AN ANSWER TO LITTLEWOOD'S PROBLEM ON BOUNDEDNESS FOR SUPER-LINEAR DUFFING'S EQUATIONS

BY

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An Answer to Littlewood's Problem on Boundedness for Super-Linear Duffing's Equations

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Abstract We consider the super-linear Duffing's equation

$$x'' + g(x) = p(t) , \qquad (*)$$

where p(t) is a continuous and periodic function in $t \in R$, and g(x) is a continuously differentiable function in $x \in R$, satisfying the super-linear condition: $g(x)/x \to \infty$, as $x \to \pm\infty$. The purpose of this paper is to prove the following theorem, which leads to an answer to Littlewood's problem on boundedness for super-linear Duffing's equations.

Theorem Every solution of the super-linear Duffing's equation (*) is bounded for $t \in R$.

I) Introduction

The long-time behavior of solutions for a periodic Hamiltonian system can be very intricate, and its study challenges one's attention . For example, J. Littlewood [1], L. Markus [2], and J. Moser [3] propose the consideration of the Duffing's equation

$$x'' + g(x) = p(t) , \qquad (1)$$

where p(t) is a continuous and 2π-periodic function in $t \in R$, and g(x) is a continuously differentiable function in $x \in R$.

The so-called Littlewood's problem on boundedness consists of the following two parts:

<u>Part A</u> Every solution of the Duffing's equation (1) is bounded for $t \in R$, if the super-linear condition

$$\text{(a)}: \qquad g(x)/x \longrightarrow \infty, \quad \text{as } x \longrightarrow \pm\infty,$$

is satisfied;

<u>Part B</u> Every solution of the Duffing's equation (1) is bounded for $t \in R$, if the sub-linear condition

$$\text{(b)}: \quad g(x)/x \longrightarrow 0, \quad g(x)\cdot \mathrm{sgn}(x) \longrightarrow \infty, \quad \text{as } x \longrightarrow \pm\infty,$$

is satisfied.

There are a few works contributing to some special cases for Part A, while there is nothing done even in a particular example for Part B. We will give here a quick survey for the results related to Part A.

In 1976, using the Moser's twist-theorem, G. Morris [4] proved that every solution of the Duffing's equation

$$x'' + 2x^3 = p(t) ,$$

is bounded for $t \in R$, where $p(t)$ is a periodic function in $t \in R$. Recently, R. Dieckerhoff and E. Zehnder in a preprint [5] proved in a similar way that every solution of the the equation

$$x'' + x^{2n+1} + \sum_{j=0}^{2n} p_j(t)x^j = 0 ,$$

is bounded for $t \in R$, where $p_j(t)(j = 1, \cdots, 2n)$ are 2π-periodic and sufficiently smooth in $t \in R$. The present author in a recent paper [6] proved an existence theorem for Lagrange's positively and negatively stable continuums passing through a given fixed-point of an area-preserving mapping, and then applied it to prove the following result: Every solution of the Duffing's equation (1) is bounded for $t \in R$, if the super-linear condition (a) holds and $p(t)$ is even[or, $p(t)$ and $g(x)$ are odd].

The purpose of this paper is to prove the following theorem, which leads to an answer to Littlewood's problem on boundedness for super-linear Duffing's equations.

<u>Theorem A</u> Every solution of the Duffing's equation (1) with the super-linear condition (a) is bounded for $t \in R$.

The method used in this paper is the same one developed in [6] with a few of modifications and simplifications, and it can be applicable to a wider class of periodic Hamiltonian systems of second order. However, compared with the Moser's twist-theorem used in [4] and [5], our method provides us with less information about the periodic solutions and the almost periodic solutions for the equation in the consideration.

II) Preliminaries

We first introduce some notations and some lemmas.

