Slowly Converging Mean Curvature Flow on Warped Manifolds

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Dedication

To my parents and friend of my youth

Abstract

In this thesis, we analyze mean curvature flow on warped manifolds which possesses handy parameterization over smooth graphs, and explicit perimeter minimizer. We first provide the long-time existence of the flow by using the theory of quasilinear parabolic equation. We next show a convergence analysis via the gradient flow structure by using the standard method of Lojasiewicz and Simon. We also include other classic methods from literature to obtain the quantitative convergence. In the last part, we establish the sharp quantitative convergence of the considered mean curvature flow to their long-time limits, and the center manifold reduction based on the Lyapunov-Schmidt reduction of energy functionals. In particular, we can construct slowly converging mean curvature flow, or that converging with algebraic rates.

Contents

Chapter 1

Introduction

In this thesis, we study the quantitative convergence of volume-preserving mean curvature flow $E(t)$ on warped manifold $\mathcal{M} = \mathbb{S}(R) \times_\varphi M$ given by the law [17, 3],

$$V(t) = \overline{H}(t) - H(t), \tag{1.1}$$

where $V(t)$ and $H(t)$ are normal velocity and mean curvature of $E(t)$, $\overline{H} = \fint_{\partial E} H \, \mathrm{d}\sigma$ is the averaged mean curvature, and σ is the surface measure on $\partial E(t)$. Here the warped product $B \times_\varphi Q$ between two manifolds (B, g_B) and (Q, g_Q) is defined as the product manifold with a warped metric given by $g := g_B + \varphi^2 g_Q$.

Equation (1.1) arises as the geometric gradient flow of the perimeter functional on $\mathcal{M}$ defined by (see, for example, [36, Chapter 1.3.2]),

$$P_\mathcal{M}(E) := \sup \left\{ \int_E \mathrm{div}_\mathcal{M}(\psi) \, \mathrm{dvol}_\mathcal{M} : \psi \in C^1(\mathcal{M}, T\mathcal{M}) \text{ and } g_\mathcal{M}(\psi, \psi) \leq 1 \right\},$$

where $\mathrm{div}_\mathcal{M}$ is the divergence operator, $\mathrm{vol}_\mathcal{M}$ is the volume measure, and $C^1(\mathcal{M}, T\mathcal{M})$ is the space of C^1 vector fields. We refer readers to notes [41, 10, 24, 26] and references therein for the study of mean curvature related flows. These equations have natural and rich applications in variational problems concerning the shape of perimeter minimizers among suitable class of objects, including minimal surfaces and isoperimetric problems [34, 7, 36], and also in physics and computer vision, including crystal surface [28, 6], phase separation [4], and image regularization [42].

Mean curvature flow on Riemannian manifolds was considered in many works [39, 3, 2, 25, 30] in recent decades. For warped manifolds in our case, Borisenko and Miquel [2] studied mean curvature flows on $M \times_\varphi \mathbb{R}$ in which they showed the long-time existence of graph $E(t) = (x, u(x,t))$ following classical estimates of mean curvature flow of graphs [13, 14]. In addition, they also discussed several technical difficulties of extending their approach to the long-time existence on $\mathbb{R} \times_\varphi M$. The later was solved by Huang, Zhang, and Zhou [25] in which they showed the long-time existence of smooth graph $E(t) = (u(x,t), x)$.

For quantitative long-time convergence of volume-preserving mean curvature flow, Escher and Simonett [17] considered volume-preserving mean curvature flow in Euclidean spaces $\mathbb{R}^n$, in which they showed that the curvature flow $E(t)$ converges to a ball (i.e., minimizer of the perimeter among constant volume) exponentially fast for initial sets with smooth and nearly spherical boundaries. In the paper [12], de Gennaro, Diana, Kubin, and Kubin considered volume-preserving mean curvature flow and surface diffusion in flat torus $\mathbb{T}^n$, in which they showed the exponential convergence of flows to stable critical points of perimeter on flat torus [1].

In these works on quantitative convergence of mean curvature flows, one frequently associates the long-time convergence of mean curvature flow with the corresponding quantitative isoperimetric inequality [22, 33, 9, 12] which characterizes the stability of perimeter-minimizer, or the isoperimetric region:

$$P_{\mathcal{M}}(E) - I_{\mathcal{M}}(m) \geq C\alpha(E),$$

for all sets of finite perimeter E with a uniform constant C. Here $\alpha(E)$ quantifying the asymmetry of the set E, or the distance to $\mathrm{Iso}_{\mathcal{M}}(m)$, the family of isoperimetric sets of volume m, and

$$I_{\mathcal{M}}(m) = \inf_E \left\{ P_{\mathcal{M}}(E) : \mathrm{vol}(E) = m \right\}$$

is the minimal perimeter. A formal relation is that gradient flow structure of mean curvature flow provides a steepest deformation of the perimeter, with this quantitative stability which establishes an optimal quantification of the deformation, one can expect an sharp quantification of convergence rates. As our references, the sharp quantitative stability of Yamabe functional was established in [16] and the sharp convergence rates for Yamabe flows were shown in [5].

Our consideration of volume-preserving mean curvature flow (1.1) on the warped manifold $\mathcal{M} = \mathbb{S}(R) \times_\varphi M$ is inspired by the sharp quantitative isoperimetric inequality for general manifolds by Chodosh, Engelstein, and Spolaor [8] given as follows.

Theorem 1 (quantitative isoperimetric inequality)**.** *Given closed and analytic Riemannian manifold $\mathcal{M}$ with dimension $2 \leq n \leq 7$. Let $0 < m \leq \mathrm{vol}(\mathcal{M})$, there exist constants C and $\gamma \geq 0$ such that*

$$P_{\mathcal{M}}(E) - I_{\mathcal{M}}(m) \geq C\alpha_{\mathcal{M}}(E)^{2+\gamma},$$

where $\alpha_{\mathcal{M}}$ is the Fraenkel asymmetry, given by

$$\alpha_{\mathcal{M}}(E) = \inf\big\{\mathrm{vol}(E\Delta\Sigma) : \Sigma \in \mathrm{Iso}_{\mathcal{M}}(m)\big\},$$

where Δ here denotes the symmetrical difference of sets.

Remark 1. *For Euclidean spaces $\mathcal{M} = \mathbb{R}^n$ and flat torus $\mathcal{M} = \mathbb{T}^n$, one has the sharp quantitative stability with $\gamma = 0$ (see, for example, [22]).*

In particular, in their argument to prove the sharpness of Theorem 1, Chodosh, Engelstein, and Spolaor [8, Section 4] consider isoperimetric inequality on

$$\mathcal{M} = \mathbb{S}(R) \times_{\varphi} \mathbb{S}^n, \text{ where } \varphi : \mathbb{S}(R) \to [1, \infty).$$

which is the special case of our considered manifold where $M = \mathbb{S}^n$. For this manifold, sets of the form $E_* = (\mathrm{s}, \mathrm{s} + \pi R) \times \mathbb{S}^n$ for $\mathrm{s} \in \mathbb{S}(R)$ are the isoperimetric region of half volume [8, Lemma 4.1] when R is large enough. If φ achieves strict minimal values at 0 and πR, we further have

$$E_* = (0, \pi R) \times \mathbb{S}^n,$$

providing us an explicit form of perimeter-minimizer, and one would expect mean curvature flow (1.1) to converge to E_* as $t \to \infty$.

For quantitative stability of E_*, if we deform the critical point E_* via a flow $E_*(\varepsilon) = (\varepsilon, C_\varepsilon) \times \mathbb{S}^n$ with C_ε being the constant that makes sure $\mathrm{vol}(E_*(\varepsilon)) = \frac{1}{2}\mathrm{vol}(N)$ for each $\varepsilon > 0$, one can explicit calculate

$$\alpha_{\mathcal{M}}(E_*(\varepsilon)) = 2\,\mathrm{vol}(\mathbb{S}^n)\int_0^\varepsilon \varphi(s)\,\mathrm{d}s, \text{ and } P_{\mathcal{M}}(E_*(\varepsilon)) - P_{\mathcal{M}}(E_*) \leq C_n\big(\varphi(\varepsilon)^n - 1\big).$$

Here $\gamma \neq 0$ corresponds to the case where φ is degenerately convex in the local neighborhood of 0, i.e., $\varphi(x) \approx 1 + x^p$ for some $p > 2$. In this scenario, we get

$$P_{\mathcal{M}}(E_*(\varepsilon)) - P_{\mathcal{M}}(E_*) \leq C\varepsilon^p \approx C\alpha_{\mathcal{M}}(E_*(\varepsilon))^p =: C\alpha_{\mathcal{M}}(E_*(\varepsilon))^{2+\gamma} \text{ where } \gamma > 0.$$

Compared to Euclidean spaces and flat torus where we have sharp component $\gamma = 0$, the component $\gamma > 0$ corresponds to the case where the isoperimetric set is less stable.

In the following, we focus on the quantitative long-time convergence of (1.1). We will prove that when $\gamma > 0$, we can only expect a optimal convergence rate $(1+t)^{-1/\gamma}$, whereas we have convergence rate $e^{-\alpha t}$ when $\gamma = 0$ for $\alpha > 0$.

To simply the analysis, we make the following assumptions.

Assumption 1. *We assume that* (1.1) *obeys the following.*

1. *$\varphi : [-\pi R, \pi R] \to [1, \infty)$ is real analytic, πR-periodic such that $\varphi(x) = \varphi(x + \pi R)$, and achieves minimal at $\varphi(0) = 1$.*

2. *The initial value E_0 satisfies $\partial E_0 = \mathrm{s}_0 \cup (\pi R + \mathrm{s}_0)$ for some constant s_0, where $\pi R + \mathrm{s}_0$ denotes the translation on $\mathbb{S}(R)$ with distance πR.*

3. *The Ricci curvature of M is non-negative.*

Since φ and ∂E_0 are both πR-periodic, by this symmetry, we have

$$\overline{H}(t) = 0 \text{ as long as the flow exists,}$$

reducing (1.1) *to a standard mean curvature flow on warped manifolds. The Ricci curvature condition is used for proving the long-time existence [25].*

For analyzing mean curvature flow (1.1), we parameterize the flow $E(t)$ as the graph of a smooth function u as follows (u is also referred to as the height function [25]),

$$E(t) = \bigcup_{x \in M} \big[u(x,t), u(x,t) + \pi R\big] \times \{x\}, \tag{1.2}$$

Taking this form of solution into (1.1), the graph u obeys a graphical mean curvature flow as follows (see [30, Section 4.3]).

$$\frac{\partial u}{\partial t} - \frac{\sqrt{\varphi(u)^2 + |\nabla u|^2}}{\varphi(u)^2} \mathrm{div}\left(\frac{\nabla u}{\sqrt{\varphi(u)^2 + |\nabla u|^2}} \right) + n \frac{\varphi'(u)}{\varphi(u)} = 0, \tag{1.3}$$

where div and ∇ are the divergence operator and gradient operator on manifold M. The long-time existence is due to Huang, Zhang, and Zhou [25].

