involve (at most) one-shot deviations from  $\sigma \in \Sigma$ , and for the histories and payoffs such strategies yield.

D6  $\sigma_j^*(\sigma_j, q_j^*, t)$

Let  $\sigma_j \in \Sigma_j$ ,  $q_j^* \in S_j$ .  $\sigma_j^*(\sigma_j, q_j^*, t) \in \Sigma_j$  is defined as follows:

$$\sigma_j^*(\sigma_j, q_j^*, t)(s) = \sigma_j(s) \quad \text{all } s \neq t$$

$$\sigma_j^*(\sigma_j, q_j^*, t)(H(t-1)) = q_j^* \quad \text{all } H(t-1) \in S^{t-1}$$

$\sigma_j^*(\sigma_j, q_j^*, t)$  is identical to  $\sigma_j$  in all periods except possibly in period  $t$ ;

in period  $t$  it requires player  $j$  to play  $q_j^*$  after all  $(t-1)$  period histories.

D7  $u_j(Q, q_j^*, t)$

Let  $Q = \{q(t)\}_{t=1}^{\infty} \in \Omega$  and  $q_j^* \in S_j$ .

$$u_j(Q, q_j^*, t) = \sum_{s=1}^{t-1} \delta^s \pi_j(q(s)) + \delta^t \pi_j(q_j^*, q_{-j}(t))$$

I adopt the convention  $u_j(Q, q_j^*, \infty) = V_j(Q)$ .

D8  $H(\sigma, q_j^*, t)$

Let  $\sigma \in \Sigma$ ,  $q_j^* \in S_j$  and  $q^* = (q_j^*, q_{-j}(\sigma)(t))$ .

$$H(\sigma, q_j^*, t) = (q(\sigma)(1), \dots, q(\sigma)(t-1), q^*).$$

Let  $\sigma = (\sigma_j, \sigma_{-j}) \in \Sigma$ . Observe that  $\tilde{V}_j(\sigma_j^*, \sigma_{-j}^*, t) = u_j(Q(\sigma), q_j^*, t) + \delta^t \tilde{V}_j(\sigma |_{H(\sigma, q_j^*, t)})$ .

This expression yields player  $j$ 's payoff from using a strategy which requires him to make (at most) a one-shot deviation from  $\sigma$ , by playing  $q_j^*$  in period  $t$ .

Lemma 2. Suppose  $\sigma$  is an N.E. Then

$$u_j(Q(\sigma), q_j^*, t) + \delta^t \tilde{V}_j(\sigma |_{H(\sigma, q_j^*, t)}) \leq \tilde{V}_j(\sigma)$$

for all  $q_j^* \in S_j$ ,  $j \in N$ ,  $t = 1, 2, \dots$ .

Proof. Consider  $\sigma_j^*(\sigma_j, q_j^*, t)$ .

Q. E. D.

Theorem 3. For all  $i = 0, 1, \dots, n$ ,  $\sigma(Q^i, Q^1, \dots, Q^n)$  is an N.E. iff

$$u_j(Q^i, q_j^*, t) + \delta^t V_j(Q^j) \leq V_j(Q^i) \quad (1)$$

for all  $q_j^* \in S_j - \{q_j^i(t)\}$ ,  $j \in N$ ,  $i = 0, 1, \dots, n$ ,  $t = 1, 2, \dots$ .

Proof. For notational convenience let  $\sigma^i \equiv \sigma(Q^i, Q^1, \dots, Q^n)$ . Since for  $q_j^* \neq q_j^i(t)$ ,  $\sigma^i |_{H(\sigma^i, q_j^*, t)} = \sigma^j$ , by Lemma 2, the inequalities (1) are necessary.

To obtain sufficiency, consider  $\sigma^i$  and any  $\sigma_j^* \in \Sigma_j$ . Let

$\{q^*(t)\}_1^\infty = Q(\sigma_j^*, \sigma_{-j}^i)$ .  $\sigma_j^*$  leads to a sequence (possibly empty, possibly infinite)

of deviations from the behavior specified by  $\sigma^i$ . I now define  $T(r)$ , where  $T(r)$  is the period in which the  $r^{\text{th}}$  deviation occurs. Let  $H^*(t) = q^*(1), \dots, q^*(t)$  and consider the functions  $\beta(t)$  as specified in D5\*. Define  $A = \{t \geq 1 : \beta(t+1)(H^*(t)) = 1\}$ .