Denote by B_r the open disk in the (x,y)-plane with radius r centered at the origin, and denote its boundary by S_r or ∂B_r, where ∂ is the boundary operator for sets in R^2. Let A(c,d) be the open annulus between S_c and S_d $(d > c > 0)$. And let the closure of a set M be denoted by $\overline{M}$.

Let

$$x = r\cos\theta \quad , \quad y = r\sin\theta \ , \quad (r \geq 0) \ .$$

Then denote by L_a the vertical line $r = a$ in the Cartesian (r,θ)-plane, and denote by T(c,d) the open strip between L_c and L_d $(d > c)$.

Therefore, L_c and L_d and T(c,d) are the liftings of S_c, S_d and A(c,d) to the (r,θ)-plane, respectively.

<u>Lemma 1</u> Let E be a connected open set in the (x,y)-plane with

$$E \cap S_c \neq \emptyset \ , \quad E \cap S_d \neq \emptyset \ , \quad (d > c > 0) \ ,$$

and let I be a connected closed set connecting S_c and S_d with

$$J \cap E = \emptyset .$$

Then there exists a connected closed set J in $\partial E \cap \bar{A}(c,d)$ connecting S_c and S_d .

(This Lemma can be proved in the same way as employed in the proof of the Proposition 3 in [6], and its proof is thus omitted here.)

Let C^* be a continuous and simple curve in $\bar{T}(c,d)$ connecting L_c and L_d $(c < d)$. Then C^* separates the open strip $T(c,d)$ in two disjoint simple and unbounded domains; i. e. ,

$$T(c,d) \setminus C^* = D_a(C^*) \cup D_b(C^*) ,$$

where $D_a(C^*)$ is a simple domain extending to infinity in the positive direction along the θ-axis in $T(c,d)$, and $D_b(C^*)$ is a simple domain extending to infinity in the negative direction along the θ-axis in $T(c,d)$. For simplicity, a set M in $T(c,d)$ is said to be above C^* (or below C^*) if $M \subset D_a(C^*)$ holds (or $M \subset D_b(C^*)$ holds) . Hence, $D_a(C^*)$ is is above C^*, and $D_b(C^*)$ is below C^*.

A set M in $T(c,d)$ is said to be bounded above (or, below) if M is below (or, above) a continuous and simple curve C' in $\bar{T}(c,d)$ connecting L_c and L_d .

Let Q be a simple domain in $T(c,d)$ with

$$\partial Q = M \cup P \cup N \quad ,$$

where M, P and N are disjoint, P is a connected closed set in $T(c,d)$, $M \subset T(c,e)$ and $N \subset T(e,d)$ are continuous curves $(c < e < d)$.

<u>Lemma 2</u> Let Q be defined as above. Then we have:

(i) If M and N are bounded above, and P is below C^*, then any point

$$p \in L_e \cap D_a(C^*)$$

does not belong to Q ;

(ii) If M and N are bounded below, and P is above C*, then any point

$$p \in L_e \cap D_b(C^*)$$

does not belong to Q .

<u>Proof</u> We only prove the conclusion (i), and omit the proof of (ii) since it is similar to that of (i).

In the case of (i), we think of P as the top boundary of Q, M as the left boundary of Q, and N as the right boundary of Q. It follows that

$$Q \setminus C^* = \left(\bigcup_{j=1}^{\infty} Q_{a,j}\right) \cup \left(\bigcup_{k=1}^{\infty} Q_{b,k}\right) ,$$

where $Q_{a,j}$ is a simple domain (or an empty) above C* (j=1,2, ···), and $Q_{b,k}$ is a simple domain (or an empty) below C* (k=1,2, ···).

Hence, any point p in $D_a(C^*)$ does not belong to $Q_{b,k}$ (k=1,2, ···).

Since P is below C*, then $Q_{a,j}$ is bounded by C*, M and N (j=1,2, ···). Note that M and N are bounded above in T(c,e) and T(e,d), respectively. It follows that for any j, $Q_{a,j}$ is a bounded domain in T(c,e) or T(e,d). Hence , any point p on L_e does not belong to $Q_{a,j}$ (j=1,2, ···).

