C_∞-REGULARITY OF SOLUTIONS AND INTERFACES OF THE POROUS MEDIUM EQUATION

By

Juan Luis Vázquez

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UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455

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C^∞-REGULARITY OF SOLUTIONS AND INTERFACES OF THE POROUS MEDIUM EQUATION

JUAN LUIS VÁZQUEZ*†

0. Introduction.

This paper is concerned with the question of optimal regularity of *local*, nonnegative solutions of the *porous medium equation*

$$u_t = (u^m)_{xx}, \quad m > 1. \tag{0.1}$$

We prove that the nonstationary parts of their interfaces are C^∞ curves. Moreover, though the function

$$v = \frac{m}{m-1} u^{m-1}, \tag{0.2}$$

usually called the *pressure*, has discontinuous derivatives, v_x and v_t, across any such piece of interface γ, it can be extended in a C^∞ way from each side of γ up to γ. This C^∞ regularity has been recently established for nonnegative solutions of the initial-value problem for (0.1), cf. [AV1], [HK]. We will show here that such regularity is in fact true for any nonnegative solution of (0.1) defined in a rectangle S of $\mathbb{R}^2$ as long as we stay away from the parabolic boundary of S.

To put these results into proper perspective let us first devote some lines to comment on the general class of equations to which (0.1) belongs, i.e., the type of (possibly nonlinear) parabolic equations which are used to describe the evolution of diffusion processes. The paradigm for such equations is the classical *heat equation*

$$u_t = \Delta u, \tag{0.3}$$

for which the main mathematical questions have been posed and most of them solved in the last 100 years, giving rise at the same time to a wealth of new analytical tools and results. The linear character of equation (0.3) and its uniform parabolicity are features that make for the success of the mathematical treatment, but, at the same time, they imply a series of consequences that very much affect the plausibility of the equation as a

*Depto. de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.

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model for realistic diffusion processes. In particular the Strong Maximum Principle implies that any nonnegative solution of (0.3), defined in a domain Ω of $\mathbf{R}^N \times \mathbf{R}$ and which is not identically 0 in Ω must be positive everywhere in Ω, a fact whose translation into physical terms is *infinite speed* of propagation. This well-known drawback is present in more general linear, uniformly parabolic equations of the form

$$u_t = \Sigma(a_{ij}u_{x_i})_{x_j} + \Sigma(b_i u)_{x_i} + cu. \tag{0.4}$$

with suitable coefficients $a_{ij}(x,t)$, $b_i(x,t)$, $c(x,t)$ defined in Ω.

On the contrary, infinite-speed propagation does not occur in the nonlinear heat equation

$$u_t = \Delta(u^m), \;\; m > 1, \tag{0.5}$$

N-dimensional version of (0.1), which appears in a number of diffusion processes, like flow of gases in porous media [M], where its given name comes from, nonlinear heat propagation in ionized gases at high temperatures [ZR] and others, cf. e.g., [P] or [V3]. In all of these applications only nonnegative solutions are considered, a restriction that we shall keep in this paper.

Equation (0.5) is not only nonlinear but also degenerate parabolic, since the diffusion coefficient $D(u) = mu^{m-1}$ vanishes at the points where $u = 0$. A consequence of this degeneracy is the *finite propagation property* (f.p.p.) whose simplest formulation is perhaps the following: if u is a solution of (0.3) whose initial data $u(x,0)$ are compactly supported, then for every $t > 0$ the function $u(\cdot,t)$ is also compactly supported. Therefore, an interface or *free boundary*, Γ, arises as the boundary of the positivity set

$$P[u] = \{(x,t) \in \mathbf{R}^N \times (0,\infty) : u(x,t) > 0\}$$

as a subset of $Q = \mathbf{R}^N \times (0,\infty)$. The section of Γ at time $t > 0$, $\Gamma(t)$, represents the *propagation front* at that time, of fundamental importance in describing the diffusion process. The f.p.p. was already established in 1958 by Oleĭnik and her collaborators, wee [OKC], in the case of one spatial dimension and is easily extended to $N > 1$.

Since the equation is parabolic in the positivity set $P[u]$ of any solution u, it follows by more or less standard arguments that $u \in C^\infty(P[u])$. On the contrary the smoothness of the solutions near their interfaces as well as the smoothness and growth properties of these interfaces are currently a major subject of investigation, cf. [CF2], [CF3], [CVW].