Theorem 2 (long-time existence)**.** *There exists $\alpha_0 > 0$ such that, for all initial value $u_0 \in C^\infty(M)$ satisfying*

$$\max_M |u_0| < \alpha_0 \text{ and } \max_M \frac{\sqrt{\varphi(u_0)^2 + |\nabla u_0|^2}}{\varphi(u_0)} < \left[1 - \frac{1}{\varphi(\alpha_0)^2}\right]^{-\frac{1}{2}} \tag{1.4}$$

there exists a unique solution $u \in C^\infty(M \times (0, \infty))$ that solves the graphical mean curvature flow (1.3) *with initial value u_0. As the consequence, the set $E(t)$ defined according to* (1.2) *solves mean curvature flow* (1.1)*.*

To enhance the readability, we provide a detailed derivation of the long-time existence in the last Chapter of the thesis (Chapter 5).

Our results are given as follows. We denote

$$\mathrm{Iso}_{\mathcal{M}} := \begin{cases} \{(\mathrm{s}, \mathrm{s} + \pi R) \times M\}_{\mathrm{s} \in \mathbb{S}(R)} & \text{if } \varphi \text{ is constant} \\ \{(0, \pi R) \times M\} & \text{if } \varphi \text{ is strictly convex at } 0 \end{cases}$$

to be the isoperimeterical regions of half volume.

Theorem 3 (sharp quantitative convergence)**.** *Let $E(t)$ be the long-time solution of mean curvature flow* (1.1)*, we have*

1. *If φ is constant, $E(t) \to E_*$ exponentially fast as $t \to \infty$ for some $E_* \in \mathrm{Iso}_{\mathcal{M}}$.*
2. *If φ is strongly convex at* 0*, $E(t) \to E_*$ exponentially fast as $t \to \infty$ where $E_* \in \mathrm{Iso}_{\mathcal{M}}$.*
3. *Otherwise, $E(t) \to E_*$ with sharp algebraic rate $(1+t)^{\frac{1}{2-p}}$ as $t \to \infty$ where $E_* \in \mathrm{Iso}_{\mathcal{M}}$, where $p > 2$ is the leading order of the Taylor expansion of φ at* 0*.*

Here convergence is measured using distance $\mathrm{dist}(E(t), E_*) := \mathrm{vol}(E(t) \Delta E_*)$*.*

In all cases, the set $K := \{(\mathrm{s}, \mathrm{s} + \pi R) \times M\}_{\mathrm{s} \in \mathbb{S}(R)}$ constitutes an important class of solutions for mean curvature flow (1.1). We establish the following exponential contractibility of K, which gives us a multiscale dynamical behavior of mean curvature flow (1.1).

Theorem 4 (exponential contractibility)**.** *Let $E(t)$ be the long-time solution of mean curvature flow* (1.1) *with suitable initial value $E_0 \notin K$ but close to K, then $E(t)$ converges to K exponentially fast as $t \to \infty$ in the sense that*

$$\operatorname{dist}(E(t), K) \sim e^{-\alpha t} \text{ as } t \to \infty,$$

where the distance is measured by $\operatorname{dist}(E, K) := \inf_{\Sigma \in K} \operatorname{vol}(E \Delta \Sigma)$.

Our main tool is the Lojasiewicz inequality due to Stanislaw Lojasiewicz [31] (also see [18] and the references therein), and the Lyapunov-Schmidt reduction used by Leon Simon for proving the infinite dimensional generalization of the Lojasiewicz inequality, i.e., Lojasiewicz-Simon inequality [38, Chapter 3.14 and Chapter 3.16]. Lojasiewicz-Simon inequalities have rich applications in long-time behavior of physical models [19] and geometric flows [11, 10].

The rest of the thesis is organized as follows.

- In Chapter 2, we provide some preliminary results for analyzing mean curvature flow

- In Chapter 3, we conduct convergence analysis of mean curvature flow based on its gradient flow structure by using Lojasiewicz-Simon inequalities.

- In Chapter 4, we provide the sharp convergence based on Lyapunov-Schmidt reduction.

- In Chapter 5, we provide the long-time existence of the flow.

- Chapter 6 is the conclusion.

Chapter 2

Preliminary dicussion

In this section, we provide a preliminary discussion on mean curvature flow (1.1), and graphical mean curvature flow (1.2).

2.1 A-prior information

We collect two a-prior estimates that will be used in the following analysis. The avoidance principal is a common feature of mean curvature flow as the corollary of comparison principal. The gradient estimate is due to [25].

Lemma 5 (avoidance principal)**.** *Let $u(t)$ be a C^2 solution to graphical mean curvature flow* (1.3)*, and $\nu_0(t)$ and $\nu_1(t)$ be another two C^2 solutions to the graphical mean curvature flow.*

$$\nu_0(0) \leq u(0) \leq \nu_1(0) \implies \nu_0(t) \leq u(t) \leq \nu_1(t).$$

Lemma 6 (gradient estimate)**.** *Given initial value $u_0 \in C^1(M)$ satisfying* (1.4)*. Let $u : M \times (0,T) \to \mathbb{R}$ be a C^3 solution of* (1.3) *with initial value u_0, there exists constant C independent of T such that*

$$|\nabla u(x,t)| \leq C \textit{ for all } (x,t) \in M \times (0,T).$$

2.2 Calculus of perimeter functional

On graphical parameterization, the perimeter $P_{\mathcal{M}}(E)$ can be computed using the area of graphical boundary given by

$$\mathcal{F}(u) := \int_M \sqrt{\varphi(u)^2 + |\nabla u|^2}\, \varphi(u)^{n-1}\, \mathrm{dvol}.$$

In the following we refer $\mathcal{F}$ to as the perimeter functional. By regarding $\mathcal{F}$ as functional on $H^1(M)$, we let

$$D\mathcal{F}(u)[v] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{F}(u+tv), \quad D^2\mathcal{F}(u)[v,w] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \mathcal{F}(u+tv+sw)$$

as its first-order and second-order Gateaux derivatives in $H^1(M)$. According to first variation and second variation formula [36], $D\mathcal{F}(u), D^2\mathcal{F}(u)$ consist of first-order and second-order differentials of u when $u \in C^2(M)$. Hence,

$$D\mathcal{F} : C^{2+\alpha}(M) \to C^\alpha(M) \text{ and } D^2\mathcal{F} : C^{2+\alpha}(M) \to C^\alpha(M) \times C^\alpha(M),$$

In this case, we identify $D\mathcal{F}(u)$ and $D^2\mathcal{F}(u)$ as functions and linear operators via duality formula,

$$D\mathcal{F}(u)[v] = \int_M D\mathcal{F}(u) v\, \mathrm{dvol}, \quad D\mathcal{F}(u)[v,w] = \int_M \langle v, D^2\mathcal{F}(u) w\rangle\, \mathrm{dvol}.$$

By using a direct calculation, we have

$$D\mathcal{F}(u) = \varphi(u)^n \left[\frac{n\varphi'(u)}{\sqrt{\varphi(u)^2 + |\nabla u|^2}} - \frac{1}{\varphi(u)} \mathrm{div}\left(\frac{\nabla u}{\sqrt{\varphi(u)^2 + |\nabla u|^2}} \right) \right]. \tag{2.1}$$

To obtain the second order differentiation, we use the second variation of perimeter from geometric calculus of variations to get the following.

Lemma 7. *Let u_∞ be a critical point of $D\mathcal{F}$, we have*

$$D^2\mathcal{F}(u_\infty) w = -\Delta w + n\varphi''(u_\infty) w.$$

where Δ denotes the Laplace operator on M. Here we note that $\varphi(u_\infty) = 1$ by our assumption.

Proof. Let $E = [u_\infty, u_\infty + \pi R] \times M$. Given smooth function $u : M \to \mathbb{S}(R)$, we define

$$E(t) = \bigcup_{x \in M} \big(tu(x) + [u_\infty, u_\infty + \pi R]\big) \times \{x\},$$

which is a deformation of E via the flow map $S(x,t) := tu(x)$. Then by second variation on general manifolds [36], we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P_{\mathcal{M}}(E(t)) = -\int_{\partial E} u \mathcal{J} u \,\mathrm{d}\sigma_{\partial E},$$

where $\mathcal{J}$ is the Jacobi operator,

$$\mathcal{J} u = \Delta_{\partial E} u + (|A_{\partial E}|^2 + \mathrm{Ric}(N_{\partial E}, N_{\partial E}))u,$$

where $\nabla_{\partial E}$ is the tangential gradient, $A_{\partial E}$ is norm of second fundamental form, Ric is the Ricci curvature tensor of $\mathcal{M}$, $N_{\partial E}$ is the unit outer normal, and $\sigma_{\partial E}$ is the surface measure on ∂E.

We deduce that

$$D^2 \mathcal{F}(u_\infty) = -\mathcal{J}.$$

Furthermore, for warped manifolds we have $\Delta_{\partial E} = \Delta$. From the definition of second fundamental form $A_{\partial E}(v, w) = \langle v, \nabla_w N_{\partial E} \rangle$. Since $N_{\partial E}$ is constant, we have $A_{\partial E} = 0$. From the computation in [25, Lemma 2.2], we also have

$$\mathrm{Ric}(N_{\partial E}, N_{\partial E})) = -n\varphi''(0)/\varphi(0) = -n\varphi''(0).$$

By assembling these components into $\mathcal{J}$, we get the second-order Gateaux derivative. $\square$

By combining first-order and second-order information of critical points, we classify local minimizers as follows,

1. If φ is constant, the space of constant functions $u_\infty = \mathrm{const}$ constitutes a family of minimizers to $\mathcal{F}$. These type of minimizers are called *integrable*.

2. If φ is strictly convex around 0, the function $u_\infty = 0$ is the unique minimizer around 0, and this type of minimizer $u_\infty = 0$ is called *non-integrable*.

These two classics of minimizers leads to two different long-time behavior of mean curvature flow. In the first case, graphical mean curvature flow (1.2) converges to a constant function as $t \to \infty$, and in the second, it converges to 0.

2.3 Linearization

In this part, we provide a formal analysis for *non-integrable* minimizers based on the principal of linearization.

Consider linearizing graphical mean curvature flow (1.2) around minimizer $u_\infty = 0$

$$\frac{\partial u}{\partial t} + D^2\mathcal{F}(0)u + Q(u) = 0,$$

where $Q(u) = \mathrm{Evo}(u) - D^2\mathcal{F}(0)u$, and Evo standards for the evolution term of graphical mean curvature flow (1.2).

If started closed to 0, $Q(u)$ is very small. The behavior of the flow is dominated by the linearized equation

$$\frac{\partial \overline{u}}{\partial t} + D^2\mathcal{F}(0)\overline{u} = 0 \text{ where } D^2\mathcal{F}(0)w = -\Delta w + n\varphi''(0)w.$$

For linearized flow, we let $K := \ker(\Delta) = \mathrm{span}(1)$, and $K^\perp$ be the $L^2(M)$ orthogonal complement. As M is compact and non-negatively curved, we have a spectral gap for the first eigenvalue of $-\Delta$. Hence, $\pi_{K^\perp}\overline{u}$ converges exponentially fast to 0 in the linearized flow. The kernel-project part $\pi_K\overline{u}$ is determined by the warping function φ, namely,

1. If φ is strongly convex at 0, $\overline{u}$ converges to 0 exponentially fast. In this case, graphical mean curvature flow u converges to 0 exponentially fast.