Note that any point p in $D_a(C^*)$ does not belong to C*. It follows that any point in $D_a(C^*) \cap L_e$ does not belong to Q .

The proof of Lemma 2 is thus completed.

A connected closed set M in R^2 is said to be simply connected if for any bounded open set in R^2, " $\partial E \subset M$ " yields " $E \subset F$ " .

Let

$$F : R^2 \longrightarrow R^2 ,$$

be a homeomorphism. A set K in R^2 is said to be positively (or negatively) stable for F in the Lagrange's sense if for any point p in K the sequence

$$\left\{ F^k(p) \right\} , \quad \text{for } k \in Z^+ , \text{ (or, for } k \in Z^-) ,$$

is bounded .

Now, we state the following basic Lemma, which was proved in [6] .

<u>Lemma 3</u> If $F : R^2 \longrightarrow R^2$, is an area-preserving homeomorphism, then for any given fixed-point z_0 of F, we have a positively stable set $K_r^+ = K_r^+(z_0, F)$ and a negatively stable set $K_r^- = K_r^-(z_0, F)$, $(r > |z_0|)$, with

1*) K_r^+ and K_r^- are simply connected closed sets in $\bar{B}_r$;

2*) Both K_r^+ and K_r^- pass through z_0 and intersect S_r ;

3*) K_r^+ and K_r^- are positively and negatively invariant with respect to F; i. e. ,

$$F(K_r^+) \subset K_r^+ \quad , \quad F^{-1}(K_r^-) \subset K_r^- \quad ;$$

4*) $K_r^+ \subset K_s^+$ and $K_r^- \subset K_s^-$, for any $s > r$.

III) <u>Twist Property</u>

It is well-known that the Duffing's equation (1) is equivalent to a Hamiltonian system

$$x' = H_y'(t,x,y) \ , \quad y' = - H_x'(t,x,y) \ , \tag{2}$$

where the Hamiltonian function H is defined by

$$H(t,x,y) = \tfrac{1}{2}y^2 + \int_0^x g(x)dx - p(t)x \ , \quad \text{for } (t,x,y) \in R^3 \ .$$

Let $(u,v) \in R^2$, and let

$$x = x(t,u,v) \ , \quad y = y(t,u,v) \ , \tag{3}$$

be the solution of (2), satisfying the initial condition:

$$x(0,u,v) = u \ , \quad y(0,u,v) = v \ .$$

From now on, assume the super-linear condition (a) is satisfied. Then it can be shown that the solution (3) exists for $t \in R$, and it is continuously differentiable in $(t,u,v) \in R^3$. Moreover, the Poincare map

$$F: \quad (u,v) \longmapsto \left(x(2\pi, u, v),\ y(2\pi, u, v) \right) ,$$

is an area-preserving diffeomorphism, and possesses an infinite set of fixed-points extending to infinity in R^2, (see [7]).

Set

$$u = r\cdot\cos\theta , \quad v = r\cdot\sin\theta , \quad (r \geq 0) ,$$

and put (3) in the form

$$x = \rho\cos\phi , \quad y = \rho\sin\phi ,$$

where

$$\rho = \rho(t,r,\theta) = \sqrt{x^2(t,u,v) + y^2(t,u,v)} .$$

Hence, $\rho(t,r,\theta)$ is continuous in $(t,r,\theta) \in R \times [0, \infty) \times R$ and 2π-periodic in θ. It is not hard to prove that there is a positive constant a_0 such that $\rho(t,r,\theta) > 0$, whenever $r > a_0$ and $(t,\theta) \in [0,2\pi] \times R$. In this case, (2) is equivalent to the following system

$$\begin{cases} \rho' = [\rho\cos\phi - g(\rho\cos\phi) + p(t)]\sin\phi , \\ \phi' = \frac{1}{\rho} g(\rho\cos\phi)\cos\phi - \sin^2\phi + \frac{1}{\rho} p(t)\cos\phi . \end{cases}$$

