THE GENERIC EXISTENCE OF RATIONAL EXPECTATIONS EQUILIBRIUM IN THE HIGHER DIMENSIONAL CASE

by

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Discussion Paper No. 82-163, June 1982

*I would like to thank Professor R. Anderson for noticing an error in a previous version of this paper, and to thank the National Science Foundation, the California Institute of Technology, and the University of Minnesota School of Management for partial support.

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1. Introduction

The study of the generic existence of rational expectations equilibria has concentrated on three cases, according to whether the dimension of the space of "states of private information" is less than, equal to, or greater than the price space. In the first case, Allen proved in [2] that fully revealing equilibria exist generically. In the second case, Jordan and Radner constructed in [10] an open set of economic environments permitting no equilibrium. There also exist open sets of environments with fully revealing equilibria in this case. The present paper shows that in the third case there is a residual set of environments which permits equilibria which are arbitrarily close to fully revealing.

Suppose that traders in a market possess private information concerning future events of common interest. A typical example is inside information about the future return on financial assets. A trader's private information can be modelled as a vector s^i which he observes before trade occurs. The state of information $s = (s^i)_{i=1}^{I}$ then describes all of the private information present in the market. Each trader i attempts to infer from market prices as much as possible about the private information $(s^j)_{j \neq i}$ observed by others. A rational expectations equilibrium associates with each state of information s a price vector $p(s)$ which clears markets when each trader chooses his demand to maximize expected utility conditioned on his private information and the information revealed by the price. However, as

Radner observed in [13], continuous changes in the price vector p(s) for different states of information can cause discontinuous changes in the partition induced by the function p(·), and thus discontinuous changes in traders' demands. Green [4] and Kreps [11] showed with explicit examples that this discontinuity can prevent the existence of equilibrium. Subsequent research has sought to determine whether this absence of equilibrium is pervasive or is confined to a "negligible" set of such examples. This question was to some extent anticipated by Green, whose example is robust to perturbations of the density function describing the uncertainty in the environment, but not to perturbations of traders' characteristics. The first result was obtained by Radner [14], who showed that if there are only finitely many possible states of information, then generically there exists an equilibrium price function p(·) which is one to one, and thus completely reveals the state of information to every trader. In [2], Allen generalized this result to the case in which the space of states of information has a lower dimension than the price space. Strictly speaking, Allen did not parameterize states of information as we have, but dealt directly with the manifold of endowment - utility profiles which would obtain if each trader could observe all private information. In [1] Allen proved that if this manifold has less than half the dimension of the price space then a one to one equilibrium price function exists generically. Allen further established in [2] that if this manifold has dimension less than the price space then generically an equilibrium price function exists which is one to one almost everywhere, and is thus completely revealing with probability one. If the price and state of information have the same dimension, Jordan and Radner [10] showed that equilibria fail to exist on an open set of environments. In particular, an open set of environments was constructed which would permit only fully revealing equilibria, but for which no fully revealing equilibria exist.

The remaining case, in which the state of information has higher dimension than the price, is intuitively the most compelling. The dimension of the space of states of information $(s^i)_{i=1}^I$ can be interpreted as the number of indices or measurements, adjusted for functional dependence such as overlaps in the private information of different traders and accounting identities, which are needed to describe all the private information relevant to future events of interest to traders. In general one would expect this number to considerably exceed the number of commodities traded. For example, in a stock market the commodities traded represent claims to the future profits of firms, whose current operating characteristics would require many numbers to describe. In this case, the present author proved in [9], in the context of a stock market model, that equilibria which reveal all decision relevant private information are nonexistant generically. Moreover, unless the functions which represent traders' utilities of future wealth have special risk tolerance properties, such equilibria will be nonexistant for a generic set of the density functions which determine the joint distribution between states of information and the future returns on securities. Thus in the higher dimensional case, the study of rational expectations equilibria is necessarily the study of partially revealing equilibria.

Despite the apparent generality of the partial revelation case, no general techniques have been developed for the study of partially revealing equilibria. The reason for this is easily explained. If the equilibrium reveals the state of information s completely, the equilibrium conditions which determine the price $p(s)$ are independent of the characteristics of the environment in different states. This "local control" of equilibria makes possible the direct application of the equilibrium manifold theory developed by Debreu [3], Smale [15], and subsequent authors. Conversely, if several states are associated with the same price, a trader's price-conditional expected utility is a convex combination of his expected utilities in each state which he cannot exclude on the basis of his own private information. Hence the equilibrium conditions in each state may involve the characteristics of the environment in all other states having the same price. If the level sets of the price function were smooth manifolds varying smoothly with the price, this might not be a significant handicap. However this regularity cannot be ensured, due to the same informational discontinuity which is responsible for the existence problem. The only natural restriction on price functions is Borel measurability, which is needed to define conditional expectations. Indeed, the equilibria constructed in the present paper are discontinuous on a dense set of states, and the discontinuities are fundamental to the existence of these equilibria.

The conceptual basis of the results in this paper and in [9] is contained in the informational decentralization theory developed by Hurwicz [8], Mount and Reiter [12], and others. One object of this theory is to measure the amount of information required by an informationally decentralized message process to achieve prespecified allocations. If the relation between environments and equilibrium messages is sufficiently regular, the dimension of the space of equilibrium messages is one index of the information used by the process. Some regularity conditions are essential to give dimension an informational capacity interpretation, since any euclidean space is Borel isomorphic to the unit interval. In [9] this regularity was obtained from the above mentioned regularity of full revelation equilibria, and such equilibria where shown to be generically nonexistant because the dimension of the price is to low to carry the information needed to achieve the fully revealing equilibrium portfolio allocations. We will show below that in the absence of full revelation no regularity is imposed and the dimension of the price ceases to be an informational constraint.

To motivate our construction of equilibria, suppose there are two traders, one of whom is "informed" in the sense that he observes the entire state of information directly, and the other is "uninformed" in the sense that he has no private information and must rely exclusively on the price. We will partition the space of states S into two subsets S_a and S_b, and construct a function $g\colon S_a \to S_b$ and an equilibrium price function p

on S such that for each observed price $q \in p(S)$, $p^{-1}(q)$ is a pair $\{s_a, g(s_a)\}$. If the dimension of S exceeds the price dimension, which will be our hypothesis, this requires $p(\cdot)$ to be discontinuous since it must be one to one on S_a and S_b separately. The informed traders' expected utility is smooth in s, but the behavior of the uninformed traders' expected utility, which is an average of his expected utility in s_a and $g(s_a)$, depends on the behavior of g. Hence if g is discontinuous it is possible for p to be discontinuous. The discontinuities necessary in a one-to-one dimension reducing function occur on a dense subset of the domain, but the "jumps" may be arbitrarily small. For this reason the distance $\sup\{||s_a-g(s_a)||: s_a \in S_a\}$, which measures the uninformed trader's uncertainty about the true state of information, can be made arbitrarily small. These remarks are merely intended to suggest the possibility of our result. The explicit construction of such equilibria is accomplished in section 3 below, and is informally described in sections 3.5, 3.16, 3.19, and 3.21.

2. The Model

The result depends on certain transversality conditions which in turn require that the set economic environments encompassed by the model be sufficiently rich. To demonstrate that this richness can be obtained without introducing any artificial generality, we pose the result in stock market context, which is perhaps the most traditional setting for the study of equilibrium under uncertainty with differential information. There are two periods, present and future, and in the present traders choose portfolios to maximize their expected utility. Private information is exogenous and is observed by traders before trade takes place.

The equilibria we obtain reveal private information almost completely, so that in equilibrium all traders will have nearly the same expectations. For this reason, we can use a theorem of Hart [5 , Theorem 3.3] to avoid imposing exogenous bonds on short sales. Hence we will define a trader's utility of future wealth on the entire real line and not be concerned with the sign of asset prices and returns. However, future asset values will be confined to a compact set, so only a compact set of wealth values and asset prices will be relevant.

2.1 Definitions: There are I traders, indexed by the superscript i. The i^{th} trader has a utility of future wealth u^i: $R \to R$. For each i, let U^i denote the set of utility functions satisfying

A.1) u^i is C^∞ with $Du^i(w) > 0$ and $D^2u^i(w) < 0$ for all $w \in R$.

The set U^i will be endowed with the topology of uniform C^∞ convergence on compacta (called the "weak $C^\infty(R, R)$ topology" by Hirsch [, pp. 34-36]). Let $\mathcal{U} = \Pi_i U^i$, with the product topology.

There are J risky assets, indexed by the subscript j, $1 \leq j \leq J$, and a single riskless asset, indexed by the subscript 0. The i^{th} trader has an endowment of assets $\omega^i = (\omega_0^i, \omega_1^i, \ldots, \omega_J^i) \in R^{J+1}$, and we assume

A.2) for some $j \geq 1$, $\Sigma_i \omega_j^i \neq 0$.

We normalize prices by setting the price of the riskless asset at unity, so the space of current prices and future returns is $\{1\} \times R^J$. A future return vector is denoted $r = (1, r_1, \ldots, r_J)$. That is, if the future return is r, the future value of a portfolio $x \in R^{J+1}$ is $rx = x_0 + \Sigma_{j \geq 1} r_j x_j$. A current price vector is denoted $p = (1, p_1, \ldots, p_J)$.

For each i, the i^{th} trader's private information set is $S^i = [0, 1]^{n^i}$, where $n^i \geq 0$, with the convention that $[0, 1]^0 = \{0\}$. Let $S = \Pi_i S^i$, with generic element $s = (s^i)$. Elements of S are called states of information. The future returns of risky assets are assumed to lie in a compact cube $P \subset R^J$, and are distributed jointly with states of information according to a density function on $S \times P$. Let $\mathcal{F}$ denote the set of all functions $f\colon\ S \times P \to R_+$ 1/ satisfying

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1/ R_+ denotes the nonnegative real numbers.

A.3) f is C^∞, and for each $s \varepsilon S$ there is some $r \varepsilon P$ with $f(s, r) > 0$.

As indicated in (A.3), the symbol r will also be used to denote elements of P, which are J-vectors. Given a density function $f \varepsilon F$, the probability of any Borel set $E \subset S \times P$ is $\int_E f(s, r)d\mu$, where μ is Lebesgue measure on $S \times P$. 2/

The set F will be endowed with the topology of C^∞ uniform convergence. The endowment profile will remain fixed, so the space of environments is $\mathcal{E} = \mathcal{U} \times F$. A subset $\mathcal{E}^*$ of $\mathcal{E}$ is <u>residual</u> if it is a countable intersection of open and dense subsets of $\mathcal{E}$.

2.2 <u>Remarks</u>: The use of higher order derivatives of utility and denisty functions simplifies certain aspects of the proof. I do not know the minimal level of differentiability which would still permit the result. Assumption (A.2) avoids the trivial equilibrium in which no traders hold risky assets. The definition of S as a product of individual information sets, together with the assumption that the probability distribution on S has a density function, formally precludes any overlap in the individual sets S^i. Such overlaps and other forms of functional dependence between coordinates of different information sets could easily be incorporated at the cost of additional notation. The compactness of S and P ensures the existence of expected utility and also permits differentiation of expected utility under the integral. It also ensures that certain transversality conditions are open in $\mathcal{E}$.

2/ For convenience we do not require the probability of $S \times P$ to be one, since this normalization does not affect expected utility maximization.

We can now define a rational expectations equilibrium and state the principal result.

2.3 Definitions: For each i let $\sigma^i: S \to S^i$ denote the projection $\sigma^i: s \to s^i$. Given $((u^i)_i, f) \in \mathcal{E}$, a rational expectations equilibrium (REE) for $((u^i)_i, f)$ is a Borel measurable function $\pi: S \to \{1\} \times P$ such that for a.e. $s \in S$, there is a portfolio profile $(x^i)_i$ satisfying

i) for each i, x^i maximizes $E\{u^i(rz)|\sigma^i, \pi\}(s)$ subject to $\pi(s)(z - \omega^i) = 0$; and

ii) $\Sigma_i(x^i - \omega^i) = 0$,

where the conditional expectation in (i) is of course with respect to the probability distribution specified by f. For any $\varepsilon > 0$, an REE π is revealing within ε if there is a subset $S^* \subset S$ whose complement $S\setminus S^*$ has Lebesgue measure zero such that for any $s, s' \in S^*$, if $\pi(s) = \pi(s')$ then $||s - s'|| < \varepsilon$, where $||\cdot||$ denotes the euclidean norm.

2.4 Theorem: Suppose that for some i, $\Sigma_{k\neq i}n^k > J$. Then there is a residual set $\mathcal{E}^* \subset \mathcal{E}$ such that for each $((u^i)_i, f) \in \mathcal{E}^*$ and each $\varepsilon > 0$, $((u^i)_i, f)$ has an REE which is revealing within ε.

2.5 Remarks: The hypothesis states that there is some trader whose private information set lacks at least $J + 1$ of the total sources of information in the environment. To become completely informed, such a trader needs to learn from market prices more information variables than the number of prices. Although complete revelation cannot generally be attained in this case, approximately complete revelation is generically possible.

Approximating full information is less important to a trader than approximating his optimal fully informed protfolio choice. Corollary 2.7 below states that the latter sense of approximation is consistent with the former. We define in 2.6 a full communication equilibrium for a state of information as an equilibrium price and portfolio profile that could result if all traders publicly communicated their private information. Since S is compact, approximately fully revealing equilibria uniformly approximate full communication equilibria.

2.6 Definition: Given $((u^i)_i, f; s) \in \mathcal{E} \times S$, a full communication equilibrium (FCE) for $((u^i)_i, f; s)$ is a price $p \in \{1\} \times P$ and a portfolio profile $(x^i)_i \in R^{I(J+1)}$ such that

i) for each i, x^i maximizes $\int_P u^i(rz)f(s, r)dr$ subject to $p(z - \omega^i) \leq 0$; and

ii) $\Sigma_i(x^i - \omega^i) = 0$.