2. If φ is degenerately convex at 0, $\overline{u}$ remains invariant. In this case, graphical mean curvature flow u has certain flexibility in K, since $D^2\mathcal{F}(0)$ is no longer coercive in the kernel K. Here the kernel-project part $\pi_K u$ is driven by the high-order term $Q(u)$.

For analysis of mean curvature flow u when φ is degenerately convex at 0, a desirable approach is constructing a decomposition:

$$u = \xi + \omega \text{ where } \xi(t) = \pi_K u(t) \text{ and } \omega(t) = \pi_{K^\perp} u(t)$$

where the mean curvature flow can be then decomposed into a system of parabolic equations, by projecting the flow onto K and $K^\perp$,

$$\frac{\partial \xi}{\partial t} + \pi_K Q(\xi + \omega) = 0, \quad \frac{\partial \omega}{\partial t} - \Delta\omega + \pi_{K^\perp} Q(\xi + \omega) = 0. \tag{2.2}$$

In this system, we expect ω to converge exponentially fast, and ξ can be analyzed separately from ω. We refer readers to Lunardi [32, Chapter 9] for rigorous discussion.

Chapter 3

Quantitative convergence as gradient flow

In this section, we focus on long-time quantitative behaviors of mean curvature flow (1.3) as gradient flows. Comparing the evolution operator in graphical mean curvature flow to (2.1), graphical mean curvature flow (1.3) can be then regarded as a gradient-like flow of $\mathcal{F}$ as follows.

Theorem 8 (gradient flow formulation)**.** *Graphical mean curvature flow* (1.2) *can be written as a gradient system as follows,*

$$\frac{\partial u}{\partial t} + \mathcal{N}(u) D\mathcal{F}(u) = 0 \text{ where } \mathcal{N}(u) = \sqrt{\varphi(u)^2 + |\nabla u|^2}\, \varphi(u)^{-n-1}. \tag{3.1}$$

where $\mathcal{N}_0 \leq \mathcal{N}(u) \leq \mathcal{N}_1$ with positive constants $\mathcal{N}_0$ and $\mathcal{N}_1$ bounded away from zero due to gradient estimate (Lemma 6).

Before start, we recall the Gronwall inequality that will be frequently used in the analysis (see for example [37, Theorem 4]).

Lemma 9 (Gronwall inequality)**.** *Let $\varrho : [0,T] \to [0,\infty)$ as continuous function that satisfies*

$$\varrho(t) \leq \varrho_0 + \int_0^t \mu(s)\omega(\varrho(s))\,\mathrm{d}s,$$

for a constant $\varrho_0 \geq 0$, where $\mu : [0,T] \to [0,\infty)$ is continuous, and $\omega : [0,\infty) \to (0,\infty)$

is continuous and monotone increasing, then we have

$$\varrho(t) \le \mathcal{X}^{-1}\left[\mathcal{X}(\varrho_0) + \int_0^t \mu(s)\,\mathrm{d}s\right] \text{ where } \mathcal{X}(t) = \int_t^\infty \frac{1}{\omega(s)}\,\mathrm{d}s.$$

3.1 Lojasiewicz-Simon inequalities

We first recall the following classical Lojasiewicz gradient inequality in finite dimension spaces due to Stanislaw Lojasiewicz [31] (also see [18] and the references therein).

Lemma 10 (Lojasiewicz gradient inequality)**.** *Let $d \ge 1$ be an integer, $U \subseteq \mathbb{R}^d$ be an open subset, and $f : U \to \mathbb{R}$ be an analytic function. If $x_\infty \in U$ is a point such that $\nabla f(x_\infty) = 0$, then there are constants $C_0 \in (0,\infty)$, and $\sigma \in (0,1]$, and $\gamma \in (0,1]$ such that*

$$C_0|f(x) - f(x_\infty)|^{1-\frac{\gamma}{2}} \le \|\nabla f(x)\|$$

for all $x \in B_\sigma(x_\infty) := \{x \in \mathbb{R}^d : \|x - x_\infty\| < \sigma\} \subseteq U$.

Remark 2. *The minimal exponent $1 - \frac{\gamma}{2}$ in Lemma 10 is called as the Lojasiewicz exponent.*

Remark 3. *The critical exponent $\gamma = 1$ corresponds to the case where f is uniformly convex. In particular, if $\lambda \mathrm{id} \le \nabla^2 f$ with $\lambda > 0$, we have*

$$\big|f(x) - f(x_\infty)\big| \le (2\lambda)^{-1}|\nabla f(x)|^2,$$

where x_∞ being a strict minimizer of f.

The following Lojasiewicz distance inequality can be established as a corollary of the Lojasiewicz gradient inequality [18], used in quantitative stability of minimizer for variational problems (see for example [16, 8]).

Corollary 11 (Lojasiewicz distance inequality)**.** *Let $d \ge 1$, and $U \subseteq \mathbb{R}^d$ be open. Let $f : U \to \mathbb{R}^d$ be a C^1 function. If there exists x_∞ such that $\nabla f(x_\infty) = 0$, and if f obeys the Lojasiewicz gradient inequality in $B_\sigma(x_\infty)$, then there exists constants $C_1 \in (0,\infty)$, $\delta \in (0, \sigma/4]$, and $\alpha = 2/\gamma$ such that*

$$C_1 \mathrm{dist}(x, B_\sigma(x_\infty) \cap \mathrm{Crit}(f))^\alpha \le |f(x) - f(x_\infty)|, \text{ for all } x \in B_\delta(x_\infty).$$

where $\mathrm{Crit}(f)$ is the set of critical points of f.

We provide the Lojasiewicz-Simon inequalities for perimeter functional $\mathcal{F}$ as follows. The proof is given in Section 3.3. Let

$$\mathcal{F}(u|u_\infty) = \mathcal{F}(u) - \mathcal{F}(u_\infty) \text{ be the energy gap.}$$

Theorem 12 (Lojasiewicz-Simon)**.** *Let u_∞ be a local minimizer of $\mathcal{F}$, there exist constants C_0, C_1, $\gamma \in (0,1]$, and $\sigma \in (0,1]$ such that*

$$C_1 \mathrm{dist}(u, B_\sigma(u_\infty) \cap \mathrm{Crit}(\mathcal{F}))^{1/\gamma} \leq \mathcal{F}(u|u_\infty) \leq C_0 \|D\mathcal{F}(u)\|_{L^2}^{2/(2-\gamma)}$$

for all $u \in B_\sigma(u_\infty)$ in $H^1(M)$, where dist *is measured by using the $H^1(M)$ norm.*

3.2 Rough upper-bounds

Let $u(t)$ be a long-time solution to (3.1). According to the Lojasiewicz-Simon gradient inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u(t)|u_\infty) = -\int_M \mathcal{N}(u(t))\big|D\mathcal{F}(u(t))\big|^2 \,\mathrm{dvol} \leq -\mathcal{N}_0\|D\mathcal{F}(u(t))\|_{L^2}^2 \leq -C\mathcal{F}(u(t)|u_\infty)^{2-\gamma}.$$

By using Gronwall inequality (Lemma 9), we have the following rate of decay for the energy

$$\mathcal{F}(u(t)|u_\infty) \leq \begin{cases} Ce^{-\alpha t} & \text{if } \gamma = 1 \\ C(1+t)^{\frac{1}{\gamma-1}} & \text{if } 0 < \gamma < 1. \end{cases}$$

for some positive constants α and C. In addition, for *non-integrable* minimizer $u_\infty = 0$, we have

$$\mathrm{dist}(u, B_\sigma(u_\infty) \cap \mathrm{Crit}(\mathcal{F})) = \|u - u_\infty\|_{H^1}.$$

By using the Lojasiewicz-Simon distance inequality, we get a quantitative convergence rate

$$\|u - u_\infty\|_{H^1} \leq \begin{cases} Ce^{-\alpha\gamma t} & \text{if } \gamma = 1 \\ C(1+t)^{\frac{\gamma}{\gamma-1}} & \text{if } 0 < \gamma < 1. \end{cases}$$

However this bound is far from being sharp. We refer readers to Sections 5 for detailed discussion of sharp quantitative convergences.

3.3 Splitting method via Lyapunov-Schmidt reduction

In this part, we prove the Lojasiewicz-Simon inequalities for perimeter functional $\mathcal{F}$ (Theorem 12). We first prove them with exponent $\gamma = 1$ if φ is strongly convex around 0.

Proof of Theorem 12 when φ is strongly convex. By a second order Taylor expansion,

$$\mathcal{F}(u) = \mathcal{F}(u_\infty) + D\mathcal{F}(u_\infty)(u - u_\infty) + \big\langle u - u_\infty, D^2\mathcal{F}((1-\theta)u + \theta u_\infty)(u - u_\infty)\big\rangle,$$

with some $0 < \theta < 1$. By the strong convexity of φ and since $D\mathcal{F}(u_\infty) = 0$, there exists a constant λ such that

$$\mathcal{F}(u|u_\infty) = \big\langle u - u_\infty, D^2\mathcal{F}((1-\theta)u + \theta u_\infty)(u - u_\infty)\big\rangle \geq \lambda \|u - u_\infty\|_{H^1}^2,$$

for all u in a local neighborhood of u_∞, which is the Lojasiewicz-Simon distance inequality.

To prove the Lojasiewicz-Simon gradient inequality, let $\widehat{u}(t)$ be the solution of the gradient flow of $\mathcal{F}$ with initial value u, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(\widehat{u}) = 2\big\langle D\mathcal{F}(\widehat{u}), D^2\mathcal{F}(\widehat{u})D\mathcal{F}(\widehat{u})\big\rangle \geq 2\lambda\|D\mathcal{F}(\widehat{u})\|_{L^2}^2 = -2\lambda\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\widehat{u}).$$

By Gronwall inequality, $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\widehat{u}) \to 0$ exponentially fast. Hence,

$$\|D\mathcal{F}(u)\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u) = \int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(\widehat{u})\,\mathrm{d}t \geq -2\lambda\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\widehat{u})\,\mathrm{d}t = 2\lambda\mathcal{F}(u|u_\infty),$$

which is the Lojasiewicz-Simon gradient inequality. □

The degeneracy of $D^2\mathcal{F}(u_\infty)$ occurs if φ is not strongly convex at 0, making the above analysis non-applicable. To overcome the degeneracy, Leon Simon introduces his *splitting* method via the Lyapunov-Schmidt reduction [38, Chapter 3.12]. Let K be the kernel of $D^2\mathcal{F}(u_\infty)$, and

$$\pi_K u = \frac{1}{\mathrm{vol}(M)}\int_M u(x)\,\mathrm{dvol}$$

be the projection operator onto the kernel, and consider the following map

$$\mathcal{N} : C^{2+\alpha}(M) \to C^\alpha(M) \text{ where } \mathcal{N}(u) := \pi_K u + D\mathcal{F}(u).$$

This map satisfies $\mathcal{N}(u_\infty) = u_\infty$, and

$$D\mathcal{N}(u_\infty)[w] = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{N}(tw) = -\Delta w + \pi_K w,$$

Here $D\mathcal{N}(u_\infty)$ is a non-degenerate linear operator. By Schauder estimate, $D\mathcal{N}(u_\infty)$ establishes an isomorphism from $C^{2+\alpha}(M)$ to $C^\alpha(M)$. By the inverse function theorem between Banach spaces $C^{2+\alpha}(M)$ and $C^\alpha(M)$, $\mathcal{N}$ is invertible at a local neighborhood of $u_\infty \in C^\alpha(M)$.