It follows that $\rho(t,r,\theta)$ and $\phi(t,r,\theta)$ are continuously differentiable in $t \in [0,2\pi]$, $r \in (a_0, \infty)$ and $\theta \in R$, with $\rho(0,r,\theta) = r$ and $\phi(0,r,\theta) = \theta$. Moreover, we have

$$\phi(t,r,\theta+2\pi) = \phi(t,r,\theta) + 2\pi .$$

Hence, for $r > a_0$, the above Poincare map F can be written in the form

$$\rho_1 = \rho(2\pi, r, \theta) , \quad \phi_1 = \theta + h(r,\theta) , \tag{4}$$

where $\rho(2\pi,r,\theta)$ and $h(r,\theta) = \phi(2\pi,r,\theta) - \theta$, are continuous in $(r,\theta) \in (a_0, \infty) \times R$, and 2π-periodic in $\theta \in R$.

The following Lemma 4 is just the Lemma 4.3 in [7], which expresses the twist property of the Poincare map F for the super-linear Duffing's equation (1).

Lemma 4 Assume the super-linear condition (a) holds. Then, for any positive integer m, there is a constant a_1, ($a_1 > a_0$), such that

$$h(r,\theta) < -2m\pi ,$$

whenever $r > a_1$ and $\theta \in R$.

IV) Main Theorem

Now we are in a position to prove the following main theorem of this paper .

Theorem A If the super-linear condition (a) is satisfied, then every solution of the Duffing's equation (1) is bounded for $t \in R$.

Proof Using Lemma 3, we have

$$K_r^+ = K_r^+(z_0, F) , \quad K_r^- = K_r^-(z_0, F) ,$$

where F is the Poincare map of (2) with (a), and z_0 is a fixed-point of F, $(r > |z_0|)$.

It suffices to prove that for any constant $d > 0$, there is a constant c, $[\, c > \max(d, |z_0|)\,]$, such that

$$B_d \subset K_c^+ \cap K_c^- \tag{5}$$

holds.

(In fact, it follows from (5) that

$$F^k(B_d) \subset F^k(K_c^+) \subset K_c^+ \subset \bar{B}_c ,$$

holds for $k \in Z^+$, and

$$F^k(B_d) \subset F^k(K_c^-) \subset K_c^- \subset \bar{B}_c ,$$

holds for $k \in Z^-$. Hence, for any point $(u,v) \in B_d$, the solution (3) of the system (2) is bounded for $t \in R$. Since d is arbitrary, we arrive at the desired result of Theorem A .)

In order to prove (5), we first claim that for any constant $d>0$,

$$B_d \subset K_c^+ \tag{5'}$$

holds for some constant $c>0$.

The proof of (5') is rather long, and thus we will devide it into the following seven steps.

<u>Step 1</u>: Assume the contrary; i. e., (5') is false. Then there is a constant r_o, $(r_o > |z_o|)$, such that

$$(B_r \setminus K_r^+) \cap B_{r_o} \neq \emptyset , \quad \text{for} \quad r > r_o . \tag{6}$$

Without loss of generality, assume

$$r_o > a_o > |z_o| ,$$

where a_o is a constant chosen in III).

Now, choose constants r_1 and r_2, $(r_o < r_1 < r_2)$, satisfying

$$B_{r_o} \subset F(\overline{B}_{r_1}) \subset B_{r_2} . \tag{7}$$

Since F possesses infinitely many fixed-points extending to infinity in R^2, we can take a fixed-point z_1, with sufficiently large $|z_1|$. Hence, without destroying (7), we can take $r_2 = |z_1|$, and thus we have

$$z_1 \in S_{r_2} .$$

Then we get a constant

$$h_o = \max |h(r_2,\theta)| , \quad \text{for} \quad \theta \in R . \tag{8}$$

Let m^* and m be two positive integers satisfying

$$2\pi m^* > h_o , \quad m > m^* + 2 . \tag{9}$$

Using Lemma 4, we get a constant d_2 such that

$$h(r,\theta) < -2m\pi , \quad \text{for} \quad r \geqq d_2 , \ \theta \in R . \tag{10}$$

