Understanding Sensemaking in Undergraduate Science Classrooms

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Dedication

This dissertation is dedicated to my four-year old son, LeFan, who started his life at the same time I started this journey.

Abstract

This three-paper dissertation explores sensemaking in undergraduate science classrooms. The first two studies explore the sensemaking of mathematical equations in science and the third study investigates students' sensemaking of one scientific phenomenon. Together, these studies provide new resources for research and contribute to theory on the sensemaking of mathematical equations and scientific phenomena.

In the first study, an extensive and systematic literature review was performed to identify possible types of sensemaking for mathematical equations in science. Relevant publications around sensemaking in both the science education and mathematics education literature were reviewed. The resulting Sci-Math Sensemaking Framework includes nine categories for different types of sensemaking, five in the mathematics dimension and four in the science dimension. This novel framework provides researchers with an analytical tool to explore sensemaking in the instruction of mathematical equations across the scientific disciplines.

In the second study, the Sci-Math Sensemaking framework was used to understand the sensemaking opportunities provided by four undergraduate biology instructors teaching mathematical equations in the context of population growth. A multiple-case study was conducted with each instructor's lesson(s) described and the different sensemaking opportunities identified. A comparison across instructors revealed that instructors provided different types of mathematics sensemaking and science sensemaking in their class, and they organized these different types of sensemaking opportunities in different ways. Instructors combined sensemaking across categories in the Sci-Math Sensemaking Framework in three different ways, two of which had not been previously identified in the literature.

The third study investigated undergraduate biology students' sensemaking of the scientific phenomenon of mutation. A qualitative analysis was performed on students' written responses to open-ended questions on how mutations arise in animal populations. Although the questions should have elicited an explanation involving a biological process, many students responded with only a definition of mutation. It is suggested that instructional approaches should place greater emphasis on understanding the process-related features of mutation.

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1 Introduction

The fundamental goal of science education is to prepare scientifically literate citizens who can explain scientific phenomena and make evidence-based decisions (Bybee, 1997). However, students at all education levels and disciplines struggle with making sense of scientific phenomenon, such as evolution (Brumby, 1984; Gregory, 2009; Nehm & Ridgway, 2011), diffusion (Chi, 2005), and energy (Lancor, 2014). Opfer, Nehm and Ha (2012) proposed that students might resort to intuitive ideas if they lack a deep understanding of scientific phenomena including an understanding of the underlying processes and mechanisms.

In addition to a focus on understanding underlying processes and mechanisms that explain scientific phenomena, science education researchers have called for a greater integration of mathematics in science courses (American Association for the Advancement of Science, 2011; Bialek & Botstein, 2004; Feser, Vasaly, & Herrera, 2013; Gross, Brent, & Hoy, 2004). Although mathematics has been widely used to represent scientific phenomena or the patterns in science, the teaching and learning of mathematics in science classroom is less well-studied. Historically, mathematics has been interwoven into physics and engineering education (Bialek & Botstein, 2004; Redish & Kuo, 2015). Recently, more attention has been paid to the role of mathematics in chemistry education (Towns, Bain, & Rodriguez, 2019) and the increase in "sophisticated blending of mathematics and biology" (p. 13) has strengthened the calls for biologists with mathematics competency (Steen, 2005). However, students tend to have difficulty in solving quantitative problems in undergraduate science class (Feser et al., 2013; Hester, Buxner, Elfring, & Nagy, 2014; Taasoobshirazi & Glynn, 2009) and they tend to plug quantities into equations without a conceptual understanding (Stewart, 1983; Tuminaro & Redish, 2007). If mathematics is not mindfully integrated into science courses, it might disadvantage subsets of students who are less prepared for mathematics and lead students to drop out of science, as students' confidence in their ability to do mathematics affects their choice of STEM-related major (Moakler & Kim, 2014).

This dissertation addresses these issues in undergraduate science education through the lens of sensemaking (Bing & Redish 2007; Kapon, 2016; Odden & Russ, 2019) and adds to the literature on sensemaking of science. First from a blended sensemaking perspective, the first two chapters examine sensemaking of scientific knowledge and mathematical knowledge around equations in science to describe instruction related to mathematical equations in science classrooms. Second, from the cognitive perspective, the third chapter characterizes how students make sense of a scientific phenomenon by examining whether students make sense of the phenomenon as an entity or as a process. Findings from this dissertation inform future study of students' sensemaking and suggest improvements for undergraduate science education

1.1 Organization of the dissertation

This three-paper dissertation explores science sensemaking situated in the teaching of undergraduate science. Chapter 2 - 4 each contains individual but related studies about the sensemaking of science. Chapter 5 includes the key findings from across three studies, and the implication and future directions based on the work in this dissertation.

Chapter 2 addresses the teaching of mathematical equations in science through the development of a novel framework for characterizing sensemaking of mathematical equations in science, the Sci-Math Sensemaking framework. A systematic literature review was performed using publications on sensemaking of mathematical equations and sensemaking of concepts in both science education and mathematics education.

Chapter 3 uses the Sci-Math Sensemaking Framework developed in Chapter 2 to analyze the instructional practice of mathematical equations in biology. Using a case study method, this study explores sensemaking opportunities included in lessons on population growth by four different instructors. The similarity and variation in types of sensemaking opportunities by different instructors were presented and discussed.

Chapter 4 explored students' sensemaking of a biological phenomenon (mutation) from a cognitive perspective. This study used a qualitative method to analyze student written responses to open ended questions on mutation. The results provided evidence about students' sensemaking of the mechanism (process) of mutation after instruction on mutation.

2 Development of the Sci-Math Sensemaking Framework: Categorizing Sensemaking of Mathematical Equations in Science

2.1 Introduction

Mathematical equations are used to represent scientific phenomenon and communicate scientific ideas (Bialek & Botstein, 2004; Brush, 2015; Gingras, 2001; Lazenby & Becker, 2019; Steen, 2005). Students are expected to be able to engage in sensemaking with these equations, to interpret the mathematical and scientific meaning represented by the equation (Bialek & Botstein, 2004; Heisterkamp & Talanquer, 2015; Kuo, Hull, Gupta, & Elby, 2013; Sevian & Talanquer, 2014). However, studies on students solving quantitative problems show that they often solve problems by relying on algorithmic procedures without making connections between the mathematical equation and the scientific phenomenon (Bing & Redish, 2009; Stewart, 1983; Taasoobshirazi & Glynn, 2009; Tuminaro & Redish, 2007). This tendency to solve problems algorithmically has been associated with a failure to transfer problem solving techniques to novel contexts or more complex problems (Becker & Towns, 2012; Nakhleh, 1993; Ralph & Lewis, 2018; Schuchardt & Schunn, 2016; Stamovlasis, Tsaparlis, Kamilatos, Papaoikonomou, & Zarotiadou, 2005). The reliance on algorithmic problem-solving strategies has been attributed to the different opportunities provided for sensemaking of mathematical equations in science during instruction (Bing & Redish, 2008; Lythcott, 1990; Schuchardt & Schunn, 2016). To successfully develop and understand the impact of providing different sensemaking opportunities, it is first necessary to understand the types of sensemaking that can occur. However, a consistent and coherent framework of sensemaking of mathematical equations in science has not yet been developed. This paper presents such a framework based on a review of the literature on instruction of mathematical equations in science and on students' problem-solving using mathematical equations in a science context.

2.1.1 Scientists Use Mathematical Knowledge to Represent Ideas about Scientific Phenomenon

Canonical mathematical equations in science have been developed based on both an understanding of scientific phenomena and of the mathematical concepts represented in the equations. Scientists use mathematical equations to formulate theories deduced from observations or experimentation, or to represent patterns they observe (Brush, 2015; de Ataíde & Greca, 2013; De Berg, 1992; Ghosh, 2009; Pospiech, 2019; Steen, 2005; Wigner, 1960). For example, Newton's second law of motion is often represented as $\vec{F}_{net} = m\vec{a}$. The development of $\vec{F}_{net} = m\vec{a}$ was based on observations from multiple scientists, as well as many experiments in making sense of the scientific phenomenon (Ghosh, 2009). The equation represents a "central principle of classical mechanics" (Gierer, 2004), a theory that has been used to explain the motions of objects. In biology, the mathematical expression for the population growth of bacteria in optimal growth conditions $N_t = N_0 2^x$ is based on scientists' biological knowledge of the relationship among the initial population size (N_0), the number of generations (x) and the final population size (N_t) after a specific time period. This biological understanding was combined with knowledge of how to arrange the mathematical variables and mathematical operations (i.e. multiplication, exponents) to match the quantitative relationships to the patterns found in the phenomenon. Whether equations are developed to represent a theory or a pattern, both scientific knowledge and mathematical knowledge are embedded in these equations.

Mathematical equations are often referred to as models of scientific phenomena, “a representation of structure in a physical system or process” (Hestenes, 2010, p. 18). The development of mathematical models of scientific phenomena as engaged in by scientists is a multi-faceted, multi-step process known as the modeling cycle (Diaz Eaton et al., 2019; Gouvea & Passmore, 2017; Halloun, 2007; Hestenes, 2010). The steps in the modeling cycle include identifying the task or relations to be represented, mathematizing the physical entities, structuring the equation to express the pattern, interpreting the equation with reference to the scientific process, and validating the equation (Borromeo Ferri, 2006; Dukerich, 2015; Gouvea & Passmore, 2017; Redish, 2017). If the mathematical model is validated in one situation, the process is repeated in other situations. If the model is not found to match the data, either the idea is rejected or the model is modified (Halloun, 2007; Hestenes, 2010). The modeling process as engaged in by scientists provides multiple opportunities for making sense of both the connections between the mathematical equation and the phenomenon and the mathematical ideas.

2.1.2 Blended Sensemaking as a Lens for Investigating Students’ Quantitative Problem-solving

Blended sensemaking is described as the process of combining separate cognitive resources to generate a new merged, blended understanding (Fauconnier & Turner, 1998). Scientific knowledge and mathematical knowledge are two cognitive resources that can be activated during sensemaking of mathematical equations associated with scientific phenomena. The sensemaking of these equations can occur with respect to only scientific ideas or only mathematical ideas if only one of these resources is activated, or sensemaking can be blended, making use of both cognitive resources (Bain, Rodriguez, Moon, & Towns, 2019; Eichenlaub & Redish, 2019). For example, in the sensemaking of $\vec{F}_{net} = m\vec{a}$, the resource from mathematics (e.g. procedures of doing calculation, knowledge of multiplication, knowledge of the mathematics notation) can be blended with resources from science (e.g. the net force causes the acceleration) and together form a blended mental space that enables proper interpretation and application of this equation. Experts' understanding of physics equations includes the blending of mathematical forms and physical intuition, but novices tend to treat mathematical equations as a calculation tool without connection to the physics knowledge (Eichenlaub & Redish, 2019). In the above equation, the vector notation above the "*F*" and the "*a*" have mathematical meaning indicating that those quantities have direction. This mathematical understanding can be combined with the physics knowledge that acceleration will occur in the same direction as the net force. Students often fail to access both of these resources and thus often do not pay attention to the directionality embedded in the equation (Flores-García, Terrazas, González-Quezada, Pierce, & Soto, 2008).

Categorization of blended sensemaking has been used to describe students' quantitative problem solving (Bain, Rodriguez, Moon, & Towns, 2018; Bain, Rodriguez, Moon, et al., 2019; Bing & Redish, 2007, 2009; Brahmia, Boudreaux, & Kanim, 2016; Greca & Moreira, 2002; Hu & Rebello, 2013; Kuo, Hull, Gupta, & Elby, 2013; Tuminaro & Redish, 2007). Bing and Redish (2007) identified two types of blending in students' problem-solving processes, single-scope blend (a one-way mapping) and double-scope blend (a back and forth integration). The difference between these two types of blending lies in whether the sensemaking uses only one cognitive resource or moves back and forth between the science and mathematics cognitive resources. In one study (Bing & Redish, 2007), when students were reasoning about the direction of air drag for falling objects using the equation $F_v = -bv$, they started their sensemaking of the equation in the physics space, mapping the variables onto aspects of the scientific phenomenon and then used their physics knowledge to reason that friction (F_v) operates in the opposite direction of velocity (v). They then discussed the mathematical rule that multiplying two negatives together yields a positive outcome. Finally, they showed a blending of the knowledge from physics and mathematics when they reasoned that positive is up, and therefore the direction of friction for falling objects is up (Bing & Redish, 2007). In chemistry, students were found to make sense of equations starting from either a chemistry or mathematics space and then pull in concepts from the other discipline to complete the problem-solving process (Bain, Rodriguez, Moon, et al., 2019). These authors proposed that the quality of blended sensemaking was dependent on whether students applied a superficial or sophisticated conceptual understanding in chemistry space. However, details were not provided on what kind of conceptual understanding should be considered as superficial or sophisticated, or on the difference between low and high quality blended sensemaking.

Students' difficulties with quantitative problem-solving has been attributed to difficulties with making sense of the conceptual knowledge embedded in the mathematical equations. In physics, chemistry, and biology, students struggle to apply mathematical equations they have learned in class to novel or more complex scenarios (Becker & Towns, 2012; Nakhleh, 1993; Ralph & Lewis, 2018; Schuchardt & Schunn, 2016; Stamovlasis, Tsaparlis, Kamilatos, Papaoikonomou, & Zarotiadou, 2005). This difficulty has been attributed to students' tendency to apply memorized algorithmic procedures instead of making sense of connections between the mathematical equations and the modeled scientific phenomenon (Bing & Redish, 2009; Stewart, 1983; Taasoobshirazi & Glynn, 2009; Tuminaro & Redish, 2007). When students spontaneously apply blended sensemaking, they are able to overcome being stuck and solve more complex problems (Bing & Redish, 2007). One instructional approach that provides opportunities for students to connect the scientific phenomenon to the mathematics is model-based instruction (Blum & Borromeo, 2009). If instruction encourages engagement in mathematical modeling and sensemaking, students show improved quantitative problem solving for novel and more complex problems (Becker, Rupp, & Brandriet, 2017; Lazenby & Becker, 2019; Schuchardt & Schunn, 2016).

2.1.3 Viewing Instruction of Mathematics in Science through the Lens of Blended Sensemaking

Sensemaking opportunities provided by the instructors in the classroom serve a critical role in students' learning (Li & Schoenfeld, 2019; Lo, Marton, Pang, & Pong, 2004; Marton, Runesson, & Tsui, 2004). What the instructor does and says create the opportunities or the environment for students to make sense of something, i.e. the conditions for students to learn specific content or skills (Marton et al., 2004). Mathematics instruction in the United States of America tends to focus on procedures without connection to mathematical concepts, while teachers in high-achieving countries focus more on conceptual understanding than procedures (Hill, Rowan, & Ball, 2005). Science instruction that focuses on high level thinking (e.g. doing scientific inquiry) as compared to low level thinking (e.g. rote memorization) is also associated with better learning outcomes for students (Tekkumru-Kisa, Stein, & Schunn, 2015). Besides the effect on students' learning outcomes, instruction also affects students' perception of what is valued in science learning and their understanding of the nature of scientific knowledge (Bing & Redish, 2009; Eichenlaub & Redish, 2019; Hutchison & Hammer, 2009; Kang, Windschitl, Stroupe, & Thompson, 2016; Russ, 2018; Russ, Coffey, Hammer, & Hutchison, 2009; Tuminaro & Redish, 2007). For example, evaluation in introductory physics courses tends to focus on students' correct calculation rather than their understanding of the meaning of the equations (Eichenlaub & Redish, 2019). In this instructional environment, students can develop the perception that mathematics is merely a tool to do calculation in physics, and they may devalue conceptual understanding of mathematical equations.