Definition 1 (reduction operator)**.** *There exists a constant σ such that we can define the reduction operator of Lyapunov-Schmidt as*

$$\Psi : B_\sigma(u_\infty) \cap C^{0,\alpha}(M) \to C^{2,\alpha}(M) \textit{ where } \Psi(u) := (\pi_K + D\mathcal{F})^{-1}(u)$$

where π_K is the projection onto K, and $D\mathcal{F}$ is the gradient of the perimeter functional.

We collect several facts on Ψ from [38, Chapter 3.12] given as below.

Lemma 13. *Let $f = \mathcal{F} \circ \Psi$, we have $\frac{1}{2}|f'(\xi)| \leq \|D\mathcal{F} \circ \Psi(\xi)\| \leq 2|f'(\xi)|$ for $\xi \in B_\sigma(u_\infty)$ when σ is small enough.*

We let $u_K = \Psi(\pi_K u)$ be the reduction of a function u. One of the most important facts about the Lyapunov-Schmidt reduction is the following due to Simon [38, Chapter 3.12]. It can be seen as a uniform strong convexity condition towards the center manifold $\Psi(K)$.

Lemma 14. *Let Ψ be the Lyapunov-Schmidt reduction operator of $\mathcal{F}$, then there exists two constants C_0 and C_1 such that, for all $u \in C^2(M)$ in a local neighborhood of u_∞, we have*

$$C_1\|u - u_K\|_{H^1}^2 \leq \mathcal{F}(u|u_K) \leq C_0\|D\mathcal{F}(u)\|_{L^2}^2, \tag{3.2}$$

Remark 4. *This Lyapunov-Schmidt reduction already establishes Lojasiewicz-Simon inequalities for integrable minimizers, because $f := \mathcal{F} \circ \Psi = \mathrm{const}$ by definition.*

We apply inequality (3.2) to proving Lojasiewicz-Simon inequalities for *non-integrable* minimizers when φ is not uniformly convex. Here the idea is to split the energy estimates into, namely, a non-degenerate part where we can apply (3.2), and a finite-dimensional part that can be handled using Lojasiewicz inequalities.

Proof of Theorem 12. Since φ is analytic, by Lojasiewicz inequality in Lemma 10, let $f = \mathcal{F} \circ \Psi$, there exist constants C and γ such that

$$|f(\pi_K u) - f(u_\infty)|^{1-\frac{\gamma}{2}} \le C|f'(\pi_K u)|.$$

In addition, by Lemma 13, there exists two positive constants C_0 and C_1 such that

$$C_0\|D\mathcal{F}(u_K)\|_{L^2} \le |f'(\pi_K u)| \le C_1\|D\mathcal{F}(u_K)\|_{L^2}.$$

Hence, by combining (3.2), we have

$$\begin{aligned}\mathcal{F}(u|u_\infty)^{1-\frac{\gamma}{2}} &= \big(\mathcal{F}(u|u_K) + \mathcal{F}(u_K|u_\infty)\big)^{1-\frac{\gamma}{2}} \le C\|D\mathcal{F}(u)\|_{L^2}^{2-\gamma} + C|f'(\pi_K u)| \\ &\le C\|D\mathcal{F}(u)\|_{L^2} + C\|D\mathcal{F}(u_K)\|_{L^2}.\end{aligned} \tag{3.3}$$

for constants C. By using Taylor expansion, we have

$$D\mathcal{F}(u_K) = D\mathcal{F}(u) - D^2\mathcal{F}((1-\theta)u + \theta u_K)(u - u_K).$$

for $0 < \theta < 1$. Hence, we have

$$\|D\mathcal{F}(u_K)\|_{L^2} \le \|D\mathcal{F}(u)\|_{L^2} + C\|u - u_K\|_{H^1} \le C\|D\mathcal{F}(u)\|_{L^2}$$

where the second inequality is due to (3.2). Therefore,

$$\mathcal{F}(u|u_\infty)^{1-\frac{\gamma}{2}} \le C\|D\mathcal{F}(u)\|_{L^2} + C\|D\mathcal{F}(u_K)\|_{L^2} \le C_0\|D\mathcal{F}(u)\|_{L^2}.$$

For the distance inequality, we use Lojasiewicz distance inequality (Corollary 11) to get

$$\mathcal{F}(u_K|u_\infty)^\gamma \ge C|f(\pi_K u) - f(u_\infty)|^\gamma \ge C' \inf\big\{|\pi_K u - \xi|^2 : \xi \in \mathrm{Crit}(f)\big\}.$$

for constants C, C' and γ. By combining (3.2), we have

$$\begin{aligned}\mathcal{F}(u|u_\infty)^{\frac{\gamma}{2}} &\ge \big(\mathcal{F}(u|u_K) + \mathcal{F}(u_K|u_\infty)\big)^{\frac{\gamma}{2}} \\ &\ge C\|u - u_K\|_{H^1}^\gamma + C\inf\big\{|\pi_K u - \xi|^\gamma : \xi \in \mathrm{Crit}(f)\big\}\end{aligned}$$

when the minimizer of f is *non-integrable*, we have $\mathrm{Crit}(f) = \{0\}$. By using triangle inequality, we get

$$\mathcal{F}(u|u_\infty)^{\frac{\gamma}{2}} \ge C\|u - u_K\|_{H^1}^\gamma + C|\pi_K u|^\gamma \ge C\|u - u_K\|_{H^1}^\gamma + C\|u_K\|_{H^1}^\gamma \ge C\|u\|_{H^1}^\gamma,$$

which concludes the proof. $\square$

Chapter 4

Sharp quantitative convergence

In this section, we focus on the sharp quantitative convergence for *non-integrable* minimizers where $u_\infty = 0$. We refer readers to Theorem 20 below for *integrable* case as the corollary of center manifold reduction. The Lojasiewicz exponent in Remark 2 is critical for sharp convergence rates of gradient flows. For non-integrable minimizers, it is related to the AS_p condition [45, 5], or the Adams–Simon positivity condition of p-th order. By analyticity, we can expand the warping function φ in a power series

$$\varphi(\xi) = \varphi(0) + \sum_{j \geq p} \varphi_p(\xi) = 1 + \sum_{j \geq p} \varphi_p(\xi),$$

where φ_j is a degree j homogeneous polynomial and p is chosen so that $\varphi_p \neq 0$.

Definition 2 (AS_p condition)**.** *Given $p \geq 2$, we say that the non-integrable minimizer u_∞ of $\mathcal{F}$ satisfies the AS_p condition, if the warping function $\varphi_p(\xi)$ is (strictly) positive for some $\xi = \pm 1$.*

4.1 Analysis from solitons

In this part, we provide a convergence analysis of mean curvature flow (1.1) based on mean curvature solitons. Solitons are special (in particular, self-similar) solutions of geometric flows, which give us partial information of the equation and are critical to their analysis [2, 25, 30].

For warped manifold $\mathcal{M} = \mathbb{S}(R) \times_\varphi M$, translational constant functions $u(t) = \xi(t)$ constitute an important class of solitons where $\xi(t) \in \mathbb{R}$ is the solution to the following

ODE

$$\frac{\partial \xi}{\partial t} + n\frac{\varphi'(\xi)}{\varphi(\xi)} = 0. \tag{4.1}$$

By applying the Gronwall inequality (Lemma 9), we have the following convergence rate for mean curvature solitons converging to non-integrable minimizers.

Proposition 15. *Let $\xi(t)$ be the solution of (4.1). If $u_\infty = 0$ satisfies AS_p, mean curvature solitons converges to the set $\{0\} \times M$, and we have*

$$|\xi(t)| \sim \begin{cases} Ce^{-\alpha t} & \text{if } p = 2 \\ C(1+t)^{\frac{1}{2-p}} & \text{if } p > 2, \end{cases}$$

with positive constants α and C.

Proof. According our assumption on φ, there exists a constant β such that

$$-n\varphi'(\xi) \leq \frac{\partial \xi}{\partial t} \leq -\beta n \varphi'(\xi)$$

when we start $\xi(0) = \xi_0$ in a local neighborhood of 0. By Gronwall inequality, we have

$$\mathcal{X}^{-1}\big(nt + \mathcal{X}(\xi_0)\big) \leq \xi(t) \leq \mathcal{X}^{-1}\big(\beta nt + \mathcal{X}(\xi_0)\big) \text{ where } \mathcal{X}(t) = \int_t^\infty \frac{1}{\varphi'(s)}\,\mathrm{d}s.$$

Here the function $\mathcal{X}^{-1}(t)$ can be explicitly calculated as follows with given warping functions φ,

$$\mathcal{X}^{-1}(t) = \begin{cases} Ce^{-\alpha t} & \text{if } p = 2 \\ C(1+t)^{\frac{1}{2-p}} & \text{if } p > 2, \end{cases}$$

for positive constants C and α. $\square$

We can extend the convergence rate of mean curvature solitons to general positive initial data via the avoidance principal (Lemma 5). In particular, if $u(0) = u_0 > 0$. Let

$$\nu_0 = \min_{x\in M} u_0(x) \text{ and } \nu_1 = \min_{x\in M} u_1(x),$$

we can get

$$\nu_0(t) \leq u(t) \leq \nu_1(t),$$

where $\nu_\alpha(t)$ are the solitons of mean curvature flow with initial value ν_α. From the analysis of the mean curvature flow solitons, we have $u(t)$ remains positive, and

$$C_0(1+t)^{\frac{1}{2-p}} \leq |\nu_0(t)| \leq \|u(t)\|_{L^2} \leq |\nu_1(t)| \leq C_1(1+t)^{\frac{1}{2-p}},$$

which enable us to conclude the existence of slow converging mean curvature flows.

Theorem 16 (sharp quantitative convergence for positive solutions). *Let $u(t)$ be a long-time solution of graphical mean curvature flow with positive initial value, we have*

$$\|u(t)\|_{L^2} \sim \begin{cases} Ce^{-\alpha t} & \text{if } p = 2 \\ C(1+t)^{\frac{1}{2-p}} & \text{if } p > 2, \end{cases}$$

for positive constants α, and C

For non-positive initial data, we have a control of upper bound given by

$$\|u(t)\|_{L^2} \leq \max\{|\nu_0(t)|, |\nu_1(t)|\} \leq C(1+t)^{\frac{1}{2-p}}$$

for some constant C. To get sharp convergence for non-positive initial data, we need more powerful tools.

Remark 5. *Here we point out that slow convergence may fail for non-positive initial date. Consider $M = \mathbb{S}^1$, and choose $u_0 \in C^\infty(M)$ such that*

$$u_0(x+\pi) + u_0(x) = 0.$$

One can check that $\pi_K u(t) = 0$ for all t. Hence, the flow stays perpendicular to the center manifold, and we have $u(t) \to 0$ exponentially fast for all even warping function.