2.7 Corollary: For any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $((u^i)_i, f) \in \mathcal{E}^*$ and any REE π which is revealing within δ, for a.e. $s \in S$ there exists a portfolio profile

$(x^i(s))_i$ such that $(\pi(s), (x^i(s))_i)$ satisfies 2.3 (i - ii) and an FCE $(p^*, (x^{*i})_i)$ for $((u^i)_i, f, s)$ such that $||x^i(s) - x^{*i}(s)|| < \varepsilon$ for all $1 \leq i \leq I$.

Proof: The result follows from a straightforward modification of Lemma 3.1 and the proof of 3.3 (iii) below.

2.8 Remarks: The endowment is fixed in the definition of $\mathcal{E}$ in order to ensure that except for assumption (A.2), the endowment plays no special role in the result. In a model of general stochastic exchange environments, such as in [1] and [2], the endowment varies with the state of information, so ω^i is part of the i^{th} trader's private information. Our reliance on perturbations in $\mathcal{U} \times \mathcal{F}$ to achieve equilibrium should facilitate the adaptation of Theorem 2.4 to such models.

3. Proof of the Theorem

We first need to record, in Lemma 3.3 below, some properties of full communication equilibria.

3.1 Lemma: Let $\{s_n\}_{n=1}^{\infty}$ be a convergent sequence in S and for any $1 \le i \le I$, let $\{(u_n^i, f_n)\}_{n=1}^{\infty}$ be a convergent sequence in $U^i \times F$. Renormalizing if necessary, we can assume without loss of generality that $\int_P f_n(s_n, r)dr = 1$ for each n. Let $a = \min \{p\omega^i : p \in P\}$ and for each n, let $X_n = \{x \in R^{J+1}: \int u_n^i(rx) f_n(s_n, r)dr \ge u^i(a)\}$. For each n, let $r_n = \int r f_n(s_n, r)dr$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in R^{J+1} with $x_n \in X_n$ for each n and $||x_n|| \to \infty$, where $||\cdot||$ denotes the euclidean norm. Then $r_n x_n \to \infty$.

proof: Let $x_a = (a, 0, \ldots, 0)$ and for each n, let $z_n = x_n - x_a$. Then for each n $\int u_n^i(a + rz_n)f_n(s_n, r)dr \ge u_n^i(a)$ so since u_n^i is concave, $r_n z_n \ge 0$ for all n. Suppose by way of contradiction that, taking a subsequence if necessary, $\lim r_n z_n < \infty$. Let $z°$ be a cluster point of the sequence $z_n/||z_n||$, and let $r° = \lim r_n$. Since $r_n z_n \ge 0$ for all n, $r°z° = 0$. Let $u^i = \lim u_n^i$, $f = \lim f_n$, and $s = \lim s_n$. Since u_n^i is concave $\{z: \int u_n^i(a + rz)f_n(s_n, r)dr \ge u_n^i(a)\}$ is convex and contains zero. Since it contains z_n, it also contains $z_n/||z_n||$. Hence $\int u^i(a + rz°)f(s, r)dr \ge u^i(a)$. Since u_n^i is strictly concave, we must have $r°z° > 0$, and this contradiction proves the Lemma.

<u>3.2 Definitions</u>: Let $((u^i)_i, f) \in \mathcal{U} \times \mathcal{F}$ and let $s \in S$. For each i, let $\omega_y^i = (\omega_1^i, \ldots, \omega_J^i)$. A price and profile of risky asset portfolios $(p, (y^i)_i) \in P \times R^{IJ}$ is an FCE for $((u^i)_i, f, s)$ if $(p, (x^i)_i)$ is an FCE as defined in 2.6 above, where $x^i = (\omega_0^i - p(y^i - \omega_y^i), y^i)$ for each i. Let $e((u^i)_i, f, s) = \{(p, (y^i)_i) \in P \times R^{IJ}$: $(p, (y^i)_i)$ is an FCE for $((u^i)_i, f, s)\}$. Let $Y = R^J \setminus \{0\}$ and let P° denote the interior of P.

<u>3.3 Lemma</u>: For each $((u^i)_i, f))$,

i) for each $s \in S$, $(p, (y^i)_i) \in e((u^i)_i, f, s)$ if and only if

a) $\int (r-p) Du^i(p\omega + (r-p)y^i) f(s,r) dr = 0$ for each i, and

b) $\Sigma_i (y^i - \omega_y^i) = 0$; and

ii) for each $s \in S$, $e((u^i)_i, f, s)$ is a nonempty subset of $P^\circ \times Y^I$.

In addition,

iii) e is upper semi-continuous.

<u>Proof</u>: (i) is immediate. That $e((u^i)_i, f, s)$ is nonempty is a consequence of Hart [5 , Theorem 3.3 and pp. 308-310]. Let $(p, (y^i)_i) \in e((u^i)_i, f, s)$ and assume that $\int f(s,r) dr = 1$. If $p \in \partial P$ there exists a portfolio $y \in Y$ with $(r - p)y > 0$ for all $r \in \overset{\circ}{P}$. Hence for each i,

$$\lim_{n \to \infty} \int u^i(p\omega + (r - p)ny) f(s,r) dr = \lim_{w \to \infty} u(w), \text{ so}$$

p cannot be an equilibrium price. Hence $p \in P^\circ$. Suppose that for some k, $y^k = 0$. Then $Du^k(p\omega + (r-p)y^k) = Du^k(p\omega)$ for all r, so (ia) implies that $p = \int r f(s,r) dr$. But since u^i is strictly concave, this implies $y^i = 0$ for all i, so (ib) contradicts (A.2). Hence $y^i \in Y$ for all i, which proves (ii).

To prove (iii), let $\{(u_n^i)_i, f_n, s_n\}_{n=1}^{\infty}$ be a convergent sequence in $\mathcal{U} \times \mathcal{F} \times S$ and let $(p_n, (y_n^i)) \in e((u_n^i), f_n, s_n)$ for each n. We again assume that $\int f_n(s_n,r)dr = 1$. It is immediate from (i) that e has a closed graph, and since $p_n \in P$ for each n, it suffices to prove that the sequence $\{(y_n^i)_i\}$ is bounded. For each i, n, let $x_n^i = (\omega_o^i - p_n(y_n^i - \omega_y^i), y_n^i)$. Then $\int u_n^i(rx_n^i)f_n(s_n, r)dr \geq u_n^i(p_n\omega^i)$ for each i, n. Let $r_n = \int rf_n(s_n,r)dr$ for each n. Then $\Sigma_{i=1}^I r_n x_n^i = r_n \Sigma_{i=1}^I \omega^i$, so by Lemma 3.1, the sequence $\{(x_n^i)_i\}$ is bounded, which completes the proof.

<u>3.4 Definitions</u>: By reindexing traders if necessary, we can assume that $\Sigma_{i>1} n_i \geq J + 1$. Let S_2 denote the last $J + 1$ coordinates of S and let S_1 denote the first $(\Sigma_i n_i) - (J + 1)$, so that $S = S_1 \times S_2$. Let $i° = \max\{i: S^i$ has no coordinates in common with $S_2\}$. Then $i° \geq 1$. If $i > i°$, trader i will be said to be <u>fully informed</u>, and if $i \leq i°$, trader i will be said to be <u>partially informed</u>.

3.5 <u>Remarks</u>: The remainder of the proof is in four parts, the first of which is an investigation of the following system of equations.

$$\int (r-p) Du^i(p\omega^i + (r-p)y_a^i)(f(s_1,s_a;r) + \gamma f(s_1,s_b;r))dr = 0,$$

$$i \leq i^\circ ;$$

$$\int (r-p) Du^i(p\omega^i + (r-p)y_a^i) f(s_1,s_a;r) dr = 0, \quad i > i^\circ ;$$

$$\Sigma_i (y_a^i - \omega_y^i) = 0 ;$$

$$\int (r-p) Du^i(p\omega^i + (r-p)y_b^i) f(s_1,s_b;r) dr = 0, \quad i > i^\circ;$$

$$\Sigma_{i \leq i^\circ} (y_a^i - \omega_y^i) + \Sigma_{i > i^\circ} (y_b^i - \omega_y^i) = 0 .$$

These equations describe an equilibrium in the two states of information (s_1, s_a) and (s_1, s_b) if the fully informed traders know the state and the partially informed traders assign probability $\int f(s_1, s_a; r)dr / \int [f(s_1, s_a; r) + \gamma f(s_1, s_b;r)]dr$ to the state (s_1, s_a) and the complementary probability to the other state. The need for the factor γ will be explained in 3.16 below. Note that s_1 and the price are the same in both states, so the partially informed traders cannot observe any distinction between the two states. Of course if $s_a = s_b$, we have an F C E. In Lemma 3.15 we will conclude that for a residual set of environments, at almost every state of information s° these equations determine the variables $(s_b; p_2,\ldots,p_J; (y_a^i)_i, (y_b^i)_{i>i^\circ})$ as functions of (s_1, s_a, p_1) near $(s_1, s_a) = s^\circ$ and $p_1 = p_1^\circ$, where p_1° is a full communication equilibrium price for the first risky asset. The intervening Lemmas establish transversality conditions which yield Lemma 3.15 as a consequence of the implicit function theorem.

3.6 Definition: Let $\mathcal{U}_1$ denote the set of utility I-tuples $(u^i)_i$ such that there exist constants $(a^i)_i$ not all zero, constants $(b^i)_i$, nonzero constants $(c^i)_i$, and a nonempty open interval $N \subset R$ such that

$$\Sigma_i a^i Du^i(b^i + c^i d) = 0 \quad \text{for all} \quad d \in N.$$

3.7 Lemma: There is a dense set $\mathcal{U}^* \subset \mathcal{U}$ with $\mathcal{U}_1 \subset \mathcal{U}\backslash\mathcal{U}^*$.

Proof: According to Definition 3.6, if $(u^i)_i \in \mathcal{U}_1$ then $\Sigma_i a^i (c^i)^{k-1} D^k u^i(b^i + c^i d) = 0$ for all $k \geq 1$ and all $d \in N$.

Define the map $\alpha\colon \mathcal{U} \times (R^I\backslash\{0\}) \times R^I \times (R\backslash\{0\})^I \times R \to R^{3I+2}$ by $\alpha((u^i), (a^i), (b^i), (c^i), d) = (\Sigma_i a^i (c^i)^{k-1} D^k u^i (b^i + c^i d))_{k=1}^{3I+2}$

A straightforward application of transversal density (e.g. [, Lemma 4.6]) shows that the set $\mathcal{U}^* = \{(u^i)_i\colon \alpha((u^i)_i, \cdot) \pitchfork \{0\}\}$ is dense in $\mathcal{U}$. If $\alpha((u^i)_i, \cdot) \pitchfork \{0\}$ then $\alpha((u^i)_i, \cdot)$ is never zero, so $(u^i)_i \notin \mathcal{U}_1$. Hence $\mathcal{U}_1 \subset \mathcal{U}\backslash\mathcal{U}^*$, which completes the proof.

3.8 Lemma: Let $(u^i)_i \in \mathcal{U}^*$ and let $(p, (y^i)_i) \in P \times Y^I$. Then for any open set $V \subset P$ there exist return vectors $r_k \in V$, $1 \leq k \leq IJ$ such that the vectors

$$((r_k - p)Du^i(p\omega^i + (r_k - p)y^i))_{i=1}^I \qquad 1 \leq k \leq IJ$$

span R^{IJ}.

Proof: Suppose by way of contradiction that the set $\{((r-p)Du^i(p\omega^i + (r-p)y^i))_{i=1}^I : r \in V\}$ lies in an IJ-1 dimensional subspace of R^{IJ}. Then there exist vectors $a'^i \in R^J$, $1 \leq i \leq I$ not all zero with $\Sigma_i a'^i(r-p)Du^i(p\omega^i+(r-p)y^i) = 0$ for all $r \in V$. Let $r \in V$ with $a'^i(r-p) \neq 0$ for some i and $(r-p)y^i \neq 0$ for all i. Let $\varepsilon > 0$ with $r+\delta(r-p) \in V$ for all $\delta \in (-\varepsilon, \varepsilon)$. For each i let $a^i = a'^i(r-p)$, let $b^i = p\omega^i$, and let $c^i = (r-p)y^i$. Then for all $d \in (1-\varepsilon, 1+\varepsilon)$

$$\Sigma_i da^i Du^i(b^i + c^i d) = 0$$

so dividing by d gives

$$\Sigma_i a^i Du^i(b^i + c^i d) = 0 \quad \text{for all } d \in (1-\varepsilon, 1+\varepsilon).$$

Hence $(u^i)_i \in \mathcal{U}_1 \subset \mathcal{U} \setminus \mathcal{U}^*$, and this contradiction completes the proof.

3.9 <u>Definition</u>: For each $((u^i)_i, f) \in \mathcal{E}$ define $\Gamma((u^i)_i, f; \cdot): S \times P \times Y^I \to R^{J(I+1)}$ by $\Gamma((u^i)_i, f; s, p, (y^i)_i) =$

$$\int (r-p)Du^i(p\omega^i + (r-p)y^i)f(s, r)dr, \quad 1 \leq i \leq I,$$

$$\Sigma_i (y^i - \omega_y^i).$$

3.10 Lemma: Let $\mathcal{E}_1 = \{((u^i)_i, f) \in \mathcal{E}: \Gamma((u^i)_i, f; \cdot) \pitchfork \{0\}\}$. Then $\mathcal{E}_1$ is open and dense.