Let  $R = |A|$  be the number of deviations. If  $A \neq \emptyset$ ,  $T(1) = \min A$ .

Set  $T(R+1) = \infty$ . Finally, for  $r = 2, 3, \dots, R$ ,  $T(r) = \min\{t \geq T(r-1) : t \in A\}$ .



Let  $t(1) = T(1)$  and  $t(r) = T(r) - T(r-1)$ ,  $r = 2, 3, \dots, R+1$ . It may now be checked that:

$$\begin{aligned} \tilde{V}_j(\sigma_j^*, \sigma_{-j}^i) &= u_j(Q^i, q_j^*(T(1)), t(1)) \\ &\quad + \sum_{r=1}^R \delta^{T(r)} u_j(Q^j, q_j^*(T(r+1)), t(r+1)) \end{aligned}$$

where  $q_j^*(T(1)) \neq q_j^i(t(1))$ ,

$$q_j^*(T(r)) \neq q_j^j(t(r)) \quad , \quad r = 2, 3, \dots$$

and  $Q^i \equiv \{q^i(t)\}_1^{\infty}$ .

Finally, use the inequalities (1) to obtain  $\tilde{V}_j(\sigma_j^*, \sigma_{-j}^i) \leq V_j(Q^i) = \tilde{V}_j(\sigma^i)$ . Hence  $\sigma^i$  is an N.E.

Q. E. D.

Theorems 1 and 3 immediately imply:

Theorem 4.  $\sigma(Q^0, Q^1, \dots, Q^n)$  is a P.E. iff the inequalities (1) are satisfied.

#### 4. Optimal Penal Codes

In this section I define an optimal penal code, and establish my main result: there exists a simple penal code which is an optimal penal code. Thus optimal penal codes are shown to exist and are also characterized. An important implication of this theorem is that a path  $Q^0$  is the outcome of a P.E. iff it is the outcome of some perfect SSP. Hence the P.E. paths of  $G^{\infty}(\delta)$  may be completely analyzed in terms of simple profiles.

D9  $\Sigma^P$  denotes the set of perfect equilibria of  $G^{\infty}(\delta)$ .  $\Omega^P = \{Q(\sigma) : \sigma \in \Sigma^P\}$

is the set of perfect equilibrium outcome paths.

I now define an optimal penal code (OPC).

D10 An optimal penal code is an n-vector of strategy profiles  $(\hat{\sigma}^1, \dots, \hat{\sigma}^n)$

such that for all i,

$$\hat{\sigma}^i \in \Sigma^P \quad \text{and} \quad \tilde{V}_i(\hat{\sigma}^i) = \min\{\tilde{V}_i(\sigma) \mid \sigma \in \Sigma^P\}.$$

Note that  $\hat{\sigma}^i$  will in general be different from  $\hat{\sigma}^j$ ,  $j \neq i$ ; the "worst" equilibrium from one player's point of view need not be "worst" from another's.

OPC's are important in that they lead to a simple characterization of the set of P.E. outcome paths:

"Suppose an OPC  $(\hat{\sigma}^1, \dots, \hat{\sigma}^n)$  exists. Then  $Q^0 \in \Omega^P$  iff

$$u_j(Q^0, q_j^*, t) + \delta^t \tilde{V}_j(\hat{\sigma}^j) \leq V_j(Q^0) \quad \text{for all } q_j^* \in S_j, \quad j \in N, \quad t = 1, 2, \dots ."$$

i.e.,  $Q^0$  is a P.E. path iff no one-shot deviations from  $Q^0$  yield a higher payoff given that a deviant j's post-deviation payoff is  $\tilde{V}_j(\hat{\sigma}^j)$ . This result is proved below.

See Theorem 8.