A very detailed study of the smoothness question has been done in the case of the one-dimensional Cauchy problem for (0.1) in $Q = \mathbf{R} \times (0,\infty)$ with initial conditions

$$u(x,0) = u_0(x) \text{ for } x \in \mathbf{R}, \tag{0.5}$$

where u_0 is a nonnegative and locally integrable function. To make the picture simpler it is usually assumed that u_0 is continuous in $\mathbf{R}$, strictly positive in a bounded interval $I = (a, b)$, and vanishes identically outside I.

Under these hypotheses there is a unique continuous and nonnegative function $u(x,t)$ defined in Q which satisfies (0.1) in the sense of distributions and takes on the initial value u_0. For every $t > 0$ the positivity set is an interval of the form

$$P(t) = (s_1(t), s_2(t)), \tag{0.7}$$

where $x = x_1(t)$ and $x = s_2(t)$ are two monotone, Lipschitz-continuous curves called respectively left-hand and right-hand interface. Moreover there exist times t_1^* and t_2^*, called *waiting times*, such that

$$s_1(t) = a \text{ if } 0 < t < t_1^* \tag{0.8a}$$

$$s_2(t) = b \text{ if } 0 < t < t_2^* \tag{0.8b}$$

$$s_i \in C^1(t_i^*, \infty) \text{ and } (-1)^i s_i'(t) > 0 \text{ for } t > t_i^*, \quad i = 1, 2 \tag{0.8c}$$

The relationship between the solution and the interfaces $s_i(t)$ is most conveniently formulated in terms of the pressure variable $v = (m/(m-1))u^{m-1}$. In fact $v \in C^\infty(P[u])$ and can be continued as a C^1 function to the moving parts of the interfaces, i.e., $\{(x,t) : x = s_i(t),\ t > t_i^*\}$. Moreover, on these parts of the interface we have

$$s_i'(t) = -v_x(s_i(t)_1 t) \tag{0.9}$$

$$v_t = v_x^2 \tag{0.10}$$

These results were proved in the period from 1969 to 1979 by Aronson, Knerr, Caffarelli and Friedman, c.f. [A1], [A2], [A3], [K], [CF1]. Recently Höllig and Kreiss [HK] and, independently, Aronson and Vazquez [AV1] have succeeded in giving a proof a the C^∞ regularity of v and $s_i(t)$ near a moving interface. The result is

THEOREM A. *Under the above assumptions $s_i \in C^\infty(t_i^*, \infty)$ and v is a C^∞ function in $P[u]$ which can be extended as a C^∞ function to the moving parts of the interfaces.*

Several comments will help us understand the optimality of this result. Firstly, though the interfaces are of course smooth in their stationary part, $0 \leq t \leq t_i^*$, they are not necessarily so at the waiting time. Aronson, Caffarelli and Vazquez [ACV] have proved that for *most* of the initial data which exhibit a stationary free boundary, e.g. if $v_0(x) = o((b-x)^\alpha)$, with $\alpha = 2/(m-1)$, as $x \to b$, $x < b$, the interface, in this case $s_2(t)$, has a *corner point* at $t = t_2^*$, i.e.,

$$D^+ s_2(t_2^*) > 0 = D^- s_2(t_2^*) \tag{0.11}$$

(D^+ and D^- denote side derivatives). This means that Lipschitz-continuity is the optimal global regularity for interfaces with positive waiting times.

In case a corner point occurs at $P_2^* = (t_2^*, s_2(t_2^*)), t_2^* > 0$ another result related to the lack of smoothness of v was proved in [ACV], Theorem 2: v_{xx} is not bounded in any neighborhood of P_2^* and, in fact, v_{xx} is not even $O(1/(b-x))$ as $(x,t) \to P_2^*$ with $t < t_2^*$. (See [A3] and [ACK] for other results on the behavior of s and v_{xx}.)

Even in case $D^+ s_2(t_2^*) = 0$ it may occur that s_2 is not C^∞ at $t = t_2^*$. This is the case of the self-similar solutions constructed in [LOT] which behave for $t > t_2^*$ like $c(t-t_2^*)^\alpha$ for $\alpha > 1$.