4.2 Lyapunov-Schmidt reduction

As we have seen in the proof of Lojasiewicz-Simon inequality in Theorem 12, the Lyapunov-Schmidt reduction provides a clear decomposition of $D\mathcal{F}$ around critical points of $\mathcal{F}$. In the following, we leverage Lyapunov-Schmidt reduction to establish the exponential contractibility of center manifold $\Psi(K)$, and obtain a sharp quantitative convergence rate.

The established method may be applied to general gradient-like flows. Let $K := \ker(D^2\mathcal{F}(u_\infty))$, we note that

$$\dim(K) = 1, \text{ and } \mathcal{F}(u) \text{ is locally convex at 0 when stricted to } K.$$

Let Ψ be the corresponding Lyapunov-Schmidt reduction operator. By identifying K with $\mathbb{R}$, and using this convexity condition on kernel, the function $\pi_K u + D\mathcal{F}(u)$ is smooth and strictly monotone increasing. As the result, we have the following lemma.

Lemma 17. $\Psi : (-\sigma, \sigma) \to \mathbb{R}$ *is strictly monotone increasing when restricted to* $B_\sigma(0) \cap K$.

The critical observation to establish the exponential contractibility is Lemma 14, which states that there exists constants λ and C such that, for all u in a local neighborhood of u_∞,

$$\lambda \|u - u_K\|_{H^1}^2 \leq \mathcal{F}(u) - \mathcal{F}(u_K) \leq C \|D\mathcal{F}(u)\|_{L^2}^2 \tag{4.2}$$

where $u_K = \Psi(\pi_K u)$ denotes the reduction of u.

According to Otto and Reznikoff [35], this condition is a uniform convexity condition towards the subspace $\Psi(K)$. Therefore, according to our discussion in the quantitative convergence rate, one can expect exponential convergence rate to the center manifold, i.e., $\inf_{\xi \in \Psi(K)} \|u(t) - \xi\| \leq Ce^{-\alpha t}$ for some constants C and $\alpha > 0$. In fact, we have exponential decay to close the gap

$$\omega(t) := u(t) - u_K(t).$$

We first establish the following estimation of the reduced gradient flow.

Lemma 18. *Let* $u(t)$ *be a solution of* (3.1). *Given* $\delta > 0$, *there exists a constant* $C(\delta)$ *such that if* $|\pi_K u(t)| < \delta$ *for all* $t > 0$, *then*

$$-C(\delta)\big[\mathcal{F}\big(u(t)\big) - \mathcal{F}\big(u_K(t)\big)\big] \leq \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\big(u_K(t)\big).$$

where $u_K = \Psi(\pi_K u)$. *Furthermore,* $C(\delta) \to 0$ *as* $\delta \to 0$.

Proof. Let $\omega(t) := u(t) - u_K(t)$. By the distance inequality in (4.2), it is sufficient to prove that

$$-C(\delta)\|\omega\|_{H^1}^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\big(u_K\big),$$

with a constant $C(\delta)$ that satisfies the condition. By using chain rule, we can calculate,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u_K) = -D\mathcal{F}(u_K)\Psi'\big(\pi_K u\big) \times \pi_K\big(\mathcal{N}(u)D\mathcal{F}(u)\big). \tag{4.3}$$

In this equation, $\Psi'(\xi) \geq 0$ by Lemma 17.

Since $\mathcal{N}_0 \leq \mathcal{N}(u) \leq \mathcal{N}_1$, by mean value theorem, we have

$$\pi_K\big(\mathcal{N}(u)D\mathcal{F}(u)\big) = \overline{\mathcal{N}}(t)\pi_K D\mathcal{F}(u)$$

for $\mathcal{N}_0 \le \overline{\mathcal{N}}(t) \le \mathcal{N}_1$. In addition, by Taylor expansion at u_K, we have

$$\begin{aligned}\pi_K D\mathcal{F}(u) &= D\mathcal{F}(u_K) + \pi_K D^2\mathcal{F}(u_K)\omega + O(\|\omega\|_{H^1}^2)\\ &= D\mathcal{F}(u_K) + D^2\mathcal{F}(u_K)\pi_K\omega + O(\|\omega\|_{H^1}^2)\end{aligned}$$

Since $D^2\mathcal{F}(u_K)$ is positive definite with constant functions as eigenvectors for $u_K \in K$, it commute with the projection π_K. Now, by the definition of Lyapunov-Schmidt reduction, we have

$$D^2\mathcal{F}(u_K)\pi_K\omega = D^2\mathcal{F}(u_K)\big(\pi_K u - \Psi(\pi_K u)\big) = D^2\mathcal{F}(u_K)D\mathcal{F}(u_K) \tag{4.4}$$

Therefore, by taking the Taylor expansion into (4.3) and considering (4.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u_K) = -\overline{\mathcal{N}}(t)\Big[\big(1 - D^2\mathcal{F}(u_K)\big)\Psi'\big(\pi_K u\big)D\mathcal{F}(u_K)^2 + \Psi'\big(\pi_K u\big)D\mathcal{F}(u_K)O(\|\omega\|_{H^1}^2)\Big].$$

Since the Hessian $D^2\mathcal{F}(u_K)$ degenerates at the critical point u_∞, we can control $\pi_K u$ small enough to make sure $D^2\mathcal{F}(u_K) < 1$.

Furthermore, $\Psi'\big(\pi_K u\big)D\mathcal{F}(u_K) = (\mathcal{F}\circ\Psi)'(\pi_K u)$. Since $\mathcal{F}\circ\Psi$ is analytic, we can make $(\mathcal{F}\circ\Psi)'$ arbitrarily small by making $\pi_K u$ close to zero. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u_K) \ge -C(\delta)\|w\|_{H^1}^2.$$

The constant $C(\delta)$ can be made arbitrarily small by choosing δ small accordingly. ☐

Remark 6 (control $\pi_K u(t)$)**.** *For mean curvature flow* (1.1)*, the projection $\pi_K u(t)$ can be controlled via the avoidance principal: If we start the initial value within $\|u_0\|_{L^\infty} < \delta$, then $|\pi_K u(t)| < \delta$ for all $t > 0$.*

Theorem 19 (exponential contractibility)**.** *Let $u : M \times (0,\infty) \to \mathbb{R}$ be the solution of* (3.1) *with initial value u_0. There exists $\delta > 0$ such that, if $|\pi_K u_0| < \delta$, then there exist positive constants C and α such that*

$$\|u(t) - u_K(t)\|_{H^1} \le Ce^{-\alpha t}.$$

where u_∞ is the long-time asymptotic limit, and $u_K(t) = \Psi(\pi_K u(t))$.

Proof. Given u and $\overline{u}$, we denote the energy gap $\mathcal{F}(u|\overline{u}) = \mathcal{F}(u) - \mathcal{F}(\overline{u})$. Along the gradient flow, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\big(u(t)|u_\infty\big) \le -\mathcal{N}_0\|D\mathcal{F}(u(t))\|_{L^2}^2 \le -C_0\mathcal{F}\big(u(t)|u_K(t)\big).$$

for a constant C_0. Now, if we split $\mathcal{F}\big(u(t)|u_\infty\big) = \mathcal{F}\big(u(t)|u_K(t)\big) + \mathcal{F}\big(u_K(t)|u_\infty\big)$, we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\big(u(t)|u_K(t)\big) + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\big(u_K(t)|u_\infty\big) \le -C_0\mathcal{F}\big(u(t)|u_K(t)\big). \tag{4.5}$$

Therefore, if we choose suitable δ such that $C_1 := C(\delta) < C_0$. Then by Lemma 18, we have

$$-C_1\mathcal{F}\big(u(t)|u_K(t)\big) \le \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\big(u_K(t)|u_\infty\big).$$

By taking this inequality into (4.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}\big(u(t)|u_K(t)\big) \le -(C_0 - C_1)\mathcal{F}\big(u(t)|u_K(t)\big).$$

Let $\alpha = C_0 - C_1$, we can use the Gronwall inequality to show that

$$\mathcal{F}\big(u(t)|u_K(t)\big) \le \mathcal{F}\big(u_0|u_\infty\big)e^{-\alpha t}.$$

Finally, we conclude the proposition by the distance inequality in (4.2). $\square$

We note that this Lyapunov-Schmidt reduction method applies to the integrable case, and we can get exponential convergence rate to a constant function as follows.

Theorem 20 (quantitative convergence for $\varphi = 1$)**.** *If $\varphi = 1$. Let $u(t)$ be the solution of graphical mean curvature flow constructed in Theorem 2. There exists a constant function u_∞ such that*

$$\|u(t) - u_\infty\|_{L^2} \le Ce^{-\alpha t}$$

for some constants C and α.

The exponential contractibility gives us the following center manifold reduction result.

Theorem 21 (center manifold reduction)**.** *There exists a constant $\delta > 0$ such that if $\|u_0\|_{L^\infty} < \delta$, then there exist $\xi_0 \in \mathbb{R}$ and constants C and α such that*

$$\|u(t) - \Psi(\xi(t))\|_{H^1} \le Ce^{-\alpha t}$$

where $\xi(t)$ is the solution to $\partial_t\xi + \mathcal{M} \circ \Psi(\xi) = 0$ with initial value $\xi(0) = \xi_0$.

Proof. Considering the exponential contractility in Theorem 19, it suffices to prove that

$$|u_K(t) - \Psi(\xi(t))| \leq Ce^{-\alpha t}.$$

for some positive constants α and C.

We trace $u_K(t)$ backwards in time, and let $\xi(s,t)$ be the solution of the following equation at time $(s-t)$ with initial value $\pi_K u(t)$ for $0 \leq s < t$,

$$\frac{\partial \xi(s,t)}{\partial s} + \mathcal{M} \circ \Psi(\xi(s,t)) = 0 \text{ and } \xi(t,t) = \pi_K u(t). \tag{4.6}$$

We note that according to comparison principle (compared to mean curvature solitons), $|\pi_K u(t)| \leq \delta(1+t)^{\frac{1}{2-p}}$ for a constant δ such that $\|u\|_{L^\infty} \leq \delta$. Hence, according to the reverse time ODE (4.6), we can find a constant C independent of t such that $|\xi(s,t)| < C\delta$ for all $0 \leq s < t$.