I now define an SPC. Let  $Q^i \in \Omega$ ,  $i \in N$ .

D11.  $\sigma^i(Q^1, \dots, Q^n) \equiv \sigma(Q^i, Q^1, \dots, Q^n)$ .

D12. The simple penal code  $\text{SPC}(Q^1, \dots, Q^n)$  is the n-vector of strategy profiles  $(\sigma^1(Q^1, \dots, Q^n), \dots, \sigma^n(Q^1, \dots, Q^n))$ .

Thus the i<sup>th</sup> strategy profile of  $\text{SPC}(Q^1, \dots, Q^n)$  specifies:

- (1) play  $Q^i$  until some player deviates singly from  $Q^i$ ;
- (2) for any  $j \in N$ , play  $Q^j$  if the j<sup>th</sup> player deviates singly from  $Q^k$ ,  $k \in N$  where  $Q^k$  is an ongoing, previously specified punishment; continue

with  $Q^k$  if no deviations occur or if two or more players deviate simultaneously.

Notice that an SPC is defined by an  $n$ -vector of outcome paths (as opposed to  $(n+1)$  for an SSP) and that the  $n$ -vector of simple profiles which define a simple penal code differ only in the initial path or punishment they prescribe.

D13.  $\text{SPC}(Q^1, \dots, Q^n)$  is said to be perfect iff  $\sigma^i(Q^1, \dots, Q^n)$  is perfect, all  $i \in N$ .

Simple penal codes, like SSP's, are appropriately named. In particular, by Theorem 4,  $\text{SPC}(Q^1, \dots, Q^n)$  is perfect iff no one-shot deviation by any player  $j$  from  $Q^i$ ,  $i \in N$ , yields him a higher payoff given that he and all other players will conform with  $Q^j$  after the deviation.

D14. Let  $\bar{Q}_i \in \Omega$ ,  $i \in N$ .  $\text{SPC}(\bar{Q}^1, \dots, \bar{Q}^n)$  is an optimal simple penal code if it is an optimal penal code.

D14 is offered to avoid legitimate confusion; an optimal SPC is optimal among the class of all penal codes, simple or otherwise.

The centrality of OPC's and SPC's to the theory developed here is attested to by the results to follow. Theorem 7 asserts that an optimal SPC exists if an OPC exists. Theorem 10 shows that if an OPC exists, perfect SSP's suffice to obtain all P.E. paths. Theorem 8 has already been referred to above. Finally, Theorem 11 establishes that an optimal penal code exists. The results obtained in Section 3 permit brisk and elementary proofs of all the propositions above.

Suppose until Theorem 10 that  $\Sigma^P \neq \emptyset$ , and let  $\bar{V}_i = \inf\{V_i(\sigma) : \sigma \in \Sigma^P\}$ ,  $i \in N$ .

Lemma 5. If  $Q \in \Omega^p$ , then

$$u_j(Q, q_j^*, t) + \delta^t \bar{V}_j \leq V_j(Q) \quad (2)$$

for all  $q_j^* \in S_j$ ,  $j \in N$ ,  $t = 1, 2, \dots$ .

Proof. Consider any  $\sigma \in \Sigma^p$  such that  $Q = Q(\sigma)$ . Since  $\sigma \in \Sigma^p$ ,  $\sigma|_{H(\sigma, q_j^*, t)} \in \Sigma^p$ .

Hence  $\tilde{V}_j(\sigma|_{H(\sigma, q_j^*, t)}) \geq \bar{V}_j$ . Now see Lemma 2.

Q. E. D.

The next lemma provides necessary and sufficient conditions in terms of one-shot deviations for  $(\bar{Q}^1, \dots, \bar{Q}^n) \in \Omega^n$  to define an optimal SPC. A remark after the proof explains why it is not redundant in view of Theorem 4. The following identity is used freely below:

$$\tilde{V}_j(\sigma^i(Q^1, \dots, Q^n)) = V_j(Q^i).$$

Lemma 6. Let  $\bar{Q}_i \in \Omega$  satisfy  $V_i(\bar{Q}^i) = \bar{V}_i$ ,  $i \in N$ . Then  $(\bar{Q}^1, \dots, \bar{Q}^n)$  defines an optimal simple penal code iff

$$u_j(\bar{Q}^i, q_j^*, t) + \delta^t \bar{V}_j \leq V_j(\bar{Q}^i) \quad (3)$$

for all  $q_j^* \in S_j$ ,  $i \in N$ ,  $j \in N$ ,  $t = 1, 2, \dots$ .