Instructional opportunities for student sensemaking of equations is not necessarily synonymous with specific teaching practices. For example, although making sense of mathematical rules often occurs through instructors delivering explanations via lecture (Njini, 2012), Baig and Halai (2006) presented a student-centered learning activity to make sense of four rules for working with fractions. Marton et al. (2004) argued that the understanding of what learners are expected to learn needs to occur before an effective teaching method can be identified. However, attention is rarely paid to the sensemaking opportunities created by what the instructor is doing or saying.

Many frameworks or protocols have been proposed to describe or measure instructional practice or discourse in mathematics or science classrooms, e.g., Classroom Observation Protocol for Undergraduate STEM (Smith, Jones, Gilbert, & Wieman, 2013), Classroom Discourse Observation Protocol (Kranzfelder et al., 2019), Instructional Quality Assessment (Boston, 2012), Mathematical Quality of Instruction (Learning Mathematics for Teaching Project, 2011), Reformed-Oriented Teaching Observation Protocol (Sawada et al., 2002). Very few frameworks specifically discuss instruction of mathematics in science classrooms. One framework that has been developed is the Mathematics Integrated into Science: Classroom Observation Protocol, MISCOP (Judson, 2013). This framework seeks to characterize the extent of integration of mathematics and science and the overall quality of instruction to evaluate the quality of integration of mathematics in science. The opportunities for student sensemaking of mathematical equations in science provided by instructors are not addressed.

This paper establishes the Sci-Math Sensemaking Framework for categorizing sensemaking of mathematical equations in science on the science and mathematics dimensions. Categories within the framework are identified based on a literature review using manuscripts from both the science education and mathematics education communities. This framework supplies researchers with a common language for discussing opportunities instructors provide for sensemaking of mathematical equations in science as well as student use of sensemaking when working with these equations.

2.2 Methods

The objective for this literature review is to identify ideas expressed in the literature about the different types of mathematics sensemaking and science sensemaking of mathematical equations in science. A snowballing approach that began with recent reviews of the literature was used (Wohlin, 2014). The procedure is shown in Figure 2.1 and includes: identification of an initial set of manuscripts; a backward screening on the reference lists of the starting set of manuscripts; a forward screening on the publications citing the starting set of manuscripts; and iteration of backward and forward screening on the included publications (Wohlin, 2014).

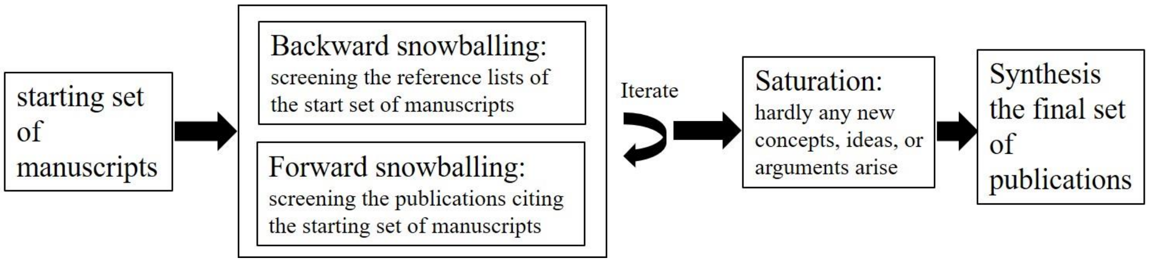

Figure 2.1. Summary of the snowballing approach.

2.2.1 Identifying the Starting Set of Manuscripts

Guided by the theory of blended sensemaking of mathematical equations in science, five topics from the field of education research were chosen to search for the starting set of manuscripts (Table 2.1). Three of the topics covered mathematics in three science fields commonly taught in schools: chemistry, biology, and physics. Mathematics sensemaking and science sensemaking were the other two topics to provide publications that discussed sensemaking in each discipline. Within each topic, the starting set of manuscripts were identified using a key word search in google scholar or based on recommendations from experts. When more than one reference was identified through these methods, the publications that contained the greatest number of citations were retained to provide breadth and depth of literature in the initial backward snowballing. Out of these, the most recent manuscript was chosen in each topic area to uncover the most recent publications in the field. Five manuscripts published during 2015-2019 were identified that cover the following topics: a) mathematics in physics, b) mathematics in chemistry, c) mathematics in biology, d) science sensemaking, e) mathematics sensemaking. (Table 2.1).

2.2.2 Iteration 1

2.2.2.1 Backward snowballing

For each of the five starting manuscripts, the publications listed in the references were screened based on the following inclusion and exclusion criteria. The inclusion criteria were: published between 1986-2019; written in English; included components of mathematical knowledge of equations or scientific knowledge with respect to equations; published papers, conference proceedings, book chapters or dissertations. The exclusion criteria were: published earlier than 1986; a whole book; written in a non-English language; not about mathematical equations; provided only a broad description of teaching strategies or student problem solving abilities or student epistemologies; not focused on mathematical equations; from the same research group referring to the same theories and findings as other references. The foundational research symposium on mathematical sensemaking (Hiebert, 1986) was published in 1986 and research in mathematics in science emerged after research in mathematics education. Therefore, the criteria for publication year was set as no earlier than 1986. References were initially screened by title, and then abstract. References which remained were read in their entirety to reach the final decision about inclusion. Forty-six references were identified through backwards snowballing.

2.2.2.2 Forward snowballing

Forward snowballing was performed on the five manuscripts in the starting set. Citations of each of these manuscripts were located through the “cite” function under Google Scholar. The inclusion criteria were the same as for backward snowballing. Of the five starting manuscripts, the book chapter for mathematics sensemaking (Rittle-

Johnson & Schneider, 2015) has been cited 222 times, the paper of science sensemaking (Odden & Russ, 2019) has been cited 20 times, the dissertation of mathematics in biology (Anita Schuchardt, 2016) has been cited once, while the book chapter for mathematics in physics (Pospiech, 2019) and the paper of math in chemistry (Bain, Rodriguez, & Towns, 2019a) have not been cited yet. Because of the recent publication dates for the starting manuscripts, the forward snowballing was not expected to produce many citations. The book chapter for mathematics sensemaking was the most cited but most of the citations are not about knowledge of mathematical equations and thus only four publications were retained from these citations. A total of fifty manuscripts were retained from iteration 1 snowballing (46 from backward snowballing and 4 from forward snowballing).

2.2.3 Iteration 2

A second iteration of backward and forward snowballing was performed on all 50 publications retained from iteration one (See Appendix).

2.2.3.1 Backward snowballing

The inclusion criteria for backward snowballing during iteration 2 was the same as during iteration 1 but an additional criterion was added: references had to use a new theoretical framework about mathematical equations to guide their research or analysis. Six publications were identified from the backward snowballing.

2.2.3.2 Forward snowballing

Publications that cited any of the 50 manuscripts retained from iteration one were identified using Google Scholar. If the citation list for the publication identified in the first iteration contained fewer than 30 cites, all publications were screened by title,

Table 2.1

Initial set manuscripts and number of publications retained after Iteration 1(I-1) and 2(I-2) snowballing

Topic	Initial set of manuscripts	Summary	Number of references cited	Publications retained after snowballing	
				I-1	I-2
Mathematics in physics	Pospiech, G. (2019). Framework of mathematization in physics from a teaching perspective. In G. Pospiech, M. Michelini, & B. Eylon (Eds.), *Mathematics in physics education* (pp. 1-33).	The chapter summarizes the roles of mathematics in physics and reviews mathematical modeling and mathematics as a language of physics. Discusses research studies on mathematics in physics education.	124	13	3
Mathematics in chemistry	Bain, K., Rodriguez, J. M. G., & Towns, M. H. (2019). Chemistry and mathematics: Research and frameworks to explore student reasoning. *Journal of Chemical Education*, *96*(10), 2086-2096.	The paper reviews frameworks that can guide research of mathematics in chemical contexts.	104	13	1
Mathematics in biology	Schuchardt, A. (2016). *Learning biology through connecting mathematics to scientific mechanisms: Student outcomes and teacher supports* (Doctoral dissertation, University of Pittsburgh).	The dissertation introduces a framework developed from a literature review categorizing the ways mathematics is included in science classrooms. Studies are presented on students' learning of mathematics in biology.	163	10	0
Science sensemaking	Odden, T. O. B., & Russ, R. S. (2019). Defining sensemaking: Bringing clarity to a fragmented theoretical construct. *Science Education*, *103*(1), 187-205.	The paper summarizes existing approaches to describing sensemaking in science education, defines science sensemaking and distinguishes sensemaking from other activities in science education.	79	2	0
Mathematics sensemaking	Rittle-Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural knowledge of mathematics. In Kadosh, R. C., & Dowker, A. *(Eds.) Oxford handbook of numerical cognition* (pp.1118-1134). Oxford, United Kingdom: Oxford University Press.	The chapter reviews studies on the definitions of and relations between two types of mathematical knowledge, procedural and conceptual. Measures of procedural and conceptual knowledge and	100	12	4
Total				50	8

abstract and then by reading the whole reference. If the citation list contained more than 30 cites, an initial filtering step was performed using the key words "equation" or "sensemaking", or "blend", or "model". If references contained these key words, they were then screened one-by- one using the same criteria as for the backward snowballing. Two additional references were identified from the forward snowballing in iteration two.

2.2.4 Criteria for Saturation

One criterion for determining when a literature review is complete is whether new meaningful information arises by including more references (vom Brocke et al., 2015). Saturation was considered to be reached if the ratio between the total number of included references to the total number of references examined is very low in iteration 2 compared to that of iteration 1. In iteration 1, fifty references were included after screening 813 references. In iteration 2, eight references were included after screening 11,118 references. Because the ratio fell from 6.2% to 0.07% and no new themes on sensemaking of equations arose in retained publications from iteration 2, we concluded that the review was saturated.

2.2.5 Themes Developed from the Retained Publications

The retained references were read and themes on sensemaking of mathematical equations in science were noted as they arose, and short descriptions were generated. These themes and descriptions were presented to and discussed over several iterations with members of an educational research group resulting in the retention of nine themes. These themes formed the nine categories of the Sci-Math Sensemaking Framework.

2.3 Results

Mathematical equations in science contain conceptual knowledge about mathematics based on the arrangement of the symbols and the operations contained within the equations. These equations also contain connections to a scientific phenomenon. To enable characterization of the sensemaking that is occurring in a science classroom along both of these dimensions, the nine themes identified from the literature review were classified as either science sensemaking (referring to making sense of connections to the scientific phenomenon) or mathematics sensemaking (referring to making sense of mathematics). The science sensemaking dimension of the Sci-Math Sensemaking Framework contains four categories while the mathematics sensemaking dimension contains five categories (Table 2.2).

2.3.1 Categories within the Science Sensemaking Dimension

There is a rich history of studying how students make sense of science (Berland et al., 2016; diSessa, 1993; Ford, 2012; Kapon, 2016; Odden & Russ, 2019; Russ, Scherr, Hammer, & Mikeska, 2008). Science sensemaking is defined as “the process of building an explanation to resolve a perceived gap or conflict in knowledge” (Odden & Russ, 2019, p. 187). Therefore, science sensemaking of mathematical equations used in science aims at understanding the scientific knowledge represented by the equations. Sensemaking of mathematical equations in science can occur during classroom instruction of equations or during the process of students interpreting or developing or applying equations (Etkina, Warren, & Gentile, 2006; Hestenes, 2010; Lazenby & Becker, 2019; Redish & Kuo, 2015; Schuchardt & Schunn, 2016). The four categories describing science sensemaking through mathematical equations in science are Sci-Label, Sci-Description, Sci-Pattern, and Sci-Mechanism (Table 2.2). The four categories are ordered theoretically to represent an increasingly sophisticated understanding of the scientific phenomenon.

2.3.1.1 Sci-Label sensemaking

Sci-Label sensemaking refers to connecting variables in mathematical equations with characteristics of objects or processes within the scientific phenomena. These characteristics could refer to quantifiable aspects of specific objects in the phenomenon (e.g., number of sperm types, mass) or they could refer to a quantity that characterizes a process within the phenomenon (e.g., time, temperature, force) (Becker et al., 2017; Becker & Towns, 2012; Bing & Redish, 2007; Geyer & Kuske-Janßen, 2019; Hansson, Hansson, Juter, & Redfors, 2015; Hu & Rebello, 2013; Izsák, 2004; Kuo et al., 2013; Lehavi et al., 2017; Pietrocola, 2009; Quale, 2011; Redish, 2017; Redish & Kuo, 2015; J.-M. G. Rodriguez, Bain, Hux, & Towns, 2019; Schuchardt & Schunn, 2016; Svoboda & Passmore, 2013; Tuminaro & Redish, 2007; Wink & Ryan, 2019).

For example, in the equation $F_{net} = ma$, the variables F, m and a are defined as the net force applied to an object, the mass of the object and the net acceleration of the object. The first step in students' interpretation of equations or their application of equations to solve problems is often labeling the variables. For example, when students were asked how they would explain the equation v=v_0+at to their friends, one student answered that "I think the first thing you'd need to go over would be the definitions of each variable and what each one means" (Kuo et al., 2013, p. 46). Labeling variables in the mathematical equations has been identified in multiple studies on students'

Table 2.2

Categories synthesized from literature to capture the sensemaking opportunities of mathematical equations in science classrooms

Dimension	Category	Short definition	Selected References
Science sensemaking	Sci-Label	Connects variables in mathematical equations to quantifiable characteristics of objects or processes in the scientific phenomenon, i.e. the definition or scientific meaning of the variable (e.g., m=mass)	Hansson et al., 2015; Hestenes, 2010; Izsak, 2004; Kuo et al., 2013; Quale, 2011; Redish & Kuo, 2015
	Sci-Description	Uses a mathematical equation to provide a quantifiable measure of a scientific phenomenon or object within the phenomenon. (e.g. equations for diversity index, the equation for speed)	Bain et al., 2019b; Brahmia et al., 2016; Lehavi et al., 2017; Lehrer & Schauble, 2010.
	Sci-Pattern	Emphasizes the qualitive trend or pattern among variables in the mathematical equation situated within the scientific phenomenon (e.g., in the equation $F = ma$, acceleration is proportional to the force on an object)	Baxter et al., 2014; Michelsen, 2015; Redish, 2017; Rodriguez et al., 2019;
	Sci-Mechanism	Emphasizes connections to a mechanism that explains how or why a scientific phenomenon occurs (e.g., for the equation $\vec{a} = \vec{F}_{net}/m$, the net force distributed over mass causes the acceleration of an object in the same direction)	Etkina et al., 2006; Hestenes, 2010; Redish, 2017; Schuchardt & Schunn, 2016
Mathematics sensemaking	Math-Procedure	Emphasizes the predetermined steps or algorithms for problem solving	Hiebert & Lefevre, 1986; Hansson et al., 2015; Peled & Segalis, 2005
	Math-Rule	Focuses on generalizable statements that guide calculation (e.g. the probability of two events occurring simultaneously is equal to the product of the individual probabilities)	Bing & Redish, 2007; Hansson et al., 2015; Potgieter & Blignaut, 2017; Schuchardt & Schunn, 2016

	Math-Structure	Focuses on the form of the equation, the numbers and arrangement of symbols and operations (e.g., $\square + \square$ as two components adding together)	Bain et al., 2019; McNeil & Alibali, 2004; Pospiech, 2019; Redish, 2017; Sherin, 2011
	Math-Relation	Emphasizes quantitative relationships between variables in the equations (e.g. $\mathrm{v} = 9.8\,\mathrm{m/sec^2} * \mathrm{t} + \mathrm{v_0}$ says that if v_0 is 0, v will be 9.8 times bigger for every unit increase in t.	Carlson et al., 2002; Lehavi et al., 2017; Rodriguez et al., 2018; Sherin, 2001
	Math-Concept	Refers to a network of knowledge that enables explanation of the what, how and why of a mathematical idea (e.g., conceptually, probability is the proportion of desired events out of all possible events)	Even, 1990; Hiebert & Lefevre, 1986; Peled & Segalis, 2005; Rittle-Johnson & Schneider, 2015

interpretation of equations when solving quantitative problems in science (Becker & Towns, 2012; Bing & Redish, 2007; Hu & Rebello, 2013; Redish & Gupta, 2010; Rodriguez et al., 2019; Tuminaro & Redish, 2007). Literature on students' development of mathematical equations also shows that one of the first steps is selecting and quantifying characteristics of the scientific phenomenon as variables (Izsák, 2004; Quale, 2011; Schuchardt & Schunn, 2016; Svoboda & Passmore, 2013).