Finally, observe that because of equation (0.9) v_x has a jump discontinuity of magnitude $|s_i'(t)|$ accross the interface $x = s_i(t)$. Because of (0.10), v_t is also discontinuous. Hence Lipschitz-continuity is the best two-sided regularity for v on a moving interface

The proofs of the C^∞ regularity given in [HK] and [AV1] are very different in spirit. While the former uses an iterative calculation with weighted norms, the latter relies on a barrier argument. In fact, the proof in [AV1] is more general in the sense that, being of a purely *local* nature, it applies to any initial data which exhibit an interface, even if they are not compactly supported, typically if $u_0(x) = 0$ for $x \geq 0$ and $u_0(x) > 0$ for $x < 0$ (and u_0 does not grow faster than $|x|^\alpha$ as $x \to -\infty$, this latter restriction being necessary for the sake of existence). Moreover the result applies not only to the left-hand and right-hand interfaces but also to any inner interfaces, which occur whenever the initial support is disconnected and which are discussed in [V1].

The local nature of [AV1]'s proof suggests the following

Conjecture. The C^∞ result should hold not only for the solutions of the Cauchy problem (or *global* solutions), but for any solution of (0.1) defined in a subdomain of Q (a *local* solution), independently of its boundary behavior.

It is the purpose of this paper to show that the conjecture is true. This is done by reexamining the process of proof of Theorem A and localising every argument in it. The work is divided into the following steps:

(1) Preliminaries and statement of the main result.

(2) Estimates for v_x, v_{xx} and v_t. The work for the Cauchy problem is due to Aronson and Bénilan, [A1], [A2] and [AB].

(3) Equations of the interface. Reviewing work by Aronson, Knerr, Caffarelli and Friedman, [A2], [K], [CF1]

(4) C^1 regularity of the interface, cf. [CF1] and also [ACK].

(5) C^∞ regularity, as in [AV1].

To be precise, the localisation effort is mainly confined to Section 2, where a new estimate for v_{xx}, Lemma 2.3, is derived. Most of Section 3 and Sections 4 and 5 repeat arguments valid in the local setting. While we omit the details of the proof of C^1-regularity, by now well-known, we explain the ideas of the proof of C^∞-regularity in some detail. We also introduce a new barrier argument at the end of Section 5.

To end this rather long introduction, let us mention that Angenent [An] has recently improved Theorem A by showing that, under the hypotheses of that theorem, both s_1 and s_2 are analytic functions of t for $t > \max(t_1^*, t_2^*)$. This result is not local and needs in an essential way the property of compact support. On the other hand, the question of analyticity of v is still open even in this specific situation.

Other properties of the interfaces, in particular asymptotic behavior, are discussed in [V1], [V2], [BV] and [AV1]. A complete treatment of the mathematical theory for the porous medium equation will appear in [AV2].

1. Preliminaries and statement of the main result.

By a *local solution* of $u_t = (u^m)_{xx}$ we will understand a continuous and nonnegative function u defined in a domain S of $\mathbf{R}^2$, such that for every test function $\phi \in C_0^\infty(S)$ the following holds

$$\iint_\Omega (u\phi_t + u^m \phi_{xx})dx\, dt = 0. \tag{1.1}$$

More precisely we say that u is *a solution of* $u_t = (u^m)_{xx}$ *in* S. By shrinking the domain we may always assume that u is continuous in the closure of S, $\overline{S}$. We will also restrict ourselves to a rectangular domain of the form $S = (-a, a) \times (0, T)$. Since the equation is invariant under translations these restrictions will entail no loss of generality in our results.

Local solutions are obtained when solving for instance the initial value problem in $\mathbf{R} \times (0, \infty)$, cf. [OKC], [CF1] or [DK], a Dirichlet boundary-value problem, cf. [AP], or a Neumann boundary-value problem, cf. [B1] and [AR]. Of course, when considering discontinuous initial data we should be using a time interval of the form (τ, T) with $0 < \tau < T$, but again this does not effect our results.

An important tool in manipulating local solutions is the comparison principle. It is convenient at this stage to introduce the following concepts. A *supersolution* of equation (0.1) in S is a continuous, nonnegative function $\hat{u}$ defined in S such that (1.1) holds with $\leq$ sign instead of $=$ for all nonnegative $\phi \in C_0^\infty(S)$. A *subsolution* is defined likewise by replacing $=$ by $\geq$. By the parabolic boundary of S we mean the set

$$\partial_p S = ([-a, a] \times \{0\}) \cup (\{-a\} \times [0, T]) \cup (\{a\} \times [0, T]) \tag{1.2}$$

The next result was established in [ACP], Proposition 9 and Theorem 12.