Proof. Suppose  $\sigma(\bar{Q}^i, \bar{Q}^1, \dots, \bar{Q}^n) \in \Sigma^p$ . By Lemma 5, (3) is satisfied. Suppose (3) holds; by Theorem 4,  $\sigma(\bar{Q}^i, \bar{Q}^1, \dots, \bar{Q}^n) \in \Sigma^p$ , all  $i \in N$ .

Q. E. D.

Note that in contrast with Theorem 4, the inequalities above must necessarily hold for  $q_j^* = \bar{q}_j^i(t)$  also. This is because the continuation of  $\bar{Q}^i$  after  $t$  (this is what a player gets if he doesn't deviate, i.e., plays  $q_j^* = \bar{q}_j^i(t)$ ) is also perfect if  $\bar{Q}^i$  is, and therefore cannot be more severe for player  $j$  than  $\bar{Q}^j$ .

Theorem 4 dealt with arbitrary perfect paths  $Q^i$  and  $Q^j$ ; it is then possible that a player for whom  $q_j^i(t)$  is a one-period best response (to  $q_{-j}^i(t)$ ) might find that the costs of signalling a deviation by playing  $q_j^* \neq q_j^i(t)$  outweigh the benefits of a more attractive post-deviation future. Note also that it is sufficient to require that (3) hold only for  $q_j^*$  such that  $q_j^*$  is a one-period best response to  $\bar{q}_{-j}^i(t)$ , provided, of course, that a best response exists.

The next two results are central.

Theorem 7. An optimal penal code exists iff an optimal simple penal code exists.

Proof. Sufficiency is obvious. To prove necessity, let  $(\hat{\sigma}^1, \dots, \hat{\sigma}^n)$  be an OPC and let  $\bar{Q}^i = Q(\hat{\sigma}^i)$ . Lemma 5 applied to  $\bar{Q}^i \in \Sigma^p$  yields the inequalities (2), all  $i \in N$ . Since  $\bar{V}_i = V_i(\bar{Q}^i)$ , Lemma 6 now implies that  $(\bar{Q}^1, \dots, \bar{Q}^n)$  defines an optimal SPC.

Q. E. D.

Theorem 8 indicates how an OPC leads to a very simple characterization of the set of P. E. paths.

Theorem 8. Suppose an OPC exists.  $\Omega^0 \in \Sigma^p$  iff

$$u_j(Q^0, q_j^*, t) + \delta^t \bar{V}_j \leq V_j(Q^0)$$

for all  $q_j^* \in S_j$ ,  $j \in N$ ,  $t = 1, 2, \dots$ .

Proof. Necessity follows from Lemma 5. To prove sufficiency, let  $\bar{Q}^1, \dots, \bar{Q}^n$  define an optimal SPC. Consider  $\sigma(Q^0, \bar{Q}^1, \dots, \bar{Q}^n) \equiv \sigma^0$ . By Theorem 4, the inequalities above, together with those obtained from Lemma 6, imply that  $\sigma^0 \in \Sigma^p$ .

Q. E. D.

Corollary 9. Suppose an OPC exists and let  $(\bar{Q}^1, \dots, \bar{Q}^n)$  define an optimal SPC. Then  $Q^0 \in \Omega^p$  iff  $\sigma(Q^0, \bar{Q}^1, \dots, \bar{Q}^n) \in \Sigma^p$ .

The next theorem, which asserts that SSP's suffice to obtain all P.E. paths (if an OPC exists), follows immediately from Corollary 9.

Theorem 10. Suppose an OPC exists. Then  $Q^0 \in \Omega^p$  iff there exist  $Q^i \in \Omega$ ,  $i \in N$  such that  $\sigma(Q^0, Q^1, \dots, Q^n) \in \Sigma^p$ .