Moreover, we can take the above d_2, together with

$$d_2 > r_2 \quad \text{and} \quad B_{r_2} \subset F(\overline{B}_{d_2}) , \tag{11}$$

and a fixed-point

$$z_2 \in S_{d_2} \ .$$

Now, choose constants d_1 and d_0 $(d_2 < d_1 < d_0)$ satisfying

$$B_{d_2} \subset F(\overline{B}_{d_1}) \subset B_{d_0} \ . \qquad (12)$$

Step 2: Set

$$B_{d_0} \setminus K^+_{d_0} = \bigcup_{i=1}^{\infty} E_i \ ,$$

where $E_i \subset B_{d_0}$ is a simple domain (or an empty set), with

$$E_i \cap E_j = \emptyset \ , \quad \text{for } i \neq j \ .$$

It follows from (6) that there is at least one simple domain, say E_1 for definiteness, satisfying

$$E_1 \cap S_{r_0} \neq \emptyset \ .$$

Since $K^+_{d_0}$ is a simply connected closed set in $\overline{B}_{d_0}$ intersecting S_{d_0}, then

$$A_1 = (\overline{E}_1 \cap S_{d_0}) \setminus K^+_{d_0}$$

is a non-degenerate open arc of S_{d_0}. Note that E_1 is a simple domain . Then there is a continuous and simple curve

$$C : \quad z = z(t) \ , \quad (0 \leq t \leq 1) \ ,$$

in $(E_1 \cup A_1) \cap A(r_0, d_0)$, connecting S_{r_0} and S_{d_0}. It follows that

$$C \cap K^+_{d_0} = \emptyset \ . \qquad (13)$$

Perturbating C slightly if necessary, we can assume that C does not pass through the above-chosen fixed-points z_1 and z_2 .

Step 3: Set

$$z(t) = (\ \eta(t)\cos@(t),\ \eta(t)\sin@(t)\ ) \ , \quad (0 \leq t \leq 1),$$

where $\eta(t) = |z(t)|$, $(r_0 \leq |\eta(t)| \leq d_0)$. Then @(t) is dedermined up to 2π , and

$$\theta = @(t) = @_o(t) + 2j\pi \quad , \qquad (j \in Z) \ ,$$

where $@_o(t), (0 \leq t \leq 1)$, is a single-valued continuous function with $@_o(0) = \theta_o$, $(0 \leq \theta_o < 2\pi)$.

Then, in the closed strip $\overline{T}(r_o, d_o)$, we have an infinite number of continuous and simple curves

$$C^{(j)} \ : \quad r = \eta(t) \ , \qquad \theta = @_o(t) + 2j\pi \ , \qquad (0 \leq t \leq 1),$$

$(j \in Z)$, which are liftings of C. Note that for each $j \in Z$, $C^{(j)}$ is a translation of $C^{(o)}$ along the θ-axis by a displacement $2j\pi$. Since $K^+_{d_o}$ is a connected set joining S_{r_o} and S_{d_o}, then from (13) we get

$$C^{(j)} \cap C^{(i)} = \emptyset \ , \qquad \text{for} \quad i \neq j \ .$$

Therefore, $C^{(j)}$, $C^{(j+1)}$, L_{r_o} and L_{d_o} bound a unique simple and bounded domain $V^{(j)}$ in $T(r_o, d_o)$, for each $j \in Z$. Note that $V^{(j)}$ is a translation of $V^{(o)}$ along the θ-axis by a displacement $2j\pi$. Hence, we have

$$(r,\theta) \in V^{(j)} \quad \text{iff} \quad (r, \theta - 2j\pi) \in V^{(o)} \ .$$