LEMMA 1.1. *Let u_1 be a supersolution of (0.1) in S and u_2 a subsolution and assume that $u_1 \geq u_2$ on $\partial_p S$. Then $u_1 \geq u_2$ in S.*

Another important tool is the possibility of approximating a solution with smooth functions. We have, cf. [DK]

LEMMA 1.2. *Let u be a solution of (0.1) in S. Then there exists a sequence of C^∞ and strictly positive solutions of (0.1) in S, $\{u_n\}$, such that $u_{n+1} \geq u_n \geq u$ and $\{u_n\}$ converges to u uniformly in $\overline{S}$.*

The approximation by smooth functions is closely related to the smoothness of the solutions when the equation does not degenerate. From the construction in [OCK] it follows that

LEMMA 1.3. $u \in C^\infty(P[u])$.

We now turn to the statement of our regularity result. By *interface* of a solution u we mean the set

$$\Gamma = \partial(P[u]) \cap S, \tag{1.3}$$

where ∂ means boundary. The interface will consist of different connected components, typically a left-hand and a right-hand interface under the assumptions discussed in the introduction. But also inner interfaces may occur consisting in up to countably many components, cf. the discussion in [V1]. It will be convenient to think of the interface as consisting of pieces of the following types

a) *stationary interface piece.* It has the form

$$\{x_0\} \times (0, t_0] \text{ with } 0 < t_0 < T \tag{1.4.a}$$

or

$$\{x_0\} \times (0, T). \tag{1.4.b}$$

b) *smooth moving interface piece.* It is a monotone curve given by

$$x = s(t), \;\; t \in I, \tag{1.5}$$

where I is an interval from t_1 to t_2, $0 \leq t_1 < t_2 \leq T$, with the endpoints included unless $t_1 = 0$ or $t_2 = T$. The function $s \in C^1(I)$ and s' does not vanish on I.

Our result is

THEOREM B. *The interface of a local solution of the porous medium equation (0.1) consists only of pieces of the above two types. Moreover for every point $(\overline{x}, \overline{t}) \in S$ on a moving interface there exists a small neighborhood $N \subset S$ such that $\Gamma \cap N$ can be described as a curve $x = s(t)$, $t \in J$,*

$$v \in C^\infty(N \cap \overline{P[u]}), \tag{1.6}$$

$$s \in C^\infty(J) \text{ and } s'(t) = -v_x(s(t), t), \tag{1.7}$$

$$s''(t) = m s'(t) v_{xx}(s(t), t), \tag{1.8}$$

the values of v_x and v_{xx} being the limits as $(x', t') \to (s(t), t)$ with $(x', t') \in P[u]$.

Remark. Formula (1.8) is a direct consequence of (1.6), (1.7) since $v_{xt}(s(t), t) = (m + 1)v_x(s(t), t)v_{xx}(s(t), t)$. It has the interest of directly relating the convexity of s to that of v.

2. First estimates.

As an easy consequence of the comparison result, Lemma 1.1, we get an L^∞-estimate for u

LEMMA 2.1. *For any solution of (0.4) in S we have*

$$\|u\|_{L^\infty(S)} \le \|u\|_{L^\infty(\partial_p S)}. \tag{2.1}$$

Derivative estimates are most conveniently formulated in terms of the pressure $v = mu^{m-1}/(m-1)$, which satisfies the equation

$$v_t = (m-1)vv_{xx} + v_x^2 \tag{2.2}$$

in a weak sense in S, cf [A2], and, of course, classically in $P[u]$. Aronson ([A1], Lemma) proved in 1969 a local estimate for v_x in terms of $M = \|u\|_{L^\infty(S)}$, t and the spatial distance

$$d(x) = \min\{a - x,\ x + a\} \tag{2.3}$$

by using the Bernstein technique, which consists in, first, a change of variable $v = \phi(w)$ for a concave function ϕ of the form $\phi(r) = Cr(4 - r)$, and then an estimate of the value of $|v_x|$ at a maximum of $z = \zeta^2 w_x^2$, ζ being a cutoff function. A careful inspection of that proof (as done in [HV], Lemma 3.1) reveals that the dependence of v_x is as follows

LEMMA 2.2. *With the above notations there exists a constant $c_1 = c_1(m) > 0$ such that*

$$v_x^2(x,t) \le c_1\left(\frac{N}{t} + \frac{N^2}{d^2}\right), \tag{2.4}$$

where $N = \sup\{v(x, 0) : |x| \le a\}$.