Similarly, during instruction of mathematical equations in science, the first step is often defining the variables in the equation (Anita Schuchardt, 2016; Hansson et al., 2015; Lehavi et al., 2017; Schuchardt & Schunn, 2016). In their study on the role of mathematics in physics lessons in upper-secondary school, Hansson et al. (2015) presented a description of a lecture on electric fields. The physics teacher introduced the equation $F = EQ$ with "F is the force that the electron senses in the electric field" before proceeding to the manipulation of the equation (Hansson et al., 2015, p. 628). It has been proposed that this mapping of variables in the equation to objects in the phenomenon sets the foundation for problem solving (Anita Schuchardt, 2016; Kuo et al., 2013; Redish & Kuo, 2015; J.-M. G. Rodriguez et al., 2019). Because defining important variables relevant to the scientific phenomena is an essential step during the sensemaking of the mathematical equations, but does not go beyond making label connections, this category is placed on the first level of sensemaking in the Sci-Math Sensemaking Framework.

2.3.1.2 Sci-Description sensemaking

The Sci-Description sensemaking category captures the use of mathematics to provide a measure of properties of physical objects, or of scientific phenomena or systems (Bain, Rodriguez, & Towns, 2019b; Brahmia et al., 2016; Geyer & Kuske-

Janßen, 2019; Lehavi et al., 2017; Lehrer & Schauble, 2010; Pospiech, 2019; Wink & Ryan, 2019). For example, density is a measure of the property of a substance. The density equation $\rho = m/V$ is an invented quantifiable characteristic derived from the two direct measures, mass and volume (Pospiech, 2019). This category differs from Sci-Label because in Sci-Label the focus is on establishing only the connection between a variable and the name of a characteristic of a scientific phenomenon, e.g. ρ is density, m is mass and V is volume, while in Sci-Description the entire equation describes how a measure such as density is calculated. Descriptive equations are found across the scientific disciplines. In biology, the Shannon index equation is a description of the biodiversity of a biological system. In chemistry, the equation for the equilibrium constant is a measure of the state of a reaction at equilibrium (Bain, Rodriguez, & Towns, 2019b). Many statistical equations are descriptions of features of a system, e.g., mean, standard deviation (Lehrer & Schauble, 2010).

Few studies address Sci-Description sensemaking (Brahmia et al., 2016; Lehavi et al., 2017; Lehrer & Schauble, 2010). Lehavi et al. (2017) describe a class discussion on the definition of speed, “the change in distance versus time” (p. 99), where the instructor tried to focus students’ attention on the definition of speed as a derived measure. However, students had difficulty with the idea that speed is a variable described by the equation while time and distance are direct measurements. Several other studies discussed instances where students derived descriptive equations from data. In biology, Lehrer & Schuable (2010) provide examples of students inventing mathematical equations to describe characteristics of a population of organisms (e.g., variation, average growth, measure of a healthy aquatic system). In physics, Brahmia et al. (2016) listed several examples of students' inventing equations to describe the features of the motion of cars, such as how fast cars move and how rapidly cars speed up.

2.3.1.3 Sci-Pattern sensemaking

The category of Sci-Pattern sensemaking emerged from multiple studies suggesting mathematical equations in science represent patterns in scientific phenomena (Bain, Rodriguez, & Towns, 2019b; Baxter, Ruzicka, Beghetto, & Livelybrooks, 2014; Becker et al., 2017; Becker & Towns, 2012; Etkina et al., 2006; Geyer & Kuske-Janßen, 2019; Gupta & Elby, 2011; Hestenes, 2010; Hu & Rebello, 2013; Karam & Krey, 2015; Kuo et al., 2013; Michelsen, 2015; Pospiech, 2019; Quale, 2011; Redish, 2017; Rodriguez et al., 2019; Sherin, 2006; Svoboda & Passmore, 2013). The Sci-Pattern category captures sensemaking of the qualitive trend or pattern among variables in the mathematical expression, how properties of a system vary with respect to one another. For example, possible sensemaking opportunities for science patterns in the equation $\rho = m/V$ are: 1) for the same substance, the larger its volume, the larger its mass, or 2) for objects with the same volume, the larger its density, the larger its mass. This type of sensemaking differs from Sci-Description because the emphasis is on understanding the qualitative relationships among variables in the equation as opposed to understanding that the equation is providing a descriptive measurement of a specific characteristic of an object or system.

The idea of sensemaking of scientific patterns embodied within a mathematical expression in science is presented both in theoretical and empirical literature about mathematics in science. For example, in chemistry, mathematical equations are often used to describe features of stable and dynamic chemical systems (Bain, Rodriguez, Moon, et al., 2019; Bain, Rodriguez, & Towns, 2019b; J.-M. G. Rodriguez et al., 2019). The Gibbs free energy equation, $\Delta G = \Delta H - T\Delta S$, describes the relationship among change in entropy, change in enthalpy and the change in free energy in a chemical reaction. One student referred to how the change in enthalpy (ΔH) and the change in entropy (ΔS) lead to a negative change in the Gibbs free energy (ΔG) when explaining the formation of lipid membranes (Redish, 2017). Sci-Pattern sensemaking is a common focus in learning activities designed by teachers to integrate mathematics into science (Baxter et al., 2014; Michelsen, 2015). For example, one group developed a learning module for students to investigate the relationship between the coefficient of friction and braking distance for cars (Michelsen, 2015). In this curriculum, the learning objective was for students to develop a qualitative understanding of the pattern that for cars with the same initial speed, the wetter the road, the longer the breaking distance.

2.3.1.4 Sci-Mechanism sensemaking

Mathematical equations can depict a causal relationship among objects within the phenomenon (Anita Schuchardt, 2016; Etkina et al., 2006; Hestenes, 2010; Lazenby, Rupp, Brandriet, Mauger-Sonnek, & Becker, 2019; Redish, 2017; Redish & Kuo, 2015; Schuchardt & Schunn, 2016). The causal relationship in the equation is the scientific mechanism that explains how or why a scientific phenomenon occurs (Machamer, Darden, & Craver, 2000). A pattern only provides information on which scientific entities are related, but a mechanism shows why the relationship among entities behaves in that way. The equation for Ohm's law $\mathrm{I} = \mathrm{U}/\mathrm{R}$, can be interpreted or taught as presenting the cause for the current, that the voltage difference between two points (U) applied across the resistance (R) causes the current (I) in a conductor (Sci-Mechanism). Alternatively, Sci-Pattern sensemaking focuses on the correlation among current, voltage, and resistance without knowing the cause for this pattern. Often in scientific research, the pattern in a phenomenon is discovered and studied before the mechanism responsible for the pattern. Therefore, Sci-Mechanism is placed at the fourth level of the Science Sensemaking dimension.

Discussion of Sci-Mechanism sensemaking is not common in the literature (Etkina et al., 2006; Hestenes, 2010; Redish, 2017; Schuchardt, 2016; Schuchardt & Schunn, 2016). Causal relationships among variables in equations is often not explicit in canonical forms because the form of the equation hides the causal relationship. For example, Newton's second law is often structured as $\mathrm{F_{net}} = \mathrm{ma}$, and interpreted as a pattern, the total force on a system in a specific direction is proportional to the acceleration in that direction. If, however, the equation is structured as $\vec{a} = \vec{F}_{net}/m$, the arrangement of the variables fosters a mechanistic interpretation, the net force distributed over the mass of an object results in acceleration of the object, while the vector indicates the direction of the net force (Redish, 2017). In biology, one curriculum restructured a mathematical equation used to predict offspring outcomes from an expression of probability rules to "number of different offspring outcomes = (number of egg types) * (number of sperm types)" (Schuchardt & Schunn, 2016). This restructuring shifts the sensemaking focus to the mechanism for inheritance: any egg type can randomly join with any sperm type to produce offspring. Students who were instructed in this curriculum showed improved quantitative skills and conceptual understanding compared to students who were instructed in the use of mathematical algorithms or rules (Schuchardt & Schunn, 2016).

2.3.2 Categories within the Mathematics Sensemaking Dimension

Mathematical equations in science do not just contain scientific meaning, they also contain mathematical meaning that can be accessed independently (Bain, Rodriguez, Moon, et al., 2019; Kuo et al., 2013; B. L. Sherin, 2001). Therefore, the Sci-Math Sensemaking Framework includes a separate mathematics sensemaking dimension. Categories of sensemaking of mathematics were derived from literature on mathematics education and the use of mathematics in science. Five categories emerged from the literature review that captured opportunities for sensemaking of mathematical knowledge. The definitions of these categories are provided in Table 2.2 and each category will be illustrated below.

2.3.2.1 Math-Procedure sensemaking

The Math-Procedure category captures sensemaking that focuses on the procedural knowledge or algorithms for using mathematical equations to solve problems (Bain, Rodriguez, & Towns, 2019b; Baroody, Feil, & Johnson, 2007; Becker et al., 2017; Bing & Redish, 2007; Case & Gunstone, 2003; Fan & Bokhove, 2014; Gupta & Elby, 2011; Haapasalo & Kadijevich, 2000; Hiebert & Lefevre, 1986; Hu & Rebello, 2013; Jacobs, Franke, Carpenter, Levi, & Battey, 2007; Karam, 2014; Kuo et al., 2013; Lehavi et al., 2017; Peled & Segalis, 2005; Pietrocola, 2009; Pospiech, 2019; Radmehr & Drake, 2019; Redish, 2017; Redish & Kuo, 2015; Rittle-Johnson & Schneider, 2015; Schuchardt, 2016; Schuchardt & Schunn, 2016; Star, 2005; Tsaparlis, 2007; Tuminaro & Redish, 2007; Uhden, Karam, Pietrocola, & Pospiech, 2012; Wells, Hestenes, & Swackhamer, 1995; Wink & Ryan, 2019). Procedural knowledge was first defined by Hiebert and Lefevre (1986), and has come to mean knowing the sequential steps in solving problems without having conceptual understanding (Baroody et al., 2007; Haapasalo & Kadijevich, 2000; Jacobs et al., 2007; Star, 2005). Peled and Segalis (2005) presented the subtraction procedure that students can engage in when solving the equation $310 - 164 = \boxed{?}$ in a mathematics classroom. These steps included 1) borrowing from the tens column, 2) subtracting 4 from the ones column, 3) borrowing from the hundreds column, 4) taking away 6 from the tens column, and 5) taking away 1 from the hundreds column. The focus on procedural knowledge in mathematics classes has been criticized as one reason for students' difficulty in understanding the meaning of equations or adopting efficient problem-solving strategies for new or complex problems (Cañadas, Molina, & del Río, 2018; Jacobs et al., 2007; Peled & Segalis, 2005). Similarly, in science classrooms, researchers have found that students tend to rely on algorithms without conceptual understanding of the science in solving problems and teachers tend to focus instruction on using mathematical procedures to do calculation (Bain, Rodriguez, & Towns, 2019b; Bing & Redish, 2007; Hansson et al., 2015; Hu & Rebello, 2013; Kuo et al., 2013; Lehavi et al., 2017; Redish, 2017; Redish & Gupta, 2010; Redish & Kuo, 2015; Schuchardt & Schunn, 2016; Tuminaro & Redish, 2007; Wink & Ryan, 2019). One description captured a high school physics instructor teaching the equation $F = EQ$. After linking the variables in the equation to science entities, the teacher presented the steps of how to solve the problem mathematically and asked students to work on similar textbook problems (Hansson et al., 2015). This category has been placed at the first level in the math sensemaking dimension.

2.3.2.2 Math-Rule sensemaking

The Math-Rule sensemaking category identifies sensemaking of generalizable statements induced from fundamental mathematics principles which are used to guide calculation or decision-making (Baroody et al., 2007; Bing & Redish, 2007; Dixon, Deets, & Bangert, 2001; Haapasalo & Kadijevich, 2000; Hansson et al., 2015; Hiebert & Lefevre, 1986; Moss & Case, 1999; Njini, 2016; Radmehr & Drake, 2019). For example, the knowledge of divisibility rules such as a dividend is divisible by 5 if the last digit is 0 or 5 enables quick decision-making of whether a dividend is divisible (Potgieter & Blignaut, 2017). Rules can be used to guide the step-by-step calculation, however, compared to procedural knowledge, mathematical rules are more generalizable. For example, the rule pertaining to the order of mathematical operations applies to all types of calculation in all problem-solving processes, while the step-by-step procedure for different problems may vary depending on problem type. Because mathematical rules have greater generalizability than mathematical procedures but can still be employed without understanding the other levels, they are placed at the second level of the mathematics sensemaking dimension.

In the literature, references to using mathematical rules to make sense of mathematical equations in science is often found in descriptions of students' problem solving process rather than during instruction (Bing & Redish, 2007; Hansson et al., 2015; Hu & Rebello, 2013; Schuchardt & Schunn, 2016). When trying to understand the relation between the direction of velocity and the viscous force represented by the equation $F_v = -bv$, students referred to the mathematic rule that "two negatives cancel out, (Bing & Redish, 2007). Similarly, Hansson et al. (2015) described how one student manipulated the equation ($mgh = \frac{mv^2}{2}$) by using the rule of division of fractions. In science classrooms, Schuchardt and Schunn (2016) describe an often-used approach to teaching inheritance where the instructor presents the probability rule that "If both events are required then multiply the probability of the two events together" to help students make sense of calculating the probability of producing offspring with specific combinations of genes.

