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By

Denis Bashkirov and Alexander A. Voronov

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455-0436
Phone: 612-624-6066 Fax: 612-626-7370
URL: http://www.ima.umn.edu

THE BV FORMALISM FOR L_∞-ALGEBRAS

DENIS BASHKIROV AND ALEXANDER A. VORONOV

Abstract. The notions of a BV_∞-morphism and a category of BV_∞-algebras are investigated. The category of L_∞-algebras with L_∞-morphisms is characterized as a certain subcategory of the category of BV_∞-algebras. This provides a Fourier-dual, BV alternative to the standard characterization of the category of L_∞-algebras as a subcategory of the category of dg cocommutative coalgebras or formal pointed dg manifolds. In particular, the *coalgebra codifferential* on $S(\mathfrak{g}[1])$ encoding the structure of an L_∞-algebra on a graded vector space $\mathfrak{g}$ turns into a *square-zero differential operator with linear coefficients* on the *algebra* $S(\mathfrak{g}[-1])$. The functor assigning to a BV_∞-algebra the L_∞-algebra given by higher derived brackets is also shown to have a left adjoint.

Introduction

One way to approach the notion of a strongly homotopy Lie algebra is via the language of formal geometry, see [KS06]. Namely, it is known that the data of an L_∞-algebra $\mathfrak{g}$ is equivalent to that of a formal pointed differential graded (dg) manifold $\mathfrak{g}[1]$. The corresponding L_∞ structure is then encoded in the cofree dg cocommutative coalgebra $S(\mathfrak{g}[1])$ of distributions on $\mathfrak{g}[1]$ supported at the basepoint. The idea of Batalin-Vilkovisky (BV) formalism in physics suggests that it might be useful to study what the L_∞ structure looks like from a Fourier-dual perspective [Los07], namely, the point of view of the standard dg commutative algebra structure on $S(\mathfrak{g}[1])$. In fact, we show that an L_∞ structure on $\mathfrak{g}$ translates into a special kind of commutative homotopy Batalin-Vilkovisky (BV_∞) structure on $S(\mathfrak{g}[-1])$ and, moreover, does it in a functorial way. Geometrically, we can say that we describe a formal pointed dg manifold $\mathfrak{g}[1]$ as a pointed BV_∞-manifold $(\mathfrak{g}[-1])^*$ of a special kind. We also show that the functor that assigns to an L_∞-algebra $\mathfrak{g}$ the BV_∞-algebra $S(\mathfrak{g}[-1])$ is left adjoint to a "functor" that assigns to a BV_∞-algebra the L_∞-algebra given by higher derived brackets. This fact may be interpreted geometrically as a statement that the functor $\mathfrak{g}[1] \mapsto (\mathfrak{g}[-1])^*$ from formal pointed dg manifolds to pointed BV_∞-manifolds has a right adjoint.

The correspondence between L_∞ and BV_∞ structures that we establish in the paper is to a large extent motivated by the technique of higher derived brackets. The origins of higher derived brackets can be traced back to the iterated commutators of A. Grothendieck, see Exposé VII_A by P. Gabriel in [SGA3], and J.-L. Koszul [Kos85], used in the algebraic study of differential operators, though the subject really flourished later in physics under the name of higher "antibrackets" in

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the works of J. Alfaro, I. A. Batalin, K. Bering, P. H. Damgaard and R. Marnelius [AD96, BBD97, BM98, BM99a, BM99b, BDA96, Ber07] on the BV formalism. A mathematically friendly approach was developed by F. Akman's [Akm97, Akm00] and generalized further by T. Voronov [Vor05a, Vor05b], who described L_∞ brackets derived by iterating a binary Lie bracket not necessarily given by the commutator. There have been various versions of higher derived brackets introduced in other contexts, such as the ternary bracket of [RW98] for Courant algebroids and its higher-bracket generalization of [FM07, Get10] for dg Lie algebras or the A_∞ products of [Bör14] for dg associative algebras and the twisted L_∞ brackets and A_∞ products of [Mar13]. The notion of a homotopy BV algebra was studied by K. Bering and T. Lada [BL09], K. Cieliebak and J. Latschev [CL07], I. Gálvez-Carrillo, A. Tonks and B. Vallette [GCTV12], O. Kravchenko [Kra00], and D. Tamarkin and B. Tsygan [TT00].

The BV formalism as a replacement of the dg-coalgebra language seems to be even more natural for studying Lie-Rinehart pairs $(\mathfrak{g}, A)$, see [Hue98, Vit13].

We review the notion of a BV_∞-algebra in Section 1 and describe the BV_∞ structure on $S(\mathfrak{g}[-1])$ in Section 2. A characterization of BV_∞-algebras arising this way is presented in Section 3. In Section 4 we prove the first main result of the paper: a description of the category of L_∞-algebras as a certain subcategory of the category of BV_∞-algebras. Section 5 is dedicated to the second main result, the adjunction theorem.

Conventions and Notation. We will work over a ground field k of characteristic zero. A differential graded (dg) vector space V will mean a complex of k-vector spaces with a differential of degree one. The degree of a homogeneous element $v \in V$ will be denoted by $|v|$. In the context of graded algebra, we will be using the Koszul rule of signs when talking about the graded version of notions involving symmetry, including commutators, brackets, symmetric algebras, derivations, *etc.*, often omitting the modifier *graded*. For any integer n, we define a *translation* (or *n-fold desuspension*) $V[n]$ of V: $V[n]^p := V^{n+p}$ for each $p \in \mathbb{Z}$. For a dg vector space V, will also consider the dg $k[[\hbar]]$-module $V[[\hbar]]$ of formal power series in a variable $\hbar$ of degree 2 with values in V. We will also sometimes refer to differential operators of order $\leq n$ as differential operators of order n.

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1. Homotopy BV algebras

We will utilize a strictly commutative version of the notion of a homotopy BV algebra, also known as a generalized BV algebra, due to Kravchenko [Kra00], which is less general than the full-fledged homotopy versions of [TT00] and [GCTV12]. Nevertheless, we will take the liberty to use the term BV_∞-algebra, following a trend set by several authors [CL07, TTW11, BL13, Vit13]. The following definition gives a graded version of Grothendieck's notion of a differential operator in commutative algebra.

Definition 1.1. Let $n \geq 0$ be an integer. A k-linear operator $D : V \to V$ on a graded commutative algebra V is a said to be a *differential operator of order* $\leq n$

if for any $n+1$ elements $a_0, \dots, a_n \in V$, we have

$$[[\dots[D, L_{a_0}], \dots], L_{a_n}] = 0,$$

where the L_a is the left-multiplication operator

$$L_a(b) := ab$$

on V and the bracket $[-,-]$ is the graded commutator of two k-linear operators.

Definition 1.2. A BV$_\infty$-*algebra* is a graded commutative algebra V over k with a k-linear map $\Delta : V \to V[[\hbar]]$ of degree one satisfying the following properties:

$$\Delta = \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \Delta_n,$$

where Δ_n is a differential operator of order (at most) n on V,

$$\Delta^2 = 0, \qquad \text{and} \qquad \Delta(1) = 0,$$

The continuous (in the $\hbar$-adic topology), $k[[\hbar]]$-linear extension of Δ to $V[[\hbar]]$ will also be denoted $\Delta : V[[\hbar]] \to V[[\hbar]]$ and called a BV$_\infty$ *operator*.

Recall that we assumed that $|\hbar| = 2$, thus $|\Delta_1| = 1$, $|\Delta_2| = -1$, and generally, $|\Delta_n| = 3 - 2n$ for $n \geq 1$. Note that Δ_1 will automatically be a differential in the usual sense, *i.e.*, define the structure of a dg commutative algebra on V. If $\Delta_n = 0$ for $n \geq 3$, we recover the notion of a *dg BV algebra*, see [KS95, Akm97, BK98, Hue98, Man99, Kra00, TT00]. If moreover $\Delta_1 = 0$, we get the notion of a *BV algebra*, also known as a *Beilinson-Drinfeld algebra*, see [BD04, Gwi12, CG14]. BV algebras arose as part of the BV formalism in physics. A basic geometric example of (a $\mathbb{Z}/2\mathbb{Z}$-graded version of) a BV algebra is the algebra of functions on a smooth supermanifold with an odd symplectic form and a volume density, see [Sch93, Get94]. An example of such a supermanifold is the odd cotangent bundle $\Pi T^* M$ of a (classical, rather than super) smooth manifold M with a volume form, where $\Pi T^* M$ denotes the translation $T^*[-1]M$ modulo 2 of the cotangent bundle. A Lie-theoretic version of this example is the graded symmetric algebra $S(\mathfrak{g}[-1])$, also known as the exterior algebra $\Lambda(\mathfrak{g})$, of a Lie algebra $\mathfrak{g}$, with the Chevalley-Eilenberg differential as Δ_2. We will describe an L_∞ generalization of this example is Section 2. More generally, one can view the BV$_\infty$ structure considered in this paper as a homotopy version of the algebraic structure arising in BV geometry.