Let

$$z_1 = (r_2 \cos\beta_o, r_2 \sin\beta_o) \ , \qquad z_2 = (d_2 \cos\xi_o, d_2 \sin\xi_o) \ .$$

Since C does not contain the fixed-points z_1 and z_2 , we can take β_o and ξ_o , such that

$$(r_2, \beta_o) \in V^{(o)} \ , \qquad (d_2, \xi_o) \in V^{(o)} \ .$$

It follows that for each $j \in Z$,

$$z_1^{(j)} = (r_2, \beta_o + 2j\pi) \ , \qquad z_2^{(j)} = (d_2, \xi_o + 2j\pi) \ , \qquad (14)$$

are liftings of z_1 and z_2 in $V^{(j)}$, respectively.

<u>Step 4</u>: Let

$$F^* \ : \ T(a_o, \infty) \longrightarrow T(0, \infty) \ ,$$

be the mapping defined by (4); i. e., F^* is a lifting of F restricted in $R^2 \setminus \overline{B}_{a_0}$. Since z_1 and z_2 are fixed-points of F, we get

$$F^*(z_1^{(o)}) = z_1^{(k)} , \quad F^*(z_2^{(o)}) = z_2^{(n)} , \tag{15}$$

for some integers k and n.

Then we claim :

$$k > n + 4 . \tag{16}$$

In fact, from (4), (14) and (15), we have

$$2\pi k = h(r_2, \beta_0) , \quad 2\pi n = h(d_2, \xi_0) .$$

Then, from (8) and (10), we have

$$2\pi k \geq - h_0 , \quad n < - m .$$

It follows from (9) that

$$k > - m^* > -m + 2 > n + 2 ,$$

which yields (16), since k, m^*, m and n are integers.

<u>Step 5</u>: It may be noted that

$$E_1 \cap S_{r_1} \neq \emptyset , \quad E_1 \cap S_{d_1} \neq \emptyset ,$$

and

$$E_1 \cap K^+_{d_0} = \emptyset .$$

Then, from Lemma 1, we get a connected closed set

$$J \subset (\partial E_1 \cap K^+_{d_0}) \cap \overline{A}(r_1, d_1) ,$$

connecting S_{r_1} and S_{d_1}. From (13), we have

$$C \cap J = \emptyset .$$

It follows that we have the liftings $J^{(j)}$ ($j \in Z$) of J satisfying

$$J^{(j)} \subset V^{(j)} \cap \overline{T}(r_1, d_1) ,$$

where $J^{(j)}$ is a connected closed set joining L_{r_1} and L_{d_1}, for each $j \in Z$. Note that

$$J^{(o)} \subset V^{(o)} \cap \overline{T}(r_1,d_1) .$$

<u>Step 6</u>: We want to prove

$$F^*(J^{(o)}) \subset D_a(C^{(k-1)}) \cap D_b(C^{(n+2)}) , \tag{17}$$

which is a key point in our proof .

For this aim, we set

$$T(r_1,d_1) \setminus J^{(o)} = G_a \cup G_o \cup G_b ,$$

where G_a, G_o and G_b are disjoint, $G_o \subset V^{(o)}$ is an open set (or empty), $G_a \subset D_a(C^{(o)})$ and $G_b \subset D_b(C^{(1)})$ are simple and unbounded domains. It can be seen that

$$\partial G_a = M_a \cup J_a^{(o)} \cup N_a ,$$

where $J_a^{(o)} \subset J^{(o)}$ is a connected closed set connecting L_{r_1} and L_{d_1}, $M_a \subset L_{r_1}$ and $N_a \subset L_{d_1}$ are open half-lines above $C^{(o)}$; i. e., $J_a^{(o)}$ is the bottom boundary of G_a, M_a is the left boundary of G_a, and N_a is the right boundary of G_a .