2.3.2.3 Math-Structure sensemaking

In the Math-Structure sensemaking category, the focus is on understanding the arrangement of symbols and operations of the mathematical equations (Bain, Rodriguez, & Towns, 2019a, 2019b; Becker & Towns, 2012; Bing & Redish, 2007; Brahmia et al., 2016; Cañadas et al., 2018; Hestenes, 2010; J. Hiebert & Lefevre, 1986; Hu & Rebello, 2013; Izsák, 2004; Jacobs et al., 2007; Karam, 2014; Karam & Krey, 2015; Kirshner, 1989; Kuo et al., 2013; Moss & Case, 1999; Pospiech, 2019; Redish, 2017; Redish & Kuo, 2015; Rodriguez et al., 2019; Rodriguez, Santos-Diaz, Bain, & Towns, 2018; Schuchardt, 2016; Sherin, 2001). The idea of mathematical structure has often been discussed in science education literature from the perspective of symbolic form, which was proposed by Sherin (2001) as “the particular arrangement of symbols in an equation [that] expresses a meaning that can be understood” (p. 480). Because interpretation of the mathematical structure depends on knowing the symbolic arrangements in particular contexts, greater sensemaking is required than when applying mathematical rules. Therefore, Math-Structure sensemaking is placed at the third level in the Mathematics sensemaking dimension.

Math-Structure sensemaking emphasizes the number and location of the variables and operations in the equation. Compared to the typical addition structure, $3 + 4 + 5 + 3 = \_$, mathematics students are more likely to offer an incorrect answer when the equation is structured as $3 + 4 + 5 = 3 + \_$ (McNeil & Alibali, 2004). In the development of mathematical representations of scientific phenomena, knowledge of mathematical structures provides resources for scientists to organize mathematical symbols and operations to represent the target relationship in the phenomenon (Pospiech, 2019; Redish & Kuo, 2015). Sherin (2001) proposed that students use "symbolic form" to make sense of physics equations. Equations in the symbolic form of $\square + \square$ has a structure of two components adding together. Sherin (2001) provides an example of how students use their knowledge of the mathematical structure of equations to express an idea from their observations of a physical phenomenon that friction consists of two components.

Students tend to memorize the structure of canonical equations without conceptual understanding of the mathematics which leads to their difficulty in choosing or developing a meaningful equation for the target scientific phenomenon (Bain, Rodriguez, & Towns, 2019; Becker & Towns, 2012; Redish, 2017; Rodriguez et al., 2018; Sherin, 2001). For example, Bain et al. (2019) show that students tend to conflate the ideas of rate constant and equilibrium constant because of similarities in the structure of the equations. One student expresses the difficulty as "*It just seems that everything is the same almost, and it's hard to distinguish each equation and each principle*" (Bain et al., 2019, p. 1573).

2.3.2.4 Math-Relation sensemaking

Math-Relation sensemaking refers to understanding the quantitative relationships expressed in the equation (Bain, Rodriguez, & Towns, 2019a; Baroody et al., 2007; Becker et al., 2017; Becker & Towns, 2012; Cañadas et al., 2018; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Dixon et al., 2001; Hestenes, 2010; Izsák, 2004; Izsák & Jacobson, 2017; Jacobs et al., 2007; Karam, 2014; Kuo et al., 2013; Lazenby & Becker, 2019; Lehavi et al., 2017; Levy & Wilensky, 2009; Moss & Case, 1999; Pietrocola, 2009; Pospiech, 2019; Redish, 2017; Redish & Kuo, 2015; Rodriguez et al., 2019; Rodriguez et al., 2018; Schuchardt, 2016; Sherin, 2001; Smidt & Weiser, 1995; Thompson & Carlson, 2017; Tuminaro & Redish, 2007; Uhden et al., 2012; Von Korff & Sanjay Rebello, 2014; Wink & Ryan, 2019). Carlson et al. (2002) defined covariational reasoning as attending to the way in which two variables change with respect to one another. For example, in the equation $y=2x$, the math-relation embedded in the equation is that y increases 2-fold for every unit increase in x. An understanding of quantitative relationships is often built on an understanding of the mathematical structure of the equation (Bassok, Chase, & Martin, 1998). However, the mathematical structure is often not referred to during sensemaking of quantitative relationships, perhaps because the knowledge of mathematical structure is intuitive and not explicitly available to students. Because Math-Relation sensemaking of an equation is built on understanding the mathematical structure, Math-Relation is placed on the fourth level in the mathematics sensemaking dimension.

Math-Relation sensemaking (coordination of relationship between quantities) has been confounded in the literature with Sci-Pattern sensemaking (coordination of the relationship between properties of a scientific phenomenon) (e.g., Carlson et al, 2002). We have distinguished them in this framework because as in the example of the equation for a line, $y = 2x$, quantitative coordination can occur separately from any knowledge of connection to real-world measures. Moreover, when discussing sensemaking of mathematical equations in science classrooms, it has been observed that students tend to limit their sensemaking to the Math-Relation sensemaking space (Becker & Towns, 2012; Izsak, 2004; Lehavi et al., 2017; Svoboda & Passmore, 2013; Wink & Ryan, 2019). When two students are developing a mathematical equation to represent the relationship of turns of a crank handle to the distance a weight is moved, they discuss the phenomenon entirely in terms of quantitative relationships “Zero inches moves 4 point 5 inches per crank. And the weight starting at 14 inches only moves 3 inches.” (Izsak, p. 494) There is no discussion of the physics of the phenomenon. In another study (Lehavi et al., 2017), the teacher expressed concern that “for students who hold the mathematical conceptualization, time, speed and distance are merely three quantities related by an equation” even when the teacher tried to use a teaching strategy to move students into physical understanding of the equation. In this case, students were limiting their sensemaking to the quantitative relationships between time, speed and distance. They were not connecting to the Sci-Pattern sensemaking that speed, time and distance are physical properties of a scientific phenomenon that have a logical relationship to one another: if speed increases, the distance traveled in a specific time period will also increase.

2.3.2.5 Math-Concept sensemaking

Math-Concept sensemaking focuses on a network of knowledge that enables explanation of the what, how and why of a mathematical idea, referred to as conceptual knowledge (Baroody et al., 2007; Even, 1990; Fan & Bokhove, 2014; Fuson et al.,1997; Haapasalo & Kadijevich, 2000; Hiebert & Lefevre, 1986; Hu & Rebello, 2013; Jacobs et al., 2007; Moss & Case, 1999; Peled & Segalis, 2005; Radmehr & Drake, 2019; Star, 2005; Thompson & Carlson, 2017). For example, a conceptual understanding of probability for two independent events A and B cooccurring includes understanding what probability means, why the individual probabilities for the two events are multiplied, and when to perform this calculation and why. Sensemaking of mathematical concepts is the prerequisite for reasoning and justification in mathematics problem solving (Peled & Segalis, 2005).

Students struggle with conceptual understanding of various mathematical ideas in mathematics classrooms (Even, 1990; Jacobs et al., 2007; Moss & Case, 1999).Incorrect or incomplete understanding of the concepts can lead to adoption of incorrect procedures and rules, or difficulty in solving novel problems(Jacobs et al., 2007). For example, students sometimes/often provide 93 as the answer for $57+36=\boxed{?}+34$, instead of 59. This error indicates that they are treating the equals sign as a signal to carry out the calculation that precedes it instead of treating it as an indicator of a relationship between the two sides of the equation (Jacobs et al., 2007). Concept-based reasoning in mathematics can lead to more efficient problem solving. Peled and Segalis (2005) investigated students' problem solving with subtraction. When students were asked to solve a word problem for the time difference between "one week, 5 days, and 18 hours" and “2 weeks, 3 days and 4 hours”, students who applied a more conceptual strategy were more successful than those who applied a rules based approach that dictated that all units of time needed to be converted to the same unit.

Math-Concept sensemaking in science classrooms is relatively underexplored. One biology curriculum developed by Schuchardt (2016) seeks to have students understand the concept of probability in the context of inheritance as the number of desired events out of all possible events. Students who participated in this curriculum showed improved ability to solve novel and complex probability problems situated in inheritance compared to students who were not exposed to this curriculum. Reflecting the mathematics education literature, Math-Concept is placed at the highest level of the mathematics sensemaking dimension.

2.4 Discussion

2.4.1 The Sci-Math Sensemaking Framework is Informed by Research across Multiple Fields

The categories of sensemaking identified in this paper are drawn from literature from several fields including studies of mathematics in science education, and mathematics education. Research from physics, chemistry and biology was synthesized to identify the four categories in the science sensemaking dimension of the Sci-Math Sensemaking Framework. Therefore, the types of science sensemaking that have been identified are expected to apply across different disciplines. Additionally, evidence from both mathematics education and science education was used to generate each category in the mathematics sensemaking dimension of the framework. These categories represent a synthesis of ideas from both fields. Therefore, the Sci-Math Sensemaking Framework is expected to provide a common structure for education studies on mathematics in science contexts.

2.4.1.1 Sensemaking opportunities on the science dimension

Categorization of the science sensemaking opportunities when working with mathematical equations in science offers a framework for exploring the different types of sensemaking that occurs in science classrooms. The four sensemaking categories on the science dimension emerged from literature on the process of mathematical modeling in science (Etkina et al. 2006; Hestenes, 2010; Izsak, 2004; Lazenby & Becker, 2019; Lehrer & Schauble, 2010; Levy & Wilensky, 2009; Michelsen, 2015; Redish, 2005, 2017; Redish & Gupta, 2009; Redish & Kuo, 2015; Uhden et al., 2012), descriptions of students' sensemaking of mathematics in science during problem solving (Hu et al., 2013; Rodriguez, et al., 2018; Sherin, 2001) and new instructional approaches to teaching mathematics in science (Lehrer & Schauble, 2010; Michelsen, 2015; Schuchardt & Schunn, 2016).

2.4.1.2 Sensemaking opportunities on the mathematics dimension

The categories identified in the mathematics sensemaking dimension of the Sci-Math Sensemaking Framework provide a nuanced description of the multiple types of mathematics sensemaking that can occur when working with mathematical equations in science. The teaching and learning of mathematical procedures, rules, structure, relation and concepts have been widely discussed in mathematics education studies (Baroody, 2003; Cañadas et al., 2018; Hiebert & Lefevre, 1986; Moss & Case, 1999; Peled & Segalis, 2005; Star, 2005). The use of mathematics in science offers the opportunity for students to practice mathematics as mathematicians, thus developing their mathematical thinking (Schoenfeld, 1992). In science, development of canonical mathematical expressions in science involves selection of a mathematical structure, including the specific variables and their arrangement, that best represents a specific scientific idea (Borromeo Ferri, 2006; Dukerich, 2015; Diaz Eaton et al., 2019; Gouvea & Passmore, 2017; Halloun, 2007; Hestenes, 2010). However, during instruction of mathematical equations in science, the rich knowledge in mathematics is often neglected (Hansson et al., 2015; Lazenby & Becker, 2019; Redish & Kuo, 2015; Svoboda & Passmore, 2013). By including a separate mathematics sensemaking dimension, the Sci-Math Sensemaking Framework emphasizes the importance of mathematics sensemaking as a means for students to grapple with the represented science concepts (Bain, Rodriguez, Moon, et al., 2019; Brahmia et al., 2016; B. L. Sherin, 2001).

2.4.2 Relationship among Different Categories of Sensemaking

The categories within the dimensions of the Sci-Math Sensemaking Framework have been organized to represent increasingly sophisticated levels of sensemaking from Sci-Label to Sci-Mechanism in the science sensemaking dimension and from Math-Procedure to Math-Concept in the mathematics sensemaking dimension. These levels have been theorized based on the referenced literature and on logic. For example, in the literature of science education, mechanistic reasoning is thought to reflect a deeper understanding of the scientific phenomenon than sensemaking of the labels of the entities or of the pattern in the phenomenon. (Machamer et al., 2000; Illari & Williamson, 2012; Ross et al., 2008). Logically, understanding Sci-Mechanism requires identifying associations between variables in the mathematical equation and properties of the scientific phenomenon. However, the placement of some of these levels (e.g., Sci-Description below Sci-Pattern) needs to be confirmed by additional research.

During the interpretation or instruction of one equation, multiple types of sensemaking may occur simultaneously. With reference to the density equation, Sci-Description sensemaking can only occur after the referents of the variables have been understood (Sci-Label sensemaking), and Sci-Description sensemaking can occur together with understanding the qualitative patterns represented in the equation (Sci-Pattern). Similarly, in the mathematics dimension, Math-Concept is the most advanced type of sensemaking, but Math-Concept sensemaking may occur in conjunction with application of procedures and rules and sensemaking of mathematical structures.

In science classrooms, little priority is placed on Sci-Mechanism or Math-Concept sensemaking of mathematical equations (Bing & Redish, 2009; Schuchardt & Schunn, 2016; Stamovlasis et al., 2005). Sensemaking at these higher levels has shown promise with elevating students' understanding of science concepts and their ability to solve quantitative problems (Mestre, Docktor, Strand, & Ross, 2011; Schuchardt & Schunn, 2016; Taasoobshirazi & Glynn, 2009). However, this does not mean that instruction of mathematical equations needs to always occur at these higher levels. For example, if the goal is to rapidly develop students' ability to quickly solve problems of the same type then Sci-Label and Math-Procedure may be most efficient. Additionally, some equations can only enable a Sci-Description or Sci-Pattern sensemaking (e.g., density equation or diversity index).

2.4.3 Sci-Math Sensemaking Framework to Identify Opportunities for Blended Sensemaking

Specifying the sensemaking occurring in each of the two dimensions of the Sci-Math Sensemaking Framework will permit identification and description of opportunities provided for blended sensemaking (Fauconnier & Turner, 1998). For example, students who rely on algorithms without connection to the scientific knowledge embodied in the equations (Becker & Towns, 2012; Bing & Redish, 2009; Case & Gunstone, 2003; Kuo et al., 2013; Stewart, 1983) are using Sci-Label sensemaking to identify the variables combined with Math-Procedure to solve the problem following a prescribed step-by-step process. These two types of sensemaking provide little opportunity for blended sensemaking and students using these two types of sensemaking have difficulty applying the mathematical equation to different contexts (Redish, 2017; Stewart, 1983; Taasoobshirazi & Glynn, 2009). On the other hand, when students combine sensemaking of mathematical structures of an equation with sensemaking of mechanisms responsible for the scientific phenomenon, they are blending two types of sensemaking (Math-Structure and Sci-Mechanism). Students may start sensemaking of mathematical equations from either the mathematics sensemaking dimension or the science sensemaking dimension. In chemistry, accessing sensemaking of an equation from either dimension could result in students moving to the other dimension for a richer understanding of the equation (Bain et al., 2018). Instruction that provides opportunities for blended sensemaking has been shown to improve students' understanding of the scientific phenomenon and their ability to solve complex and novel quantitative problems (Schuchardt & Schunn, 2016). Additionally, those with more experience in a field are more likely to apply blended sensemaking to, and be more successful at solving quantitative problems, than those with less experience (Redish, 2017).

2.4.4 Relationship between Sensemaking Opportunities and Pedagogical Strategies

The types of sensemaking opportunities of mathematical equations in science is often related to the pedagogical strategies that are used. Evidence from innovative instructional approaches synthesized in this reference shows that instruction which has students develop mathematical equations to model scientific phenomena can create opportunities for students to engage in higher levels of sensemaking, including Sci-Description (Lehrer & Schauble, 2010), Sci-Pattern (Baxter et al., 2014), Sci-Mechanism (Schuchardt & Schunn, 2016), Math-Structure (Izsák, 2004), and Math-Concept (Schuchardt & Schunn, 2016). However, this relationship is not absolute. Sensemaking can occur in classes taught by different methods. Students can spontaneously engage in Sci-Mechanism and Math-Structure sensemaking after having equations provided to them during instruction (Mestre et al., 2011; Redish, 2017; Stewart, 1983; Taasoobshirazi & Glynn, 2009). By separating sensemaking from pedagogical strategies, it is possible to investigate whether different teaching methods can promote or limit the type of sensemaking opportunities that occur during students' quantitative problem solving.