We come now to the main estimate, which shows that v_{xx} is locally bounded from below.

LEMMA 2.3. *Under the above assumptions we have*

$$v_{xx} \ge -\frac{1}{(m+1)t} - c_2\frac{N^2}{d} \quad \text{in } \mathcal{D}'(S) \tag{2.5}$$

for some positive constant c_2 depending only on m.

Remark. If instead of a local solution we have a global solution in a strip $\mathbf{R} \times (0, T)$, then we may take $d \to \infty$ and thus recover the estimate

$$v_{xx} \geq -\frac{1}{(m+1)t}$$

one-dimensional version of the bound for Δv, obtained by Aronson and Bénilan [AB].

Proof of the Lemma. By Lemma 1.2 we may assume that u is strictly positive and C^∞ in S. Then $p = v_{xx}$ satisfies in S the equation

$$L(p) \equiv p_t - (m-1)vp_{xx} - 2mv_x p_x - (m+1)p^2 = 0. \tag{2.6}$$

Let us now pick a point $(x_0, t_0) \in S$ and set $r = d(x_0)$. We shall find a lower bound for v_{xx} in the subrectangle $S_1 = (x_0 - r, x_0 + r) \times (0, t_0]$. Shifting the x-axis we will assume that $x_0 = 0$ to simplify things. We consider in S_1 the function

$$\overline{p}(x,t) = -\frac{1}{(m+1)t} - \frac{A}{(r^2 - x^2)^2} \tag{2.7}$$

We shall find an A of the form $A = c_2(m)r^2N$ so that $L(\overline{p}) \leq 0$ in S_1. Since $\overline{p} = -\infty$ on $\partial_p S_1$ we may conclude that $p \geq \overline{p}$ in S_1. In particular

$$p(x_0, t_0) \geq -\frac{1}{(m+1)t_0} - \frac{c_2 r^2 N}{r^4} = -\frac{1}{(m+1)t_0} - \frac{c_2 N}{r^2},$$

i.e., (2.5). Hence we only need to check that $L(\overline{p}) \leq 0$. We have

$$L(\overline{p}) = \frac{1}{(m+1)t^2} + (m-1)v\left\{\frac{4A}{w^3} + \frac{24\,Ax^2}{w^4}\right\} + \frac{8mv_x Ax}{w^3}$$
$$-(m+1)\left(\frac{1}{(m+1)t} + \frac{A}{w^2}\right)^2 = (*),$$

where we set $\omega = r^2 - x^2$. Taking into account that

$$|v_x| \leq c_1^{\frac{1}{2}}\left(\frac{N}{t} + \frac{N^2}{d^2}\right)^{\frac{1}{2}} \leq \left(c_1\frac{N}{t}\right)^{\frac{1}{2}} + c_1^{\frac{1}{2}}\frac{N}{d},$$

also that $v \leq N$, $|x| \leq r$ and $\omega \leq r^2$ in S_1, $(*)$ gives

$$-\omega^4 L(\overline{p}) \geq A\{A(m+1) + \frac{2}{t}\omega^2 - 28(m-1)r^2N$$
$$- 8mc_1^{\frac{1}{2}}r\omega N d^{-1} - 8mc_1^{\frac{1}{2}}\omega r N^{\frac{1}{2}}t^{-\frac{1}{2}}\}.$$

But

$$8mc_1^{\frac{1}{2}}r\omega N^{\frac{1}{2}}t^{-\frac{1}{2}} \leq \frac{2\omega^2}{t} + 8m^2c_1r^2N,$$

and

$$\frac{\omega}{d} = \frac{r^2 - x^2}{r - |x|} = r + |x| \leq 2r,$$

hence $-\omega^4 L(p) \geq A\{A(m+1) - c(m)r^2N\}$ and we get $L(\overline{p}) \leq 0$ with $A \geq c_2(m)r^2N_\#$

COROLLARY 2.4. *The following precise version of estimate (2.4) holds*

$$v_x^2 \le \frac{2N}{(m+1)t} + \frac{2CN^2}{d^2} \tag{2.8}$$

Proof. It follows from elementary calculus that whenever a real function v satisfies $0 \le v \le M$ and $v'' \ge -c$, then $(v')^2 \le 2NC_{\#}$.