Example 1.3. ([KV08, BL13, Vit13]) Let M be a smooth graded manifold and $C^\infty(M, S(T[-1]M)[1])$ be the graded Lie algebra of (global, smooth) multivector fields on M with respect to the Schouten bracket. When M is a usual, ungraded manifold, $S(T[-1]M)[1]$ is the exterior-algebra bundle $\bigwedge TM$, in which a k-vector field, or a section of $\bigwedge^k TM$, has degree $k-1$. A *generalized Poisson structure* on a graded manifold M is a multivector field P of degree one such that $[P, P] = 0$. A generalized Poisson structure on M may be expanded as $P = P_0 + P_1 + \dots$ with $P_n \in C^\infty(M, S^n(T[-1]M)[1])$. For $n \geq 1$, the generalized Lie derivative $\Delta_n = [d, i_{P_n}]$, where $i_{(-)}$ is the interior product, defines an nth-order differential operator of degree $3 - 2n$ on the de Rham algebra $(\Omega(M), d)$, where $\Omega(M) := C^\infty(M, S(T^*[-1]M))$. If we assume that $P_0 = 0$ to avoid differential operators of order zero, then $\Delta = \Delta_1 + \Delta_2 \hbar + \dots + \Delta_n \hbar^{n-1} + \dots : \Omega(M) \to \Omega(M)[[\hbar]]$ defines a BV$_\infty$ structure on $\Omega(M)$, known as the *de Rham-Koszul* BV$_\infty$ structure.

2. From homotopy Lie algebras to homotopy BV algebras

The construction of this section belongs essentially to C. Braun and A. Lazarev, see [BL13, Example 3.12]. Consider an L_∞*-algebra* $\mathfrak{g}$, *i.e.*, a graded vector space $\mathfrak{g}$ and a codifferential on the cofree graded cocommutative coalgebra $S(\mathfrak{g}[1])$ on $\mathfrak{g}[1]$ with respect to the "shuffle" comultiplication:

$$\delta(x_1 \dots x_m) := \sum_{n=0}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} (x_{\sigma(1)} \dots x_{\sigma(n)}) \otimes (x_{\sigma(n+1)} \dots x_{\sigma(m)}),$$

where $x_1, \dots, x_m \in \mathfrak{g}[1]$, $\mathrm{Sh}_{n,m-n}$ is the set of $(n, m-n)$ *shuffles*: permutations σ of $\{1, 2, \dots, m\}$ such that $\sigma(1) < \dots < \sigma(2) < \dots < \sigma(n)$ and $\sigma(n+1) < \dots < \sigma(n+2) < \dots < \sigma(m)$, and $(-1)^{|x_\sigma|}$ is the *Koszul sign* of the permutation of $x_1 \dots x_m$ to $x_{\sigma(1)} \dots x_{\sigma(m)}$ in $S(\mathfrak{g}[1])$. Here a *codifferential* is a coderivation $D : S(\mathfrak{g}[1]) \to S(\mathfrak{g}[1])$ of degree one such that $D^2 = 0$ and $D(1) = 0$. Given that a coderivation is determined by its projection to the cogenerators, we can write

$$D = D_1 + D_2 + D_3 + \dots,$$

where D_n is the extension as a coderivation of the nth symmetric component $l_n : S^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ of the projection $S(\mathfrak{g}[1]) \xrightarrow{D} S(\mathfrak{g}[1]) \to \mathfrak{g}[1]$. An explicit relation between D_n and l_n will be useful: for $x_1, \dots, x_m \in \mathfrak{g}[1]$

$$D_n(x_1 \dots x_m) = \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} l_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) x_{\sigma(n+1)} \dots x_{\sigma(m)}, \tag{1}$$

if $m \geq n$, and $D_n(x_1 \dots x_m) = 0$ otherwise.

Theorem 2.1 (C. Braun and A. Lazarev)**.** *Given an* L_∞*-algebra* $\mathfrak{g}$*, the free graded commutative algebra* $S(\mathfrak{g}[-1])$ *becomes a* BV_∞*-algebra under the* BV_∞ *operator*

$$\Delta := \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n D_n. \tag{2}$$

Proof. Since $D_n : S^m(\mathfrak{g}[1]) \to S^{m-n+1}(\mathfrak{g}[1])$ is a degree one map, it turns into a map $D_n : S^m(\mathfrak{g}[-1]) \to S^{m-n+1}(\mathfrak{g}[-1])$ of degree $3 - 2n$ under the new grading.[1] For each n,

$$\sum_{i+j=n} D_i D_j = 0,$$

because this sum is exactly the component of D^2 which maps $S^m(\mathfrak{g}[1])$ to $S^{m-n+2}(\mathfrak{g}[1])$. The map D_n will also be a differential operator of order n, because of the following lemma, which expresses an important relation between coderivations and differential operators and may be observed directly from Equation (1).

Lemma 2.2. *The coderivation of the coalgebra* $S(\mathfrak{g}[1])$ *extending a linear map* $S^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ *becomes a differential operator of order* $\leq n$ *on the algebra* $S(\mathfrak{g}[-1])$*.*

□

[1] Strictly speaking, the use of D_n to denote the two maps is abuse of notation, because they differ by powers of the double suspension operator $\mathfrak{g}[1] \to \mathfrak{g}[-1]$, but we prefer to keep it this way, because double suspension does not affect signs.

The construction of this section seems to be math-physics folklore in the case when $(\mathfrak{g}, d, [-,-])$ is a dg Lie algebra: the differential $\Delta = D_1 + \hbar D_2$ defines a dg BV algebra structure on $S(\mathfrak{g}[-1])$. The operator Δ is essentially the homological Chevalley-Eilenberg differential:

$$\begin{aligned}\Delta(x_1 \dots x_m) &= \sum_{i=1}^{m} (-1)^{|x_1 \dots x_{i-1}|} x_1 \dots dx_i \dots x_m \\ &+ \hbar \sum_{1 \le i < j \le m} (-1)^{|x_{\sigma(i,j)}|+|x_i|} [x_i, x_j] x_1 \dots \widehat{x_i} \dots \widehat{x_j} \dots x_m,\end{aligned}$$

where $\sigma(i,j)$ is the corresponding shuffle, the x_i's in $\mathfrak{g}$ are treated as elements of $\mathfrak{g}[-1]$, and, following standard conventions, $d = l_1$ and $[x_i, x_j] = (-1)^{|x_i|} l_2(x_i, x_j)$.

Remark. An A_∞-analogue of the above construction has been proposed by J. Terilla, T. Tradler, and S. Wilson in [TTW11]: for an A_∞-algebra V, the tensor algebra $T(V[-1])$ (equipped with the shuffle product) is provided with a BV$_\infty$-structure. There is also an interesting generalization to the ∞-version of a Lie-Rinehart pair considered by L. Vitagliano [Vit13].

Remark. We will also need a certain $\hbar$-enhancement of the construction of a BV$_\infty$-algebra from an L_∞-algebra. Suppose the graded $k[[\hbar]]$-module $\mathfrak{g}[[\hbar]]$ for a graded vector space $\mathfrak{g}$ is provided with the structure of a topological L_∞-algebra over $k[[\hbar]]$ with respect to $\hbar$-adic topology. Then the same formula (2) defines a BV$_\infty$-structure on $S(\mathfrak{g}[-1])$ over k. There is a subtlety, though: each operator D_n is a formal power series in $\hbar$ now, and in the $\hbar$-expansion

$$\Delta = \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \Delta_n,$$

there are contributions to Δ_n from D_1, D_2, ..., and D_n. This still guarantees that Δ_n is a differential operator of order at most n on $S(\mathfrak{g}[-1])$ satisfying the conditions of Definition 1.2.