Since $\partial_t(\pi_K u) + \pi_K \mathcal{M}(u_K + \omega) = 0$, we can use Lipchitz continuity to get

$$\begin{aligned}
|\pi_K u(s) - \xi(s,t)| &\leq \int_s^t |\pi_K \mathcal{M}(u_K(\sigma) + \omega(\sigma)) - \mathcal{M} \circ \Psi(\xi(\sigma,t))| \,\mathrm{d}\sigma \\
&\leq \int_s^t |\mathcal{M} \circ \Psi(\pi_K u(\sigma)) - \mathcal{M} \circ \Psi(\xi(\sigma,t))| + C_0 \|\omega(\sigma)\|_{H^1} \,\mathrm{d}\sigma \\
&\leq \mu \int_s^t |\pi_K u(\sigma) - \xi(\sigma,t)| \,\mathrm{d}\sigma + C_0 \int_s^t \|\omega(\sigma)\|_{H^1} \,\mathrm{d}\sigma
\end{aligned}$$

By using Gronwall inequality, there exist two constants C and μ (the Lipschitz constant μ can be made close to zero if δ is small enough) such that

$$|\pi_K u(s) - \xi(s,t)| \leq C \int_s^t e^{\mu(\sigma - s)} \|\omega(\sigma)\|_{H^1} \,\mathrm{d}\sigma.$$

By applying Theorem 19 and Remark 6 to graphical mean curvature flow, $\|\omega(s)\|_{H^1}$ decays exponentially fast. By using the continuity $\mathcal{F}(u|u_K) \leq \Lambda \|u - u_K\|_{H^1}^2$, there exists two constants C and $\alpha > \mu$ (by controlling δ small enough) such that

$$\sup_{s<\sigma<t} Ce^{\alpha(\sigma - s)} \|\omega(\sigma)\|_{H^1} \leq \|\omega(s)\|_{H^1}$$

Hence, there exists a constant C such that

$$|\pi_K u(s) - \xi(s,t)| \leq C\|\omega(s)\|_{H^1} \int_s^t e^{(\mu - \alpha)(\sigma - s)} \,\mathrm{d}\sigma \leq C\|\omega(s)\|_{H^1}.$$

Choosing $s = 0$, we get $|\pi_K u_0 - \xi(0,t)| \le C\|\omega(0)\|_{H^1}$. Therefore, $\{\xi(0,t)\}_t$ is bounded. By using the compactness of closed bounded sets, there exists a subsequence $t_n \to \infty$ such that $\xi(0,t_n) \to \xi_0$. By continuity, we let $\xi(t)$ be the solution of the ODE with initial value ξ_0, it then follows,

$$|\pi_K u(t) - \xi(t)| \le C\|\omega(t)\|_{H^1} \le Ce^{-\alpha t}.$$

Finally, we use the Lipschitz continuity of Ψ to conclude. □

Corollary 22 (slow convergence for general data)**.** *Let $u : M \times (0,\infty) \to \mathbb{R}$ be the long-time solution to graphical mean curvature flow with initial value u_0. If $\sup_{x\in M} |u_0(x)| < \delta$ for a small constant δ and $|\pi_K u_0| > C\|\omega_0\|_{H^1}$, we have*

$$\|u(t)\|_{L^2} \sim \begin{cases} Ce^{-\alpha t} & \text{if } p = 2 \\ C(1+t)^{\frac{1}{2-p}} & \text{if } p > 2, \end{cases}$$

for positive constants α and C.

Proof. According to the proof of the previous theorem, we have

$$|\xi_0| \ge |\pi_K u_0| - C\|\omega_0\|_{H^1} > 0,$$

which means that

$$\|u(t) - \Psi(\xi(t))\|_{H^1} \le \|u(t) - u_K(t)\|_{H^1} + |u_K(t) - \Psi(\xi(t))| \le Ce^{-\alpha t}.$$

Hence $u(t)$ has the same rate of convergence as the reductioned flow $\Psi(\xi(t))$. □

Chapter 5

Long-time existence

In this section, we provide a derivation of the long-time existence of graphical mean curvature flow via the standard theory of quasilinear parabolic equations [15].

Throughout this part, (M, g) is a compact n-dimensional Riemannian manifold without boundary. To simply the notation, the Einstein summation convention between upper and lower indices will be used. We let ∇ be the Levi-Cevita covariant derivative of differential forms,

$$\nabla : \Omega^N(M) \to \Omega^{N+1}(M),$$

where $\Omega^N(M)$ denotes the space of N-forms. Levi-Cevita covariant derivative provide us an intrinsic way of differentiating functions and tensors. We also define the norm of differential as follows,

$$|\mathbf{a}| = g^{j_1 k_1} \cdots g^{j_N k_N} \mathrm{a}_{j_1 \cdots j_N} \mathrm{a}_{k_1 \cdots k_N} \text{ where } \mathbf{a} = \mathrm{a}_{j_1 \cdots j_N} \, \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_N}.$$

where g^{jk} is the inverse of the matrix $g_{jk} := g(\partial_j, \partial_k)$ and (∂_j) denotes the frame of a local coordinate (x_j). Owing to their intrinsic properties, these geometric quantities are invariant under change of coordinates. Therefore, we can put any coordinate to compute geometric quantities of interest.

5.1 Set-up

To see graphical mean curvature flow (1.3) is quasilinear parabolic, we derive its non-divergence form as follows.

Corollary 23 (non-divergence form)**.** *On local coordinate* (x_j)*, graphical mean curvature flow* (1.3) *can be written in a quasilinear parabolic form as follows*

$$\frac{\partial u}{\partial t} - \mathscr{A}(x,u,\nabla u)^{jk}\frac{\partial^2 u}{\partial x_j \partial x_k} + \mathscr{B}(x,u,\nabla u) = 0, \tag{5.1}$$

where

$$\begin{aligned}\mathscr{A}(x,u,\nabla u)^{jk} &= \frac{1}{\varphi(u)^2}\left(g^{jk} - \frac{1}{\nu^2}g^{j\alpha}g^{k\beta}\frac{\partial u}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta}\right)\\ \mathscr{B}(x,u,\nabla u) &= \Gamma^\ell_{jk}\left(g^{jk} - \frac{1}{\nu^2}g^{j\alpha}g^{k\beta}\frac{\partial u}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta}\right)\frac{\partial u}{\partial x_\ell} + \frac{\varphi'(u)}{\varphi(u)}\left(n+1-\frac{\varphi(u)^2}{\nu^2}\right).\end{aligned}$$

Here $\nu := \sqrt{\varphi(u)^2 + |\nabla u|^2} = \sqrt{\varphi(u)^2 + g^{jk}\partial_j u \partial_k u}$*, and* Γ^ℓ_{jk} *is the Christoffel symbol of the Levi-Cevita connection. In addition, the second-order coefficient satisfies*

$$0 \le \frac{1}{\varphi(u)^2 + |\nabla u|^2}\left(g^{jk}\xi_j\xi_k\right) \le \mathscr{A}(x,u,\nabla u)^{jk}\xi_j\xi_k \le \frac{1}{\varphi(u)^2}\left(g^{jk}\xi_j\xi_k\right).$$

for all $\xi \in \mathbb{R}^n$*.*

Proof. Graphical mean curvature flow (1.3) is equivalent to the following quasi-linear parabolic equation

$$\frac{\partial u}{\partial t} - \frac{1}{\varphi(u)^2}\left(\Delta u - \frac{1}{\nu^2}\langle \nabla^2 u \cdot \nabla u, \nabla u\rangle\right) + \frac{\varphi'(u)}{\varphi(u)}\left(n+1-\frac{\varphi(u)^2}{\nu^2}\right) = 0, \tag{5.2}$$

where $\nu = \sqrt{\varphi(u)^2 + |\nabla u|^2}$, Δ is the Laplacian operator on M, and $\nabla^2 u$ is the Hessian of u on M. Let (x_j) be a local coordinate. On this local coordinate, the operators can be explicitly calculated as follows,

$$\Delta u = \sum_{j,k=1}^n \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_j}\left(\sqrt{g}g^{jk}\frac{\partial u}{\partial x_k}\right), \quad \nabla^2 u = \sum_{j,k=1}^n \left(\frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{\ell=1}^n \Gamma^\ell_{jk}\frac{\partial u}{\partial x_\ell}\right) \mathrm{d}x_j \otimes \mathrm{d}x_k.$$

Here Laplacian Δu is the trace of the Hessian $\nabla^2 u$ in the sense that

$$\Delta u = \mathrm{tr}(\nabla^2 u) = \sum_{j,k=1}^n g^{jk}\left(\frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{\ell=1}^n \Gamma^\ell_{jk}\frac{\partial u}{\partial x_\ell}\right).$$

By taking this coordinate expression into (5.2), one can see that the second order term is

$$\sum_{j,k=1}^n \mathscr{A}(x,u,\nabla u)^{jk}\frac{\partial^2 u}{\partial x_j \partial x_k} := \frac{1}{\varphi(u)^2}\sum_{j,k=1}^n\left(g^{jk} - \sum_{\alpha,\beta=1}^n \frac{1}{\nu^2}g^{j\alpha}g^{k\beta}\frac{\partial u}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta}\right)\frac{\partial^2 u}{\partial x_j \partial x_k}.$$

The lower order terms $\mathscr{B}(x,u,\nabla u)$ can be compute similarly.

To see (5.1) is parabolic, let $\mu^j = \sum_\alpha g^{j\alpha}\partial_\alpha u$. By using Cauchy inequality, we have

$$0 \le \sum_{j,k=1}^{n} (\mu^j\mu^k)(\xi_j\xi_k) \le \Big[\sum_{j,k=1}^{n} g^{jk}\partial_j u\,\partial_k u\Big]\Big[\sum_{j,k=1}^{n} g^{jk}\xi_j\xi_k\Big] = |\nabla u|^2 \sum_{j,k=1}^{n} g^{jk}\xi_j\xi_k \text{ for all } \xi \in \mathbb{R}^n.$$

Hence, the coefficient in the second order term of (5.2) satisfy

$$0 \le \frac{1}{\varphi(u)^2 + |\nabla u|^2}\sum_{j,k=1}^{n} g^{jk}\xi_j\xi_k \le \sum_{j,k=1}^{n} \mathscr{A}(x,u,\nabla u)^{jk}\xi_j\xi_k \le \frac{1}{\varphi(u)^2}\sum_{j,k=1}^{n} g^{jk}\xi_j\xi_k,$$

which gives parabolicity. □

5.2 Linear parabolic equations

In this part, we provide an overview of Hölder estimates for linear parabolic equations [15, 29]:

$$\frac{\partial u}{\partial t} - Lu := \frac{\partial u}{\partial t} - \sum_{j,k=1}^{n} A^{jk}(x,t)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u = 0,$$

where $\lambda_0 I_{d\times d} \le A(x,t) \le \lambda_1 I_{d\times d}$ with positive constants λ_0, λ_1.

Hereinafter, we denote

$$Q_R(x,t) := B_R(x) \times (t - R^2, t),$$

to be the parabolic cylinder, and $Q_R := Q_R(0,0)$. We denote the parabolic boundary

$$\partial_{\mathrm{p}} Q_R(x,t) = \partial B_R(x) \times (t - R^2, t) \cup B_R(x) \times \{t - R^2\}.$$

Given $0 < \alpha < 1$, we define parabolic Hölder space $C^\alpha(Q_R)$ as the space of continuous functions such that (see, for example, [29, Chapter 8] and [27, Section 1.4])

$$|u(p,t) - u(q,s)| \le C\big(|p-q|^\alpha + |t-s|^{\alpha/2}\big),$$

for some constant C, and the parabolic Hölder semi-norm as

$$[u]_\alpha(Q_R) := \sup_{(p,t)\neq(q,s)} \frac{|u(p,t) - u(q,s)|}{|p-q|^\alpha + |t-s|^{\alpha/2}}.$$

Remark 7. *Hereinafter, we adopt* $\|\cdot\|(\Omega)$ *to emphsis the norm measured on the domain* Ω*, while* $\|\cdot\|$ *denotes the norm measured on the whole domain, or when the context is clear.*

We say that u is C^2 and $u \in C^2(Q_R)$, if it is continuously differentiable in time and second-order continuously differentiable in space. We say $u \in C^{2,\alpha}(Q_R)$, if in addition,

$$[u]_{2+\alpha} := [\partial_t u]_\alpha + [\nabla^2 u]_\alpha < \infty.$$

We recall the following Schauder estimates of parabolic PDEs [29].