Proof: Using 3.3 (iii), it is straightforward to show that $\mathcal{E}_1$ is open. To prove that $\mathcal{E}_1$ is dense let $\{K_n\}_{n=1}^{\infty}$ be a collection of compact sets with $S \times P \times Y^I = \cup_n K_n$, and for each n let $\mathcal{E}_{1n} = \{((u^i)_i, f) \in \mathcal{E}: \Gamma((u^i)_i, f; \cdot) \pitchfork \{0\}$ on $K_n\}$. Then for each n, $\mathcal{E}_{1n}$ is open, and $\mathcal{E}_1 = \cap_n \mathcal{E}_{1n}$ so it suffices to show that $\mathcal{E}_{1n}$ is dense for each n. Given n, let $((u^i)_i, f) \in \mathcal{E}$ and let $(s°, p°, (y°^i)_i) \in K_n$. By Lemma 3.7 we can assume that $(u^i)_i \in \mathcal{U}^*$. Let $V = \{r \in P: f(s, r) > 0\}$ and let the return vectors $\{r_k\}_{k=1}^{IJ}$ be given by Lemma 3.8. For each k let N_k be a compact neighborhood of r_k with $N_k \subset V$ and $N_k \cap N_{k'} \neq \emptyset$ if $k \neq k'$. Moreover the neighborhoods N_k should be chosen small enough so that for each $(r'_k)_k$ with $r'_k \in N_k$ for all k, the vectors

$$((r'_k - p)Du^i(p\omega^i + (r'_k - p)y^i))_{i=1}^I, \quad 1 \leq k \leq IJ$$

span R^{IJ}. Let $N°$ be a compact neighborhood of $s° \in S$ such that for each $s \in N°$ and each $r \in \cup_k N_k$, $f(s, r) > 0$. For each k let $h_k: S \times P \to R_+$ be a smooth function with

i) $h_k(s°, r_k) = 1$, and

ii) $h_k(s, r) = 0$ for all $(s, r) \notin N° \times N_k$.

Then there is some $\varepsilon > 0$ such that $f + \Sigma_k \lambda_k h_k \in F$ for all $(\lambda_k)_k \in (-\varepsilon, \varepsilon)^{IJ}$. Define the function

$$\Psi_1: (-\varepsilon, \varepsilon)^{IJ} \times S \times P \times Y^I \to R^{J(I+1)} \text{ by}$$

$$\Psi_1((\lambda_k)_k, s, p, (y^i)_i) = \Gamma((u^i)_i, f + \Sigma_k \lambda_k h_k; s, p, (y^i)_i) .$$

It follows directly from the definitions of the functions h_k that $D\Psi_1$ is surjective at $(0, s°, p°, (y^{\circ i})_i)$ and thus $\Psi_1 \pitchfork \{0\}$ on an open set $A \times B$ where $0 \in A \subset (-\varepsilon, \varepsilon)^{IJ}$ and B is a neighborhood of $(s°, p°, (y^{\circ i})_i)$. Let $\{B_m\}_{m=1}^M$ be a finite open cover of K_n consisting of such neighborhoods B_m with associated functions Ψ_m. Then there is some $\varepsilon' > 0$ with $f + \Sigma_m \Sigma_k \lambda_{mk} h_{mk} \in F$ for all $(\lambda_{mk}) \in (-\varepsilon', \varepsilon')^{IJM}$. Define the function $\Psi^*: (-\varepsilon', \varepsilon')^{IJM} \times S \times P \times Y^I \to R^{J(I+1)}$ by $\Psi^*((\lambda_{mk})_{mk}, s, p, (y^i)_i) = \Gamma((u^i)_i f + \Sigma_{mk} \lambda_{mk} h_{mk}; s, p, (y^i)_i)$. Then $D\Psi^*$ is surjective at $(0, s, p, (y^i)_i)$ for each $(s, p, (y^i)_i) \in K_n$ so there is some $\varepsilon'' > 0$ such that $\Psi^* \pitchfork \{0\}$ on $(-\varepsilon'', \varepsilon'')^{IJM} \times K_n$. Hence by transversal density [7] $\Psi((\lambda_{mk})_{mk}, \cdot) \pitchfork \{0\}$ on K_n for all $(\lambda_{mk})_{mk}$ in a dense subset of $(-\varepsilon'', \varepsilon'')$. This proves that $\mathcal{E}_{1n}$ is dense, which proves the lemma.

3.11 <u>Lemma:</u> For any $((u^i)_i, f) \in \mathcal{E}$, any $(s, p, z^i) \in S \times P \times Y$ and any $1 \leq i \leq I$, let F denote the derivative of the function $y^i \to \int (r-p) Du^i (p\omega^i + (r-p) y^i) f(s, r) dr$ evaluated at z^i. Then F is nonsingular.

Proof: Up to a positive scalar multiple F is represented by the matrix

$$(E\{D^2u^i(p\omega^i + (r-p)z^i)(r_j - p_j)(r_k - p_k)\})^J_{j,k=1},$$

where E is the expectation operator determined by $f(s, \cdot)$. This matrix determines the quadratic form on R^{J+1} defined by $h \to E\{D^2u^i(p\omega^i + (r-p)z^i)((r-p)h)^2\}$, which by assumption (A.1) is negative definite, so F is nonsingular.

3.12 Definitions: For each $((u^i)_i, f) \in \mathcal{E}$ define $\Gamma_2((u^i)_i, f; \cdot)\colon S_1 \times S_2 \times S_2 \times P \times Y^{2I-i°} \to R^{J(2I-i°+2)}$ by $\Gamma_2((u^i)_i, f; s_1, s_a, s_b, p, (y^i_a)_i, (y^i_b)_{i>i°}) =$

$$\int (r-p)Du^i(p\omega^i + (r-p)y^i_a)(f(s_1, s_a; r) + f(s_1, s_b; r))dr,$$

$$i \leq i°$$

$$\int (r-p)Du^i(p\omega^i + (r-p)y^i_b)f(s_1, s_b; r)dr,\ i > i°,$$

$$\Sigma_i(y^i_a - \omega^i_y),$$

$$\int (r-p)Du^i(p\omega^i + (r-p)y^i_a)f(s_1, s_a; r)dr,\ i > i°$$

$$\Sigma_{i\leq i°}(y^i_a - \omega^i_y) + \Sigma_{i>i°}(y^i_b - \omega^i_y).$$

For each $(s°, p°, (y^{°i})_i) \in S \times P \times Y^I$ define $\Gamma'_2((u^i)_i, f; s°, p°, (y^{°i})_i)$ as the derivative of $\Gamma_2((u^i)_i, f; \cdot)$ with respect to $(s_b; p_2,\ldots,p_J; (y^i_a)_i, (y^i_b)_{i>i°})$ at $(s^°_1, s^°_2, s^°_2; p°, (y^{°i})_i, (y^{°i})_{i>i°})$. Let

$C^J = \{c \in R^{J+1}: (\Sigma c_j^2)^{1/2} = 1\}$ and for each $1 \le \ell \le J(2I-i^\circ+2)$ let $\Gamma'_{2\ell}$ denote the ℓ^{th} column of the matrix representing Γ'_2. For example, $\Gamma'_{2(J+2)}$ represents the derivative of Γ_2 with respect to p_2. For each $((u^i)_i, f) \in \mathcal{E}$ define $\Gamma^*((u^i)_i, f; \cdot): S \times P \times Y^I \times C^J \times R^{J(2I-i^\circ+1)-1} \to R^{J(3I-i^\circ+3)}$ by

$$\Gamma^*((u^i)_i, f; s, p, (y^i)_i; c) =$$

$$\Gamma((u^i)_i, f; s, p, (y^i)_i),$$

$$\Sigma_{\ell=1}^{J(2I-i^\circ+2)} c_\ell \Gamma'_{2\ell}((u^i)_i, f; s, p, (y^i)_i) .$$

3.13 Lemma: Let $\mathcal{E}_2 = \{((u^i)_i, f) \in \mathcal{E}: \Gamma^*((u^i)_i, f; \cdot) \pitchfork \{0\}\}$. Then $\mathcal{E}_2$ is residual.

Proof: Let $((u^i)_i, f) \in \mathcal{E}$ and let $(s^\circ, p^\circ, (y^{\circ i})_i; c^\circ) \in \Gamma^{*-1}(0)$. Since $(c_j^\circ)_{j=1}^{J+1} \in C^J$, renumbering coordinates of s_2 if necessary we can assume that $c_1^\circ \neq 0$. By Lemmas 3.7 and 3.10 respectively we can assume that $(u^i)_i \in \mathcal{U}^*$ and $((u^i)_i, f) \in \mathcal{E}_1$. Let $V = \{r \in P: f(s, r) > 0\}$, let the return vectors $\{r_k\}_{k=1}^{IJ}$ be defined by Lemma 3.8, and let the compact neighborhoods N° of s° and N_k of r_k for each k be a defined in the proof of Lemma 3.10. For each k let h_k: $S \times P \to R$ be a smooth function satisfying

i) $h_k(s, r) = 0$ for all $(s, r) \notin N° \times N^*_k$;

ii) $h_k(s°, \cdot) \equiv 0$; and

iii) $\frac{\partial}{\partial s_{21}} h_k(s°, r_k) = 1.$

There is some $\varepsilon > 0$ such that $f + \Sigma_k \lambda_k h_k \in F$ for all $(\lambda_k)_k \varepsilon (-\varepsilon, \varepsilon)^{IJ}$. Define ψ_1 on $(-\varepsilon, \varepsilon)^{IJ} \times S \times P \times Y^I \times C^J \times R^{J(2I-i°+i)}$ by $\psi_1((\lambda_k)_k, s, p, (y^i)_i, c) = \Gamma((u^i)_i, f + \Sigma_k \lambda_k h_k; s, p, (y^i)_i)$, and referring to the definition of Γ_2, let

$$\left.\begin{array}{l} \psi_2((\lambda_k)_k, s, p, (y^i)_i, c) \\ \\ \psi_3(\quad) \\ \\ \psi_4(\quad) \\ \\ \psi_5(\quad) \end{array}\right\} \text{denote the} \left\{\begin{array}{l} \text{first } IJ \text{ rows of} \\ \quad \Gamma_2'((u^i)_i, f + \Sigma_k \lambda_k h_k; s, p, (y^i)_i, c) \\ \text{next } J \text{ rows of } " \\ \\ \text{next } J(I-i°) \text{ rows of } " \\ \\ \text{last } J \text{ rows of } ". \end{array}\right.$$

Define $\psi: (-\varepsilon, \varepsilon)^{IJ} \times S \times P \times Y^I \times C^J \times R^{J(2I-i°+1)} \to R^{J(3I-i°+3)}$ by

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \end{pmatrix}$$

For each $c \in C^J \times R^{J(2I-i°+1)}$, let c^1 denote the first $(J+1)$ coordinates of c, pertaining to the derivative of Γ_2 with respect to s_b, let c^2 denote the next $J-1$ coordinates, pertaining to $(p_2,\ldots,p_J)$, let c^3 denote the next $Ji°$ coordinates, pertaining to $(y_a^i)_{i\leq i°}$, let c^4 denote the next $J(I-i°)$ coordinates, pertaining to $(y_a^i)_{i>i°}$, and let c^5 denote the last $J(I-i°)$ coordinates, pertaining to $(y_b^i)_{i>i°}$. Then the matrix $D\psi(0, s°, p°, (y^{\circ i}), c°)$ can be partitioned as

$$
\begin{array}{c c c c c c c c}
 & (s,p,y) & (\lambda_k)_k & c^1 & c^2 & c^3 & c^4 & c^5 \\
\psi_1 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\psi_2 & A_2 & B_2 & C_2 & D_2 & E_2 & 0 & G_2 \\
\psi_3 & A_3 & 0 & C_3 & D_3 & E_3 & F_3 & 0 \\
\psi_4 & A_4 & 0 & C_4 & D_4 & 0 & F_4 & 0 \\
\psi_5 & A_5 & 0 & C_5 & D_5 & E_5 & 0 & G_5
\end{array}.
$$

The fact that $B_1 = 0$ follows from (ii), and since $((u^i)_i, f) \in \mathcal{E}_1$, A_1 is surjective. Since $c_1 = c_1^1 \neq 0$, B_2 is surjective by our choice of the functions h_k. It is immediate from the definitions that E_3 and G_5 are surjective, and Lemma 3.11 implies that F_4 is surjective. To prove that $D\psi$ is surjective, let $z = (z_1, z_2, z_3, z_4, z_5) \in R^{J(3I-i°+3)}$. Let

a) $x_A \in R^{\Sigma n_i + J(I+1)}$ with $A_1 x_A = z_1$;

b) $x_F \in R^{J(I-i^\circ)}$ with $F_4 x_F = z_4 - A_4 x_A$;

c) $x_E \in R^{J_{i^\circ}}$ with $E_3 x_E = z_3 - A_3 x_A - F_3 x_F$;

d) $x_G \in R^{J(I-i^\circ)}$ with $G_5 x_G = z_5 - A_5 x_A - E_5 x_E$;

e) $x_B \in R^{IJ}$ with $B_2 x_B = z_2 - A_2 x_A - E_2 x_E - G_2 x_G$.

If $x = (x_A, x_B, x_C, x_D, x_E, x_F, x_G)$ with $x_C = 0$ and $x_D = 0$, then $D\psi x = z$. Hence $\psi \pitchfork \{0\}$ on a neighborhood of $(0, s^\circ, p^\circ, (y^{\circ i})_i, c^\circ)$. For any compact subset $K \subset R^{J(2I-i^\circ+1)-1}$, the set $\{((u^i)_i, f) \in \&:$ $\Gamma^*((u^i)_i, f; \cdot) \pitchfork \{0\}$ on $S \times P \times Y \times C^J \times K\}$ is easily proved to be open as a consequence of 3.3(iii). Proceeding by analogy with the proof of Lemma 3.10 shows that $\&_2$ is residual.