To summarize, given an L_∞-algebra $\mathfrak{g}$, we obtain a canonical BV$_\infty$-algebra structure on $S(\mathfrak{g}[-1])$. There is also a construction going in the opposite direction.

3. From homotopy BV algebras to homotopy Lie algebras

Suppose we have a BV$_\infty$-algebra V. Then for each $n \ge 1$, the following *higher derived brackets*

$$\begin{aligned}l_n^\hbar(a_1, \dots, a_n) &:= [[\dots[\Delta, L_{a_1}], \dots], L_{a_n}]1 \\ &= \sum_{k=n}^{\infty} \hbar^{k-1} [[\dots[\Delta_k, L_{a_1}], \dots], L_{a_n}]1 \end{aligned} \tag{3}$$

on $V[[\hbar]]$, their $\hbar$-modification

$$L_n := \frac{1}{\hbar^{n-1}} l_n^\hbar, \tag{4}$$

and their "semiclassical limit"

$$(5)\qquad \begin{aligned} l_n(a_1, \dots, a_n) &:= \lim_{\hbar\to 0} L_n(a_1, \dots, a_n) \\ &= \lim_{\hbar \to 0} \frac{1}{\hbar^{n-1}}[[\dots[\Delta, L_{a_1}], \dots], L_{a_n}]1 \\ &= [[\dots[\Delta_n, L_{a_1}], \dots], L_{a_n}]1 \end{aligned}$$

on V turn out to be L_∞ brackets, according to the results in this section below. Just for comparison, note that $l_1^\hbar = L_1 = \Delta$, whereas $l_1 = \Delta_1$. Observe also that we have a linear (or strict) L_∞-morphism

$$\begin{aligned} (V[[\hbar]][-1], l_n^\hbar) &\to (V[[\hbar]][1], L_n), \\ v &\mapsto \hbar v, \end{aligned}$$

which becomes an L_∞-isomorphism after localization in $\hbar$. Thus, we can think of the L_∞ structure given by the brackets L_n as an $\hbar$-translation of the L_∞ structure given by $l_n^\hbar$.

One can express Δ through $l_n^\hbar$ via the following useful formula

$$(6)\qquad \Delta(a_1 \dots a_n) = \sum_{j=1}^{n} \sum_{\sigma \in \mathrm{Sh}_{j,n-j}} (-1)^{|a_\sigma|} l_j^\hbar(a_{\sigma(1)}, \dots, a_{\sigma(j)}) a_{\sigma(j+1)} \dots a_{\sigma(n)}$$

for $a_1, \dots, a_n \in V$, which is easy to prove by induction on n using Equation (7) below, starting with $n = 1$ for $l_1^\hbar = \Delta$.

Theorem 3.1 (Bering-Damgaard-Alfaro). *For a* BV_∞*-algebra* V*, the higher brackets* $l_n^\hbar$*,* $n \geq 1$*, defined by* (3) *endow the suspension* $V[[\hbar]][-1]$ *with the structure of an* L_∞*-algebra over* $k[[\hbar]]$*. Moreover, the bracket* $l_{n+1}^\hbar$ *measures the deviation of* $l_n^\hbar$ *from being a multiderivation with respect to multiplication.*

Remark. This result was first observed by the physicists [BDA96] and proven in a more general context by T. Voronov [Vor05a, Vor05b]. The L_∞ structure was also rediscovered by O. Kravchenko in [Kra00].

Proof. Using the Jacobi identity for the commutator of linear operators along with the fact that L_a and L_b (graded) commute, it is easy to check that the higher brackets $l_n^\hbar$ are symmetric on $V[[\hbar]]$:

$$l_n^\hbar(a_{\pi(1)}, \dots, a_{\pi(n)}) = (-1)^{|a_\pi|} l_n^\hbar(a_1, \dots, a_n)$$

for all $a_1, \dots, a_n \in V[[\hbar]]$, where $(-1)^{|a_\pi|}$ is the Koszul sign, see Section 2. Since $|\Delta| = 1$, the degree of $l_n^\hbar$ as a bracket on $V[[\hbar]]$ will be the same. We can extend the $k[[\hbar]]$-linear operators $l_n^\hbar : S^n(V)[[\hbar]] \to V[[\hbar]]$ to coderivations $D_n : S(V)[[\hbar]] \to S(V)[[\hbar]]$ and consider the total coderivation

$$D = D_1 + D_2 + \dots.$$

on $S(V)[[\hbar]]$. The differential property $D^2 = 0$ for this coderivation is equivalent to the series of *higher Jacobi identities*:

$$\sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|a_\sigma|} l_{m-n+1}^\hbar(l_n^\hbar(a_{\sigma(1)}, \dots, a_{\sigma(n)}), a_{\sigma(n+1)}, \dots, a_{\sigma(m)}) = 0$$

for all $a_1, \dots, a_m \in V[[\hbar]]$, $m \geq 1$. The physicists [BDA96] and T. Voronov [Vor05a] in a more general situation checked these identities using the following key observation for an arbitrary odd operator Δ on $V[[\hbar]]$, not necessarily squaring to zero:

$$\sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|a_\sigma|} l^\hbar_{m-n+1}(l^\hbar_n(a_{\sigma(1)}, \dots, a_{\sigma(n)}), a_{\sigma(n+1)}, \dots, a_{\sigma(m)}) \\ = [[\dots[\Delta^2, L_{a_1}], \dots], L_{a_m}]1.$$

Given that $\Delta^2 = 0$, the higher Jacobi identities follow.

The deviated multiderivation property, more precisely,

$$(7) \quad l^\hbar_{n+1}(a_1, \dots, a_i, a_{i+1}, \dots, a_{n+1}) = l^\hbar_n(a_1, \dots, a_i \cdot a_{i+1}, \dots, a_{n+1}) \\ - (-1)^{(1+|a_1|+\dots|a_{i-1}|)|a_i|} a_i l^\hbar_n(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1}) \\ - (-1)^{(1+|a_1|+\dots|a_i|)|a_{i+1}|} a_{i+1} l^\hbar_n(a_1, \dots, a_i, a_{i+2}, \dots, a_{n+1})$$

of the higher brackets may be derived from the identity

$$[Q, L_{ab}] = [[Q, L_a], L_b] + (-1)^{|Q|\cdot|a|} L_a[Q, L_b] + (-1)^{(|Q|+|a|)|b|} L_b[Q, L_a]$$

for an arbitrary (homogeneous) linear operator Q on $V[[\hbar]]$. Applying this to $Q = [[\dots[\Delta, L_{a_1}], \dots], L_{a_n}]$, we see that $l^\hbar_{n+1}$ measures the deviation of $l^\hbar_n$ from being a derivation in the last variable. Since the higher brackets are symmetric, we obtain the same property in each variable. □

Corollary 3.2 (T. Voronov [Vor05a]). *Given a* BV_∞*-algebra* V*, the brackets* L_n*,* $n \geq 1$*, defined by* (4) *endow the graded* $k[[\hbar]]$*-module* $V[[\hbar]][1]$ *with the structure of an* L_∞*-algebra over* $k[[\hbar]]$*. Likewise, the brackets* l_n*,* $n \geq 1$*, defined by* (5) *endow the graded vector space* $V[1]$ *with the structure of an* L_∞*-algebra over* k*. Moreover, the brackets* l_n *are multiderivations of the graded commutative algebra structure.*

Proof. The statements of the corollary are obtained as the "semiclassical limit" of the statements of Theorem 3.1, and so is the proof. Note the change of suspension to desuspension from the theorem to the corollary. This corresponds intuitively to the statement that the semiclassical limit of the space $V[[\hbar]][-1]$ is $\hbar V[-1] = V[1]$. Concretely, the desuspension guarantees that the degree of the nth higher bracket l_n on $V[2]$ is still one: indeed $|\Delta_n| = 3 - 2n$, when Δ_n is considered as an operator on V; therefore, the degree of l_n as a multilinear operation on $V[2]$ will be $3 - 2n + 2(n-1) = 1$.