Theorem 11 establishes the existence of an OPC. A comment of Jacques Cr mer [3] greatly simplified my original proof and recently Harris [5] has independently of Cr mer provided a very similar argument. All these proofs exploit in an essential way the idea of an SPC and the one-shot deviation arguments developed above.

Notice that we have reached this point without need for any assumptions.

I now invoke the following:

A1  $S$  is a compact topological space.

A2  $\pi : S \rightarrow \mathbb{R}^n$  is continuous.

Endow  $\Omega = S^{\infty}$  with the product topology. Then  $V : \Omega \rightarrow \mathbb{R}^n$  is continuous and by Tychonoff's Theorem  $\Omega$  is compact.

A3  $G = (\{S_i\}_{i=1}^n; \{\pi_i\}_{i=1}^n)$  has a Nash equilibrium  $q^e \in S$ .

A3 does not play a central role in the argument; it serves merely as a simple sufficient condition to ensure that  $\Sigma^p \neq \emptyset$ . Denote by  $\sigma^e$  the strategy profile defined by:

$$\begin{aligned} \sigma^e(1) &= q^e \\ \sigma^e(t)(H(t-1)) &= q^e, \quad \text{all } H(t-1) \in S^{t-1} \text{ and } t = 2, 3, \dots \end{aligned}$$

It may be easily checked that  $\sigma^e \in \Sigma^p$ .

Theorem 11. Assume A1 - A3. Then an OPC exists.

Proof. By A3,  $\Sigma^P \neq \emptyset$  and  $\bar{V}_i$  is well defined,  $i \in N$ . Consider  $\{\sigma^{i\nu}\}_{\nu=1}^{\infty}$ ,  $Q^{i\nu} = Q(\sigma^{i\nu})$  such that  $\sigma^{i\nu} \in \Sigma^P$  and  $\lim_{\nu \rightarrow \infty} V_i(Q^{i\nu}) = \bar{V}_i$ . Since  $\Omega$  is compact,  $\{Q^{i\nu}\}$  may be taken to be a convergent sequence. Let  $\bar{Q}^i = \lim_{\nu \rightarrow \infty} Q^{i\nu}$ . Since  $V_i(\cdot)$  is continuous,  $V_i(\bar{Q}^i) = \bar{V}_i$ . To complete the proof I show that  $(\bar{Q}^1, \dots, \bar{Q}^n) \in \Omega^n$  defines an optimal SPC.

Suppose not. By Lemma 6,  $u_j(\bar{Q}^i, q_j^*, t) + \delta^t \bar{V}_j > V_j(\bar{Q}^i)$ , some  $i, j, t, q_j^*$ . Since  $Q^{i\nu} \rightarrow \bar{Q}^i$ , by continuity, for  $\nu$  large enough,  $u_j(Q^{i\nu}, q_j^*, t) + \delta^t \bar{V}_j > V_j(Q^{i\nu})$ . By Lemma 5,  $Q^{i\nu} \notin \Omega^P$ , a contradiction.

Q. E. D.

Remark. In certain contexts, for reasons of tractability or otherwise (see the next section), it might be appropriate to restrict attention to penal codes which only pre-scribe outcome paths or punishments contained in some subset  $\psi$  of  $\Omega$ .

D15  $\Sigma(\psi)$

Consider  $\psi \subseteq \Omega$ .  $\sigma \in \Sigma(\psi)$  iff  $\sigma \in \Sigma$  and

- (i)  $Q(\sigma) \in \psi$
- (ii) for any  $H(t) = (q(1), \dots, q(t)) \in S^t$  such that  $q(t) \neq \sigma(t)(q(1), \dots, q(t-1))$ ,  $Q(\sigma|_{H(t)}) \in \psi$ .

Optimal penal codes and simple penal codes relative to  $\psi \subseteq \Omega$  are defined below.