3.14 <u>Lemma</u>: Let $((u^i)_i, f) \in \&_1 \cap \&_2$. Then there is a subset $S^\circ \subset S$ such that $S \setminus S^\circ$ is a null set[1/] and for each $s^\circ \in S^\circ$, and each p°, $(y^{\circ i})_i$ with $\Gamma((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i) = 0$, $\Gamma_2'((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i)$ is nonsingular.

<u>Proof</u>: Since $((u^i)_i, f) \in \&_1$, $\Gamma((U_i^i), f; \cdot) \pitchfork \{0\}$ so $M_1 = \Gamma((u^i)_i, f; \cdot)^{-1}(0)$ is a $\Sigma_i n^i$ dimensional submanifold of $S \times P \times R^{JI}$. Let S_1 the set of regular values of the

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[1/] A null set is a closed nowhere dense set having Lebesgue measure zero.

projection of M_1 on S. By 3.3(ii) the projection is onto S, so by Sard's theorem, $S\setminus S_1$ is a null set. Since $((u^i)_i, f) \in \mathcal{E}_2$, $\Gamma^*((u^i)_i, f; \cdot) \pitchfork \{0\}$, so $M_2 = \Gamma^*((u^i)_i, f; \cdot)^{-1}(0)$ is a $(\Sigma_i n^i)-1$ dimensional submanifold of $S \times P \times Y^I \times C^J \times R^{J(2I-i^\circ+1)-1}$. Let S_2 be the projection of M_2 on S. Then S_2 has Lebesgue measure zero since $\dim M_2 < \dim S$, and using 3.3(iii), its intersection with S_1 is easily verified to be closed in S_1. Hence $S_2 \cup S\setminus S_1$ is a null set. Let $S^\circ = S\setminus(S_2 \cup S\setminus S_1)$. Let $s^\circ \in S^\circ$ and let $(p^\circ, (y^{\circ i})_i)$ with $\Gamma((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i) = 0$. Suppose by way of contradiction that Γ_2' is singular. Then there exists $c = (c^1, c^2) \in R^{J+1} \times R^{J(2I-i^\circ+1)-1}$ with $\Sigma_{\ell=1}^{J+1} c_j^1 \Gamma_{2\ell}' + \Sigma_{\ell=J+2}^{J(2I-i^\circ+1)-1} c_{\ell-(J+2)}^2 \Gamma_{2\ell}' = 0$ and $c \neq 0$, where $\Gamma_{2\ell}'$ denotes the ℓ^{th} column of Γ_2'. Since $s^\circ \in S_1$, $D\Gamma((u^i)_i, f; s^\circ, \cdot)$ is nonsingular at $p^\circ, (y^{\circ i})_i$, so by the definition of Γ_2' we must have $c^1 \neq 0$. But then $c' = [(\Sigma_{j=1}^{J+1}(c_j^1)^2)^{\frac{1}{2}}]^{-1} c \in C^J \times R^{J(2I-i^\circ+1)-1}$ and $\Gamma^*((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i, c') = 0$, which contradicts the fact that $s^\circ \notin S_2$. This completes the proof.

3.15 Lemma: There is a residual set $\mathcal{E}^* \subset \mathcal{E}_1 \cap \mathcal{E}_2$ such that for each $((u^i)_i \in \mathcal{E}^*$ there is a subset $S^* \subset S$ such that $S\setminus S^*$ is a null set and for each $s^\circ \in S^*$ and each $(p^\circ, (y^{\circ i})_i)$ with $\Gamma((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i) = 0$ there are open neighborhoods V_1 of s_1°, V_2 of s_2°, V_3 of p_1°, and V_4 of the point $\gamma^\circ = 1 \in R$, and a C^1 function $\phi: V \to S_2$ where $V = V_1 \times V_2 \times V_3 \times V_4$ satisfying

i) for each $(s_1, s_a, p_1, \gamma) \in V$, if $s_b = \phi(s_1, s_a, p_1, \gamma)$ then there exist $(p_2, \ldots, p_J;\ (y_a^i)_i,\ (y_b^i)_{i>i^\circ})$ with

$$\int (r-p) Du^i(p\omega^i + (r-p)y_a^i)(f(s_1, s_a, r) + \gamma f(s_1, s_b, r))dr = 0,\ i \leq i^\circ;$$

$$\int (r-p) Du^i(p\omega^i + (r-p)y_a^i) f(s_1, s_a, r) dr = 0,\ i > i^\circ;$$

$$\Sigma_i (y_a^i - \omega_y^i) = 0$$

$$\int (r-p) Du^i(p\omega^i + (r-p)y_b^i) f(s_1, s_b, r) dr = 0,\ i > i^\circ; \text{ and}$$

$$\Sigma_{i \leq i} (y_a^i - \omega_y^i) + \Sigma_{i>i^\circ}(y_b^i - \omega_y^i) = 0.$$

ii) $\phi(s^\circ, p_1^\circ, \gamma) = s_2$ for each $\gamma \in V_4$;

iii) for each $(s, p_1, \gamma) \in V$ the derivative of ϕ with respect to s_2 at (s, p_1, γ) is nonsingular; and

iv) for each $(s, p_1, \gamma) \in V$ and each $1 \leq \ell \leq J+1$ the derivative of ϕ_ℓ with respect to p_1 at (s, p_1, γ) is nonsingular.

<u>Proof</u>: Let $((u^i)_i, f) \in \&_1 \cap \&_2$, let $S^\circ \subset S$ given by Lemma 3.14, and let p°, $(y^{\circ i})_i$ with $\Gamma((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i) = 0$. Define the function $\Gamma_3: S_1 \times S_2 \times S_2 \times P \times Y^{JI} \times Y^{J(I-i^\circ)} \times R \to$ $\to R^{J(2I-i^\circ+2)}$ by

$$\Gamma_3(s_1, s_a, s_b, p), (y_a^i)_i, (y_b^i)_{i>i^\circ}, \gamma) =$$

$$\int(r-p)Du^i(p\omega^i + (r-p)y_a^i)(f(s_a,r) + \gamma f(s_b,r)dr, \qquad i \le i^\circ;$$

$$\int(r-p)Du^i(p\omega^i + (r-p)y_a^i)f(s_a,r)dr, \qquad i > i^\circ,$$

$$\Sigma_i(y_a^i - \omega_y^i),$$

$$\int(r-p)Du^i(p\omega^i + (r-p)y_b^i)f(s_b,r)dr, \qquad i > i^\circ;$$

$$\Sigma_{i\le i^\circ}(y_a^i-\omega_y^i) + \Sigma_{i>i^\circ}(y_b^i - \omega_y^i) .$$

We need to show that at $(s_1^\circ, s_2^\circ, s_2^\circ, p^\circ, (y^{\circ i})_i, \gamma^\circ)$ the derivative of Γ_3 with respect to $(s_b; p_2,\ldots,p_J; (y_a^i)_i, (y_b^i)_{i>i^\circ})$ is nonsingular. Since $\gamma^\circ = 1$, this derivative is Γ_2', which is nonsingular by Lemma 3.14. The Implicit Function Theorem now yields the function ϕ satisfying (i) and (ii). To prove (iii), suppose by way of contradiction that there is some $z = (ds_1, ds_a, ds_b, dp, (dy_a^i)_i, (dy_b^i), d\gamma) \in R^{\Sigma n_i+J(2I-i^\circ+2)}$ with $D\Gamma_3(s_1^\circ, s_2^\circ, s_2^\circ, p^\circ, (y^{\circ i})_i, (y^{\circ i})_{i>i^\circ}, \gamma^\circ)z = 0$ and $ds_1 = 0$, $ds_b = 0$, $dp_1 = 0$, $d\gamma = 0$, and $ds_a \ne 0$. Since $\gamma^\circ = 1$, s_a and s_b enter Γ_3 symmetrically, so if z' is obtained from z by exchanging ds_a and ds_b, $D\Gamma_3 z' = 0$. This contradicts the nonsingularity of the derivative of Γ_3 with respect to $(s_b; p_2,\ldots,p_J; (y_a^i)_i, (y_b^i)_{i>i^\circ})$, which proves that the derivative of ϕ with respect to s_a is nonsingular at $(s^\circ, p_1^\circ, \gamma^\circ)$. Reducing V if necessary establishes (iii).

To prove (iv), let $1 \le j \le J+1$ and suppose by way of contradiction that there is some $z = (ds_1, ds_a, ds_b, dp, (dy_a^i)_i, (dy_b^i)_i, d\gamma)$ with $ds_1 = 0$, $ds_a = 0$, $ds_{bj} = 0$, $dp_1 \ne 0$, and $D\Gamma_3(s_1^\circ, s_2^\circ, \check{s}_2, p^\circ, (y^{\circ i})_i, (y^{\circ i})_{i>i^\circ}, \gamma^\circ)z = 0$. This implies that the derivative of Γ_3 with respect to $(p, s_b^{)j(}, (y_a^{\circ i})_i, (y_b^{\circ i})_{i>i^\circ})$ is singular, where $s_b^{)j(}$ denotes s_b with the j^{th} coordinate deleted. A direct analoge of Lemmas 3.13 and 3.14 and the argument of the previous paragraph yields a residual set $\mathcal{E}_{j+2} \subset \mathcal{E}_1 \cap \mathcal{E}_2$ such that for each $((u^i)_i, f) \in \mathcal{E}_{j+2}$ there is a subset $S^j \subset S^\circ$ with $S \setminus S^j$ a null set and for each $s^\circ \in S^j$ and each $(p^\circ, (y^{\circ i})_i)$ with $\Gamma((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i) = 0$ the required derivative is nonsingular at $(s_1^\circ, s_2^\circ, s_2^\circ, p^\circ, (y^{\circ i})_i, (y^{\circ i})_{i>i^\circ}, \gamma^\circ)$. The proof is completed by setting $\mathcal{E}^* = \cap_{j=1}^{J+1} \mathcal{E}_{j+2}$, and given $((u^i)_i, f) \in \mathcal{E}^*$, setting $S^* = \cap_{j=1}^{J+1} S^j$.

<u>3.16</u> <u>Remarks</u>: Since the last component of the domain of Γ^* is not compact, $\mathcal{E}^*$ may not be open. However $\mathcal{E}^*$ does contain open subsets. For example, if $((u^i)_i, f) \in \mathcal{E}^*$ is such that <u>every</u> state of information has regular full communication equilibira, that is, $\Gamma((u^i)_i, f; s, \cdot) \pitchfork \{0\}$ for all $s \in S$, then $((u^i)_i, f)$ has an open neighborhood in $\mathcal{E}^*$. Hence $\mathcal{E} \setminus \mathcal{E}^*$ is not dense.

For fixed $(s_1^\circ, p_1^\circ, \gamma^\circ)$, the function ϕ defines s_b locally as a function of s_a. Suppose that for each s_a, the partially informed traders know that the state of information is either (s_1, s_a) or (s_1, s_b). For any $r \in P$, the conventional change of variables formula gives the conditional probability of (s_1, s_b, r) as $\alpha(s_a)f(s_1, s_b; r)/[f(s_1, s_a;r) + \alpha(s_a)f(s_1, s_b; r)]$, where $\alpha(s_a)$ is the absolute value of the Jacobian of $\phi(s_1^\circ, \cdot; p_1^\circ, \gamma^\circ)$ evaluated at s_a. To derive ϕ we assumed that the partially informed traders believed the conditional probability to be $\gamma f(s_1, s_b; r)/[f(s_1, s_a; r) + \gamma f(s_1, s_b; r)]$ (see 3.5 above). Hence these traders' expectations will not generally be rational unless γ varies with s_a. But if γ is a function of s_a, the Jacobian of the function $s_a \to \phi(s_1^\circ, s_a; p_1^\circ, \gamma(s_a))$ involves the derivatives of γ. Thus setting γ equal to the absolute value of this Jacobian yields a first order nonlinear partial differential equation (3.18(2) below) and we need to establish the existence of a local solution.

This equation is not sufficiently well-behaved to allow the usual method of characteristics, but since it constrains the derivative of γ only in a single direction, it is easily converted to an autonomous ordinary differential equation (3.18(3)). At an FCE, since $s_a = s_b$, ϕ is independent of of γ. This implies that along the FCE submanifold γ can be obtained by the implicit function theorem which yields, after some computations, $\gamma \equiv 1$. This means that the ODE is stationary along the FCE submanifold when $\gamma = 1$. For this reason, the first part of the proof of Lemma 3.18 consists of coordinate changes so that the domain of γ is represented as the FCE submanifold and its single complementary dimension. These coordinate changes are more tedious than usual since we have to verify that they preserve the Jacobian. The ODE is unstable in γ, so the solution of the PDE is obtained as the stable manifold of the ODE. The construction of the stable manifold is somewhat lengthy since the stationary set of the ODE is a surface rather than a point, so the usual method of linearization about a point is inapplicable.

3.17 Definitions: Let $((u^i)_i, f) \in \mathcal{E}^*$ and let S^* be given by Lemma 3.15. These will remain fixed from now on. In the proof of Lemma 3.15 S^* is constructed as a subset of the set S° specified in Lemma 3.14. The construction of S° in the proof of Lemma 3.14 establishes the property that for each $s \in S^\circ$ and each $(p, (y^i)_i)$ with $\Gamma((u^i)_i, f; s, p, (y^i)_i) = 0$, the derivative of $\Gamma((u^i)_i, f; \cdot)$ with respect to $(p, (y^i)_i)$ is nonsingular at $(s, p, (y^i)_i)$. We will make use of this property below.