The multiderivation property is obtained by dividing (7) by $\hbar^{n-1}$ and noticing that the left-hand side will not survive the limit as $\hbar \to 0$, because it has $\hbar$ as a factor. □

Remark. The algebraic structure which combines the graded commutative multiplication with the L_∞ structure given by the brackets l_n is a particular case of the G_∞-algebra structure, see [GJ94, Tam98, Tam99, Vor00].

Remark. The construction given by the higher brackets l_n obviously induces an operad morphism $sL_\infty \to \mathrm{BV}_\infty$, where s denotes the operadic suspension, see, *e.g.*, [MSS02]. Here BV_∞ stands for the operad describing commutative BV_∞-algebras, as opposed to the full BV_∞ operad of [TT00, GCTV12, DCV13]. This operad morphism immediately gives a functor from the category of BV_∞-algebras to that of L_∞-algebras, provided we restrict ourselves to morphisms of algebras over operads, *i.e.*, consider only linear (strict) morphisms. However, we will focus on nonlinear morphisms in the subsequent sections of the paper.

Example 3.3. The L_∞ structure given by the brackets l_n coming from the BV_∞ structure of Example 1.3 is known as the de Rham-Koszul L_∞ structure and generalizes the Koszul brackets on the de Rham complex of a manifold, [KV08, BL13, Vit13].

Example 3.4. Let $\mathfrak{g}$ be an L_∞-algebra. Then by the construction of Section 2, we get the structure of a BV_∞-algebra on $S(\mathfrak{g}[-1])$. If we apply the "semiclassical" derived brackets l_n of this section to the BV_∞-algebra $S(\mathfrak{g}[-1])$, we will get the structure of an L_∞-algebra on $S(\mathfrak{g}[-1])[1]$. Later we show in Theorem 3.6(3) that the L_∞-algebra $\mathfrak{g}$ becomes an L_∞-subalgebra of $S(\mathfrak{g}[-1])[1]$. The higher brackets on $S(\mathfrak{g}[-1])[1]$ may be viewed as extensions of the higher brackets on $\mathfrak{g}$ as multiderivations. These higher brackets *generalize the Schouten bracket* on the exterior algebra of a Lie algebra to the L_∞ case.

Our goal is to characterize those BV_∞-algebras which come from L_∞-algebras as in Section 2. Note that such a BV_∞-algebra is free as a graded commutative algebra by construction: $V = S(U)$, and that for each $n \ge 1$, the nth component Δ_n of the BV_∞ operator maps $S^m(U)$ to $S^{m-n+1}(U)$ for $m \ge n$ and to 0 for $0 \le m < n$, because of Equation (1). Since an nth-order differential operator on a free algebra $S(U)$ is determined by its restriction to $S^{\le n}(U)$, this condition on Δ_n is equivalent to the condition that Δ_n maps $S^n(U)$ to U and $S^{<n}(U)$ to 0. Interpreting differential operators on $S(U)$ as linear combinations of partial derivatives with polynomial coefficients, differential operators of the above type may also be characterized as *differential operators of order n with linear coefficients.*

Definition 3.5. A *pure* BV_∞-algebra is the free graded commutative algebra $S(U)$ on a graded vector space U with a BV_∞ operator $\Delta : S(U) \to S(U)[[\hbar]]$ such that, for each $n \ge 1$, Δ_n maps $S^n(U)$ to U and $S^{<n}(U)$ to 0.

The following theorem (Parts (1) and (2)) shows that freeness and purity are not only necessary but also sufficient conditions for a BV_∞-algebra to arise from an L_∞-algebra.

Theorem 3.6.
(1) *Given a pure* BV_∞ *algebra* $(V = S(U), \Delta)$*, the restriction of the brackets* l_n *to* $U[1] \subset S(U)[1]$ *provides* $U[1]$ *with the structure of an* L_∞*-subalgebra.*
(2) *The original pure* BV_∞ *structure on* $S(U)$ *coincides with the* BV_∞ *structure* (2) *of Section* 2 *coming from the derived* L_∞ *structure on* $U[1]$*.*
(3) *If we start with an* L_∞ *structure on a graded vector space* $U[1]$ *and construct the* BV_∞*-algebra* $S(U)$ *as in Section* 2*, then the derived brackets* l_n *on* $U[1] \subset S(U)[1]$ *return the original* L_∞ *structure on* $U[1]$*.*

Proof. The first statement we need to check is that $l_n(a_1, \dots, a_n)$ is in U whenever $a_1, \dots, a_n \in U$ and $n \ge 1$, as a priori all we know is that $l_n(a_1, \dots, a_n) \in S(U)$. The condition that Δ_n maps $S^m(U)$ to 0 for $0 \le m < n$ implies by (5) that $l_n(a_1, \dots, a_n) = \Delta_n(a_1 \dots a_n)$, which must be in U, because of the condition $\Delta_n : S^n(U) \to S^1(U) = U$.

For the second statement, we need to check that the nth-order differential operator Δ_n, the nth component of the given BV_∞ structure, is equal to the coderivation

D_n defined by (1). Recall that on the free algebra $S(U)$, an nth-order differential operator is determined by its restriction to $S^{\le n}(U)$. Given the assumption that Δ_n vanishes on $S^{<n}(U)$, it follows that Δ_n on $S(U)$ is determined by its restriction to $S^n(U)$. By the previous paragraph, its restriction to $S^n(U)$ is equal to l_n. On the other hand, this is also the restriction of the coderivation D_n to $S^n(U)$, as per formula (1). Lemma 2.2 shows that the coderivation D_n is also an nth-order differential operator. Thus, it is also determined by its restriction to $S^n(U)$.

Finally, let l_n be the L_∞ brackets on an L_∞-algebra $U[1]$ and $\tilde{l}_n$ be the higher derived brackets produced on the pure BV_∞-algebra $S(U)$ by formula (5) for $n = 1$, $2, \ldots$ We claim that $\tilde{l}_n(x_1, \ldots, x_n) = l_n(x_1, \ldots, x_n)$ for all n and $x_1, \ldots, x_n \in U$. Indeed,

$$\tilde{l}_n(x_1, \ldots, x_n) = [[\ldots[D_n, L_{x_1}], \ldots], L_{x_n}]1,$$

where D_n is the extension of l_n to $S(U)$ as a coderivation, see Equation (1). The same equation implies that $D_n : S(U) \to S(U)$ is zero on $S^{<n}(U)$. Hence all but one term $(D_n \circ L_{x_1} \circ \cdots \circ L_{x_n})(1)$ of this iterated commutator vanish. It remains to observe that by (1) this is nothing but $l_n(x_1, \ldots, x_n)$. □

Remark. A general, not necessarily pure BV_∞-structure on $S(U)$ leads to an interesting algebraic structure on $U[1]$, called an *involutive L_∞-bialgebra*, see [CFL13]. From the properadic, rather than BV prospective, this structure is described in [Val07] and [DCTT10]. The BV formalism for ordinary L_∞-bialgebras seems to be subtler: apparently, one needs to weaken the definition of a BV_∞ structure on $S(U)$ by requiring that for each $n \ge 1$, the coefficient by $\hbar^{n-1}$ in the expansion of $\Delta^2 = \frac{1}{2}[\Delta, \Delta]$ be of order $\le n-1$, rather than the expected order $\le n$, instead of asking for vanishing of each coefficient of Δ^2 in its expansion in $\hbar$.

4. Functoriality

The correspondence between BV_∞-algebras and L_∞-algebras that we studied above has remarkable functorial properties with a suitable notion of a morphism between BV_∞-algebras.

First of all, recall the definition of a morphism between L_∞-algebras.

Definition 4.1. An *L_∞-morphism* $\mathfrak{g} \to \mathfrak{g}'$ between L_∞-algebras is a morphism $S(\mathfrak{g}[1]) \to S(\mathfrak{g}'[1])$ of codifferential graded coalgebras, *i.e.*, a morphism of graded coalgebras commuting with the structure codifferentials, such that $1 \in S^0(\mathfrak{g}[1])$ maps to $1 \in S^0(\mathfrak{g}'[1])$.

Remark. Since we deal with counital coalgebras, we assume that L_∞-morphisms respect the counits. The extra condition $1 \mapsto 1$ means that we are talking about "pointed" morphisms, if we invoke the interpretation of L_∞-morphisms as morphisms between formal pointed dg manifolds, see [KS06].