2.5 Limitations

The categories of sensemaking that are presented in the Sci-Math Sensemaking Framework are drawn from published literature. Moreover, the scientific disciplines that were included were only biology, physics and science and in educational contexts. It is possible that other sensemaking opportunities will be discovered during analysis of instruction in different contexts or in investigation of scientists' use of mathematics in their work. The framework is intended to be modifiable to allow addition of new categories.

2.6 Implications

This Sci-Math Sensemaking Framework is generated from a systematic literature review that combines theoretical and empirical evidence on the teaching and learning of equations in mathematics and science. The categories developed in this study capture sensemaking opportunities of equations in science which has rarely been studied. This framework can provide a consistent way for researchers to compare sensemaking of mathematical equations in science across studies. The framework is intended to be used by researchers to examine students' interpretation and application of mathematical equations as well as the sensemaking opportunities created during class by instructors. This framework may also be used by instructors to reflect on their own teaching, to examine whether the sensemaking opportunities provided in class align with their learning objectives.

3 Blended and more: Instructors organize sensemaking opportunities for mathematical equations in different ways when teaching the same scientific phenomenon

3.1 Introduction

Mathematical equations are routinely used in science to represent scientific phenomena (Bialek & Botstein, 2004; Brush, 2015; Lazenby & Becker, 2019). Thus, policy documents in science education advocate that students should use quantitative reasoning and mathematical modeling to study complex physical systems with mathematical equations (American Association for the Advancement of Science, 2011; NGSS Lead States, 2013). However, students struggle with using equations in solving quantitative problems in science and fail to make connections between the equation and the scientific phenomenon (N. Becker & Towns, 2012; Bing & Redish, 2009; Stewart, 1983; Taasoobshirazi & Glynn, 2009; Tuminaro & Redish, 2007). These difficulties have been attributed to a lack of opportunities provided for sensemaking of mathematical equations in science during instruction (Bing & Redish, 2008; Lythcott, 1990; Schuchardt & Schunn, 2016). However, little research exists on sensemaking related to mathematical equations in science classes. Thus, this study explores sensemaking opportunities provided by college biology instructors when teaching mathematical equations in their biology classes.

3.1.1 Literature Review

Mathematics plays a critical role in scientific research and science education. Although students have been exposed to mathematics before they entered college, they still struggle with making sense of mathematical equations learned in science class.

Mathematics in mathematics classes are different than the mathematics in science classes (Redish & Kuo, 2015), and the sensemaking of mathematical equations in science needs to connect to both disciplines, the science and the mathematics. Research on science sensemaking and mathematics sensemaking has been limited to its own disciplinary community. The science education community refers to scientific sensemaking as constructing of explanations for scientific phenomena (Kapon, 2016; Odden & Russ, 2019), while the mathematics education community refers to mathematics sensemaking as developing a sense of mathematical practice in the process of learning different types of mathematical knowledge (Rittle-Johnson & Schneider, 2015; Schoenfeld, 1992).

In both science and mathematics education, there are calls to move practices away from memorization of terminology and facts and repetitive application of procedural algorithms for solving problems. Instead, the emphasis is on providing opportunities for students to engage in sensemaking and authentic disciplinary practices. The space instructors provide for students during instruction affects what knowledge students can make sense of and how students construct their knowledge during classes (Marton et al., 2004). Instruction needs to shift towards providing students experiences that foster their sensemaking of mathematics and science and away from the memorization of facts or procedures (Li & Schoenfeld, 2019).

Multiple types of scientific knowledge and mathematical knowledge are embodied in mathematical equations in science. The scientific knowledge includes the entities and processes relevant to the scientific phenomenon represented by the equations, while the mathematical knowledge includes the syntax and semantics used to express meaning (Zhao & Schuchardt, revise and resubmit). Sensemaking of equations can occur using either scientific or mathematical resources or the scientific and mathematical knowledge may be merged to generate a blended sensemaking (Bain, Rodriguez, Moon, & Towns, 2019; Eichenlaub & Redish, 2019; Fauconnier & Turner, 1998; Zhao & Schuchardt, revise and resubmit).

The idea of blended sensemaking originates from the field of cognitive science where it has been defined as a cognitive operation that draws upon two separate mental resources/spaces to create a third mental resource/space which is greater than the sum of its parts (Fauconnier & Turner, 1998). Blended sensemaking has been adopted as a lens for analyzing student problem solving while using equations in science (Bain et al., 2018; Bing & Redish, 2007; Eichenlaub & Redish, 2019; Kuo et al., 2013). In this context, blended sensemaking has come to mean that students are drawing on both their knowledge of mathematics and their knowledge of science when solving mathematical equations in science. For example, when solving a chemistry kinetics problem, a student first interprets the mathematical meaning of a negative sign before the slope of a linear equation by saying that there will be a negative slope. She then explicitly connects this mathematical idea of a decreasing slope to the chemical concept that the concentration of one reactant will get smaller and smaller because it is getting converted to another chemical. This process of connecting the mathematical meaning with the events occurring in the chemical phenomenon is referred to as blended sensemaking (Bain et al., 2018).

Students who tend to blend mathematical and chemical resources when solving equations in chemistry were more likely to identify multiple problem-solving approaches (Bain et al, 2018). In physics and biology, students who spontaneously blend resources when solving mathematical equations are often able to make progress solving complex and novel problems (Schuchardt, 2016; Redish & Kuo, 2015). However, many students even when they have the appropriate mathematical and scientific knowledge fail to blend these resources when solving mathematical problems. Instead, students often use a plug-and-chug approach and get stuck (Bing & Redish, 2007; Eichenlaub & Redish, 2019; Kuo et al., 2013). Instruction that is designed to create opportunities for blended sensemaking is related to improved student quantitative problem-solving skills and conceptual understanding (Schuchardt & Schunn, 2016). Therefore, instructors have been encouraged to create more opportunities for students to engage in blended sensemaking (Bain, et al., 2018; Eichenlaub & Redish, 2019). However, little is known about the types of sensemaking opportunities (scientific, mathematical or blended) of mathematical equations that instructors are currently providing in their science classrooms.

Currently, in the study of mathematical equations in science, blended sensemaking has generally been treated as a singular entity. However, hints have occurred that blended sensemaking may be more complex (e.g., Bain et al, 2018; Bing and Redish, 2007). Bain et al. (2018) described the blended sensemaking of two students as low quality and high quality. The “low quality” student discussed ideas about how the concentration of reactants in a reaction would change over time and stated that it is based on the dissociation equation. However, he did not make explicit connections between the mathematical meaning of the equation and the events in the chemical phenomenon. In contrast, another student, described as engaged in “high quality” blended sensemaking explicitly related the mathematical meaning embedded in the equation to events happening in the chemical phenomenon. He explicitly related the chemistry phenomenon that reactants were consumed to produce products in a reaction to first the mathematical meaning of a negative slope and then the meaning of rate in the equation.

In our prior work, we developed a framework for categorizing sensemaking of mathematical equations in science along two dimensions (mathematical and scientific). According to this framework, the low quality sensemaking described by Bain et al. (2018) is occurring solely in the science dimension, because after a reference to an equation, the sensemaking is only in the science space. Their science sensemaking is likely cued by some features of the mathematical equation, but it is not clear what these features are or if the student has consciously made sense mathematically of the equation as opposed to just perceiving a general form. Based on our framework, the second student described above would be engaged in sensemaking of the equation in the mathematical dimension and in the science dimension. He is also engaging in another level of sensemaking where he combines the mathematical sensemaking and science sensemaking around the equation. In this paper, we reserve the term "blended sensemaking" of equations for this combined sensemaking along both dimensions.

3.1.2 Conceptual Framework

This study describes the different sensemaking opportunities provided by four undergraduate instructors during their lessons on mathematical equations in biology. During instruction, an instructor can focus on different types of knowledge embodied in an equation, either separately as scientific knowledge or as mathematical knowledge, or as blended sensemaking (Bain, et a., 2018; Bing & Redish, 2007; Kuo et al., 2013) if sensemaking from both science and mathematics dimensions are merged during the instruction. The sensemaking opportunities created by instructors around a mathematical equation in biology will be identified separately in the mathematics and science space using the Sci-Math Sensemaking Framework. Then instances where instructors provide opportunities for blending both types of sensemaking of the equation will be discussed.

The Sci-Math Sensemaking Framework was developed based on an extensive literature review of published work from both science education and mathematics education fields on sensemaking of equations (Table 3.1, Zhao & Schuchardt, revise and resubmit). The Sci-Math Sensemaking Framework includes nine categories for different types of sensemaking, five in the mathematics and four in the science dimension (Table 3.1, Zhao & Schuchardt, revise and resubmit). Sensemaking opportunities in mathematics include application of a step-by-step algorithm (*Math-Procedure*), knowledge of generalizable statements guiding calculation (*Math-rule*), understanding the implications of different arrangements of symbols and operations (*Math-Structure*), identification of quantitative relationships (*Math-Relation*) and understanding the mathematical ideas embedded in the equation (*Math-Concept*).Sensemaking opportunities in science include definition of variables (*Sci-Label*), understanding an equation as a definition of a property of an object or a system that cannot be directly measured (*Sci-Description*), understanding qualitative trends in the represented phenomenon (*Sci-Pattern*), and understanding the mechanism causing the phenomenon to occur (*Sci-Mechanism*).

Table 3.1

Categories in the Sci-Math Sensemaking Framework (Zhao & Schuchardt, revise and resubmit; section of Table 2.2)

Dimension	Category	Short definition
Science sensemaking	Sci-Label	Connects variables in mathematical equations to quantifiable characteristics of objects or processes in the scientific phenomenon, i.e. the definition or scientific meaning of the variable (e.g., m=mass)
	Sci-Description	Uses a mathematical equation to provide a quantifiable measure of a parameter of a scientific phenomenon or an object within the phenomenon. (e.g. equations for diversity index, the equation for density)
	Sci-Pattern	Emphasizes the qualitive trend or pattern among variables in the mathematical equation situated within the scientific phenomenon (e.g., in the equation $\mathrm{F} = \mathrm{ma}$, acceleration is proportional to the force on an object)
	Sci-Mechanism	Emphasizes connections to a mechanism that explains how or why a scientific phenomenon occurs (e.g., for the equation $\vec{\mathrm{a}} = \vec{\mathrm{F}}_{\mathrm{net}}/\mathrm{m}$, the net force distributed over mass causes the acceleration of an object in the same direction)
Mathematics sensemaking	Math-Procedure	Emphasizes the predetermined steps or algorithms for problem solving
	Math-Rule	Focuses on generalizable statements that guide calculation (e.g. the probability of two events occurring simultaneously is equal to the product of the individual probabilities)
	Math-Structure	Focuses on the form of the equation, the numbers and arrangement of symbols and operations (e.g., $\square + \square$ as two components adding together)
	Math-Relation	Emphasizes quantitative relationships between variables in the equations (e.g. $\mathrm{v} = 9.8\,\mathrm{m/sec^2} * \mathrm{t} + \mathrm{v_0}$ says that if v_0 is 0, v will be 9.8 times bigger for every unit increase in t.
	Math-Concept	Refers to a network of knowledge that enables explanation of the what, how and why of a mathematical idea (e.g., conceptually, probability is the proportion of desired events out of all possible events)

3.2 Methods

3.2.1 Research Design

The research goal is to identify different types of sensemaking opportunities provided by different instructors teaching the same biological phenomenon with mathematical equations. According to Yin (2017), a case study method is appropriate when the goal is to provide an in-depth description of how a social phenomenon occurs. A multiple case study design was used to illustrate the range of sensemaking that can occur in science classrooms. A case was defined as all the lessons taught by one instructor that involved equations in population growth. Cases were bounded by including only instruction that included mathematical equations. Instruction covering only biological concepts without mentioning equations was excluded from the analysis. A rich description was generated for each instructor's lessons on equations after an iterative writing and revising process among authors. Specifically, this study aims to provide in-depth understanding of the current instructional approaches to equations in science from the sensemaking perspective. The research questions guiding this study are:

1) What sensemaking opportunities about mathematical equations are present during the instruction of mathematical equations on population growth by different instructors?

2) How are sensemaking opportunities about mathematical equations provided by different instructors when population growth equations are taught?

3.2.2 Participants and Instructional Context

This study draws on previously collected classroom data from 41 undergraduate biology classrooms taught by 21 instructors. The purpose of the larger study was to understand the instruction of mathematical equations in undergraduate biology classes across the nation. The instructors chosen for this study were purposefully selected because they all taught the same biological phenomenon (population growth) using mathematical equations. Therefore, this study focuses on describing the different opportunities each instructor provided for sensemaking of mathematical equations when teaching the same biological topic.

Four instructors (two female and two male) were included in this study (see Table 3.2). They have all been teaching for more than 5 years and have doctorates in biology. Amanda and John taught an introductory biology class to biology majors in a large Midwestern university, albeit in different semesters (class size 100-150 students). Dory taught a group of 12 students in a pre-college orientation experience for incoming biology majors at the same university as Amada and John. Miguel taught an upper level course for biology majors at a Western Hispanic serving university (class size approximately 90 students).

Table 3.2
Participants and instructional context for each instructor (indicates that the instructor is one of the authors)*

Instructor	University	Class size	Course level
Amanda	R1 Midwestern university	100-150	Introductory biology major
John	R1 Midwestern university	100-150	Introductory biology major
Miguel	R2 Hispanic serving university	~90	Upper level biology major
Dory*	R1 Midwestern university	12	Pre-college orientation for biology majors

3.2.3 Data Collection and Analysis

For each instructor, the lesson was audio recorded. Slides used in the class, and worksheets designed for students were collected. For two instructors (Amanda and John), the second author made notes about the instruction while the class was being recorded. The primary data source for the study was the transcripts and slides from the population growth lessons. Observation field notes served as a secondary data source to provide additional details about the nature of class activities and non-verbal interactions.

The data was analyzed in two steps. First, a rich description of the instruction was produced based on the transcripts and available slides and worksheets. The first and third author read the descriptions along with the transcripts and noted areas of disagreement which were resolved with discussion. Second, the Sci-Math Sensemaking Framework was adopted to identify the mathematical and scientific sensemaking opportunities around mathematical equations within the rich case description. Based on the rich description, sensemaking opportunities were only identified when made available to all students either through verbal statements made by instructors or students in large group discussions or on slides presented to the class. When the sensemaking was not explicit in the rich description, it was not identified. The purpose is not to provide an exhaustive list of all instances of sensemaking in the rich descriptions, rather to understand how instructors set up and afford sensemaking opportunities. The same two authors and a colleague with expertise in science education discussed the identification of sensemaking opportunities, and a consensus was reached. The identified sensemaking opportunities were added to the rich description in parentheses. Patterns in the types and the order of sensemaking opportunities were identified. A cross-case analysis was performed based on comparing and contrasting the sensemaking opportunities in the four instructors' lessons. The four instructors agreed with the presentation of their lessons as described in this manuscript.