Let $s^\circ \in S^*$, let $(p^\circ, (y^{\circ i})_i)$ satisfy $\Gamma((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i) = 0$, and let V^* be an open neighborhood of (s°, p_1°) with $V^* \subset V_1 \times V_2 \times V_3$, using the terminology of Lemma 3.15. Let $\gamma^*: V^* \to V_4$ be a C^1 function and define the C^1 function $\phi^*: V^* \to S_2$ by $\phi^*(s_1, s_2, p_1) = \phi(s_1, s_2, p_1, \gamma^*(s_1, s_2, p_1))$. Define the function $J(\gamma^*)$: $V^* \to R_+$ by setting $J(\gamma^*)(s_1, s_2, p_1)$ equal to the absolute value of the determinant of the matrix

$$\left(\frac{\partial \phi_j^*(s_1, s_2, p_1)}{\partial s_{2\ell}}\right)_{j,\ell=1}^{J+1}.$$

3.18 Lemma: For each $s^\circ \in S^*$ and each $(p^\circ, (y^{\circ i})_i)$ with $\Gamma((u^i)_i, f; s^\circ, p^\circ, (y^{\circ i})_i) = 0$ there exists a neighborhood V^* of (s°, p_1°) with $V^* \subset V_1 \times V_2 \times V_3$ and a C^1 function $\gamma^*: V^* \to V_4$ such that $\gamma^* = J(\gamma^*)$.

Proof: Let $s^\circ \in S^*$ and let $(p^\circ, (y^{\circ i})_i)$ satisfy $\Gamma((u^i), f; s^\circ, p^\circ, (y^{\circ i})_i) = 0$. The nonsingularity of the derivative of Γ with respect to $(p, (y^i)_i)$ (3.17) implies that, with a possibly smaller choice of V_1 and V_2, there is a smooth function $g: V_1 \times V_2 \to R$ such that $g(s^\circ) = p_1^\circ$ and for each $s \in V_1 \times V_2$, if $p_1 = g(s)$ then there is some $(p_2, \ldots, p_J; (y^i)_i)$ with $\Gamma((u^i)_i, f; s; p_1, p_2, \ldots, p_J; (y^i)_i) = 0$. Let $D_2 g(s^\circ)$ denote the derivative of g with respect to s_2 at s°, and let $T_2 = \{z_2 \in R^{J+1}: D_2 g(s^\circ) z_2 = 0\}$. For each $(s, p_1, \gamma) \in V$ let $D_2\phi(s, p, \gamma)$ denote the derivative of ϕ with respect to s_2 at (s, p_1, γ). Referring to the equations in 3.15 (i) which implicitly define γ we have $D_2\phi(s^\circ, p_1^\circ, \gamma) z_2 = z_2$ for each $\gamma \in V_4$ and each $z_2 \in T_2$. Hence for each $\gamma \in V_4$, the linear map $D\phi(s^\circ, p_1^\circ, \gamma): R^{J+1} \to R^{J+1}$ is the identity on the subspace T_2. For each $i > i^\circ$ define $h^i: V_2 \to R^J$ by $h^i(s_2) = y^i$ where y^i satisfies

$$\int (r-p^\circ) Du^i(p^\circ \omega^i + (r-p^\circ) y^i) f(s_1^\circ, s_2, r) dr = 0,$$

and define $h: V_2 \to R^J$ by $h(s_2) = \Sigma_{i>i^\circ} h^i(s_2)$. Then $h(s_2^\circ) = \Sigma_{i>i^\circ} y^{\circ i}$. Let $z_2^\circ \in R^{J+1}$ with $z_2^\circ \neq 0$ and $Dh(s_2^\circ) z_2^\circ = 0$.

Again referring to 3.15 (i) we have $D_2\phi(s^\circ, p_1^\circ, \gamma)z_2^\circ =$ $-\gamma^{-1}z_2^\circ$ for each $\gamma \in V_4$. By 3.15 (iii) $D_2\phi(s^\circ, p_1^\circ, \gamma)$ is a linear isomorphism, so $z_2^\circ \not\in T_2$. Let $\{z_2^1,\ldots,z_2^J\}$ be a basis for T_2. Then $\{z_2^\circ, z_2^1,\ldots,z_2^J\}$ is a basis for R^{J+1} and in terms of these coordinates $D_2\phi(s^\circ, p_1^\circ, \gamma)$ has the representation

$$\left(\begin{array}{c|c} -\gamma^{-1} & 0 \\ \hline 0 & I_J \end{array}\right)$$

for each $\gamma \in V_4$, where I_J is the $J \times J$ identity matrix. We will generalize this representation below.

Let V_2' denote V_2 with the coordinate system $\{z_2^\circ, z_2^1,\ldots,z_2^J\}$ and $s_2^\circ = 0$. Elements of V_2' will be denoted (α°, β), where $\alpha^\circ \in R$ and $\beta \in R^J$. For any C^1 function $k\colon V_1 \times V_2 \times V_3 \to S_2$, let D_2k denote the derivative of k with respect to s_2. With the coordinate change we have a composite function $k'\colon V_1 \times V_2' \times V_3 \to R^{J+1}$ whose derivative with respect to (α°, β), D_2k', is AD_2kA^{-1}, where A is the matrix representing the coordinate change $s_2 \to (\alpha^\circ, \beta)$. Hence this change of coordinates does not affect the determinant of D_2k. We now introduce an additional coordinate change which also preserves this determinant. Since $z_2^\circ \not\in T_2$, $D_2g(s^\circ)z_2^\circ \neq 0$. Therefore, viewing g as a function on $V_1 \times V_2'$ and reducing V_1 and V_3 if necessary we have a smooth function $\alpha^*\colon V_1 \times V_{22}' \times V_3 \to R$ Where V_{22}' is a neighborhood of 0 in R^J such that

$\alpha^*(s_1^\circ, 0, p_1^\circ) = 0$ and for each $(s_1, \beta, p_1) \in V_1 \times V'_{22} \times V_3$, if $\alpha^\circ = \alpha^*(s_1, \beta, p_1)$ then $g(s_1; \alpha^\circ, \beta) = p_1$. Define the coordinate change $c\colon V_1 \times V'_2 \times V_3 \to V_1 \times R^{J+1} \times V_3$ by $c(s_1; \alpha^\circ, \beta; p_1) = (s_1; \alpha, \beta; p_1)$ where $\alpha = \alpha^\circ - \alpha^*(s_1, \beta, p_1)$. Let V_2^* be a neighborhood of 0 in R^{J+1} with $V_1 \times V_2^* \times V_3 \subset$ $\subset c(V_1 \times V'_2 \times V_3)$. To show that c preserves the relevant determinant, let $k\colon V_1 \times V'_2 \times V_3 \to V'_2$ be a C^1 function and consider the composite function $k^*\colon V_1 \times V_2^* \times V_3 \to R^{J+1}$, defined by $k^*(s_1; \alpha, \beta; p_1) = (\alpha', \beta')$ where $(\alpha', \beta') =$ $= \pi_2(c(s_1, k(\bar{c}^{1}(s_1; \alpha, \beta; p_1)), p_1))$ and π_2 denotes the projection $(s_1; \alpha, \beta; p_1) \to (\alpha, \beta)$. Let D_2k^* denote the derivative of k^* with respect to (α, β). Then for each $(s_1; \alpha, \beta; p_1) \in V_1 \times V_2^* \times V_3$, $D_2k^*(s_1; \alpha, \beta; p_1) = B_1D_2kB_2^{-1}$ where B_1 is the $(J+1) \times (J+1)$ matrix

$$\left(\begin{array}{c|c} 1 & -\frac{\partial\alpha^*}{\partial\beta_1}(s_1; \bar{\alpha}^\circ, \bar{\beta}; p_1), \ldots, -\frac{\partial\alpha^*}{\partial\beta_J}(s_1; \bar{\alpha}^\circ, \bar{\beta}; p_1) \\ \hline 0 & I_J \end{array}\right)$$

with $(\bar{\alpha}^\circ, \beta) = k(c^{-1}(s_1; \alpha, \beta; p_1))$, and B_2 is the matrix

$$\left(\begin{array}{c|c} 1 & -\frac{\partial\alpha^*}{\partial\beta_1}(s_1; \tilde{\alpha}^\circ, \tilde{\beta}^\circ; p_1), \ldots, -\frac{\partial\alpha^*}{\partial\beta_J}(s_1; \tilde{\alpha}^\circ, \tilde{\beta}^\circ; p_1) \\ \hline 0 & I_J \end{array}\right)$$

where $(s_1; \tilde{\alpha}^\circ, \tilde{\beta}^\circ; p_1) = c^{-1}(s_1; \alpha, \beta; p_1)$. Since B_1 and B_2 have determinant 1, the coordinate change c does not affect the determinant of D_2k. Henceforth ϕ will be viewed as a function on $V_1 \times V_2^* \times V_3 \times V_4$ with this coordinate change.

Then for each $(s_1;\ 0,\ \beta;\ p_1,\ \gamma) \in V_1 \times V_2^* \times V_3^* \times V_4$, 3.15 (i) implies

a) $\phi(s_1;\ 0,\ \beta;\ p_1,\ \gamma) = (0,\ \beta).$

Let V^* be a neighborhood of $(s_1^\circ,\ 0,\ p_1^\circ)$ in $V_1 \times V_2^* \times V_3$ and let $\gamma^*\colon\ V^* \to V_4$ be a C^1 function, let ϕ^* be as defined in 3.17 above (in the new coordinates), and let $D_2\phi^*$ denote the derivative of ϕ^* with respect to $(\alpha,\ \beta)$. Let $D_2\gamma^*$ denote the derivative of γ^* with respect to $(\alpha,\ \beta)$ and let $D_\gamma\phi$ denote the derivative of ϕ with respect to γ. Expanding the determinant $|D_2\phi^*|$ and using Cramer's rule yields

(1) $$|D_2\phi^*| = |D_2\phi|(1 + D_2\gamma^*(D_2\phi)^{-1}D_\gamma\phi),$$

so the equation $J(\gamma^*) = \gamma^*$ is equivalent to the partial differential equation

(2) $$\gamma^* = -|D_2\phi|(1 + D_2\gamma^*(D_2\phi)^{-1}D_\gamma\phi)$$

anticipating the negativity of $|D_2\phi^*|$.

Given any $(s_1;\ 0,\ \beta;\ p_1) \in V_1 \times V_2^* \times V_3$ define $h\colon V_2^* \to R^J$ as in the first paragraph above with $(s_1^\circ,\ p_1^\circ)$ replaced by $(s_1,\ p_1)$ and let $b(s_1,\ \beta,\ p_1) \in R^J$ with $Dh(0,\ \beta)(1,\ b) = 0$. Then $D_2\phi(s_1;\ 0,\ \beta;\ p_1,\ \gamma)(1,\ b) = -\gamma^{-1}(1,\ b)$ so by (a) we have

(b) $$D_2\phi(s_1;\ 0,\ \beta;\ p_1,\ \gamma) = \left(\begin{array}{c|c} -\gamma^{-1} & 0 \\ \hline -(1+\gamma^{-1})b & I_J \end{array}\right).$$

Hence

(c) $\qquad |D_2\phi(s_1;\ 0,\ \beta;\ p_1,\ \gamma)| = -\gamma^{-1}$.

Recall from the first paragraph above that $b(s_1^\circ;\ 0,\ p_1^\circ,\ \gamma) = 0$. By (a) $D_\gamma\phi(s_1;\ 0,\ \beta;\ p_1,\ \gamma) = 0$ so it follows from (c) that setting $\gamma^*(s_1;\ 0,\ \beta;\ p_1) = 1$ solves (2) on the subspace $\alpha = 0$.

The partial differential equation (2) suggests the following system of ordinary differential equations on $V_1 \times V_2^* \times V_3 \times V_4$

$$\dot\gamma = \gamma + |D_2\phi|$$

$$(\dot\alpha,\ \dot\beta) = -|D_2\phi|(D_2\phi)^{-1}D_\gamma\phi \tag{3}$$

$$\dot s_1 = 0, \quad \dot p_1 = 0.$$

Note that according to (2), $\dot\gamma$ is the desired derivative of γ^* in the direction $(\dot s_1;\ \dot\alpha,\ \dot\beta;\ \dot p_1)$. For each $(s_1;\ 0,\ \beta;\ p_1) \in$ $\in V_1 \times V_2^* \times V_3$, $(s_1;\ 0,\ \beta;\ p_1,\ 1)$ is a stationary point of (3). For any $(s_1;\ 0,\ \beta;\ p_1,\ \gamma) \in V_1 \times V_2^* \times V_3 \times V_4$, we have from (a) that $\frac{\partial}{\partial\beta_j} D_\gamma\phi = 0$ for all $1 \le j \le J$ and $\frac{\partial}{\partial\gamma} D_\gamma\phi = 0$. Also (c) implies that $\frac{\partial}{\partial\beta_j}|D_2\phi| = 0$ for all $1 \le j \le J$. From (b) we have $\frac{\partial}{\partial\alpha} D_\gamma\phi = \frac{\partial}{\partial\gamma} D_\alpha\phi = \gamma^{-2}(1,\ b(s_1,\ \beta,\ p_1))$, and

$$(D_2\phi)^{-1} = \left(\begin{array}{c|c} -\gamma & 0 \\ \hline -\gamma(1+\gamma^{-1})b & I_J \end{array}\right).$$

Using the Taylor expansion with respect to α at $\alpha = 0$ we can write (3) as

$$
(4)\qquad
\begin{aligned}
\dot{\gamma} &= \gamma - \gamma^{-1} + a(s_1, \beta, p_1, \gamma)\alpha + r_\gamma(s_1; \alpha, \beta; p_1, \gamma) \\
\dot{\alpha} &= -\gamma^{-2}\alpha + r_\alpha(s_1; \alpha, \beta; p_1, \gamma) \\
\dot{\beta} &= -\gamma^{-2}b(s_1, \beta, p_1)\alpha + r_\beta(s_1; \alpha, \beta, p_1, \gamma)
\end{aligned}
$$

where r_γ, r_α, and r_β are remainder terms of the second order in α, and $a(s_1, \beta, p_1, \gamma) = \frac{\partial}{\partial\alpha}|D_2\phi(s_1; 0, \beta; p_1, \gamma)|$.