Now we will consider the corresponding notion of a morphism between BV_∞-algebras. We will only need this notion for BV_∞-algebras of Theorem 3.6, that is to say, BV_∞-algebras which are pure. Somewhat more generally, we will give a definition in the case when the source BV_∞-algebra is just free. A more general notion of a BV_∞-morphism for more general BV_∞-algebras can be found in [TT00]. We use the definition of a BV_∞-morphism by Cieliebak-Latchev [CL07].

Before giving the definition, we need to recall a few more notions. Fix a morphism $f : A \to A'$ between graded commutative algebras. We say that a k-linear map

$D : A \to A'$ is a *differential operator of order* $\le n$ *over* $f : A \to A'$ or simply a *relative differential operator of order* $\le n$ if for any $n+1$ elements $a_0, \dots, a_n \in A$, we have

$$[[\dots[D, L_{a_0}], \dots], L_{a_n}] = 0,$$

where $[D, L_a]$ is understood as the map $A \to A'$ defined by

$$[D, L_a](b) := D(ab) - (-1)^{|a|\cdot|D|} f(a) D(b).$$

For $f = \mathrm{id}$ we recover the standard definition Def. 1.1 of a differential operator on a graded commutative algebra.

Let $V = S(U)$ be a free graded commutative algebra and V' an arbitrary graded commutative algebra. Given a k-linear map $\varphi : S(U) \to V'[[\hbar]]$ of degree zero such that $\varphi(1) = 0$, define a degree-zero, continuous $k[[\hbar]]$-linear map $\exp(\varphi) : S(U)[[\hbar]] \to V'[[\hbar]]$, called the *exponential*, by

$$\exp(\varphi)(x_1 \dots x_m) :=$$
$$\sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1, \dots, i_k = 1 \\ i_1 + \dots + i_k = m}}^{m} \frac{1}{i_1! \dots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \varphi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots \varphi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)}),$$

where S_m denotes the symmetric group, $x_1, \dots, x_m$ are in U, and $(-1)^{|x_\sigma|}$ is the Koszul sign of the permutation of $x_1 \dots x_m$ to $x_{\sigma(1)} \dots x_{\sigma(m)}$ in $S(U)$. By convention, we set $\exp(\varphi)(1) := 1$. The reason for the exponential notation, introduced by Cieliebak and Latschev [CL07], is, perhaps, the following statement, which they might have been aware of, cf. [CFL13]. The proof is a straightforward computation.

Lemma 4.2. *If* $S \in \lambda U[[\hbar]]^0[[\lambda]]$ *or* $\lambda U((\hbar))^0[[\lambda]]$, *where* λ *is another, degree-zero formal variable, then*

$$\exp(\varphi)(e^S) = e^{\varphi(e^S)}.$$

Here we have extended φ *and* $\exp(\varphi)$ *to* $\lambda S(U)((\hbar))[[\lambda]]$ *by* $\hbar^{-1}$*- and* λ*-linearity and continuity.*

Remark. The extra formal variable λ in the lemma guarantees "convergence" of the exponential e^S. We could have achieved the same goal, if we considered completions of our algebras or assumed that λ was a nilpotent variable, varying over the maximal ideals of finite-dimensional local Artin algebras. Informally speaking, given the way the space $S(U)[\lambda, \hbar, \hbar^{-1}]$ of $S(U)$-valued polynomials in λ and Laurent polynomials in $\hbar$ is completed: $S(U)((\hbar))[[\lambda]]$, we could think of λ as being "much smaller" than $\hbar$.

The exponential, not surprisingly, has an inverse, called the *logarithm*, which we will use a little later. Given a k-linear map $\Phi : S(U) \to V'[[\hbar]]$ of degree zero such that $\Phi(1) = 1$, define a degree-zero, continuous $k[[\hbar]]$-linear map $\log(\Phi) : S(U)[[\hbar]] \to V'[[\hbar]]$ by

$$\log(\Phi)(x_1 \dots x_m)$$
$$:= \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} \sum_{\substack{i_1, \dots, i_k = 1 \\ i_1 + \dots + i_k = m}}^{m} \frac{1}{i_1! \dots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \Phi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots \Phi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)})$$

under the same notation as for the exponential. By convention, we set $\log(\Phi)(1) := 0$.

Definition 4.3 (Cieliebak-Latchev [CL07])**.** A BV_∞-*morphism* from a BV_∞-algebra $(V = S(U), \Delta)$ to a BV_∞-algebra (V', Δ') is a k-linear map $\varphi : V \to V'[[\hbar]]$ of degree zero satisfying the following properties:

(1) $\varphi(1) = 0$,
(2) $\exp(\varphi)\Delta = \Delta' \exp(\varphi)$, and
(3) φ admits an expansion

$$\varphi = \frac{1}{\hbar}\sum_{n=1}^{\infty} \hbar^n \varphi_n,$$

where $\varphi_n : V \to V'$ is a differential operator of order $\leq n$ over the morphism $S(U) \to V'$ induced by the zero linear map $U \xrightarrow{0} V'$, *i.e.*, φ_n maps $S^{>n}(U)$ to 0.

We will use the same notation for the continuous, $k[[\hbar]]$-linear extension $\varphi : V[[\hbar]] \to V'[[\hbar]]$ of the k-linear map $\varphi : V \to V'[[\hbar]]$, as well as for the corresponding BV_∞-morphism $\varphi : (V, \Delta) \to (V', \Delta')$.

Remark. A BV_∞-morphism can be regarded as a quantization of a morphism of dg commutative algebras. Indeed, φ_1 must be nonzero only on $U = S^1(U) \subset S(U)$ and by construction $\exp(\varphi_1)$ will be a graded algebra morphism. The equation $\exp(\varphi)\Delta = \Delta' \exp(\varphi)$ at $\hbar = 0$ reduces to $\exp(\varphi_1)\Delta_1 = \Delta'_1 \exp(\varphi_1)$, which implies that $\exp(\varphi_1)$ is a morphism of dg algebras with respect to the "classical limits" Δ_1 and Δ'_1 of the BV_∞ operators.

Example 4.4. A nice example of a BV_∞-morphism $S(V) \to V$ may be obtained from the projection $p_1 : S(V) \to V$ of the symmetric algebra $S(V)$ to its linear component $V = S^1(V)$ for any BV_∞-algebra V. Before talking about morphisms, we need to provide $S(V)$ with the structure of a BV_∞-algebra. To do that, we take the L_∞ structure on $V[[\hbar]][1]$ over $k[[\hbar]]$ given by the brackets L_n, see (4), and then the BV_∞ structure on $S(V)$ from the remark at the end of Section 2. To regard p_1 as a BV_∞-morphism, we compose it with the inclusion $V \subset V[[\hbar]]$ and get a k-linear map $\varphi = \varphi_1 : S(V) \to V[[\hbar]]$. By construction, $\varphi(1) = 0$. One can easily check that $\exp(\varphi) = m$, the multiplication operator $S(V) \to V$. To see that $\exp(\varphi)$ commutes with the BV_∞ operators, we observe that, for $a_1, \dots, a_n \in V$, the value of the BV_∞ operator coming from the brackets L_j on the product $a_1 \otimes \dots \otimes a_n \in S(V)$ is equal to

$$\sum_{j=1}^{n} \hbar^{j-1} \sum_{\sigma \in \mathrm{Sh}_{j,n-j}} (-1)^{|a_\sigma|} L_j(a_{\sigma(1)}, \dots, a_{\sigma(j)}) \otimes a_{\sigma(j+1)} \otimes \dots \otimes a_{\sigma(n)},$$

because of Equations (1) and (2). When we apply m to that, the tensor product (multipliciation in $S(V)$) will change to multiplication in V. The result will just be equal to $(\Delta m)(a_1 \otimes \dots \otimes a_n)$ in view of Equation (6).

Another feature of a BV_∞-morphism $\varphi : S(U)[[\hbar]] \to V'[[\hbar]]$ is that it propagates solutions $S \in \lambda U((\hbar))^2[[\lambda]]$ of the *Quantum Master Equation* (*QME*)

$$\Delta e^{S/\hbar} = 0 \tag{8}$$

to solutions of the QME in $\lambda V'((\hbar))^2[[\lambda]]$.