D16 Let  $\psi \subseteq \Omega$ . An optimal penal code relative to  $\psi$  is an n-vector of strategy profiles  $(\hat{\sigma}^1, \dots, \hat{\sigma}^n)$  such that for all  $i$ ,

$$\hat{\sigma}^i \in \Sigma^P \cap \Sigma(\psi) \quad \text{and} \quad \hat{V}_i(\hat{\sigma}^i) = \min \{ \hat{V}_i(\sigma) \mid \sigma \in \Sigma^P \cap \Sigma(\psi) \} .$$

D17 Let  $\psi \subseteq \Omega$ .  $\text{SPC}(Q^1, \dots, Q^n)$  is a simple penal code relative to  $\psi$  iff  $Q^i \in \psi$ , all  $i \in N$ .

The results obtained above for  $\psi = \Omega$  extend easily to the more general case by arguments which are completely analogous and are therefore omitted. In particular the following is true:

Theorem 12. Assume  $\psi$  is compact,  $Q(\sigma^e) \in \psi$  and A1 - A3. Then there exist  $\bar{Q}^i \in \psi$ ,  $i \in N$  such that  $\text{SPC}(\bar{Q}^1, \dots, \bar{Q}^n)$  is an optimal penal code relative to  $\psi$ .

## 5. An Application to Oligopoly

A major question in oligopoly theory is the following: what is the maximal degree of implicit collusion sustainable by credible threats? In Abreu [1] I consider this question in the context of a repeated oligopolistic game. This exercise is perhaps most interesting as a case-study on the use of "simplicity" arguments and as a demonstration of how OPC's may be explicitly determined in concrete applications. I report some of my results below.

I consider identical quantity-setting firms that produce at constant marginal cost, and investigate mainly OPC's relative to the class of symmetric punishments. (A punishment/path is symmetric if it prescribes identical action streams for all players. However, the intertemporal structure of these streams is completely unrestricted. Note that even in a symmetric game, a natural asymmetry is introduced when a player deviates; thus in general OPC's relative to the class of symmetric punishments need not be globally optimal.)

By Theorem 12 I need consider only SPC's. Since only symmetric punishments are allowed, an optimal SPC is defined by a single symmetric path.



We may therefore refer without ambiguity to an optimal symmetric punishment.

Exploiting the one-shot deviation arguments developed above for SPC's, I am able to show that an optimal symmetric punishment:

(i) has a two-phase stick-and-carrot structure. The first phase—the stick—lasts for only one period.<sup>3</sup> The second phase—the carrot—is stationary; it is the most collusive output level which the optimal symmetric punishment can itself support.

(This essentially implies that the optimal symmetric punishment may be obtained as the solution to a pair of simultaneous equations, thus reducing a potentially complex infinite dimensional problem to a trivial two-dimensional one. The almost exclusive reliance on Cournot-Nash punishments in discounted repeated games and in oligopolistic supergames in particular, cannot therefore be justified on pragmatic grounds.)

(ii) has a present discounted value of zero for a significant range of parameter values. Since firms can always obtain a zero payoff by producing nothing forever, for these parameter values the optimal symmetric punishment is globally optimal (i. e., among the class of all punishments, symmetric and otherwise). Thus, in this case, the problem of determining an OPC is completely solved.

These results illustrate a mode of analysis based on SPC's, one-shot deviations, etc. which I believe could be profitably adapted to the particular features of a given economic supergame. While globally optimal penal codes may be difficult to determine in some cases, typically there will be enough structure to derive SPC's

---

<sup>3</sup> The one-period characterization depends on the assumptions that  $\lim_{p \rightarrow 0} D(p) = \infty$ , where  $D(\cdot)$  is the industry demand function, and that marginal cost is strictly positive.

which both can be easily computed and are more severe than Cournot-Nash reversion.

I would like to record here that David Pearce made a very important contribution to this research. I started investigating stick-and-carrot punishments at his instigation. He suggested that they might fare better than Cournot-Nash punishments; as it turned out, his intuition was richly confirmed.

## 6. Conclusions

The goal of this paper was to explore the structure of infinitely repeated games with discounting. While infinitely repeated games without discounting are well understood [2, 6], less seems to be known about their no-discounting counterparts. The extreme difficulty of the latter stems from the complex patterns of intertemporal trade-offs that arise in the discounting case; these trade-offs are trivialized when only the long run matters.