Let $W = W_1 \times (W_\alpha \times W_\beta) \times W_3$ be a compact neighborhood of $(s_1^\circ, 0, p_1^\circ)$ with $W \subset V_1 \times V_2^* \times V_3$, and let $0 < \delta_4 < \frac{1}{2}$ with $[1 - \delta_4, 1 + \delta_4] \subset V_4$. Then there is some $K > 0$ such that for any $(s_1; \alpha, \beta; p_1, \gamma) \in W \times [1 - \delta_4, 1 + \delta_4]$ and each $\sigma = \alpha, \beta, \gamma$ we have

i) $||r_\sigma(s_1; \alpha, \beta; p_1, \gamma)|| < K|\alpha|^2$; and

ii) $||Dr_\sigma(s_1; \alpha, \beta; p_1, \gamma)|| < K|\alpha|$,

where the norm $||\cdot||$ applied to a vector or matrix denotes the sum of the absolute values of all of the coordinates or entries, respectively. Also, since $b(s_1^\circ, 0, p_1^\circ) = 0$, for any $\varepsilon > 0$ W_1, W_β and W_3 be chosen small enough so that $||b(s_1, \beta, p_1|| < \varepsilon$ on $W_1 \times W_\beta \times W_3$.

It is apparent from (4) that the variable γ is unstable, so it will be convenient to extend (4) to all real values of γ.

For each $\sigma = \alpha, \beta, \gamma,$ let r^*_σ be a C^1 function on $W \times R$ satisfying, for each σ,

iii) $r^*_\sigma(s_1; \alpha, \beta; p_1, \gamma) = r_\sigma(s_1; \alpha, \beta, p_1, \gamma)$ if $|\gamma - 1| \leq \delta_4$;

iv) $||r^*_\sigma(s_1; \alpha, \beta; p_1, \gamma)|| \leq 2 \max \{||r_\sigma(s_1; \alpha, \beta; p_1, \gamma')||: |\gamma' - 1| \leq \delta_4\}$;

v) $||D_\zeta r^*_\sigma(s_1; \alpha, \beta; p_1, \gamma)|| \leq 2 \max \{||D_\zeta r_\sigma(s_1; \alpha, \beta; p_1, \gamma')||: |\gamma' - 1| \leq \delta_4\}$ for each $\zeta = \gamma, \alpha, \beta_1, \ldots, \beta_J$,

for all $(s_1, \alpha, \beta; p_1, \gamma) \in W \times R$. Similarly, let $\alpha^*: W_1 \times W_\beta \times W_3 \times R \to R$ be a C^1 extension of α satisfying

vi) $|\alpha^*(s_1, \beta, p_1, \gamma)| < 2 \max \{|\alpha(s_1, \beta, p_1, \gamma')|: |\gamma' - 1| \leq \delta_4\}$

vii) $||D\alpha^*(s_1, \beta, p_1, \gamma)|| < 2 \max\{||D\alpha^*(s_1, \beta, p_1, \gamma')||: |\gamma' - 1| \leq \delta_4\}$.

Finally, let $f: R \to R$ and $g: R \to R$ be C^1 functions with the properties that

viii) $f(\gamma) = \gamma - \gamma^{-1}$ if $|\gamma - 1| \leq \delta_4$, and for each $\gamma \in R$, $Df(\gamma) \geq 3/2$; and

ix) $g(\gamma) = \gamma^{-2}$ if $|\gamma - 1| \leq \delta_4$, and for each $\gamma \in R$, $1/3 \leq g(\gamma) \leq 9$ and $|Dg(\gamma)| \leq 16$.

These extensions yield the following system of differential equations on $W \times R$, which agree with (4) on $W \times [1-\delta_4, 1+\delta_4]$

$$\dot{\gamma} = f(\gamma) + \alpha a^*(s_1, \beta, p_1, \gamma) + r^*_\gamma(s_1; \alpha, \beta; p_1, \gamma)$$

$$(5) \quad \dot{\alpha} = -\alpha g(\gamma) + r^*_\alpha(s_1; \alpha, \beta; p_1, \gamma)$$

$$\dot{\beta} = -\alpha g(\gamma) b(s_1, \beta, p_1) + r^*_\beta(s_1; \alpha, \beta; p_1, \gamma).$$

Choose $0 < \delta < 1$ so that if $||s_1 - s_1^\circ|| < \delta$, $|\alpha| < \delta$, $||\beta|| < \delta$ $|p_1 - p_1^\circ| < \delta$, and $|\gamma - 1| < \delta$ then

x) $||b(s_1, \beta, p_1)|| < 1/54$; and

xi) $||r_\sigma(s_1; \alpha, \beta; p_1, \gamma)|| < |\alpha|/6J$ for each $\sigma = \alpha, \beta, \gamma$.

and let $\delta_4 < \delta$. Let $x = (\bar{s}_1, \bar{\alpha}, \bar{\beta}; \bar{p}_1, \bar{\gamma})$ with

$||\bar{s}_1 - s_1^\circ|| < \delta$, $|\bar{\alpha}| < \delta$ $||\bar{\beta}|| < \delta$, $|\bar{p}_1 - p_1^\circ| < \delta$, $\bar{\gamma} \varepsilon R$.

Then there is a unique local solution $\gamma(\cdot, x)$, $\alpha(\cdot, x)$, $\beta(\cdot, x)$ of (5) with the initial condition x. For each t in the domain of this solution, (ix) and (xi) imply that $\frac{d}{dt}|\alpha(t, x)| \leq -|\alpha|/6$, so

$$\text{(xii)} \qquad |\alpha(t, x)| \leq |\bar{\alpha}|\bar{e}^{t/6} .$$

By (ix) and (x), $\frac{d}{dt}||\beta(t, x)|| \leq 1/3|\alpha(t, x)| \leq 1/3|\bar{\alpha}|\bar{e}^{t/2}$ so $||\beta(t, x)|| \leq ||\bar{\beta}|| + 2/3|\bar{\alpha}|$. Hence if $||\bar{\beta}|| < \delta/2$ and $|\bar{\alpha}| < \delta/2$ then $||\beta(t, x)|| < \delta$ for all t in the domain of the solution. Let $W^* = \{(s_1, \alpha, \beta, p_1) \varepsilon W: ||s_1 - s_1^\circ|| < \delta$, $|\alpha| < \delta/2$, $||\beta|| < \delta/2$, and $||p_1 - p_1^\circ|| < \delta\}$. Then for each $x \varepsilon W^* \times R$, the solution $\gamma(\cdot, x)$, $\alpha(\cdot, x)$ and $\beta(\cdot, x)$ is defined and unique for all $t \geq 0$.

Let $a^\circ = 2 \max\{|a(s_1, \beta, p_1, \gamma')|: |\gamma' - 1| \leq \delta_4\}$. For each $x \in W^* \times R$ and each t, (vi), (vii), and (xi) imply that if $\gamma(t, x) > 2(a^\circ + 1)$, $\dot{\gamma}(t, x) > 0$ and if $\gamma(t, x) < -2(a^\circ + 1)$ then $\dot{\gamma}(t, x) < 0$. Hence for each $(\bar{s}_1; \bar{\alpha}, \bar{\beta}; \bar{p}_1) \in W^*$ there is some $\bar{\gamma}_t$ with $|\bar{\gamma}_t| \leq 2(a^\circ+1)$ and $\gamma(t; \bar{s}_1; \bar{\alpha}, \bar{\beta}; \bar{p}_1, \bar{\gamma}_t) = 1$. Let $\gamma^*(\bar{s}_1; \bar{\alpha}, \bar{\beta}; \bar{p}_1)$ be a cluster point of the bounded sequence $\{\bar{\gamma}_n\}_{n=1}^{\infty}$. It follows easily that if $\gamma(t)$ is a solution of (5) with the initial condition $(s_1; \alpha, \beta; p_1, \gamma^*(s_1; \alpha, \beta; p_1))$ then $\gamma(t) \to 1$. To show that γ^* is a C^1 function we need to study the derivatives of solutions of (5).

For notational simplicity, we will prove only that γ^* is C^1 in α and β, using an argument which extends directly to s_1 and p_1. Let $x = (\bar{s}_1; \bar{\alpha}, \bar{\beta}; \bar{p}_1, \bar{\gamma}) \in W^* \times R$ and for each $t \geq 0$, let $F(t)$ denote the derivative of the right hand side of (5) with respect to (γ, α, β) at $(\bar{s}_1; \alpha(t,x), \beta(t,x); \bar{p}_1, \gamma(t,x))$. For any $d = (d_\gamma, d_\alpha, d_\beta) \in R^{J+2}$, the derivatives of $\alpha(t, \cdot)$, $\beta(t, \cdot)$ and $\gamma(t, \cdot)$ at x in the direction d are obtained as the solution to the differential equation

$$(6) \qquad \begin{pmatrix} \dot{\psi}_\gamma \\ \dot{\psi}_\alpha \\ \dot{\psi}_\beta \end{pmatrix} = \begin{bmatrix} \\ F(t) \\ \\ \end{bmatrix} \begin{pmatrix} \psi_\gamma \\ \psi_\alpha \\ \psi_\beta \end{pmatrix} .$$

with the initial condition $\psi(0) = d$ [Theorem 3.1, p. 95].
By (i-xii) there is some $M > 0$ such that

(7) $$\frac{d}{dt}|\psi_\gamma(t)| \geq (3/2 - \delta M e^{-t/6})|\psi_\gamma| - a^\circ|\psi_\alpha| - \delta e^{-t/6} M(|\psi_\alpha| + ||\psi_\beta||);$$

(8) $$\frac{d}{dt}|\psi_\alpha(t)| \leq (\delta M e^{-t/6} - 1/3)|\psi_\alpha| + \delta M e^{-t/6}(||\psi_\beta|| + |\psi_\gamma|),$$ and

(9) $$\frac{d}{dt}||\psi_\beta(t)|| \leq (1/6 + \delta M e^{-t/6})|\psi_\alpha| + \delta M e^{-t/6}(||\psi_\beta|| + |\psi_\gamma|).$$

With $\delta < (12M)^{-1}$, (8) and (9) imply

(10) $$||\psi_\alpha(t), \psi_\beta(t)|| \leq e(||d_\alpha, d_\beta|| + 2\delta M \int_0^t e^{-5/6}|\psi_\gamma(s)|ds).$$

First suppose that ψ_γ is bounded. Then by (10), ψ_α and ψ_β are bounded, so by (8), $\psi_\alpha(t) \to 0$. Then by (7), since ψ_γ is bounded, we must have $\psi_\gamma(t) \to 0$. Thus

(I) if ψ_γ is bounded then $\psi_\gamma \to 0$.

Second, let $\gamma < [12M(a^\circ+1)e^{(12M+1)}]^{-1}$. Then for each t and each $\sigma = \alpha, \beta$, (10) implies

(11) $$||\psi_\alpha(t), \psi_\beta(t)|| \leq e||d_\alpha, d_\beta|| + (2a^\circ + 2)^{-1}\max\{|\psi_\gamma(s)|: s \leq t\}.$$

Let $A(d_\alpha, d_\beta) = 4(a^\circ + 1)e||d_\alpha, d_\beta|| + 1$. Suppose that for some t, $|\psi_\gamma(t)| \geq A$, and let $t^* = \min\{t: |\psi_\gamma(t)| \geq A\}$. Then for each $t \geq t^*$, (7) and (11) imply that $\frac{d}{dt}|\psi_\gamma(t)| > A/4$, so $|\psi_\gamma(t)| \to \infty$. In summary,

(II) if $|\psi_\gamma(t)| > A(d_\alpha, d_\beta)$ for any t, $|\psi_\gamma| \to \infty$.

In particular, if $d = (1, 0, 0) = (A(0, 0), 0, 0)$, $|\psi_\gamma| \to \infty$.
Therefore

(12) $D_x\gamma(t, x)(1, 0) \to \infty$.

For each t, each $x = (s_1^\circ; \alpha, \beta; p_1^\circ, \gamma) \in W^* \times R$, and each $(d_\alpha, d_\beta) \in R^{J+1}$, let $d_\gamma^t(x; d_\alpha, d_\beta) \in R$ such that if $\psi(t)$ is a solution of (6) with the initial condition $\psi(0) = (d_\gamma^t(x; d_\alpha, d_\beta), d_\alpha, d_\beta)$ then $\psi_\gamma(t) = 0$. Then for each t, $|d_\gamma^t(x; d_\alpha, d_\beta)| < A(d_\alpha, d_\beta)$, so let $d^*(x; d_\alpha, d_\beta)$ be a cluster point of the sequence $\{d_\gamma^n(x; d_\alpha, d_\beta)\}_{n=1}^\infty$. By (12) and the linearity of the derivative, $d^*(x; d_\alpha, d_\beta)$ is the unique cluster point. It is immediate that $d^*(x; \cdot)$ is linear, and it follows from (12) and (I-II) that $d^*(\cdot; d_\alpha, d_\beta)$ is continuous.