Proposition 4.5. *If $\varphi : S(U) \to V'$ is a* BV_∞*-morphism and $S \in \lambda U((\hbar))^2[[\lambda]]$ is a solution of the QME* (8)*, then*

$$S' := \hbar\varphi(e^{S/\hbar}) \in \lambda V'((\hbar))^2[[\lambda]]$$

is a solution of the QME

$$\Delta' e^{S'/\hbar} = 0.$$

Proof. By Lemma 4.2 we have $e^{\varphi(e^{S/\hbar})} = \exp(\varphi)(e^{S/\hbar})$. Since $\exp(\varphi)$ must respect the BV_∞ operators, we get

$$\Delta' e^{\varphi(e^{S/\hbar})} = \Delta' \exp(\varphi)(e^{S/\hbar}) = \exp(\varphi)\Delta(e^{S/\hbar}) = 0.$$

□

Now we are ready to study functorial properties of the correspondence between L_∞-algebras and BV_∞-algebras from Theorem 3.6. Since the BV_∞-algebra corresponding to an L_∞-algebra is pure, we would like to concentrate on BV_∞-morphisms between such BV_∞-algebras. Among these BV_∞-morphisms, those of the following type turn out to form an interesting category.

Definition 4.6. We will call a BV_∞-morphism $\varphi : S(U) \to S(U')$ between BV_∞-algebras which are free as graded commutative algebras *pure*, if φ_n maps $S^n(U)$ to $U' \subset S(U')[[\hbar]]$ and all other symmetric powers $S^k(U)$ to 0. In other words, one can say that φ_n is a *differential operator of order n with linear coefficients over the morphism* $S(U) \to S(U')$ induced by the zero map $U \xrightarrow{0} S(U')$.

BV_∞*-algebras which are free as graded commutative algebras* form a *category* under pure BV_∞-morphisms in the following way. Given pure BV_∞-morphisms $V \xrightarrow{\varphi} V' \xrightarrow{\psi} V''$, their composition $\psi \diamond \varphi : V \to V''$ is defined by composing their exponentials:

$$\psi \diamond \varphi := \log(\exp(\psi) \circ \exp(\varphi)).$$

Under this composition, the role of identity morphism on $S(U)$ is played by $\varphi = \varphi_1 = \mathrm{id}_U$: in this case, $\exp(\varphi) = \mathrm{id}_{S(U)}$.

Proposition 4.7. *The composition $\psi \diamond \varphi$ of any pure* BV_∞*-morphisms is a pure* BV_∞*-morphism.*

Proof. First of all, we need to see that the properties (1)-(3) of a BV_∞-morphism are satisfied. Property (1) is satisfied because of our conventions on the values of exponentials and logarithms of maps at 1. Property (2) is obvious by construction. Property (3) may be established from the formula

$$(9)\quad (\psi \diamond \varphi)(x_1 \dots x_m) = \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1\\ i_1+\dots+i_k=m}}^{m} \frac{1}{i_1!\dots i_k!} \sum_{\sigma\in S_m} (-1)^{|x_\sigma|}\psi(\varphi(x_{\sigma(1)}\dots x_{\sigma(i_1)})\dots \varphi(x_{\sigma(m-i_k+1)}\dots x_{\sigma(m)})),$$

which is easily verified by exponentiating it and comparing it to $\exp(\psi) \circ \exp(\varphi)$. Indeed, the coefficient $(\psi \diamond \varphi)_n(x_1 \dots x_m)$ by $\hbar^{n-1}$ on the right-hand side will be coming from terms

$$\psi_j(\varphi_{j_1}(x_{\sigma(1)}\dots x_{\sigma(i_1)})\dots\varphi_{j_k}(x_{\sigma(m-i_k+1)}\dots x_{\sigma(m)}))$$

with $j-1+\sum_{p=1}^k(j_p-1) = j-1+\sum_{p=1}^k j_p - k = n-1$. Observe that because of Property (3) for ψ and φ, for such a term not to vanish, it is necessary that $j \ge k$ and $j_p \ge i_p$ for each p. Thus, we will have $n = j + \sum_{p=1}^k j_p - k \ge k + \sum_{p=1}^k i_p - k = m$, which is Property (3) for $\psi \diamond \varphi$. The fact that the composite BV_∞-morphism is pure is obvious from Eq. (9) and purity of ψ. □

Theorem 4.8. *The correspondence $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ of Section 2 from L_∞-algebras to BV_∞-algebras is functorial. This functor establishes an equivalence between the category of L_∞-algebras and the full subcategory of pure BV_∞-algebras of the category of BV_∞-algebras free as graded commutative algebras with pure morphisms. The functor $V = S(U) \mapsto U[1]$ of Theorem 3.6(1) provides a weak inverse to this equivalence.*

Restricting this theorem to the world of dg Lie algebras and dg BV algebras, we obtain the following corrolaries, which, surprisingly, seem to be new.

Corollary 4.9. *The functor $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ from dg Lie algebras to dg BV algebras establishes an equivalence between the category of dg Lie algebras with L_∞-morphisms and the category of dg BV algebras (V, Δ_1, Δ_2), free as graded commutative algebras $V = S(U)$ and whose BV structure is pure: Δ_2 maps U to 0 and $S^2(U)$ to U, with BV_∞-morphisms $S(U) \to S(U')$ satisfying the purity condition.*

Corollary 4.10. *The functor $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ from dg Lie algebras to dg BV algebras establishes an equivalence between the category of dg Lie algebras (with dg Lie morphisms) and the category of dg BV algebras (V, Δ_1, Δ_2), free as graded commutative algebras $V = S(U)$ and whose BV structure is pure: Δ_2 maps U to 0 and $S^2(U)$ to U, with morphisms defined as morphisms $\Phi : S(U) \to S(U')$ of graded algebras respecting the differentials Δ_1 and Δ_2 and satisfying the purity condition: Φ maps U to U'.*

Now let us prove the theorem.

Proof. We need to see that an L_∞-morphism $\mathfrak{g} \to \mathfrak{g}'$ induces a BV_∞-morphism $S(\mathfrak{g}[-1]) \to S(\mathfrak{g}'[-1])$. By definition an L_∞-morphism is a graded coalgebra morphism $\Phi : S(\mathfrak{g}[1]) \to S(\mathfrak{g}'[1])$ respecting the codifferentials and such that $\Phi(1) = 1$. As a coalgebra morphism, Φ is determined by its projection $\varphi : S(\mathfrak{g}[1]) \to \mathfrak{g}'[1]$ to the cogenerators $\mathfrak{g}'[1]$ via the following formula:

$$(10) \quad \Phi(x_1 \dots x_m) = \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1+\dots+i_k=m}}^m \sum_{\sigma \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_\sigma|} \varphi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots \varphi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)}),$$

where $\mathrm{Sh}_{i_1,\dots,i_k}$ denotes the set of $(i_1, \dots, i_k)$ shuffles, $x_1, \dots, x_m$ are in $\mathfrak{g}[1]$, and $(-1)^{|x_\sigma|}$ is the Koszul sign of the permutation of $x_1 \dots x_m$ to $x_{\sigma(1)} \dots x_{\sigma(m)}$ in $S(\mathfrak{g}[1])$. (For $m = 0$, we just have $\Phi(1) = 1$ and $\varphi(1) = 0$.) The above formula follows from iteration of the coalgebra morphism property:

$$\delta^{k-1}\Phi = \Phi^{\otimes k}\delta^{k-1}$$

along with its projection to $(\mathfrak{g}'[1])^{\otimes k}$ for each $k = 1, \dots, m$. To turn φ into a BV_∞-morphism, we need to rewrite it as a power series in $\hbar$:

$$\varphi_\hbar := \frac{1}{\hbar} \sum_{n=1}^{\infty} \hbar^n \varphi_n, \tag{11}$$

where $\varphi_n : S(\mathfrak{g}[-1]) \to S(\mathfrak{g}'[-1])$ maps all symmetric powers to 0, except for $S^n(\mathfrak{g}[-1])$, on which φ_n is the restriction of $\varphi : S(\mathfrak{g}[1]) \to \mathfrak{g}'[1]$ to $S^n(\mathfrak{g}[1])$ along with an appropriate shift in degree to make it into a linear map $S^n(\mathfrak{g}[-1]) \to \mathfrak{g}'[-1]$. Note that the degree of φ was supposed to be zero, as it was a projection of the morphism Φ of graded coalgebras. In terms of grading on $S^n(\mathfrak{g}[-1])$ and $\mathfrak{g}'[-1]$, the degree of shifted φ_n is $2 - 2n$. Multiplication by $\hbar^{n-1}$ shifts that degree back to 0, thus we see that the degree of $\varphi_\hbar$ is zero as well.