Lemma 24 (Schauder estimate)**.** *If the coefficients* A*,* b *and* c *are* $C^{0,\alpha}$*, then for all* $u \in C^{2,\alpha}(Q_R)$ *with* $0 < \alpha < 1$*, there exists a uniform constant* $C = C(n, \alpha)$ *such that*

$$[u]_{2+\alpha}(Q_{R/2}) \leq C\big([\partial_t u - Lu]_\alpha(Q_R) + \|u\|_{L^\infty}(Q_R)\big).$$

Lemma 25 (high-order Schauder estimate)**.** *If the coefficients* A, b, c *are* $C^{k,\alpha}$*, then there exists a constant* C *such that*

$$\|u\|_{k+2,\alpha}(Q_{R/2}) \leq C\big(\|u\|_{L^\infty}(Q_R) + \|\partial_t u - Lu\|_{k,\alpha}(Q_R)\big).$$

for all $u \in C^{k+2,\alpha}(Q_R)$*.*

We recall the Harnack inequality for parabolic equations in non-divergence form due to Krylov and Safonov [44].

Lemma 26 (Harnack inequality)**.** *Let* u *be a positive* C^2 *solution of* $\partial_t u - Lu = 0$ *on* Q_R*, then there exists a uniform constant* C *such that*

$$\sup_{Q_{R/2}} u \leq C \inf_{Q_{R/2}} u.$$

This Harnack inequality is a maximal principles in a quantitative way, which implies the control of oscillations, and gives us the following rough Hölder regularity. We refer readers to [20, Chapter 2.1] for the discussion.

Corollary 27 (Krylov-Safonov estimate)**.** *Let* u *be a solution of* $\partial_t u - Lu = 0$ *on* Q_R*, there exists a uniform constant* C *such that*

$$\|u\|_\alpha(Q_{R/2}) \leq C\|u\|_{L^\infty}(Q_R).$$

These estimates of linear parabolic equation are fundamental to the analysis of (non-linear) parabolic equations. For example, we can regard some nonlinear parabolic PDEs as linear parabolic PDEs by freezing some parts of the equation using approximations of the solution, and use linear theory to obtain the regularity. These estimates lies at the heart of Hilbert's XIX problem.

5.3 Function spaces

In this part, we provide the function spaces on manifolds for establishing the Hölder estimates. We let $d_M(p,q)$ be the intrinsic metric of (M,g), given by minimial length of curves connecting p and q. Building on this metric space structure, we define the Hölder space as follows [23].

Definition 3 (parabolic Hölder spaces)**.** *Given $\alpha \in (0,1)$ and $T \in (0,\infty]$. We say $u : M \times (0,T) \to \mathbb{R}$ is Hölder α continuous, if*

$$[u]_\alpha = \sup_{(p,t)\neq(q,s)} \frac{|u(p,t)-u(q,s)|}{|t-s|^{\alpha/2} + d_M(p,q)^\alpha} < \infty,$$

We denote $\|u\|_\alpha = \|u\|_{L^\infty} + [u]_\alpha$ as the parabolic Hölder norm, and $C^\alpha(M \times (0,T))$ as the parabolic Hölder space which is a Banach space with Hölder norm $\|\cdot\|_{0,\alpha}$.

We define first and second parabolic Hölder spaces as follows.

Definition 4. *Given $\alpha \in (0,1)$ and $T \in (0,\infty]$. We define parabolic Hölder semi-norms*

$$[u]_{1,\alpha} = [\nabla u]_\alpha, \quad [u]_{2,\alpha} = [\partial_t u]_\alpha + [\nabla^2 u]_\alpha.$$

We let $C^{1,\alpha}(M \times (0,T))$ be the space of functions such that

$$\|u\|_{1,\alpha} := \|u\|_\alpha + [u]_{1,\alpha} < \infty,$$

and $C^{2,\alpha}(M \times (0,T))$ be the space of functions such that

$$\|u\|_{2,\alpha} := \|u\|_{L^\infty} + \|\partial_t u\|_{L^\infty} + \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty} + [u]_{2,\alpha} < \infty.$$

Remark 8. *We recall that ∇ is the Levi-Cevita covariant derivative of differential forms.*

High-order spaces $C^{k,\alpha}(M \times (0,T))$ can be defined analogously. We refer readers to [29] for the definition on $\mathbb{R}^n$.

5.4 A-prior estimates

One of the main technical difficulties for establishing long-time existence of solutions to graphical mean curvature flow (1.3) is that it fails to be uniformly strong parabolic, and the degeneracy occurs when $\sup|\nabla u| \to \infty$. The following gradient estimate by [25] overcomes this issue by establishing the uniform parabolicity as long as the solution exists.

Lemma 28 (gradient estimate)**.** *Let φ be given by Assumption 1. Given initial value $u_0 \in C^1(M)$ that satisfies* (1.4)*. Let $u : M \times (0,T) \to \mathbb{R}$ be a C^3 solution of* (1.3) *with initial value u_0, then there exists a constant C independent of T such that*

$$|\nabla u(x,t)| \leq C \textit{ for all } (x,t) \in M \times (0,T).$$

We now state the following fundamental Hölder estimate.

Theorem 29 (fundamental Hölder estimate)**.** *Let $T > 0$. If u is a C^2 solution of graphical mean curvature flow* (5.2) *with initial value $u_0 \in C^{1,1}(M)$ satisfying the condition of the gradient estimate (Lemma 28), then there exist a uniform constant C such that*

$$\|u\|_{1,\alpha}(M \times (0,T)) \leq C$$

for some $0 < \alpha < 1$.

Proof. We first show that, given (small) $\delta > 0$, there exist constants $C(\delta)$ and $\alpha < 1$ such that

$$\|u\|_{1,\alpha}(M \times [\delta,T)) \leq C(\delta)$$

To get this estimate, we apply *interior estimates* on small geodesic balls $B_r(p)$, and then extends the estimation to the whole space through a *covering argument.*

(interior estimate) Let $(B_{2r}(p))_{p\in M}$ be the set of geodesic balls of radius $2r$, which forms an open cover of M. We choose r small enough such that the exponential map at each p forms a diffeomorphism from a local neighborhood of $0 \in T_pM$ onto $B_{2r}(p)$, which is possible since M is compact. On local geodesic coordinate (x_j), we have

$$\frac{\partial u}{\partial t} - \mathscr{A}(x,u,\nabla u) : \nabla^2 u + \mathscr{B}(x,u,\nabla u) = 0.$$

Given $e_j \in \partial B(0)$ and consider a displacement of u given by $u_\tau := u(\cdot - \tau e_j)$ where τ is a sufficiently small number. The function u_τ satisfies

$$\frac{\partial u_\tau}{\partial t} - \mathscr{A}(x + \tau e_j, u_\tau, \nabla u_\tau) : \nabla^2 u_\tau + \mathscr{B}(x + \tau e_j, u_\tau, \nabla u_\tau) = 0.$$

Let $\omega = (u_\tau - u)/\tau$. By subtracting equations of u_τ and u, and applying Taylor's theorem, we have

$$\frac{\partial \omega}{\partial t} - \mathscr{A}(x, u, \nabla u) : \nabla^2 \omega + \mathscr{C}(x) \cdot \nabla \omega + \mathscr{D}(x)\omega = 0.$$

where $\mathscr{C}$ and $\mathscr{D}$ are low-order terms that does not contain $\nabla^2 \omega$. We now apply the Hölder estimate of Krylov and Safonov to get $\|\omega\|_\alpha \le \|\omega\|_{L^\infty}$, i.e.,

$$\tau^{-1}\|u_\tau - u\|_\alpha(Q_r(p,t)) \le C\tau^{-1}\|u_\tau - u\|_\infty(Q_{2r}(p,t)) \le C \text{ for all } t \ge 4r^2.$$

Here the last inequality is due to the gradient estimate (Lemma 28). By taking the limit $\tau \to 0$ and notice that u is C^2, we have $\|\partial_j u\|_\alpha \le C$ for a uniform constant C. As this applies to all directional derivatives, we get

$$\|u\|_{1,\alpha}(Q_r(p,t)) \le C$$

for a constant C.

(covering argument) Since $(B_r(p))_{p\in M}$ forms an open cover of the compact manifold M, there exists a sub-cover finite sub-cover $(B_r(p_\ell))_{\ell=1}^N$, and we have

$$\|u\|_{1,\alpha}(M \times (t - r^2, t)) \le \sum_{\ell=1}^{N} \|u\|_{1,\alpha}(Q_r(p,t)) \le NC \text{ for all } t \ge 4r^2.$$

By covering $[3r^2, T) \times M$ with K "interval" $\big((t_\ell - r^2, t_\ell) \times M\big)_{\ell=1}^K$, we get

$$\|u\|_{1,\alpha}(M \times (3r^2, T)) \le \sum_{\ell=1}^{K} \|u\|_{1,\alpha}((t_\ell - r^2, t_\ell) \times M) \le KNC.$$

Here we get $\|u\|_{1,\alpha}(M \times [\delta, T)) \le C(\delta)$ by choosing $3r^2 = \delta$.

In the covering argument, we have $C(\delta) \to \infty$ as $\delta \to 0$. This scenario can happen when initial value is irregularity and the estimate can not be uniformly extended to the boundary. Instead, the regularity in the region $(0, \delta)$ is dominated by the initial value:

$$\|u\|_{1,\alpha}(M \times (0,\delta)) \le C(\delta)\|u_0\|_{1,\alpha}(M)$$

as the result of the maximal principle and a barrier method. We refer readers to [29, Section 9] for discussion. By combining $\|u\|_{1,\alpha}(M\times[\delta,T)) \le C(\delta)$, we get the result. $\square$

We next state the smoothness of the solution based on the Schauder estimate (Lemma 24) and high-order Schauder estimate (Lemma 25).

Theorem 30 (smoothness)**.** *Let $T>0$. If u is a solution of graphical mean curvature flow* (5.2) *such that fundamental Hölder estimate holds true, i.e., $\|u\|_{1,\alpha}(M\times(0,T)) \le C$ for some constant C and $0<\alpha<1$, we have $u\in C^\infty(M\times(0,T))$.*

Proof. This can be proved by using the boost-trapping technique [20, Chapter 3] as the consequence of high-order Schauder estimates, where we apply high-order Schauder estimates repeatedly to get

$$u\in C^{1,\alpha} \implies \mathscr{A},\mathscr{B}\in C^{0,\alpha} \implies u\in C^{2,\alpha} \implies \ldots \implies \mathscr{A},\mathscr{B}\in C^{1+k,\alpha} \implies u\in C^{1+k,\alpha}. \tag{5.3}$$

where $\mathscr{A} := \mathscr{A}(x,u,\nabla u)$ and $\mathscr{B} := \mathscr{B}(x,u,\nabla u)$ are coefficients in the quasilinear parabolic form of graphical mean curvature (5.1) on local coordinates.