My results provide a general framework for analyzing the set of P. E. outcome paths of the discounting model. Essential to this framework is the concept of an OPC, and the two related notions of an SPC and an SSP. It was shown that there exists an SPC which is an OPC. An implication of this result is that the discounting model may be completely analyzed in terms of simple strategy profiles. SSP's, like SPC's, are "simple" in the relevant sense: they may be easily checked to be perfect. In particular, only one-shot deviations from at most  $(n+1)$  outcome paths need be considered.

The theory developed here does not apply to the no-discounting case. Optimal penal codes are in general not defined, and nontrivially history-dependent

punishments need to be employed; simple strategy profiles do not suffice to generate all P. E. outcome paths.

The results reported in Section 5 illustrate how the general framework presented here can be exploited, and suggest that the optimality and simplicity approach could be fruitfully applied to other repeated economic models with discounting. In particular, the somewhat unimaginative reliance on Cournot-Nash reversion to support cooperative behavior does not seem tenable on theoretical or pragmatic grounds. These results also illustrate an important general point which was referred to in earlier sections. Optimality might require that a deviant "cooperate" (in a one-period sense) in his own punishment. While there is no theoretical presumption to the contrary, it is in a naive sense "counterintuitive" and appears to have been overlooked. Whereas in the no-discounting model, the possibility of "cooperative" deviants is a redundant nicety, it is often critical in a world with discounting.

Analogues to my theorems ought to appear in any model with discounting and a "repeated" structure. Finally, the conceptualization of strategy profiles in terms of outcome paths and deviations from prescribed outcome paths should prove useful in other contexts.

## Bibliography

- [1] Abreu, D. : "Repeated Games with Discounting: A General Theory and an Application to Oligopoly," mimeo, 1982.
- [2] Aumann, R. J. and L. Shapley: "Long Term Competition – A Game Theoretic Analysis," mimeo, 1976.
- [3] Crémer, J. : Personal Communication, 1983.
- [4] Friedman, J. W. : "A Non-cooperative Equilibrium for Supergames," Review of Economic Studies, 28 (1971), 1-12.
- [5] Harris, C. : "A Note on the Existence of Optimal Simple Penal Codes," mimeo, 1984.
- [6] Rubinstein, A. : "Equilibrium in Supergames with the Overtaking Criterion," Journal of Economic Theory, 21 (1979), 1-9.
- [7] Selten, R. : "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragerträchtigkeit," Zeitschrift für die Gesamte Staatswissenschaft 121 (1965), 301-324 and 667-689.
- [8] Selten, R. : "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory 4 (1975), 22-55.