From Lemma 2.3 and the fact that $v_t \ge (m-1)vv_{xx}$ we obtain an estimate for v_t, as in [AB].

COROLLARY 2.5. *We have*

$$v_t \ge \frac{(m-1)v}{(m+1)t} - \frac{c_3 N v}{d^2} \tag{2.9}$$

where $c_3 > 0$ depends ony on m.

Different local estimates have been obtained by Bénilan, see e.g. [B2]. A proof of the local estimate $v_{xx} \ge -C$ using the results of [B2] was suggested to the author by Souganidis [S]. I thank both of them for interesting discussions on the subject.

3. Properties of the interface.

An immediate consequence of Corollary 2.5 is the *retention* property : if $u(x_0,t_0) > 0$ at a point $(x_0,t_0) \in S$ we have $u(x_0,t) > 0$ for every $t \in (t_0,T)$. Therefore the positivity set expands in time and the sets $P(t) = \{x \in (-a,a) : u(x,t) > 0\}$ are formed by a countable union of disjoint intervals, whose end-points evolve monotonically in time.

Since on any compact subrectangle K of S, v_{xx} is bounded below by a constant $-C(K)$, the function $f(x) = v_x(x,t) + C(K)x$ is nondecreasing in $K(t) = \{x : (x,t) \in K\}$. According to [A2], this means that for any $(x_0,t_0) \in S$ there exist the limits

$$\lim_{\substack{x \to x_0 \\ x < x_0}} v_x(x,t) = v_x(x-,t) \tag{3.1.a}$$

and

$$\lim_{\substack{x \to x_0 \\ x > x_0}} v_x(x,t) = v_x(x+,t) \tag{3.1.b}$$

A local comparison argument, comparing v with *linear-front* solutions of the form

$$\overline{v}(x,t) = c[c(t-t_0) - (x-x_0)]_+, c \in \mathbf{R}^+, \tag{3.2}$$

allowed Knerr [K] to conclude that at any point (x_0,t_0) of an interface $D^+s(t_0)$ exists and

$$D^+s(t_0) = -v_x(s(t)\pm,t) \tag{3.3}$$

the sign $\pm$ being chosen to point out the derivative from the side where v is positive. Therefore, it follows from Lemma 2.2 that the interface $x = s(t)$ is Lipschitz-continuous for $t \geq \tau > 0$.

We come now to the convexity properties of $s(t)$. To fix the ideas we will assume that $x = s(t)$ is a nondecreasing interface in a time interval (t_1, t_2) with $0 \geq t_1 \geq t_2 \geq T$, which is not stationary, so that $P[u]$ will include the domain $\{(x,t) : s(t_1) < x < s(t), t_1 < t < t_2\}$. Following [CF1] we consider a point (x_0, t_0) on such an interface and adapt to it an explicit *Barenblatt* solution of the form

$$\overline{v}(x,t) = \frac{(r(t-\tau)^2 - (x-x_1)^2)_+}{2(m+1)(t-\tau)} \tag{3.4}$$

where $x_1 \in (-a, a)$ and $\tau \in [0, t_0)$ and $r(t) = Ct^{1/(m+1)}$ with $C > 0$. Choosing appropriately x_1, τ and C we may make $\overline{v}$ satisfy the following conditions

$$\overline{v}(x_0, t_0) = v(x_0, t_0) = 0 \tag{3.5.a}$$

$$\overline{v}(x_0-, t_0) = v_x(x_0, t_0) < 0 \tag{3.5.b}$$

and, finally, $\overline{v}_{xx}$ is positive in an interval $(x_1 - r(t_0 - \tau), x_1 + r(t_0 - \tau))$ strictly contained in $(-a, a)$, and on that interval and for $t = t_0$

$$\overline{v}_{xx} = -\frac{1}{(m+1)(t_0 - \tau)} \leq v_{xx}. \tag{3.5.c}$$

This last condition is different from the corresponding one in [CF1] and forces us to choose a $\tau > 0$, possibly close to t_0 in conjunction with estimate (2.5). Anyway τ will only depend on t_0 and $d(x_0)$. Because of (3.5) we conclude that $v(x, t_0) \geq \overline{v}(x, t_0)$ in $(-a, a)$ and, by the comparison result, $v(x,t) \geq \overline{v}(x,t)$ for $x \in (-a, a)$ and some $t \geq t_0$. This means that $s(t)$ is bounded below for $t \geq t_0$ by the interface to the Barenblatt solution, i.e.,