The mean value theorem and (12) imply that for each $(s_1^\circ; \alpha, \beta; p_1^\circ) \in W^*$, $\gamma^*(s_1^\circ; \alpha, \beta; p_1^\circ)$ is the unique cluster point of the sequence $\{\bar{\gamma}_n\}$, and that γ^* is continuous. To show that $d^*([s_1^\circ; \alpha, \beta; p_1^\circ, \gamma^*(s_1^\circ; \alpha, \beta; p_1^\circ)], d_\alpha, d_\beta)$ is the derivative of $\gamma^*(s_1^\circ; \alpha, \beta; p_1^\circ)$ in the direction (d_α, d_β), let $\{\lambda_m\}_{m=1}^\infty$ be a sequence of small positive numbers with $\lambda_m \to 0$ and for each m let $\gamma_m^* = \gamma^*(s_1^\circ; \alpha + \lambda_m d_\alpha, \beta + \lambda_m d_\beta; p_1^\circ)$, and let $x_m = (s_1^\circ; \alpha + \lambda_m d_\alpha, \beta + \lambda_m d_\beta; p_1^\circ, \gamma_m^*)$. Let

$\gamma_0^* = \gamma^*(s_1^\circ; \alpha, \beta; p_1^\circ)$ and let $x^\circ = (s_1^\circ; \alpha, \beta; p_1^\circ, \gamma_0^*)$.
Since $\lim_{t\to\infty} \gamma(t, x_m) = 1 = \lim_{t\to\infty} \gamma(t, x^\circ)$, the sequence $\{(\gamma(n, x_m) - \gamma(n, x^\circ))/\lambda_m\}_{n=1}^\infty$ converges to zero. Then by the mean value theorem and (I-II), for each m there is some x_m'

in the interval $[x^\circ, x_m]$ such that $(\gamma_m^* - \gamma_0^*)/\lambda_m = d^*(x_m'; d_\alpha, d_\beta)$. Then by the continuity of $d^*(\cdot; d_\alpha, d_\beta)$, $\lim (\gamma_m^* - \gamma_0^*)/\lambda_m = d^*(x^\circ; d_\alpha, d_\beta)$. That γ^* is C^1 in (s_1, p_1) as well follows from the same argument, extended by adding the equations $\dot{s}_1 = 0$ and $\dot{p}_2 = 0$ to (5) and in (6), differentiating the augmented system with respect to (s_1, p_1) also.

It is immediate that for each $(s_1; 0, \beta; p_1) \in W^*$, $\gamma^*(s_1; 0, \beta; p_1) = 1$. Since γ^* is C^1, let V^* be a neighborhood of $(s_1^\circ, 0, p_1^\circ)$ such that $\gamma^*(V^*) \subset (1-\delta_4, 1+\delta_4)$. Finally, (3) and the definition of γ^* imply that γ^* satisfies (2) on V^*, which proves the Lemma.

3.19 Remarks: We can now use γ^* and ϕ^* to construct a local REE. By way of interpretation we have thus far assumed that partially informed traders can identify the state of information up to the pair s^a, s^b, where $s^a = (s_1, s_a)$ and $s^b = (s_1, s_b)$. We now generate this information through prices by making p_1 a 1-1 function of s^a. Then $p_1(s^a)$ is an equilibrium price in s^a and s^b, where $s^b = (s_1, \phi^*(s_1, s_a, p_1(s_a)))$, and the prices of the other assets, which are also the same in both states, are given by Lemma 3.15 (which defines ϕ). The single variable p_1 is obtained as a 1-1 function of the vector s^a by splicing the binary expansions of the components of s^a and viewing the resulting sequence as the ternary expansion of p_1.

Of course we also have to verify that the weight $\gamma^*(s^a, p_1(s^a))$ determines the correct conditional probability. If we let g_2 denote the function $s^a \to s^b$ and let μ denote Lebesgue measure on S, this amounts to showing that γ^* is the derivative of the measure $\mu \circ g_2$ with respect to μ(3.20(vi) below). To avoid the double counting of states, it is also necessary to show that g_2 is 1-1 and that its domain and range are disjoint. These and some additional properties are established in Lemma 3.20.

3.20 <u>Lemma</u>: For each $s^\circ \in S^*$, there exist integers $n(s^\circ)$ and $n'(s^\circ)$ such that for any open cube $K \subset S^*$ centered at s° with side $\varepsilon < 1/n(s^\circ)$ there is a cube K' centered at s° with side $\varepsilon/n'(s^\circ)$ such that for any open and dense set $G^\circ \subset R$ there exists a Borel measurable function $g = (g_1, g_2)\colon\ K' \to G^\circ \times K$ with the following properties

i) for each $s^a \in K$, if $p_1 = g_1(s^a)$ and $s^b = g_2(s^a)$, then $s_1^b = s_1^a$, $s_{2j}^b \neq s_{2j}^a$ for each $1 \leq j \leq J + 1$, and $s_2^b = \phi^*(s^a, p_1)$;

ii) g_1 and g_2 are 1-1 and have Borel measurable inverses;

iii) $g_1(K')$ is contained in a closed nowhere dense set of Lebesgue measure zero;

iv) $\text{cl}\ g_2(K') \subset K \backslash K'$, where cl denotes closure;

v) $\mu(g_2(K')) = \mu(\text{cl}\ g_2(K'))$

where μ denotes Lebesgue measure on S; and

vi) for each Borel measurable set $E \subset K'$,

$$\mu(g_2(E)) = \int_E \gamma^*(s, g_1(s))d\mu.$$

<u>Proof</u>: Let $s^\circ \in S^*$, and let p° be a full communication equilibrium price at s°. Let G be an open neighborhood of s° and let G_1 be an open interval in R about p_1° with $G \times G_1 \subset V^*$ (Lemma 3.18). By Lemma 3.15 there is a positive integer n° and positive numbers α_j and β_j, $1 \leq j \leq J+1$ such that for any cube $K^\circ = K \times K_1$ centered at (s°, p_1°) with side $\varepsilon < 1/n^\circ$, for each $1 \leq j \leq J+1$ and each $(s, p_1) \in K^\circ$,

a) $\alpha_j |p_1 - p_1^\circ| < |\phi_j^*(s^\circ, p_1) - \phi_j^*(s^\circ, p_1^\circ)|$;

b) $|\phi_j^*(s, p_1) - \phi_j^*(s^\circ, p_1)| < \beta_j \Sigma_{\ell=1}^N |s_\ell - s_\ell^\circ|$; and

c) $|\phi_j^*(s, p_1) - \phi_j^*(s^\circ, p_1^\circ)| < \beta_j(\Sigma_{\ell=1}^N |s_\ell - s_\ell^\circ| + |p_1 - p_1^\circ|)$,

where $N = \dim S$. Let $\alpha = \min\{\alpha_j: 1 \leq j \leq J+1\}$ and $\beta = \max\{\beta_j + 2: 1 \leq j \leq J+1\}$. Let $\delta_1 = \varepsilon/2(\alpha+\beta)$. Then $\beta\delta_1 = \varepsilon/2 - \alpha\delta_1$. Now let $\delta_2 = \alpha\delta_1/N\beta = \varepsilon\alpha/2N\beta(\alpha+\beta)$. Let $n' > 2(\alpha+\beta)N\beta/\alpha$, which is independent of ε. Let K' be a compact cube in S^*, centered at s°, with side ε/n'. Note that $\varepsilon/n' < \delta_2$. Let $p_1' = p_1^\circ + \delta_1$. Then for any $s \in K'$ if $s_{2_j}^* = \phi_j^*(s, p_1')$, then using (c),

d) $|s^*_{2_j} - s^\circ_{2_j}| < \beta_j[(N\delta_2/2) + \delta_1] < \beta\delta_1 + \beta N(\delta_2/2)$

$$= (\varepsilon/2) - \alpha\delta_1 + \alpha\delta_1/2 = (\varepsilon/2) - (\alpha\delta_1/2) < \varepsilon/2$$

and using (a) and (b),

e) $|s^*_{2_j} - s_{2_j}| \geq |s^*_{2_j} - s^\circ_{2_j}| - |s_{2_j} - s^\circ_{2_j}| > (\alpha\delta_1 - \beta N(\delta_2/2)) - \delta_2/2$

$$= (\alpha\delta_1 - \delta_2)/2 = (\alpha\delta_1/2)(1-1/N\beta) > 0.$$

Then (d) and (e) ensure the existence of an open interval $G'_1 \subset K_1 \cap G_1$ with $p'_1 \in G'_1$ such that for any $p_1 \in G'_1$ and any $s \in K'$,

f) $(s_1, \phi^*(s, p_1)) \in K \backslash K'$; and

g) $s_{2_j} \neq \phi^*_j(s, p_1)$ for all $1 \leq j \leq J+1$

For each $(s, p_1) \in K^\circ$ let $d(s, p_1) = \frac{\partial}{\partial p_1} \phi^*(s, p_1)$, and let $D(s, p_1)$ denote the derivative of ϕ^* with respect to s_2 at (s, p_1). Let $z^\circ = [D(s^\circ, p^\circ_1)]^{-1} d(s^\circ, p^\circ_1)$. Since $d(s^\circ, p^\circ_1) \neq 0$, $z^\circ \neq 0$. Let A be a $(J+1) \times (J+1)$ matrix with $|\det A| = 1$ and $Az^\circ = \lambda(1,\ldots,1)$ for some $\lambda > 0$. For each $h \in R^{J+1}$, let $C(h)$ denote the closed convex hull of the set $\{d(s, p_1) + D(s, p_1)h\colon (s, p_1) K^\circ\}$, and let $H = \{h \in R^{J+1}\colon 0 \in C(h)\}$. By increasing n°, if necessary, we can assume that $-A(H) \subset R^{J+1}_{++}$ and for each $(s, p_1) \in K^\circ$, $\phi^*(s_1, \cdot, p_1)$ is 1-1, since ϕ^* is C^1.

We now construct g_1 and g_2. Write $K' = K_1' \times K_2' \subset S_1 \times S_2$ and let $K_2'' = A(K_2')$. Translating $K_1' \times K_2''$ to the origin in R^N, we can associate with each $s \in K_1' \times K_2''$ an N-tuple of binary expansions $(\{s_\ell^k\}_{k=1}^\infty)_{\ell=1}^N$. That is, $\{s_\ell^k\}$ is a sequence of 0's and 1's with $s_\ell = \Sigma_{k=1}^\infty s_\ell^k 2^{-k}$, with the usual convention to make this sequence unique. Given an open and dense set $G^\circ \subset R$, let $p_1^* \in G_1' \cap G^\circ$ and let $m > 0$ such that $[p_1^* - 3^{-m}, p_1^* + 3^{-m}] \subset G_1' \cap G^\circ$. Now let g_1': $K_1' \times K_2'' \to G^\circ$ be defined by $g_1'(s) = p_1^* + \Sigma_{k=1}^\infty a_k 3^{-(k+m)}$ where $\{a_1, a_2, \ldots\} = \{s_1^1, s_2^1, \ldots, s_N^1; s_1^2, \ldots\}$. Then g_1' is clearly 1-1 and Borel measurable. Let g_2': $K_1' \times K_2'' \to K$ be defined by $g_2'(s_1, s_2) = [s_1, \phi^*(s_1, A^{-1}s_2; g_1'(s_1, s_2))]$. Then g_2' is Borel measurable, and we need to show that g_2' is 1-1.

Let $s, s' \in K_1' \times K_2''$ with $g_2'(s) = g_2'(s')$. Then $s_1 = s_1'$ and, with $p_1 = g_1'(s)$ and $p_1' = g_1'(s')$, we have $\phi^*(s_1, A^{-1}s_2; p_1) = \phi^*(s_1, A^{-1}s_2'; p_1')$. Let $\sigma = A^{-1}s_2$ and $\sigma' = A^{-1}s_2'$. If $p_1 = p_1'$, then since $\phi^*(s_1, \cdot; p_1)$ is 1-1 on K', $\sigma = \sigma'$ so $s_2 = s_2'$. Suppose by way of contradiction that $p_1 > p_1'$. Then integrating the derivative of the function $t \to \phi^*(s_1, t\sigma + (1 - t)\sigma'; tp_1 + (1 - t)p_1')$ from 0 to 1 shows that $(p_1 - p_1')^{-1}(\sigma - \sigma') \in H$ so $s_2' - s_2 \in R_{++}^J$. By the construction of g_1', this implies $p_1' > p_1$, and this contradiction proves that g_2' is 1-1. Now define $g_1: K' \to G_1'$ by $g_1(s_1, s_2) = g_1'(s_1, As_2)$ and define $g_2: K' \to K$ by $g_2(s_1, s_2) = g_2'(s_1, As_2)$. Then g_1 and g_2 are clearly Borel measurable. It is straightforward to show that g_1^{-1} is Borel measurable, and since ϕ^* is an open map, g_2^{-1} is also measurable. This proves (i) and (ii), and (iii) and (iv) are clear.

We now show that for any Borel set $E \subset K_1' \times K_2''$,

(1) $$\mu(g_2'(E)) \leq \int_E \gamma^*(s_1; A^{-1}s_2; g_1'(s_1, s_2))d\mu$$

where μ is Lebesgue measure on R^N. It suffices to prove (1) for the case in which E is a "dyadic cube", that is, $E = \Pi_{\ell=1}^N [\Sigma_{k=1}^m s_\ell^k 2^{-k}, \Sigma_{k=1}^m s_\ell^k 2^{-k} + 2^{-(m+1)}]$, since any open set is a countable disjoint union of such cubes. Let $s^* \varepsilon E$ and let $p_1^* = g_1'(s^*)$. Define $\phi': E \to K$ by $\phi'(s_1, s_2) = (s_1, \phi^*(s_1^*, A^{-1}s_2; p_1^*))$. Let Φ denote the derivative of ϕ' at s^*, which is nonsingular, and let $\phi'': (s_1, s_2) \to \Phi^{-1}\phi'(s_1, s_2)$. Then the derivative of ϕ'' at s^* is the identity matrix. By the definition of the derivative, for any $\delta > 0$ we can choose m sufficiently large so that $\phi''(E)$ is contained in the cube

$\{(s_1 \pm \delta 2^{-(m+1)}, \ldots, s_N \pm \delta 2^{-(m+1)}): s \varepsilon E\}$. Hence $\mu(\phi''(E)) \leq \mu(E) + \delta 2^{-Nm}$. The set $g_1'(E)$ is contained in an an interval I of length $3^{-(Nm+1)}$. Since ϕ^* is C^1, there is some $\alpha^* > 0$ with

$$||\Phi^{-1}[\phi'(s_1, s_2) - (s_1, \phi^*(s_1, A^{-1}s_2; p_1))]|| < \alpha^* 3^{-(Nm+1)}$$

for all $p_1 \varepsilon I$. Hence $\Phi^{-1}(g_2'(E))$ is contained in the cube

$$\{(s_1 \pm \delta 2^{-(m+1)}, \ldots, s_N \pm \delta 2^{-(m+1)}) + (\pm\alpha^* 3^{-(Nm+1)}, \ldots, \pm \alpha^* 3^{-(Nm+1)}):$$

$s \varepsilon E\}$. Hence $\mu(\Phi^{-1}(g_2'(E))) < \mu(E) + 2^{-Nm}(\delta + \alpha^*(2/3)^{Nm})$, so $\mu(g_2'(E)) < |\det\Phi|[\mu(E) + 2^{-Nm}(\delta + \alpha^*(2/3)^{Nm})$, so using the fact that $|\det A| = 1$,

$$\mu(g_2'(E)) < \gamma^*(s_1^*, A^{-1}s_2^*; \; p_1^*)[\mu(E) + 2^{-Nm}(\delta + \alpha^*(2/3)^{Nm})].$$

Any dyadic cube of length 2^{-k} can be covered by $2^{N(m+1-k)}$ cubes of length $2^{-(m+1)}$ for any $m > k$, so the usual approximation by simple functions proves (1).