Note that by construction, the purity condition on $\varphi_\hbar$ is satisfied, and thereby we have

$$\begin{aligned} &\exp(\varphi_\hbar)(x_1 \dots x_m) \\ &\quad = \sum_{k=1}^{m} \frac{\hbar^{m-k}}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1+\dots+i_k=m}}^{m} \frac{1}{i_1! \dots i_k!} \sum_{\sigma \in S_m} (-1)^{|x_\sigma|} \varphi(x_{\sigma(1)} \dots x_{\sigma(i_1)}) \dots \\ &\qquad\qquad \varphi(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)}), \end{aligned}$$

whence, comparing this to the right-hand side of (10), we get

$$\exp(\varphi_\hbar) = \sum_{m=0}^{\infty} \hbar^m \Phi_m,$$

where Φ_m is the component of Φ of degree $-m$ in the grading given by the symmetric power, so as

$$\Phi = \sum_{m=0}^{\infty} \Phi_m.$$

We know that Φ is compatible with the structure codifferentials D and D' of $\mathfrak{g}$ and $\mathfrak{g}'$: $\Phi D = D' \Phi$. The BV_∞ operator on $S(\mathfrak{g}[-1])$ was defined as $\Delta = \sum_{m=1}^{\infty} \hbar^{m-1} D_m$, where D_m maps each $S^n(\mathfrak{g}[1])$ to $S^{n-m+1}(\mathfrak{g}[1])$; likewise for $S(\mathfrak{g}'[-1])$, see (2). Thus, the equation $\exp(\varphi_\hbar)\Delta = \Delta' \exp(\varphi_\hbar)$ is satisfied, being just a weighted sum of the components of the equation $\Phi D = D' \Phi$, where the component shifting the symmetric power down by $n \geq 0$ is being multiplied by $\hbar^n$. This completes verification of the fact that $\varphi_\hbar$ is a pure BV_∞-morphism.

Conversely, we need to see that every pure BV_∞-morphism comes from an L_∞-morphism. By Theorem 3.6 we can assume that the source and the target of this BV_∞-morphism are the BV_∞-algebras $S(\mathfrak{g}[-1])$ and $S(\mathfrak{g}'[-1])$ coming from some L_∞-algebras $\mathfrak{g}$ and $\mathfrak{g}'$. Every BV_∞-morphism is given by a formal $\hbar$-series like (11) satisfying the three conditions of Definition 4.3. Since the morphism is pure, we can "drop" the $\hbar$ from $\varphi_\hbar$ and note that the formal series

$$\varphi := \sum_{n=1}^{\infty} \varphi_n$$

will produce a well-defined linear map $S(\mathfrak{g}[1]) \to \mathfrak{g}'[1]$. Dropping the $\hbar$ results in this map also having degree zero. Now we can generate a unique morphism $\Phi : S(\mathfrak{g}[1]) \to S(\mathfrak{g}'[1])$ of coalgebras by the linear map φ. This morphism Φ will be given by formula (10). Since $\varphi_h(1) = 0$, we get $\varphi(1) = 0$ and $\Phi(1) = 1$ by the same formula. We just need to check that this morphism Φ respects the codifferentials D and D' on these two coalgebras, respectively. As in the first part of the proof, we see that the equation $\exp(\varphi_\hbar)\Delta = \Delta' \exp(\varphi_\hbar)$ implies $\Phi D = D'\Phi$. Thus, Φ is an L_∞-morphism.

We also need to check the functoriality properties of the correspondence $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$. The fact that $\mathrm{id}_{\mathfrak{g}}$ maps to the identity morphism is obvious. Now, if we have two L_∞-morphisms $\mathfrak{g} \to \mathfrak{g}' \to \mathfrak{g}''$ given by dg coalgebra morphisms $S(\mathfrak{g}[1]) \xrightarrow{\Phi} S(\mathfrak{g}'[1]) \xrightarrow{\Psi} S(\mathfrak{g}''[1])$ with $\Phi = \sum_{m=0}^\infty \Phi_m$ and $\Psi = \sum_{m=0}^\infty \Psi_m$, we note that the exponentials $\exp(\varphi_\hbar) = \sum_{m=0}^\infty \hbar^m \Phi_m$ and $\exp(\psi_\hbar) = \sum_{m=0}^\infty \hbar^m \Psi_m$ of the respective BV_∞-morphisms will compose in the same way as Φ and Ψ, the only difference being that the component decreasing the symmetric power by m gets multiplied by $\hbar^m$. $\square$

5. Adjunction

In this section, we prove a certain "adjunction" property. The quotation marks are due to the fact that in our setting, arbitrary BV_∞-algebras do not even make up a category. However, the theorem below makes sense for arbitrary BV_∞-algebras and BV_∞-morphisms.

Recall that given an L_∞-algebra $\mathfrak{g}$, we have constructed a BV_∞-algebra $S(\mathfrak{g}[-1])$ in Section 2. Conversely, given a BV_∞-algebra V, we have used the higher derived brackets L_n to induce an L_∞-stucture on $V[[\hbar]][1]$ over $k[[\hbar]]$ as in Corollary 3.2.

Note that both constructions are functorial. The fact that $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ defines a functor is the first statement of Theorem 4.8. We need to see that the construction assigning to a BV_∞-algebra V the L_∞-algebra $(V[[\hbar]][1], L_n)$ is also functorial. Given a BV_∞-morphism $\varphi : V = S(U) \to V'$, we need to construct an L_∞-morphism $V[[\hbar]][1] \to V'[[\hbar]][1]$. This construction will be accomplished in two steps.

Step 1. Compose the BV_∞-morphism $\varphi : V \to V'$ with the BV_∞-morphism $p_1 : S(V) \to V$ of Example 4.4 to get a BV_∞-morphism $\varphi \diamond p_1 : S(V) \to V'$.

Step 2. Given an L_∞-algebra $\mathfrak{g}[[\hbar]]$ over $k[[\hbar]]$ and a BV_∞-morphism $\psi : S(\mathfrak{g}[-1]) \to V'$, where $S(\mathfrak{g}[-1])$ is provided with the BV_∞ structure of the remark at the end of Section 2, we will construct a canonical L_∞-morphism $\mathfrak{g}[[\hbar]] \to V'[[\hbar]][1]$. Then we will just apply this construction to the BV_∞-morphism $S(V) \to V'$ of Step 1.

In order to construct an L_∞-morphism $\mathfrak{g}[[\hbar]] \to V'[[\hbar]][1]$, take the graded $k[[\hbar]]$-coalgebra morphism, continuous in the $\hbar$-adic topology,

$$F : S(\mathfrak{g}[1])[[\hbar]] \to S(V'[2])[[\hbar]]$$

induced by the $k[[\hbar]]$-linear map

$$f : S(\mathfrak{g}[1])[[\hbar]] \to V'[[\hbar]][2]$$

whose restriction $f|_{S^k(\mathfrak{g}[1])[[\hbar]]} : S^k(\mathfrak{g}[1])[[\hbar]] \to V'[[\hbar]][2]$ is the restriction of $\hbar^{1-k}\psi$ to $S^k(\mathfrak{g}[1])[[\hbar]$ for $k \geq 0$:

$$f|_{S^k(\mathfrak{g}[1])[[\hbar]]} = \hbar^{1-k}\psi|_{S^k(\mathfrak{g}[1])[[\hbar]]}.$$

This map takes values in $V'[[\hbar]][2]$, despite the division by a power of $\hbar$, because the restriction of ψ to $S^k(\mathfrak{g}[-1])$ is in fact equal to $\sum_{n=k}^\infty \hbar^{n-1}\psi_n = \hbar^{k-1}\sum_{n=0}^\infty \hbar^n \psi_{n+k}$. Note that since ψ is of degree zero, f will also have degree zero.