$$s(t) \geq x_1 + r(t - \tau) = \overline{s}(t) \tag{3.6}$$

Since $s(t_0) = \overline{s}(t_0)$ and, by (3.2), $D^+ s(t_0) = \overline{s}'(t_0)$, and this happens at all points (x_0, t_0), we conclude as in [CF1] or [ACV] that

$$s''(t) \geq \overline{s}''(t) \equiv \frac{m}{(m+1)(t-\tau)} \overline{s}'(t) \tag{3.7}$$

in some weak sense, so that finally we get

LEMMA 3.1. *For every compact subset K of S there exists a constant $c > 0$ such that for any nondecreasing interface contained in K*

$$s''(t) + Cs'(t) \geq 0 \tag{3.8}$$

in the sense of measures.

It easily follows from this result that $s(t)$ is a Lipschitz-continuous function with lateral derivatives $D^+s(t)$, $D^-s(t)$ at every t such that $D^-s(t) \leq D^+s(t)$. It also follows that whenever $D^+s(t_0) > 0$ at $t = t_0$ we have $D^+s(t) > 0$ for $t > t_0$, in other words once an interface starts to move it never stops. in fact the function $f(t) = s'(t)e^{Ct}$ is nondecreasing in t, cf. [CF1].

4. C^1-regularity.

Using the information gathered in previous sections, Caffarelli and Friedman set out in [CF1], §4, 5, 6, 7, to prove the C^1 regularity of moving interfaces and of the pressure in $P[u]$ up to this type of interface in the case of the initial-value problem (0.1), (0.5). Their proof was subsequently clarified in [ACK], where the following linear behaviour near a moving interface was established.

LEMMA 4.1 [ACK]. *Let (x_0, t_0) be a point on an increasing interface and let $c = s'(t_0) > 0$. The behavior of v in a neighborhood of (x_0, t_0) can be described as*

$$v(x,t) = c[c(t - t_0) - (x - x_0)]_+ + o(|x - x_0| + |t - t_0|) \tag{4.1}$$

The proof relies on a delicate blow-up argument near (x_0, t_0). Since no changes are necessary to adapt it to our situation, we omit further details and refer to the above mentioned sources. Let us only remark that [CF1] establishes the equation $v_t = v_x^2$ on the interface, so that, by (2.2), we have $vv_{xx} = 0$ on it, an estimate that will be needed in the next section.

5. C^∞-regularity.

The proof of the C^∞-regularity statements for v and s in Theorem B is done in [AV1] by

1) reducing the problem to proving that some derivatives of v are bounded near a moving free boundary and
2) obtaining these bounds by means of a type of *barrier* that blows up at Γ.

1). The first part goes as follows:

i) By virtue of equation (0.9) the C^∞ regularity for s is a consequence of the C^∞-regularity for v, starting from formula (0.9).

ii) Proving that $v \in C^\infty(N \cap \overline{P[u]})$ is equivalent to proving that for every pair of intergers $i, j \geq 0$ there exists a constant C_{ij} such that

$$|D_t^{(i)} D_x^{(j)} v| \leq C_{ij} \text{ in } N \cap P[u] \tag{5.1}$$

iii) By using equation (2.2) and differentiating it in P with respect to t and x as many times as needed, we may express all the derivatives in (5.1) in terms of only x-derivatives, $D_x^{(j)}, j \geq 0$.

2). Therefore we are left with the task of obtaining bounds like (5.1) for v, v_x, v_{xx}, v_{xxx}, etc. Now v was estimated in Lemma 2.1 and v_x in Lemma 2.2. Hence we have to begin with v_{xx}, for which a bound from below was obtained in Lemma 2.3.

(i) To obtain an *upper* bound for v_{xx} near $(\overline{x}, \overline{t}) \in \Gamma$ we consider the PDE satisfied by $p = v_{xx}$

$$L(p) \equiv p_t - (m-1)vp_{xx} - smv_x p_x - (m+1)p^2 = 0, \tag{5.2}$$

where the coefficients $(m-1)v$, $2mv_x$ and $(m+1)$ are viewed as fixed functions of x and t. Let us set $a = s'(\overline{t}) > 0$. We pick a small $\varepsilon > 0$, to be specific such that