3.3 Results

The results are first presented by individual case. The rich description of each instructor's implemented lesson on equations about population growth is presented, followed by an analysis of each case guided by the nine categories in the Sci-Math Sensemaking Framework (Table 3.1). A summary of the specific equations used by each instructor is provided in Table 3.3. Within the case descriptions the equations are referred to by number.

Table 3.3
Mathematical equations used by instructors

		Amanda	John	Miguel	Dory
Equation 1	$R_0 = \sum l_x m_x$	✓			
Equation 2	$g = \sum x l_x m_x / R_0$	✓			
Equation 3	$\lambda = {R_0}^{1/g}$	✓			
Equation 4	$N_t = N_0 \lambda^t$	✓			
Equation 5	$dN/_{dt} = rN$	✓	✓		
Equation 6	$\frac{dN}{dt} = r_{max} N \left(\frac{K-N}{K}\right)$	✓	✓		
Equation 7	$N_t = N_0 2^n$			✓	✓
Equation 8	$n = 3.3\log\left({N_t}/_{N_0}\right)$			✓	
Equation 9	$g = t/n$			✓	
Equation 10	$k = n/t$			✓	

3.3.1 The Description of Amanda's Instruction

Amanda spent 97 minutes teaching equations on population growth, 63 minutes in the first class, and 34 minutes in the second class. Amanda started her instruction in population growth by stating, "*So we're going to start looking at not just life tables now, but actually have you look at the growth of populations. How is it changing over time?*" She shared a life table of the population of northern water snakes (Figure 3.1). After naming the variables (*Sci-Label*), l_x and m_x as survivorship and fertility respectively, Amanda showed students how to calculate R_0 (Equation 1, Table 3.3). She presented the calculation as a series of steps, first explaining, "*we multiply a survivorship to each age class by the fertility, we get net offspring produced at each age*" and then, "*If you take the sum of that column, l_x times m_x, you get the net reproductive rate per generation*" (*Math-Procedure*). Amanda concluded this example by describing R_0 (which is calculated by Equation 1) as "*a population growth parameter*" (*Sci-Description*) and explaining how to interpret the parameter, because "*It [R_0] is 0.79, meaning each female produces 0.79 offspring, so not even reproducing themselves. So this population is in decline.*" She introduced a second equation (Equation 2, Table 3.3) to calculate generation time for the snake population in the same way, first defining generation time as "*the average time between the mother and the daughter's first offspring*" (*Sci-Description*) and then working step by step to show how to calculate generation time (*Math-Procedure*).

Age (years)	# survivors	Survivorship: l_x	Fertility: m_x	$l_x m_x$
0	1000	1.0000	0.0000	0.0000
1	400	0.4000	0.0000	0.0000
2	164	0.1640	0.0000	0.0000
3	100	0.1000	0.0000	0.0000
4	64	0.0640	5.5400	0.3546
5	30.7	0.0307	6.7600	0.2075
6	14.8	0.0148	7.8100	0.1156
7	7.1	0.0071	8.8900	0.0631
8	3.4	0.0034	9.1000	0.0309
9	1.6	0.0016	9.2400	0.0148
10	0	0.0000	0.0000	0.0000
				sum

Net reproductive rate per generation: $R_o = \Sigma\, l_x m_x = 0.79$

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Figure 3.1. Slide presented during Amanda’s instruction. The data table of the snake population and the equation for calculating R0 on a slide presented during instruction.

Following her presentation of the steps to calculate R_0 and generation time, Amanda went over the logistics of the weekly project, which asked students to calculate population growth parameters including generation time using data tables formatted identically to the snake example. Amanda offered step-by-step instructions,

> *What you eventually want to calculate is g, the generation time there (Sci-Label). And the other thing you're going to do is to calculate current age distribution, given a stable population. So, read through the directions carefully. I carefully typed it out step by step to help you work through this.* (*Math-Procedure*).

After students worked in groups for about 30 minutes, Amanda described how to estimate growth rates in the population. This presentation followed the same structure as the previous instruction with Amanda presenting Equation 3 and steps of calculation for a new variable λ (*Math-Procedure).* Then Amanda presented Equation 4, the geometric growth equation, by defining each variable (*Sci-Label*) and then described the biological meaning of λ by “*You have this finite rate of increase that’s greater than one, population is increasing. This last one is decreasing. If it equals one, the population is constant*”

(*Sci-Description*). Amanda then introduced the differential equations for continuous growth (Equation 5, 6) by naming the equations, geometric growth equation and continuous growth equation, and then defined the variable r_{max} (*Sci-Label*). At the end of the first class, she expressed her expectations to the students by saying "*What I want you to know are the terms in the equations. What is λ, what is R_0, what do these terms mean?* (*Sci-Label*) *What is each equation used for?*"

At the beginning of second class, Amanda reviewed the definition of r_{max}, in Equation 6 (*Sci-Label*). She showed the logistic growth equation with a new variable, carrying capacity (K in Equation 6, Table 3.3). Amanda defined K as "*how many individuals the environment can support*" and said that the entire term $r_{max}\left(1-\frac{N}{K}\right)$ is called "*the realized growth rate*" (*Sci-Label*) when resources are limited for the population. Amanda went on to explain the pattern among carrying capacity (*K*), initial population size (*N*) and the realized growth rate. She said, "*realized growth rate is going to go down depending on how close the population is to K*" (*Sci-Pattern*).

After her illustration of Equation 6, Amanda spend 17 minutes implementing an activity, presenting students three plots for different forms of Equation 6. Amanda asked students to identify where on the plots the population growth rate is zero, positive, and negative based on the labels on the x-axis and y-axis. After students worked in groups, Amanda asked students to share their answers and checked students' answers in the whole class discussion, focusing students' attention on the names of each variable in the equations shown on the axes (*Sci-Label*) and the quantitative relationship between the variables on the x-axis and y-axis as shown in the equations (*Math-Relation*). Amanda closed the lesson on population growth by reviewing the names of the parameters, λ (finite rate of increase), and R_0 (reproduction rate), and two of the variables located in Equations 6 and 5, R_{max} (intrinsic rate of increase) and r (per capita growth rate) (*Sci-Label*).

3.3.1.1 Sensemaking opportunities presented in Amanda's lesson

Amanda introduced her students to six equations in the span of 97 minutes. For the first three equations, she began with defining variables (*Sci-Label*) or describing the parameters (*Sci-Description*) and then demonstrated how to use these equations step by step (*Math-Procedure*). Amanda reinforced her focus on sensemaking as *Math-Procedure,* directing students to complete the tasks for the weekly project by following her step-by-step instruction. The importance of *Sci-Label* sensemaking during this lesson was made explicit when Amanda, at the end of the first class, stated that she expected students to know the terms in the equations.

Amanda focused on *Sci-Label* and *Sci-Description* when teaching Equations 4 and 5. During instruction of Equation 6, opportunities for *Sci-Pattern* and *Math-Relation* sensemaking were provided. Amanda first presented a description of changes in population growth as they relate to changes of population size with respect to carrying capacity. This portion of the lesson provided an opportunity for *Sci-Pattern* sensemaking because the presentation focused on describing the qualitative relationship between aspects of the biological phenomenon represented as variables in the mathematical equation. Using the same equation, Amanda engaged students in making sense of the abstract mathematical relationship between the variables (*Math-Relation*) by having them examine graphs and describe how the magnitude of the variables varied with respect to one another independently of the biological phenomenon. At the end of the instruction of equations, Amanda reinforced *Sci-Label*.

In general, Amanda's instruction of mathematical equations often started in the science sensemaking dimension, then moved to the mathematics sensemaking space and ended in the science space (i.e. *Sci-Label* to *Math-Procedure* for Equations 1-3, and *Sci-Pattern* to *Math-Relation* for Equation 6). Science sensemaking was separated from the sensemaking of mathematics.

3.3.2 The Description of John's Instruction

John's instruction of the equations on population growth totaled 15 minutes. John started the instruction of mathematical equations on population growth by saying, "*Related to the [population] density is this idea of growth rate, which is really what we're going to be talking about*." John used the growth of a fish population as an example for the exponential growth of populations, then he used an analogy from a non-scientific context, an interest-bearing bank account to focus on the mathematical relationship in exponential growth (*Math-Relation*). He said, "*you put money into a savings account then you expect that to grow. Given some parameters about how much money you got, your principal, and the interest rate, you can predict over time how that money is going to grow*." John presented a slide which showed the step-by-step procedures for calculating the increase in money between years. Using this slide, he explained the mathematical relationship among these variables with a step-by-step calculation (*Math-Relation/Math-Procedure*),

> *So, what that means is if you put a thousand dollars into that account, 5% rate means, on average each one of those thousand dollars is contributing*

a nickel to the growth of that, of your money. Right. That's what that interest rate means. So, after one year, you can predict how much money you're going to have. You're going to have a thousand fifty dollars. So, your money is growing. Let that money sit more. What you see is at year 2, you start with a thousand fifty dollars. That's how much you have in the year 1. You let it sit there. It continues to grow. And so, between year 1 and year 2, you add fifty-two dollars and fifty cents. How much do you add the first year? Fifty. Right? And so, we do this for another year. What we see is between year 2 and year 3, you add fifty-five dollars and twelve cents.

John then related the quantitative relationship among the principal and interest rate and the yearly growth of money to the mathematical relationship in the population growth phenomenon and stated "*money growing is really no different than population growth that we start out with a starting population size analogous to our principal and it's going to grow because each individual on average can contribute some average amount to that*" (*Math-Relation*). After addressing the similarity in the mathematical relationship of exponential growth between money growing and population growth, John introduced Equation 5 with an emphasis on the ever-increasing rate of population growth. He provided definitions for the variables in Equation 5, "*the N is their population size and the r value is the intrinsic rate for the population*" (*Sci-Label*).

After John presented the biology of factors that can limit population growth without referring to equations, he introduced the logistic population growth equation (Equation 6, Table 3). He defined the variable K as carrying capacity, "*the equilibrium* *population size for a habitat*" and labeled $\frac{K-N}{K}$ as "*the dampening term*" (*Sci-Label*). He then described the numerical relationships among mathematical variables in this term, stating that if "*N is small relative to the K, this entire term becomes a big number minus a small number divided by a big number*" (*Math-Relation*). Next, he situated the equation in the biological phenomenon of population growth, presenting the relationship among the biological meaning of these variables, "*which says populations are essentially going to grow exponentially with population size that are far away from the carrying capacity*" (*Sci-Pattern*). John followed up by presenting the patterns for when the population size is close to *K*, first introducing the mathematical relationship among the variables (*Math-Relation*) and then situating the relationship in the biological phenomenon of population growth (*Sci-Pattern*).

At the end of the lesson on population growth, John directed students to begin working on their weekly project which presented equations 1,2, and 4 that were not discussed in class.

3.3.2.1 Sensemaking opportunities presented in John's lesson

John presented two equations (Equation 5 and 6) during the 15-minute lecture. During the analogy of exponential growth in a non-biological context, John used *Math-Procedure* by presenting a step-by-step procedure for calculating the growth in principal to support *Math-Relation* sensemaking in the context of money growth. He reinforced the mathematical relationships between principal, interest rate and the increase in money, *Math-Relation.* After establishing the mathematical procedures and presenting the similarity of the relationship in money growth to the relationship in population growth represented by Equation 5, John provided biological definitions for each variable in Equation 5, *Sci-Label*.

During instruction of Equation 6, John began with *Sci-Label,* naming the variables and terms. He continued by creating sensemaking opportunities of *Math-Relation* closely followed by *Sci-Pattern*. Because the initial focus is on the quantitative relationship between variables in the mathematical equation without referring to the biological context, this is considered *Math-Relation*. Having established the relevant mathematical relationship, John described the qualitative relationship between the biological phenomena represented in the equation, *Sci-Pattern*.

John's instructional pattern changed with each of the equations included during the class. With Equation 5, he started in the mathematics sensemaking space, then moved to and ended in the science sensemaking space (*Math-Relation* and *Math-Procedure* to *Sci-Label*). Whereas, with Equation 6, he started in the science sensemaking space, moved to mathematics sensemaking space and ended in the science sensemaking space (*Sci-Label* to *Math-Relation* to *Sci-Pattern*). *Math-Relation* and *Math-Procedure* occurred simultaneously in the mathematics sensemaking space, while science sensemaking was always separated from the sensemaking of mathematics.

3.3.3 The Description of Miguel's' Instruction

Miguel spent 43 minutes teaching equations on population growth. He started his instruction of population growth in bacteria by directing students to work on a "thought exercise" (Figure 3.2) which asked students to draw a picture of population growth and then represent that growth mathematically.

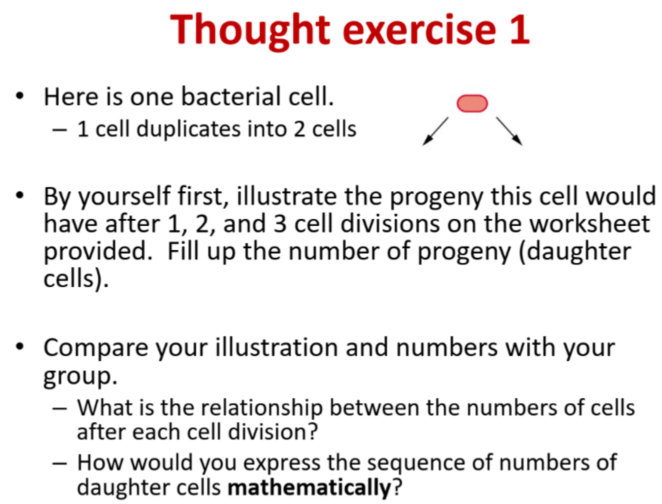


Figure 3.2. Slide presented in Miguel's instruction. Directions for building mathematical equation of the population growth of bacteria presented on a slide during instruction..

Students worked by themselves first, and then worked in small groups. After six minutes of group work, Miguel interrupted to pose questions to the entire class, focusing their attention on the relationship between the number of progeny and the number of generations by stating,

> *So now, class, look at the numbers that you put there. How is that number of cells related to the generation that you're looking at? So look, how many cells are in the first generation? How many cells are in the second generation? How many cells in the fourth generation?* (*Math-Relation*)

After students worked for four additional minutes, Miguel had each group write the equations they had developed on their whiteboards and started his presentation to the whole class. Miguel projected to the whole class while he was drawing the number of cells after multiple cell divisions and explaining his drawing. On the drawing, he labeled the number of generations and highlighted the relationship between the number of cells and the number of generations by stating,

> *OK, we have one cell. That cell is going to divide and have two progeny (Sci-Mechanism). Of course, this cell is going to divide and also have two progeny as well as this one [Miguel continues drawing cells and progeny]…. Which gives us the numbers that you came up. Before there was any division, there was only one cell. After the first generation, now there are two cells. After the second generation, now we have four cells. Eight cells after the third and 16 cells after the fourth. So put the generation here. Generation 2 to the 1, as you guys concluded very nicely. 2 to the 2, 2 to the 3, 2 to the 4. That is the sequential relationship* (*Math-Relation*).