For a rigorous proof, let us show that, for any $\delta>0$ and $k\in\mathbb{N}$, one has

$$\|u\|_{k+2,\alpha}([\delta,T)\times M) \le C_k(\delta).$$

Similar to the proof of fundamental Hölder estimate, let $(B_{2r}(p))_{p\in M}$ be the set of (small) geodesic balls of radius $2r$. On local geodesic coordinate (x_j), we have

$$\frac{\partial u}{\partial t} - \mathscr{A}(x,u,\nabla u) : \nabla^2 u + \mathscr{B}(x,u,\nabla u) = 0.$$

Since $\|u\|_{1,\alpha}(M\times(0,T)) \le C$, we have

$$\mathscr{A}(x,u,\nabla u)^{jk}, \mathscr{B}(x,u,\nabla u) \in C^{0,\alpha}(B_{2r}(p)\times(0,T)).$$

According to the Schauder estimate (Lemma 24), we get

$$\|u\|_{2,\alpha}(B_r(p)\times(t-r^2,t)) \le C \text{ for all } t\ge 4r^2.$$

By applying the *covering argument* the same as that in the fundamental Hölder estimate, we get

$$\|u\|_{2,\alpha}(M\times[\delta/2,T)) \le C(\delta) \tag{5.4}$$

for a uniform constant $C(\delta)$. Since now $\|u\|_{2,\alpha}(M \times [\delta/2, T)) \leq C(\delta)$, we have

$$\mathscr{A}(x, u, \nabla u)^{jk}, \mathscr{B}(x, u, \nabla u) \in C^{1,\alpha}(B_{2r}(p) \times [\delta/2, T)).$$

Repeating the above argument using high-order Schauder estimate (Lemma 25), we can further get

$$\|u\|_{3,\alpha}(M \times [\delta/2 + \delta/4, T)) \leq C_1(\delta).$$

Therefore, by iterating the high-order Schauder estimate according to (5.3), we have

$$\|u\|_{k+2,\alpha}\big(M \times \big[\textstyle\sum_{\ell=1}^{k} 2^{-\ell}\delta, T\big)\big) \leq C_k(\delta).$$

As $\sum_{\ell=1}^{\infty} 2^{-\ell} = 1$, we proved the result. □

Remark 9 (boundary regularity)**.** *Similar to the fundamental Hölder estimate, we can not extend Schauder estimates uniformly to the boundary by taking $\delta \to 0$, unless the initial value is smooth. However, we still have $u \in C^\infty(M \times (0, T))$, since $(0, T)$ is open.*

Finally, we show the uniqueness of the solution.

Theorem 31 (uniqueness)**.** *If a C^2 solution to graphical mean curvature flow* (1.3) *exists, then it is unique.*

Proof. We let u and u' be two solutions with the same initial value u_0. We denote graphical mean curvature flow in a quasilinear form (in each local coordinate)

$$\frac{\partial u}{\partial t} - \mathscr{A}(x, u, \nabla u) : \nabla^2 u + \mathscr{B}(x, u, \nabla u) = 0$$

Let $\omega = u - u'$. By subtracting the equations for u and u', and applying Taylor's theorem, we have

$$\frac{\partial \omega}{\partial t} - \mathscr{A}(x, u, \nabla u) : \nabla^2 \omega + \mathscr{C}(x) \cdot \nabla \omega + \mathscr{D}(x)\omega = 0.$$

where $\mathscr{C}$ and $\mathscr{D}$ are low-order terms that does not contain $D^2\omega$.

Since $\mathscr{A}$ is uniformly elliptic according to the gradient estimate (Lemma 28), by using Garding's inequality, there exists constants $C > 0$ and $G \geq 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_M \omega^2 \,\mathrm{dvol} = 2 \int_M \omega \frac{\partial \omega}{\partial t} \,\mathrm{dvol} &= 2 \int_M \omega \Big[\mathscr{A}(x, u, \nabla u) : \nabla^2 \omega - \mathscr{C}(x) \cdot \nabla \omega - \mathscr{D}(x)\omega\Big] \,\mathrm{dvol} \\ &\leq -C \int_M |\nabla \omega|^2 \,\mathrm{dvol} + G \int_M \omega^2 \,\mathrm{dvol} \leq G \int_M \omega^2 \,\mathrm{dvol}.\end{aligned}$$

By Gronwall lemma, we have $\int_M \omega^2 \,\mathrm{dvol} = 0$, which means that $\omega = 0$ since ω is continuous. $\square$

Remark 10. *This uniqueness can also be proved by using the comparison principal, or the maximum principal [29].*

5.5 Prove of existence

In this subsection, we construct long-time solutions to graphical mean curvature flow (1.3) based on the a-prior estimates presented in the above subsection. A well-developed method for constructing solutions to quasilinear equations in terms of a-prior estimates is the Leray-Schauder fixed point theorem (see for example [15, 21]).

Lemma 32 (Leray–Schauder fixed point theorem)**.** *Let $\mathcal{B}$ be a Banach space and $F : \mathcal{B} \times [0,1] \to \mathcal{B}$ be a mapping with $F_\sigma(x) := F(x,\sigma)$, if F is a compact mapping such that*

1. *$F_0 = 0$.*
2. *there exists a constant C such that, if $(x,\sigma) \in \mathcal{B} \times [0,1]$ is a fixed point such that $F(x,\sigma) = x$, then $\|x\| < C$.*

Then there exists a fixed point for the mapping F_1.

We state the following global Schauder estimate that will be used in the fixed-point iteration.

Theorem 33 (global Schauder estimate)**.** *Given $\omega \in C^{1,\alpha}(M \times (0,T))$, and let u be the solution of the following linear parabolic equation with initial value $u_0 \in C^{2,\alpha}(M)$,*

$$\frac{\partial u}{\partial t} - \sigma\left[\frac{1}{\varphi(\omega)^2}\left(\Delta u - \frac{1}{\nu(\omega)^2}\langle \nabla^2 u \cdot \nabla\omega, \nabla\omega\rangle\right) + \frac{\varphi'(\omega)}{\varphi(\omega)}\left(n+1-\frac{\varphi(\omega)^2}{\nu(\omega)^2}\right)\right] = 0$$

where $\nu(\omega) := \sqrt{\varphi(\omega)^2 + |\nabla\omega|^2}$. There exists a uniform constant such that $\|u\|_{2,\alpha}(M \times (0,T)) \leq C$.

Proof. The proof is almost identical to the covering argument used to prove Theorem 30, expect that here, we use the regularity of the initial value to the extend the estimate uniformly to boundary, i.e., global Schuader estimates. We refer readers to [29, Section 9] for discussion. $\square$

To establish the compactness, we use the following compact embedding.

Lemma 34 (compact embedding)**.** *The embedding* $C^{2,\alpha}(M\times(0,T)) \hookrightarrow C^{1,\alpha}(M\times(0,T))$ *is compact for* $T<\infty$*.*

Proof. The compact embedding is the direct result of the Arzela–Ascoli theorem. □

We now apply Leray–Schauder fixed point theorem to prove the existence of solutions to (1.3).

Theorem 35. *Given initial value* $u_0 \in C^{2,\alpha}(M)$ *that satisfies the condition of the gradient estimate (Lemma 28), there exists a* C^∞ *solution* $u : M\times(0,\infty)\to\mathbb{R}$ *to* (5.2) *with initial value* u_0*.*

Proof. According to the smoothness theorem (Theorem 30), it is sufficient to establish the existence of a $C^{1,\alpha}$ solution. Given $T>0$, we let $\mathcal{B} = C^{1,\alpha}(M\times(0,T))$. Given $(\omega,\sigma)\in\mathcal{B}\times[0,1]$, we define the following mapping

$$F : \mathcal{B}\times[0,1]\to\mathcal{B} \text{ such that } (\omega,\sigma)\to u,$$

where u is the solution to the following linear parabolic equation.

$$\frac{\partial u}{\partial t} - \sigma\left[\frac{1}{\varphi(\omega)^2}\left(\Delta u - \frac{1}{\nu(\omega)^2}\langle\nabla^2 u\cdot\nabla\omega,\nabla\omega\rangle\right) + \frac{\varphi'(\omega)}{\varphi(\omega)}\left(n+1-\frac{\varphi(\omega)^2}{\nu(\omega)^2}\right)\right] = 0,$$
$$u(x,0) = \sigma u_0(x)$$

where $\nu(\omega) := \sqrt{\varphi(\omega)^2 + |\nabla\omega|^2}$. In particular, when $\sigma=1$, the above equation recovers graphical mean curvature flow (1.3), and $u=0$ is the only trivial solution for $\sigma=0$.

If $F(u,\sigma)=u$, then u is the solution to mean curvature flow up to a time scaling with factor σ. By fundamental Hölder estimate, we have

$$\|u\|_{1,\alpha}(M\times(0,T)) \le C$$

for a constant C.

To see the mapping F is compact, we prove it takes value *compact sets* of $\mathcal{B}$, and it is *continuous*. The compactness is given by the global Schauder estimate (Lemma 33) and compact embedding (Lemma 34). To prove the continuity, we let $\big((\omega_\ell,\sigma_\ell)\big)_{\ell\in\mathbb{N}}$ be a sequence converging to (ω, σ) under metric $\|\cdot\|_{1,\alpha}$. By the compactness of $\big(F(\omega_\ell, \sigma_\ell)\big)_{\ell\in\mathbb{N}}$, there exists $u \in \mathcal{B}$ such that check this

$$u_\ell := F(\omega_\ell, \sigma_\ell) \to u \text{ in } \mathcal{B},$$

up to the extraction of a subsequence. On this subsequence, we have (point-wisely)

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \sigma\left[\frac{1}{\varphi(\omega)^2}\left(\Delta u - \frac{1}{\nu(\omega)^2}\big\langle\nabla^2 u \cdot \nabla\omega, \nabla\omega\big\rangle\right) + \frac{\varphi'(\omega)}{\varphi(\omega)}\left(n + 1 - \frac{\varphi(\omega)^2}{\nu(\omega)^2}\right)\right] \\
&\quad = \lim_{\ell\to\infty} \frac{\partial u_\ell}{\partial t} - \sigma\left[\frac{1}{\varphi(\omega_\ell)^2}\left(\Delta u_\ell - \frac{1}{\nu(\omega_\ell)^2}\big\langle\nabla^2 u_\ell \cdot \nabla\omega_\ell, \nabla\omega_\ell\big\rangle\right) + \frac{\varphi'(\omega_\ell)}{\varphi(\omega_\ell)}\left(n + 1 - \frac{\varphi(\omega_\ell)^2}{\nu(\omega_\ell)^2}\right)\right] = 0.
\end{aligned}$$

Since the solution to the limiting equation is unique, the whole sequence (u_ℓ) converges. Hence, the continuity of F is proved. As this applies to all $T > 0$, the long-time existence is guaranteed. $\square$

Chapter 6

Conclusion and Discussion

In this thesis, we studied the quantitative convergence of mean curvature flow on warped manifolds. We showed the long-time existence of the flow based on quasilinear parabolic theory, and provided the sharp quantitative convergence via Lojasiewicz-Simon inequalities and Lyapunov-Schmidt reduction. We also include other methods from literature to show the quantitative convergence. For the future directions, it would be interesting to establish the sharp quantitative convergence for general mean curvature flow. It is also interesting to extend the current reduction method based on Lyapunov-Schmidt reduction for proving the sharp convergence of other geometric flows such as Yamabe flows [5] and Allen-Cahn equations [35].

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