$$4(m+1)\varepsilon < (a - 4m\varepsilon)(a - \varepsilon) \tag{5.3}$$

and consider a small domain $D = \Big([\overline{x} - \delta, \overline{x} + \delta] \times [\overline{t} - \eta, \overline{t} + \eta]\Big) \cap P[u]$, such that in D

$$a - \varepsilon \leq -v_x \leq a + \varepsilon \tag{5.4.a}$$

and

$$vv_{xx} \leq \varepsilon, \tag{5.4.b}$$

This can be done uniformly away from $t = t^*$ by the results of §4. It follows that

$$(a - \varepsilon)(s(t) - x) < v(x, t) < (a + \varepsilon)(s(t) - x) \tag{5.4.c}$$

in D and

$$a - \varepsilon < s'(t) < a + \varepsilon \tag{5.4.d}$$

for $t_1 \equiv \overline{t} - \eta < t < t_2 \equiv \overline{t} + \eta$. Consider now the function

$$\overline{p} = \overline{p}_{\alpha,\beta}(x, t) = \frac{\alpha}{s(t) - x} + \frac{\beta}{s^*(t) - x}, \tag{5.5}$$

where $s^*(t) = s(t_1) + (a + 2\varepsilon)(t - t_1)$. It is easy to check by substituting this expression into (5.2) that $L(\overline{p}) \geq 0$ and D if $\beta \leq \beta_0(\varepsilon) = (a - 4m\varepsilon)/2(m+1)$ and $\alpha \leq \alpha_0(\varepsilon) = (\alpha - (4m-1)\varepsilon)/2(m+1)$. Moreover we may have $\overline{p} \geq p$ on $\partial_p D$ by putting $\beta = \beta_0(\varepsilon)$ and imposing that η be small compared to δ, without any other condition on α. It follows from the Maximum Principle that $p \leq \overline{p}_{\alpha,\beta}$ in D, and letting $\alpha \to 0$ we even get $p \leq \overline{p}_{0,\beta}$. Take now the subdomain $D_1 = \{(x,t) \in D : |t - \overline{t}| < \eta/2\}$. In D_1 we have

$$p(x,t) \leq \frac{\beta}{s^*(t) - x} \leq \frac{2\beta}{\varepsilon\eta} = C(a,\varepsilon). \tag{5.6}$$

This is the desired estimate for v_{xx}.

ii) An estimate for $|v_{xxx}|$ is obtained by using a similar barrier argument on the equation for $= v_{xxx}$

$$q_t - (m-1)vq_{xx} - (3m-1)v_x q_x - (4m+2)v_{xx}q = 0, \tag{5.7}$$

which has the advantage of being linear in q. There are two main differences with respect to the above argument for v_{xx}. First, we need an estimate to start with. In this case, we may obtain a bound of the form

$$v_{xxx} = O(1/(s(t) - x)) \tag{5.8}$$

As $(x,t) \to (\overline{s}(\overline{t}), \overline{t}) \in \Gamma$ with $(x,t) \in P[u]$ and $s'(\overline{t}) > 0$, (and not $o(\ldots)$ as in (5.4.b)) by using a rescaling argument near Γ plus interior regularity theory on equation (5.7), c.f. the details in [AV1]. The second difference consists in the conditions on α and β. Though, because of the linearity of (5.7), α and β need not be small we may not let $\alpha \to 0$ because of our "poor" estimate (5.8). This difficulty in the Phragmen-Lindelöf-type argument is overcome in [AV1] by an original so-called *barrier transformation*, Lemma 3.4, that allows us to pass from a barrier like (5.5) to a new barrier of the form

$$\overline{q}(x,t) = \frac{\beta + 2\alpha}{s^*(t) - x}. \tag{5.9}$$

The bound can also be obtained directly by slightly changing the barrier. Thus Friedman and the author have found that if A and B are large enough, then

$$\phi(x,t) = \frac{A\gamma}{s(t) - x - \gamma} + \frac{B}{s^*(t) - x} \tag{5.10}$$

is a barrier for $|v_{xxx}|$ in the domain

$$D_\gamma = \{(x,t) \in D : s(t) - x \geq 3\gamma\}$$

Arguing as in i) an estimate for $|v_{xxx}|$ follows, since $\gamma > 0$ is arbitrary.

iii) Bounds for the higher-order drivatives $D_x^{(j)}v, j \geq 4$, are obtained exactly in the same way as the bound for v_{xxx}, cf. [AV1].

With these estimates the proof of Theorem B is complete.

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