Based on the quantitative relationship, Miguel presented the mathematical expression that he expected students to come up with, "*2 to the n*" (Equation 7, Table 3.3). He asked the whole class after presenting the equation, "*the cells will be two to the n where n is the number of generations or cell divisions. (Sci-Label) Right? OK. Now, I have a quick question for you. Do we always start with one cell? How would you accommodate now if you don't start with one cell? (Sci-Pattern) What would you do?*" Miguel followed up on a student's answer by saying, "*Everybody heard Emily? Then you will know. Multiply it by the number of cells that you had at the beginning.*" (*Math-Structure*) Miguel added the variable N_0, resulting in the canonical equation, $N_t = N_0 2^n$, which he defined as "*the number of cells at time T is going to be equal to the number of cells initially N_0 times two to the n, where little n is the number of generations*" (*Sci-Label*).

Miguel switched to calculations using this canonical equation by asking the whole class how the equation could be manipulated to solve for n. He said, "*That's an equation* *that you guys came up with. We have an exponent. You remember what needs to happen if you want to calculate little n when you have a number to the n power, what will you need to do?*" After one student volunteered the answer, Miguel showed the class the steps of manipulating the canonical equation of population growth of bacteria to get the equation for calculating the number of generations (*Math-Procedure*, Equation 8, Table 3.3). He brought up some mathematical rules when presenting the procedures of manipulating the equation. For example, he said, "*From any kind of addition, you can move this value of* $logN_0$ *to the other side by subtracting it*" (*Math-Rule*).

After presenting the equation for calculating the number of generations, Miguel asked students to solve an application question, calculating the final population size for bacteria (Application question 1, Table 3.4). After students briefly worked in groups, Miguel asked the students to share their answers, and confirmed with students the correct answer. Miguel concluded with his expectation for students to plug numbers into the equation, saying "*All right, everybody gets the same number. Good job! So, you know, you're getting used to getting into the equation, how to plug the numbers in and use it*" (*Math-Procedure*).

Table 3.4
The application question presented to the students during Miguel's instruction

ID	Content
1	You made jam with 0.5 lb of blueberries. These berries were contaminated with 500 endospores of Clostridium botulinum. After 100 cell divisions, how many bacteria would you have in your jam?
2	Suppose 1000 bacteria are inoculated in a tube containing a minimal salts medium, where they double once an hour, and 10 bacteria are inoculated into a rich medium, where they double every 20 minutes. Which tube will have more bacteria after 2 hours? After 4 hours?
3	For Application question 2, determine and compare the specific growth rate (k) for these two cultures.

Before moving on to application question 2, Miguel introduced a new population growth parameter, generation time, described by Equation 9 (Table 3.3) (*Sci-Description*) and then asked students to solve application question 2. Application question 2 (Table 3.4) asked students to calculate the final population size for bacteria that had different generation times. Students worked in small groups to solve application question 2.

After students had worked on application question 2 in small groups for nine minutes, Miguel moved on to present a new variable, "*I want you to think about another factor that you can calculate and this factor is called a specific growth rate*" (Equation 10, Table 3.3, *Sci-Description*). He presented this variable by focusing on the structure of the equation by stating "*It's the inverse of the generation time. Generation time is the time divided by the number of generations. The specific growth rate (k) is the opposite, the number of generations divided by a particular unit of time*" (*Math-Structure*). Then he presented students the steps of calculating for growth rate, stating,

> *So, you can calculate this easily because you know that you have one generation. One generation happens in how much time? [One student answered] Six hours since the growth rate is the number of the time divided by the number of generations. That is six divided by one, which is six. (Math-Procedure)*

Miguel ended the class after he directed students to calculate for the growth rate (k) for application question 3 (Table 3.4).

3.3.3.1 Sensemaking opportunities presented in Miguel's lesson

Miguel introduced four equations (Equations 7-10) during the 43 minutes of class on population growth. Miguel began with a focus on *Math-Relation* by engaging students in developing an equation to show the relationship among numbers of cells for different numbers of generations. During his presentation to the whole class after students finished their work, Miguel introduced the mechanism of cell division to explain how the mathematical relationships were generated (*Sci-Mechanism*/*Math-Relation*). A suggestion that modifying the initial starting number of cells might lead to changes briefly touched on *Sci-Pattern* which led to the incorporation of this variable into the equation using multiplication (*Math-Structure*). To wrap up this portion of the lesson, Miguel shifted to *Sci-Label* sensemaking when he presented the canonical form of the equation and defined the variables.

In showing students how to transform Equation 7 into Equation 8, Miguel focused on the steps for manipulating the equation including explicit discussion of relevant mathematical rules, thus providing opportunities for *Math-Rule* to support *Math-Procedure*. With Equations 9 and 10 Miguel explicitly stated that these were factors or parameters that affect population growth calculated using an equation (*Sci-Description*). When introducing Equation 10, Miguel provided opportunities for *Math-Structure* and *Math-Procedure* when he compared the structure of Equation 9 and 10 as inverse of each other and demonstrated the steps for calculation using Equation 10.

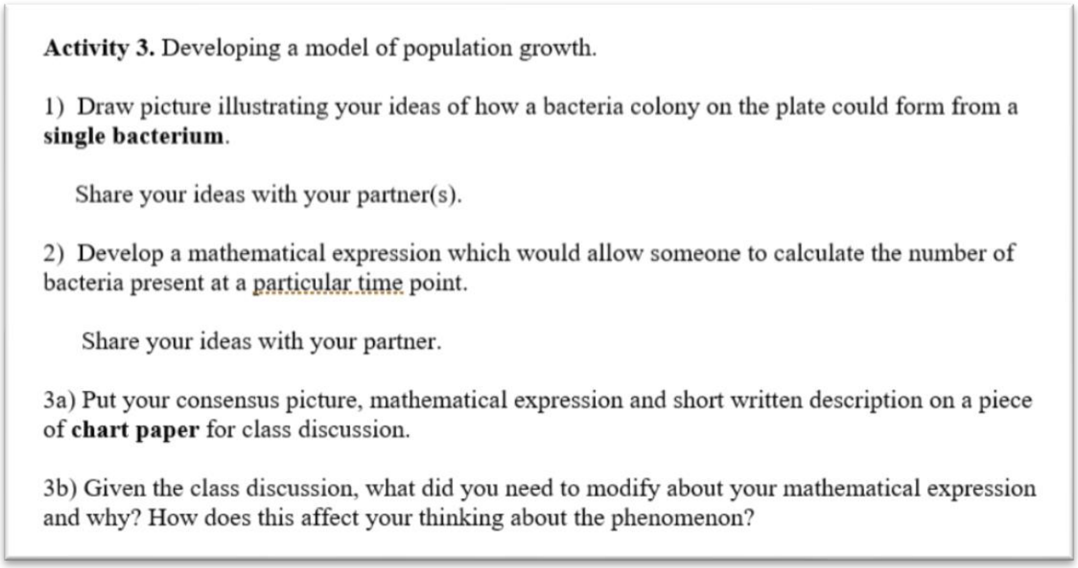
The majority of Miguel's instruction was spent in the mathematical sensemaking space (*Math-Relation*, *Math-Rule*, *Math-Procedure*). During the first portion of the lesson science sensemaking and mathematical sensemaking co-occurred, and *Math-Rule* cooccurred with *Math-Procedure* in the same mathematics sensemaking space, but during the second half of the lesson science sensemaking and mathematical sensemaking were separated.

3.3.4 The Description of Dory's Instruction

Dory's instruction of mathematical equations in population growth lasted for 111 minutes, with 35 minutes of instruction for the first part of the lesson, and 76 minutes for the second part of the lesson. Dory started with an equation building activity where students were asked to represent mathematically how a bacteria colony forms from a single bacterium (Figure 3.3). Students worked individually for 3 minutes, then Dory directed them to work in pairs by saying,

> *Go ahead and put [your initial ideas] on the chart paper. If you're using symbols or numbers, say what the symbols or numbers mean (Sci-Label). For example, X is just X, but if the X means how often the bacteria divide, we need to know that, or how many bacteria are produced. Because Xs and Ys don't help us if they don't communicate that...And I'm asking you to do both a picture, a math expression, and a verbal description. And put your names on it, and we're going to post them up and share them.*

While students discussed with their partner and drew on their chart paper. Dory moved around to interact with different groups.



Activity 3. Developing a model of population growth.

1) Draw picture illustrating your ideas of how a bacteria colony on the plate could form from a **single bacterium**.

Share your ideas with your partner(s).

2) Develop a mathematical expression which would allow someone to calculate the number of bacteria present at a particular time point.

Share your ideas with your partner.

3a) Put your consensus picture, mathematical expression and short written description on a piece of **chart paper** for class discussion.

3b) Given the class discussion, what did you need to modify about your mathematical expression and why? How does this affect your thinking about the phenomenon?

Figure 3.3. Dory's instructions for building a mathematical equation on the worksheet provided to students.

After 15 minutes, Dory guided students to put the chart paper on the wall and students took turns presenting their work. Dory asked students to present their equation and to explain the meaning of the symbols or numbers they had in their equation (*Sci-Label*). During the group discussion after each group's presentation, Dory posed questions on how their equations connected to the biology of the phenomenon of population growth, and she encouraged the rest of students to ask questions. For example, after one group presented their equation with the meaning of each variable in their equation, Dory asked them to explain the difference in their equation compared to the equation presented earlier, $s = 2^x$. Students explained to the whole class that they used x/n as the exponent rather than x (*Math-Structure*) because they tried to represent how long an individual cell takes to divide by using n (*Sci-Mechanism*). A student brought up the idea of how to account in the equation for different factors that might affect cell division *(Sci-Mechanism)* and the class discussed how this would affect the x/n term

(*Math-Structure*). Dory affirmed that "*there are things that just basically stop [cells] dividing...So I think [this equation] does allow you to do that because if the n was really long, really large, then you, basically, would not get to this [inaudible]*" (*Sci-Mechanism, Math-Structure*).

The last group that presented included a variable representing the initial number of bacteria (equivalent to N_0 in Equation 7). The presenters explained their rationale for including this variable and a multiplication function (*Math-Structure*) by connecting to the biological idea that the population does not always start at one and providing a biological pattern which would necessitate multiplication (*Sci-Pattern*). Dory affirmed the students' ideas and shared additional scenarios when the initial population size needs to be included. She prompted students to consider the effect of those changes on the predicted population size (*Sci-Pattern*) and stated that "*x in this case is the number of bacteria that you start your culture with*" (*Sci-Label*).

After all groups had presented, Dory started writing on her chart paper (Figure 3.4) and said, "*So we are going to put together maybe an amalgamation of equations.*" She wrote the equation as $\text{number of bacteria} = N_0 \times 2^{t/F}$, and labeled each variable by stating, "*T equals time elapsed, and F equals the time for binary fission to occur. But sometimes we call that generation time*" (*Sci-Label*). Based on this equation, she asked students "*Why is that two there? What does it mean?*" After a student volunteered that it was a constant, Dory prompted, "*It's a constant, but it's a two. Why is it two and not four?*" Dory confirmed the next student's response, "*Yeah. [a student talked] It's the number of bacteria produced by each bacterium*" (*Sci-Mechanism*).

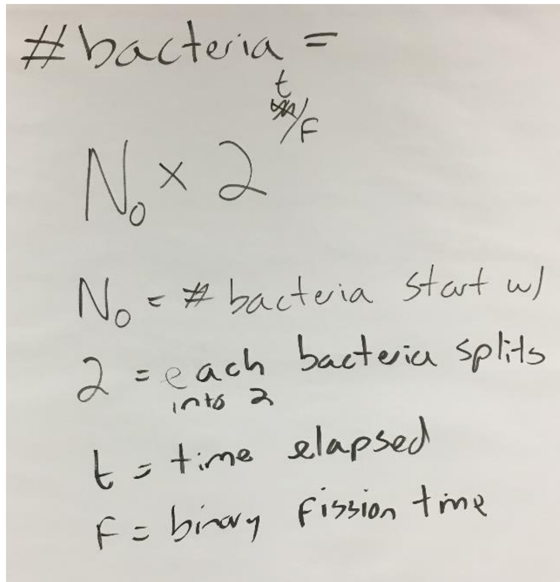


Figure 3.4. Artifact from Dory's instruction. The equation presented to students at the end of the activity on building a mathematical equation for growth of a single type of bacteria in the shaken culture (Note: Dory changed the exponent x to t to make the connection to the biological phenomenon more explicit).

Dory started the second part of the lesson by stating that she had heard students say that the equation they had developed was not a good representation of the population growth in a different condition, where two types of bacteria grew in the same space. She asked students to modify their equation to fit the new condition. Students worked independently for two minutes and then in pairs for 37 minutes, writing their new equations on poster paper. After the small group work, Dory asked students to look at other groups' posters and to share their posters as they did previously. During the whole class discussions, multiple ideas surfaced about the connections of the equation to the biological phenomenon of population growth and how the mathematical structure could be used to depict ideas and observations about the phenomenon. For example, one group presented their equation and explained that they noticed a pattern in the data that Smooth bacteria grew slower than the Wrinkly bacteria (*Sci-Pattern*) and that they used multiplication (*Math-Structure*) to represent this pattern. The whole class discussed which bacteria grew faster (*Sci-Pattern*) and how to match the multiplication (*Math-Structure*) correctly to the pattern. Dory summarized students' conversations the use of the mathematical structure of multiplication indicate that it took the Smooth bacteria a longer time to reproduce than the Wrinkly bacteria, "*Yes. You're saying that it takes a longer amount of time for the Smooth to reproduce, and therefore you have to multiply it by two, compared to the Wrinkly to indicate that*" (*Sci-Pattern* and *Math-Structure*). In another example, the student group presented the difference in growth rates of these types of bacteria by adding a subtraction. Guided by Dory's questions, such as "*what are you proposing biologically happens when you do a subtraction"*(*Math-Structure*), this group explained that in the population growth, Smooth and Wrinkly bacteria competed for resources, and Smooth bacteria were being killed by the Wrinkly as Wrinkly were better (*Sci-Mechanism*), thus a subtraction was used to better represent the population growth phenomenon (*Math-Structure*).

Dory also pointed out to the whole class that equations had boundaries and limitations (*Math-Concept*). After one group presented their equation that considered the time when the Wrinkly Spreader started reproduction and the idea of carrying capacity, Dory stated, "*You actually put some boundaries on this equation…T can't be less than 12, otherwise we get a negative time*", and "*the other limitation you put it on, this only applies in certain circumstances. Good. It's nice to have limitations on equations*" (*Math-Concept*). Dory wrapped up this activity by pointing to students that there was not a right equation and the choice of which equation to use depended on "*the ideas you're trying to express*".

3.3.4.1 Sensemaking opportunities presented in Dory's lessons

Dory engaged students in building two equations to represent population growth in 111 minutes. Both parts of the lesson followed the same format, students worked on building equations in small groups writing their ideas on poster paper. Then students presented their equation to the class.

During the presentation of equation ideas in the first whole class discussion, Dory consistently asked students to explain how the variables or the mathematical operations in their equation were connected to aspects of the biological phenomenon. In response to these prompts, students discussed how structural aspects of the equation such as including a divisor in the exponent or using multiplication (*Math-Structure*) related to the biological mechanism of cell division that causes population growth (*Sci-Mechanism*). At the end of the discussion, Dory synthesized students' ideas into a single equation and labeled the variables (*Sci-Label*). She paused at one point in the labeling to ask students to relate the constant in the equation to the biological mechanism of cell division (*Sci-Mechanism*).