- | #  | Author(s)   | Title  | #  | Author(s)  | Title  |
|----|---|--|----|--|--|
| 50 | <b>C. Johnson, W. Barrett,</b>                          | Spanning Tree Extensions of the Hadamard-Fischer Inequalities                            | 73 | <b>S. Pelikan,</b>   | A Dynamical Meaning of Fractal Dimension   |
| 51 | <b>Andrew Postlewaite, David Schmiedler,</b>            | Revelation and Implementation under Differential Information                             | 74 | <b>D. Heath, W. Sudderth,</b>                                  | Continuous-Time Portfolio Management: Minimizing the Expected Time to Reach A G                          |
| 52 | <b>Paul Blanchard,</b>                                  | Complex Analytic Dynamics on the Riemann Sphere  | 75 | <b>J.S. Jordan,</b>  | Information Flows Intrinsic to the Stability Economic Equilibrium  |
| 53 | <b>G. Levitt, H. Rosenberg,</b>                         | Topology and Differentiability of Labyrinths in the Disc and Annulus                     | 76 | <b>J. Jerome,</b>  | An Adaptive Newton Algorithm Based on Numerical Inversion: Regularization Post Condition                 |
| 54 | <b>G. Levitt, H. Rosenberg,</b>                         | Symmetry of Constant Mean Curvature Hyper-surfaces in Hyperbolic Space                   | 77 | <b>David Schmiedler,</b>                                       | Integral Representation Without Additivity   |
| 55 | <b>Ennio Stacchetti,</b>                                | Analysis of a Dynamic, Decentralized Exchange Economy                                    | 78 |  | Abstracts for the Workshop on Bayesian Analysis in Economics and Game Theory                             |
| 56 | <b>Henry Simpson, Scott Spector,</b>                    | On Failure of the Complementing Condition and Nonuniqueness in Linear Elastostatics      | 79 | <b>G. Chichilnitsky, G.M.Heal,</b>                             | Existence of a Competitive Equilibrium in $L_p$ and Sobolev Spaces                                       |
| 57 | <b>Craig Tracy,</b>                                     | Complete Integrability in Statistical Mechanics and the Yang-Baxter Equations            | 80 | <b>Thomas Seidman,</b>   | Time-dependent solutions of a Nonlinear system in Semiconductory Theory, II: Boundedness and Periodicity |
| 58 | <b>Tongren Ding,</b>                                    | Boundedness of Solutions of Duffing's Equation   | 81 | <b>Yakar Kannal,</b>   | Engaging in R&D and the Emergence of Expected Non-convex Technologies                                    |
| 59 |   | Abstracts for the Workshop on Price Adjustment, Quantity Adjustment, and Business Cycles | 82 | <b>Herve Moulin,</b>   | Choice Functions over a finite set: A Summary  |
| 60 | <b>Rafael Rob,</b>                                      | The Coase Theorem an Informational Perspective   | 83 | <b>Herve Moulin,</b>   | Choosing from a Tournament   |
| 61 | <b>Joseph Jerome,</b>                                   | Approximate Newton Methods and Homotopy for Stationary Operator Equations                | 84 | <b>David Schmiedler,</b>                                       | Subjective Probability and Expected Utility Without Additivity   |
| 62 | <b>Rafael Rob,</b>                                      | A Note on Competitive Bidding with Asymmetric Information                                | 85 | <b>I.G. Kevrekidis, R. Aris, L.D. Schmidt, and S. Pelikan,</b> | The Numerical Computation of Invariant Circles of Maps   |
| 63 | <b>Rafael Rob,</b>                                      | Equilibrium Price Distributions  | 86 | <b>F. William Lawvere,</b>                                     | State Categories, Closed Categories, and the Existence Semi-Continuous Entropy Functions                 |
| 64 | <b>William Ruckle,</b>                                  | The Linearization Projection, Global Theories  | 87 | <b>F. William Lawvere,</b>                                     | Functional Remarks on the General Concept of Chaos   |
| 65 | <b>Russell Johnson, Kenneth Palmer, George R. Sell,</b> | Ergodic Properties of Linear Dynamical Systems   | 88 | <b>Steven R. Williams,</b>                                     | Necessary and Sufficient Conditions for the Existence of a Locally Stable Message Process                |
| 66 | <b>Stanley Reiter,</b>                                  | How a Network of Processors can Schedule Its Work  | 89 | <b>Steven R. Williams,</b>                                     | Implementing a Generic Smooth Function   |
| 67 | <b>R.N. Goldman, D.C. Heath,</b>                        | Linear Subdivision Is Strictly a Polynomial Phenomenon                                   |    |  |  |
| 68 | <b>R. Glachett, R. Johnson,</b>                         | The Floquet Exponent for Two-dimensional Linear Systems with Bounded Coefficients        |    |  |  |
| 69 | <b>S. Williams,</b>                                     | Realization and Nash Implementation: Two Aspects of Mechanism Design                     |    |  |  |
| 70 | <b>S. Williams,</b>                                     | Sufficient Conditions for Nash Implementation  |    |  |  |
| 71 | <b>Nicholas Yannellis, William R. Zame,</b>             | Equilibria in Banach Lattices Without Ordered Preferences                                |    |  |  |
| 72 | <b>W. Harris, Y. Sibuya,</b>                            | The Reciprocals of Solutions of Linear Ordinary Differential Equations                   |    |  |  |