Similarly, we have

$$\partial G_b = M_b \cup J_b^{(o)} \cup N_b ,$$

where the top boundary $J_b^{(o)} \subset J^{(o)}$ is a closed connected set joining L_{r_1} and L_{d_1} , the left boundary $M_b \subset L_{r_1}$ and the right boundary $N_b \subset L_{d_1}$ are open half-lines below $C^{(1)}$.

Since $J_b^{(o)} \subset V^{(o)}$ is above $C^{(o)}$, we have

$$z_1^{(-1)} \in G_b .$$

Similarly, we have

$$z_2^{(1)} \in G_a$$

Since (4) and (15) yield

$$z_1^{(k-1)} = F^*(z_1^{(-1)}) \ , \quad z_2^{(n+1)} = F^*(z_2^{(1)}) \ ,$$

then we have

$$z_1^{(k-1)} \in F^*(G_b) \ , \quad z_2^{(n+1)} \in F^*(G_a) \ . \tag{18}$$

<u>Step 7</u>: In order to use Lemma 2, we consider the simple domain $Q = F^*(G_b)$ and its boundary

$$\partial Q = M \cup P \cup N \ ,$$

where

$$P = F^*(J_b^{(o)}), \quad M = F^*(M_b), \quad N = F^*(N_b) \ .$$

From (7) and (12), we have

$$Q \cup P \subset T(r_o, d_o), \quad M \subset T(r_o, r_2), \quad N \subset T(d_2, d_o) \ ,$$

where P is a connected closed set, M and N are continuous curves bounded above. Note that M, P and N are disjoint.

Since

$$z_1^{(k-1)} \in L_{r_2} \cap V^{(k-1)} \ ,$$

we have

$$z_1^{(k-1)} \in L_{r_2} \cap D_a(C^{(k-1)}) \ . \tag{19}$$

If

$$P \cap C^{(k-1)} \neq \emptyset \ , \quad (\text{i. e., } F^*(J_b^{(o)}) \cap C^{(k-1)} \neq \emptyset),$$

then we arrive at

$$F(J) \cap C \neq \emptyset \ .$$

Since $F(J) \subset F(K^+_{d_o}) \subset K^+_{d_o}$, we get

$$K^+_{d_o} \cap C \neq \emptyset \ ,$$

which contradicts to (13) . Therefore, we have

$$P \cap C^{(k-1)} = \emptyset \ .$$

It follows that P is either above $C^{(k-1)}$ or below $C^{(k-1)}$.

Assume P is below $C^{(k-1)}$. Then, using Lemma 2(i), we conclude that any point

$$p \in L_{r_2} \cap D_a(C^{(k-1)})$$

does not belong to Q. It follows from (19) that $z_1^{(k-1)}$ does not belong to $Q = F^*(G_b)$. However, this conclusion is in conflict with (18).

Hence, P is above $C^{(k-1)}$. It follows that $F^*(J^{(o)})$ is above $C^{(k-1)}$, since $F^*(J^{(o)}) \supset F^*(J_b^{(o)}) = P$, $F^*(J^{(o)}) \cap C^{(k-1)} = \emptyset$ and $F^*(J^{(o)})$ is a connected closed set.

In a similar way, we can prove that $F^*(J^{(o)})$ is below $C^{(n+2)}$.

Hence, we have proved (17), which implies that $C^{(n+2)}$ is above $C^{(k-1)}$. It follows that $(k-1) < (n+2)$. Hence, we have

$$k < n + 3 ,$$

which is in conflict with (16). This contradiction proves the desired conclusion (5').

In a similar way, replacing F by F^{-1}, and K_r^+ by K_r^- , we can prove that for any constant $d > 0$,

$$B_d \subset K_c^- \qquad (5'')$$

holds for some constant $c > 0$.

From (5'),(5") and 4*) of Lemma 3, we then arrive at (5). Thus we have proved Theorem A.

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