We need to check that F defines an L_∞-morphism. It is easy to see that $F(1) = 1$, because $\psi(1) = 0$. What is far less trivial is the fact that F respects the codifferentials, the structure codifferential D on $S(\mathfrak{g}[1])[[\hbar]]$ and the codifferential D' on $S(V'[2])[[\hbar]]$ induced as a continuous coderivation, see (1), by the sum of the brackets (4):

$$L_n : S^n(V'[2])[[\hbar]] \to V'[[\hbar]][2].$$

What we know is $\Delta' \exp(\psi) = \exp(\psi)\Delta$, where Δ' is the BV_∞ operator on V' and Δ is the structure codifferential D on $S(\mathfrak{g}[1])[[\hbar]]$ enhanced by $\hbar$, as in the remark at the end of Section 2. To see that this implies the equation $D'F = FD$, we need to develop some BV calculus and compare it to colagebra calculus.

Let us start with coalgebra calculus. Each side of the equation is a continuous coderivation over the coalgebra morphism F and as such determined by the projection $p_1 : S(V'[2])[[\hbar]] \to V'[[\hbar]][2]$ to the cogenerators $V'[[\hbar]][2]$ of the range. Thus, all we need to show is that $p_1 D'F = p_1 FD$, after projecting to the cogenerators. Now, for a monomial $x_1 \dots x_m \in S^m(\mathfrak{g}[1])$, we have

$$\begin{aligned}(12)\quad & p_1 D'F(x_1 \dots x_m) \\ &= \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1+\dots+i_k=m}}^{m} \sum_{\sigma \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_\sigma|} L_k(f(x_{\sigma(1)} \dots x_{\sigma(i_1)}), \dots, \\ &\qquad\qquad f(x_{\sigma(m-i_k+1)} \dots x_{\sigma(m)})),\end{aligned}$$

using the shuffle notation, see Equation (10), as well as

$$p_1 FD(x_1 \dots x_m) = f(D(x_1 \dots x_m)). \tag{13}$$

We need to show that the right-hand sides of these equations are equal, based on the equation $\Delta' \exp(\psi) = \exp(\psi)\Delta$. We will do that after we develop some BV calculus.

Turning to BV calculus, we have

$$\begin{aligned}(14)\quad & \Delta(x_1 \dots x_m) \\ &= \sum_{k=1}^{m} \hbar^{k-1} \sum_{\tau \in \mathrm{Sh}_{k,m-k}} (-1)^{|x_\tau|} l_k(x_{\tau(1)}, \dots, x_{\tau(k)}) x_{\tau(k+1)} \dots x_{\tau(m)},\end{aligned}$$

where l_k's are the L_∞ brackets on $\mathfrak{g}$, because of Equation (1). Now apply $\exp(\psi)$ to both sides, reassemble products of ψ's not containing l_k's into $\exp(\psi)$, and use (14) again to pass from l_k's back to Δ and get

$$\begin{aligned}(15)\quad & \exp(\psi)\Delta(x_1 \dots x_m) \\ &= \sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} \psi(\Delta(x_{\sigma(1)} \dots x_{\sigma(n)})) \exp(\psi)(x_{\sigma(n+1)} \dots x_{\sigma(m)}).\end{aligned}$$

Move on to computation of $\Delta' \exp(\psi)$:

(16) $$\Delta' \exp(\psi)(x_1 \dots x_m) = \sum_{n=1}^{m} \sum_{\sigma \in \mathrm{Sh}_{n,m-n}} (-1)^{|x_\sigma|} \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1\\ i_1+\dots+i_k=n}}^{n} \sum_{\tau \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_{\tau\sigma}|} l_k^\hbar(\psi(x_{\tau\sigma(1)} \dots x_{\tau\sigma(i_1)}), \dots, \psi(x_{\tau\sigma(n-i_k+1)} \dots x_{\tau\sigma(n)})) \exp(\psi)(x_{\sigma(n+1)} \dots x_{\sigma(m)}),$$

which follows from the definition of $\exp(\psi)$ and the identity (6).

Now let us compare (15) with (16), which are equal by assumption. One can show by induction on m that the top, $n = m$ terms of the two formulas must also be equal:

$$\psi(\Delta(x_1 \dots x_m)) = \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1\\ i_1+\dots+i_k=m}}^{m} \sum_{\tau \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_\tau|} l_k^\hbar(\psi(x_{\tau(1)} \dots x_{\tau(i_1)}), \dots, \psi(x_{\tau(m-i_k+1)} \dots x_{\tau(m)})).$$

It remains to pass from ψ, Δ, and $l_k^\hbar$ to f, D, and L_k, respectively, in this equation, with appropriate powers of $\hbar$, resulting in the equation

$$\hbar^{m-1} f(D(x_1 \dots x_m)) = \hbar^{m-1} \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k=1\\ i_1+\dots+i_k=m}}^{m} \sum_{\tau \in \mathrm{Sh}_{i_1,\dots,i_k}} (-1)^{|x_\tau|} L_k(f(x_{\tau(1)} \dots x_{\tau(i_1)}), \dots, f(x_{\tau(m-i_k+1)} \dots x_{\tau(m)})).$$

In view of (12) and (13), we see that $D'F = FD$. This completes Step 2.

Theorem 5.1. *Suppose $\mathfrak{g}$ is an L_∞-algebra and V is a BV_∞-algebra. There exists a canonical bijection*

$$\mathrm{Hom}_{\mathrm{BV}_\infty}(S(\mathfrak{g}[-1]), V) \cong \mathrm{Hom}_{L_\infty}(\mathfrak{g}, V[[\hbar]][1]),$$

where the L_∞-structure on $V[[\hbar]][1]$ is given by the modified brackets L_n. This bijection is natural in the L_∞-algebra $\mathfrak{g}$ and in the BV_∞-algebra V.

Proof. A correspondence from the BV_∞-morphisms on the left-hand side to the L_∞-morphisms on the right-hand side was constructed in Step 2 before the theorem in a more general case of an L_∞-algebra over $k[[\hbar]]$.

Conversely, given an L_∞-morphism $F : S(\mathfrak{g}[1]) \to S(V[2])[[\hbar]]$, we use the same conversion formula

(17) $$\varphi|_{S^k(\mathfrak{g}[1])[[\hbar]]} = \hbar^{k-1} f|_{S^k(\mathfrak{g}[1])[[\hbar]]},$$

f being the projection of F to the cogenerators $V[2][[\hbar]]$, for $k \geq 0$, as in Step 2 before the theorem, to get a BV_∞-morphism $\varphi : S(\mathfrak{g}[-1]) \to V$. Tracing the argument there backward, we see that φ is indeed a BV_∞-morphism. This establishes a bijection in the adjunction formula.

The naturality of the construction follows from the fact that, in view of (17), F and $\exp(\varphi)$ are given by almost identical formulas, with the only difference coming from insertion of powers of $\hbar$, which plays the role of grading shift. □

Corollary 5.2. *The functor* $\mathfrak{g} \mapsto S(\mathfrak{g}[-1])$ *of Sections* 2 *and* 4 *from the category of* L_∞*-algebras to the category of* BV_∞*-algebras free as graded commutative algebras with pure morphisms has a right adjoint, which is given by the functor of modified higher derived brackets* L_n*.*

Remark. This corollary generalizes the construction of a pair of adjoint functors by Beilinson and Drinfeld [BD04, 4.1.8] from the case of dg Lie and BV algebras to the case of L_∞- and BV_∞-algebras.

Remark. The results of this section extend easily to the case when an L_∞-algebra $\mathfrak{g}$ is replaced with a topological L_∞-algebra $\mathfrak{g}[[\hbar]]$ over $k[[\hbar]]$ and we use the BV_∞ structure on $S(\mathfrak{g}[-1])$ described in the remark at the end of Section 2. In particular, there is a natural bijection

$$\mathrm{Hom}_{\mathrm{BV}_\infty}(S(\mathfrak{g}[-1]), V) \cong \mathrm{Hom}_{L_\infty}(\mathfrak{g}[[\hbar]], V[[\hbar]][1]),$$

where on the right-hand side, we consider continuous L_∞-morphisms over $k[[\hbar]]$.

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E-mail address: `bashk003@umn.edu`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

E-mail address: `voronov@umn.edu`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA, and Kavli IPMU (WPI), University of Tokyo, Kashiwa, Chiba 277-8583, Japan