We now establish

(2) $\mu(\text{cl } g_2'(K_1' \times K_2'')) = \mu(g_2'(K_1' \times K_2''))$.

First note that g_1', and thus g_2', is continuous on the complement of the set $Q = \{s \varepsilon K_1' \times K_2''$: for some ℓ, s_ℓ is a dyadic rational$\}$. For each ℓ, let $\{q_\ell^n\}_{n=1}^\infty$ be an enumeration of the dyadic rationals in the ℓ^{th} edge of $K_1' \times K_2''$. Let B be a closed set with Lebesgue measure zero containing $g_1'(K_1' \times K_2'')$.

Then $\text{cl}\ g_2'(K_1' \times K_2'') \subset g_2'(K_1' \times K_2'') \cup \{(s_1, \phi^*(s_1, A^{-1}s_2;\ p_1)):$ $p_1 \in B$ and $s_\ell = q_\ell^n\}\}$. With s_ℓ fixed at q_ℓ^n, the set $\{(s_1, A^{-1}s_2;\ p_1):\ s_\ell = q_\ell^n$ and $p_1 \in B\}$ is a subset of R^N with Lebesgue measure zero. Hence (2) follows from the fact that ϕ^* is C^1.

We now prove that for any dyadic cube $E \subset K_1' \times K_2''$,

(3) $\quad \mu(g_2'(E)) \geq \int_E \gamma^*(s_1, A^{-1}s_2;\ g_1'(s_1, s_2))d\mu.$

If E has side 2^{-k}, for each $m \geq k$ let $\{E_{m_n}\}_{n=1}^{2^{N(m-k)}}$ be a cover of E by dyadic cubes with side 2^{-m} and edges assigned so that the cover is disjoint. For each m_n let $s^{m_n} \in \text{int}\ E_{m_n}$ and let $p_1^{m_n} = g_1'(s^{m_n})$. Define $g_2^{m_n}$: $E_{m_n} \to K$ by $g_2^{m_n}(s) = (s_1, \phi^*(s_1, A^{-1}s_2;\ p_1^{m_n}))$, and define g_2^m: $E \to K$ by $g_2^m(s) = g_2^{m_n}(s)$ if $s \in E_{m_n}$. Then the sequence $\{g_2^m\}$ converges pointwise to g_2' except on Q. Moreover, if $\{\bar{s}^{m_q}\}_{q=1}^{\infty}$ is a sequence in $K_1' \times K_2''$ converging to some s, and $\{s'^{m_q}\}$ is a sequence in K converging to some s', with $s'^{m_q} = g_2^{m_q}(\bar{s}^{m_q})$ for each q, then $s' \in \text{cl}\ g_2'(E)$. For each m, let c^m: $E \to R_{++}$ be defined by $c^m(s) = |\det Dg_2^{m_n}(s)|$ if $s \in E^{m_n}$. Then for each m, $\mu(g_2^m(E)) = \int_E c^m d\mu$, and

$$\lim_{m\to\infty} \int_E c^m d\mu = \int_E \gamma^*(s_1, A^{-1}s_2; g_1'(s))d\mu .$$

Let r denote the latter integral. For each m. let $F^m = \text{cl } g_2^m(E)$. Since $g_2^{m_n}$ is C^1 for each n, $\mu(F^m) = \mu(g_2^m(E))$ for each m. Let $F = \cap_{q\geq 1}(\cup_{m\geq q} F^m)$. Then $\mu(F) \geq r$. If $s' \in F$, there is a sequence $\{s'^{m_q}\}_{q=1}^{\infty}$ converging to s' with $s'^{m_q} \in g_2^{m_q}(E)$ for each q. Taking a subsequence if necessary, there is a sequence $\{\bar{s}^{m_q}\}$ in E with $\bar{s}^{m_q} \to s$ for some $s \in E$, and $s'^{m_q} = g_2^{m_q}(\bar{s}^{m_q})$ for each q. Hence $s' \in \text{cl } g_2'(E)$, so $F \subset \text{cl } g_2'(E)$. Therefore $\mu(g_2'(E)) = \mu(\text{cl } g_2'(E)) \geq \mu(F) \geq r$, which proves (3).

Since $\text{cl } g_2(K') = \text{cl } g_2'(K_1' \times K_2'')$, (2) implies (v). Since the determinant of the linear map $(s_1, s_2) \to (s_1, As_2)$ has absolute value 1, (1) and (3) imply (vi), which completes the proof.

3.21 Remarks: The final stage of the proof consists of piecing the local REE's together into a global REE. The two requirements are that the local REE's not intersect, either in S or in P, and that they cover a set of measure one in S. The nonintersection requirement forces the local REE's to be chosen inductively, but the construction is relatively straightforward.

3.22 Proof of the Theorem: Let $\varepsilon > 0$ and for any $m > 1$ let C_1 be a compact subset of S^* with $\mu(S^*\setminus C_1) < 1/4m$. Since C_1 is compact, we can define $n = \max\{n(s): s \in C_1\}$ and let $n' = \max \{n'(s): s \in C_1\}$ (Lemma 3.20). For each $s \in C_1$ let $K(s) \subset S^*$ be a cube centered at s with side less than ε/Nn. Since C_1 is compact, there is a finite subcover of the cover $\{K(s): s \in C_1\}$. By refining this subcover we obtain a finite collection of compact rectangles $\{K_\ell\}_{\ell=1}^{k_1}$ which cover C_1, have disjoint interiors, and each of which has its largest side less than twice its smallest side. For each ℓ, let s_ℓ and δ_ℓ denote respectively the center and smallest side of K_ℓ. Note that $\delta_\ell < \varepsilon/n$, and let A_ℓ be a cube centered at s_ℓ with side δ_ℓ/n'. Then for each ℓ, $\mu(A_\ell) > (2n')^{-N}\mu(K_\ell)$. Let $G_1 = R$, and let $g_1^1: A_1 \to G_1$ and $g_2^1: A_1 \to K_1$ be given by Lemma 3.20. Let $B_1 = \text{cl}\, g_2^1(A_1)$ and let $P_1 = \text{cl}\, g_1^1(A_1)$. For each $\ell > 1$, inductively define $G_\ell = G_{\ell-1}\setminus P_{\ell-1}$, $g_1^\ell: A_\ell \to G_\ell$, $g_2^\ell: A_\ell \to K_\ell$, $B_\ell = \text{cl} g_2^\ell(A_\ell)$ and $P_\ell = \text{cl}\, g_1^\ell(A_\ell)$. Then $\mu(\cup_{\ell=1}^{k_1}(A_\ell \cup B_\ell)) =$
$= \mu(\cup_{\ell=1}^{k_1}(A_\ell \cup g_2^\ell(A_\ell))) > \Sigma_{\ell=1}^{k_1} \mu(A_\ell) > (2n')^{-N}(1 - 1/4m)$,
where the equation follows from Lemma 3.20 (iv). Now let C_2 be a compact subset of $C_1\setminus \cup_{\ell=1}^{k_1}(A_\ell \cup B_\ell)$ with $\mu((C_1\setminus\cup_{\ell=1}^{k_1}(A_\ell \cup B_\ell))\setminus C_2) < 1/8m$. Since $C_2 \subset C_1$, the number n' applies to C_2 as well, so continuing the induction from

$G_{k_1+1} = G_{k_1} \setminus P_{k_1}$ we obtain another finite collection $\{A_\ell, B_\ell, P_\ell\}_{\ell=k_1+1}^{k_2}$ disjoint from the previous collection with $\mu[\cup_{\ell=k_1+1}^{k_2}(A_\ell \cup B_\ell)] > (2n')^{-N}(\mu[S^* \setminus \cup_{\ell=1}^{k_1}(A_\ell \cup B_\ell)] - 1/4m - 1/8m)$.

Continuing in this fashion yields a finite collection $\{A_\ell, B_\ell, P_\ell\}_{\ell=1}^{k}$ with $\mu[\cup_{\ell=1}^{k}(A_\ell \cup B_\ell)] \geq 1 - 1/m$. Now let C_1' be a compact subset of $S^* \setminus \cup_{\ell=1}^{k}(A_\ell \cup B_\ell)$ with $\mu[(S^* \setminus \cup_\ell (A_\ell \cup B_\ell)) \setminus C_1'] < 1/4m^2$. Choosing a possibly larger value of n' we can repeat the above treatment of C_1, of course beginning with $G_{k+1} = G_k \setminus P_k$, to obtain $\mu(\cup_{\ell=1}^{k'}(A_\ell \cup B_\ell))) \geq$ $\geq \mu(S^*) - 1/m^2$. Continuing indefinitely in this fashion yields a countable collection $\{A_\ell, B_\ell, P_\ell;\ g_1^\ell, g_2^\ell\}$ with the properties:

i) $\mu(\cup_{\ell=1}^{\infty}(A_\ell \cup B_\ell)) = 1$;

ii) for each ℓ, $B_\ell = \text{cl}\ g_2^\ell(A_\ell)$, $P_\ell = \text{cl}\ g_1^\ell(A_\ell)$, and $A_\ell \cap B_\ell = \phi$;

iii) for each ℓ, if $s_a \in A_\ell$ and $s_b \in B_\ell$ then $||s_a - s_b|| < \varepsilon$; and

iv) if $\ell \neq \ell'$, then $A_\ell \cap A_{\ell'} = \emptyset$, $B_\ell \cap B_{\ell'} = \emptyset$, $A_\ell \cap B_{\ell'} = \emptyset$, and $P_\ell \cap P_{\ell'} = \emptyset$.

Let $S^{**} = \cup_{\ell=1}^{\infty}(A_\ell \cup g_2^\ell(A_\ell))$ and let $p_1^\circ \in R \setminus \cup_\ell P_\ell$. Define $g_1^*: S \to P_1$ by

$$g_1^*(s) = \begin{cases} g_1^\ell(s) & \text{if } s \in A_\ell \ ; \\ g_1^\ell((g_2^\ell)^{-1}(s)) & \text{if } s \in g_2^\ell(A_\ell); \text{ and} \\ p_1^\circ & \text{if } s \in S \backslash S^{**} . \end{cases}$$

It follows directly from Lemma 3.20 and the definition of ϕ^* that $\pi = g_1^*$ is an REE which is revealing within ε.

4. Concluding Remarks

The result established here essentially completes the theory of the generic existence of rational expectations equilibrium, but several details remain. First, it should be determined whether $\mathcal{E}^*$ can be chosen to be open as well as residual. It was mentioned in 3.16 above that $\mathcal{E}^*$ has an open and dense intersection with the set $\{((u^i)_i, f) \in \mathcal{E}$: for <u>every</u> $s \in S$, every FCE $(p, (x^i)_i)$ for $((u^i)_i, f, s)$ is regular$\}$, so the complement of $\mathcal{E}^*$ is not dense, but the result is still not as strong as one would like. Second, the existing results at least formally leave open the case $\Sigma_{k \neq i} n^k \leq J$ for all i and $\Sigma_i n^i > J$. If $\Sigma_{k \neq i} n^k < J$ for all i, the approach taken by Allen [2] should be able to establish the generic existence of an REE π such that for each i, the function $s \to (\sigma^i(s), \pi(s))$ is injective except on a set of measure zero. That is, prices may not reveal all private information but they reveal to each trader the information of others. If $\Sigma_{k \neq i} n^k = J$ for some i, it may be possible to generalize the example in [10] to show that neither the existence nor the nonexistence of rational expectations equilibria is generic in this case. Third, the results in [1], [2], and [10] are obtained in model of general stochastic exchange environments, whereas the results here and in [9] are obtained in a stock market model. However, each result appears to be adaptable to the alternative model with only straightforward modifications.

A more fundamental issue is whether an equilibrium which relies on carefully crafted mathematical pathologies is a compelling description of market behavior. Our result may be interpreted to imply that additional conditions are needed in the definition of rational expectations equilibrium. Of course continuity could be imposed to rule out the equilibria constructed here, but the global continuity of π would be too strong a restriction in view of the discontinuities present in the Walrasian equilibrium correspondence for classical deterministic exchange environments. A more natural condition would require that π be continuous except on a null subset of S. I have not explored the existence of equilibria satisfying this additional regularity condition. However, the imposition of mathematical regularity as an economic equilibrium condition would not seem to be in the best spirit of economic analysis. A possibly more fruitful approach would be to model the implementation of rational expectations equilibria. It does not seem likely that the equilibria constructed here could be the outcome of a natural informationally decentralized allocation mechansim, but this need not be due to their discontinuity.

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