During the second part of the lesson, the focus in the discussion was on using equations to describe the differences in population growth in two kinds of bacteria. During this discussion, ideas about patterns of growth (Wrinkly bacteria grew slower than Smooth bacteria) (*Sci-Pattern*) surfaced along with biological mechanisms to explain this difference in growth (*Sci-Mechanism*). Students described how these ideas were represented in their equation by functions such as multiplication or subtraction (*Math-Structure*). The first set of presenters spontaneously explained how the pattern of growth was described by multiplication in their equation. Dory prompted the second set of presenters to explain how subtraction was used to indicate that one bacteria type grew slower than another, eliciting a description of how subtraction represented the biological mechanism of competition. During this discussion, in Dory's summaries of students' comments, she drew attention to how the choice of mathematical functions connected to students' ideas about the biological phenomenon, connection *Math-Structure* with *Sci-Pattern* or *Sci-Mechanism*. Towards the end of the discussion, Dory highlighted the *Math-Concept* of limits of equations that one student group had depicted on their poster.

Dory's lesson started with providing the *Sci-Label* space for all students. Dory engaged students in using science sensemaking (*Sci-Pattern* or *Sci-Mechanism*) to support their *Math-Structure* during the sharing of ideas in whole class discussions. Dory often wrapped up each group's presentation and discussion with summary statements that integrated the mathematical sensemaking and science sensemaking spaces.

3.4 Cross-case Analysis

The instructors all covered mathematical equations that represented aspects of the same biological phenomenon, population growth. However, the sensemaking opportunities that were provided with respect to these equations differed from instructor to instructor (Table 3.5) even though all were teaching the same biological phenomenon. The organization of the types of sensemaking also varied in interesting ways. After a brief description of the different types of sensemaking presented by different instructors, the organization of this sensemaking will be discussed in greater depth.

Table 3.5
Sensemaking opportunities during four instructors' lesson on equations in population growth

	Sensemaking category	Amanda	John	Miguel	Dory
Science sensemaking	Sci-Label	✓	✓	✓	✓
	Sci-Description	✓		✓	
	Sci-Pattern	✓	✓	✓	✓
	Sci-Mechanism			✓	✓
Mathematics sensemaking	Math-Procedure	✓	✓	✓	
	Math-Rule			✓	
	Math-Structure			✓	✓
	Math-Relation	✓	✓	✓	
	Math-Concept				✓

3.4.1 Types of Sensemaking Provided by the Four Instructors Differ

All instructors provide opportunities for multiple types of sensemaking of equations in the lessons (Table 3.5). Even though all instructors are teaching equations associated with population growth, there are differences in the number of types of sensemaking and what sensemaking opportunities instructors chose to provide in their classes. Amanda and John provide the same subset of sensemaking opportunities with the exception that Amanda also includes Sci-Description. Dory's instruction is the most different from others, with *Sci-Mechanism, Math-Structure* and *Math-Concept* present, in Dory's class but not Amanda or John's. The *Math-Procedure* sensemaking that is present in Amanda and John's class is absent from Dory's class. Miguel is a combination of these two types of instructional approaches, with eight types of sensemaking that are present in the other three instructors' classes. The one type of sensemaking is missing is *Math-Concept*, and Miguel is the only one that also has *Math-Rule*. A description of the types of sensemaking included in a lesson has the potential to make instructors aware of what is included and what is excluded so that they can determine if the instructional spaces provided for sensemaking match their learning objectives.

By providing spaces for multiple types of sensemaking, instructors expose students to different resources that students can use to develop their understanding of equations and the scientific phenomenon (Odden & Russ, 2019). However, it may be possible to try to include too much in a single lesson leading to superficial coverage of some types of sensemaking and impeding development of in-depth understanding. Miguel who covers eight types of sensemaking only touches on Sci-Mechanism briefly, while Dory returns to this type of sensemaking repeatedly through the lesson. It is likely that a balance is needed between in-depth coverage and presentation of multiple types of sensemaking. Future studies are needed to characterize the affordances and constraints of these two approaches to covering types of sensemaking of mathematical equations in science instruction for students' sensemaking. While the description of the different types of sensemaking is interesting, the ways these sensemaking opportunities are organized are more important.

3.4.2 Sensemaking Opportunties Were Organized Differently

Instructors present the types of sensemaking opportunities in different sequences (Figure 3.5). For example, John and Miguel startd with sensemaking in the mathematics dimension and then move on to sensemaking in the science dimension while Amanda and Dory started with sensemaking in the science dimension and progress to sensemaking in the mathematics dimension. One of the key differences between instructors was whether different types of sensemaking occurred within the same activity and whether or not those different types of sensemaking were used to support one another. These differences have implications for whether or not students were provided with the opportunity to engage in blended sensemaking synergistically to achieve a novel outcome (Fauconnier & Turner, 1998). Three types of co-occurrence of sensemaking within the same activity observed, Blended (which has been described by other researchers during student problem-solving) (Bain et al., 2018; Bing & Redish, 2007; Kuo et al., 2013) and two novel types, coordinated and adjacent. All three of these will be described in more detail below.

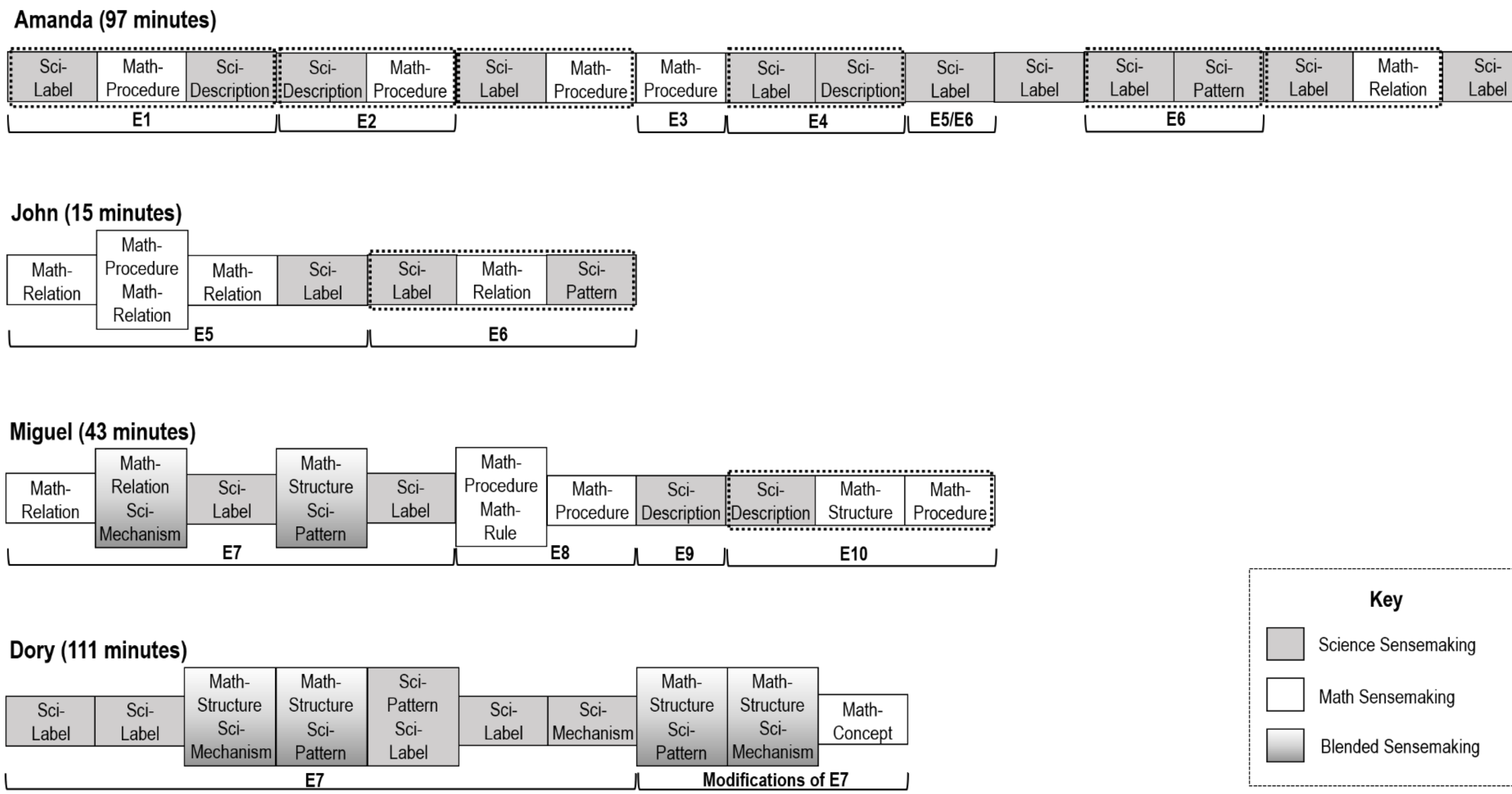


Figure 3.5. The organization of sensemaking opportunities identified during each instructor's lesson on equations about population growth. The types of sensemaking opportunities are presented in sequential order. Mathematics sensemaking is placed in a white box, and science sensemaking is placed in a gray box. Sensemaking opportunities that cooccurred are placed in a single box. A gradient background indicates a co-occurrence of sensemaking from both mathematics and science dimensions. Each sensemaking type corresponds to a single activity except where a dotted box indicates that different types of sensemaking opportunities occur in the same activity. E= Equation. Instructional time is not represented in the figure.

3.4.2.1 Blended Sensemaking

Blended sensemaking has been used to analyzed sstudents' problem-solving process (Bain et al., 2018; Bing & Redish, 2007, 2008; Eichenlaub & Redish, 2019; Kuo et al., 2013). This study provides evidence for blended sensemaking during instruction. Miguel and Dory both have instances where science sensemaking and mathematics sensemaking co-occur (gradient boxes in Figure 3.5). Moreover, in these instances, one type of sensemaking is used to support or explain the other type of sensemaking. This is the blended sensemaking that Bain et al. (2018) refer to as high quality sensemaking in students' problem sovling. For example, Dory uses mathematics sensemaking of the equation to support science sensemaking of the equation as she prompts students to reason about how the mathematical structures depict the scientific ideas. Miguel also has students engage in blended sensemaking when he references the scientific mechanism for population growth (cell division) to explain the mathematical exponent function in the equation. While Dory spends a considerable portion of the whole-class discussion on blended sensemaking, Miguel's references to blended sensemaking tend to be brief occupying no more than a few minutes. Students who spontaneously engage in blended sensemaking during quantitative problem-solving have greater success (Bain et al., 2018; Eichenlaub & Redish, 2019; Kuo, et al., 2013; Schuchardt & Schunn, 2016). Students' struggle with solving quantitative problems in science has been attributed to students isolating their scientific knowledge from the calculation process (Eichenlaub & Redish, 2019). Providing explicit connections between mathematics and science sensemaking during instruction may allow more students to develop blended sensemaking and thus have improved facility with problems which differ from those seen in instruction (Schuchardt, 2016; Schuchardt & Schunn, 2016).

3.4.2.2 Coordinated Sensemaking

Sensemaking from the same dimension co-occursin John and Miguel's lessons (Figure 3.5). Because the sensemaking involves coordination of two types of sensemaking within the same dimension to make sense of a mathematical equation, we refer to this as "coordinated sensemaking". In Miguel's lesson, two types of mathematical sensemaking (*Math-Procedure* and *Math-Rule*) co-occur and one is used to support the other one. Miguel cues the mathematical rules to explain the steps of manipulating the equation. John presents steps of calculation to support the understanding of the mathematical relationship of exponential growth. This type of coordination of sensemaking does not fit the blended sensemaking that has been described in the literature of making sense of equations (e.g., Bain et al., 2018; Bing & Redish, 2007).

This type of coordinated sensemaking shares similarities with the definitions of sensemaking in mathematics which focus on generating coherence (Li & Schoenfeld, 2020). However, in mathematics, sensemaking is defined as the coherence between formal (procedural) and conceptual mathematics (Li & Schoenfeld, 2020). The type of coordination seen here does not precisely fit the mathematical definition. Future research needs to be done to determine if coordinated sensemaking within a single dimension supports students understanding of the mathematics behind an equation and whether that impacts problem-solving.

3.4.2.3 Adjacent Sensemaking

All instructors have instructional episodes where different types of sensemaking, either within the same dimension or across dimensions, are presented sequentially within the same activity but are not explicitly connected to one another. We are labeling these instances as “adjacent sensemaking”. One example of adjacent sensemaking across dimensions occurs when Amanda and Miguel both connect the variables in the equations to quantifiable characteristics in the scientific phenomenon (*Sci-Label*), and then immediately show the steps of how to solve the equations to calculate an answer (*Math-Procedure*). However, the *Sci-Label* is not used to explain or support the way the mathematical procedures are performed. This type of instruction seems to capture an often-described process for instruction of equations in science which focuses on learning the terminology and doing calculations (Hansson et al., 2015). In another example, John provided *Sci-Pattern* and *Math-Relation* within one activity without offering learning opportunities for students about how the pattern in population growth is related to the quantitative relationship as represented by the equation. It is important to recognize the existence of adjacent sensemaking because the proximity of the mathematical and science sensemaking within the lesson gives students the opportunity to independently make connections across the dimensions. Thus, these students may blend sensemaking of the equations. However, because the connection is not made explicit, many students might not be blending across the mathematics and science sensemaking dimensions. If future research reveals that adjacent sensemaking occurs more commonly than blended sensemaking in instruction of equations in science classes, this may explain why very few students use blended sensemaking when solving problems (Bain et al., 2018; Bing & Redish, 2009).

3.5 Conclusion

By showing that instances of sensemaking can be identified and by providing new ways of describing blended and unblended sensemaking, this study offers a new and valuable way to explore these nuances in enacted lessons of equations in science. In this study, four instructors provide different types of sensemaking opportunities in their classes even when teaching equations related to the same biological phenomenon. By emphasizing different sensemaking aspects of the equations, instructors make available different resources to students that have the potential to affect students' science and mathematics knowledge, an avenue for future research. This study also identifies different ways instructors organize the different sensemaking opportunities, as distinct events contained within separate activities or as co-occurring events within the same activity. In addition to the previously described blended sensemaking (Bain, et al., 2018; Bing & Redish, 2007), this study identified two other types of co-occurrence of sensemaking, adjacent and coordinated. Coordinated and blended sensemaking makes explicit connections between two types of sensemaking where one type of sensemaking is used to support or explain another type of sensemaking. However, adjacent and separate sensemaking events are not explicitly connected. Explicitly addressing the links between different type of knowledge embodied in equations makes these connections available to all students. Based on prior work (Eichenlaub & Redish, 2019; Kuo et al., 2013; Schuchardt & Schunn), the ability to move back and forth between connected sensemaking resources has the potential to improve students' ability to solve more complex and novel quantitative problems in science. Research on the effect of instruction that presents separated, adjacent, coordinated and blended sensemaking on students' problem-solving will allow instructors to choose the type of sensemaking opportunity that best achieves their instructional goals.

3.6 Future Directions

In these four case studies, the contexts of two of the four instructors (Miguel and Dory) differed from the context of Amanda and John, which may influence instructors' choice on what sensemaking opportunities to provide. This paper does not attempt to distinguish the effect of context on presentation of sensemaking opportunities or choice of pedagogical strategies. Additional research is needed to explore these effects.

We also note that the pedagogical strategies chosen by the four instructors differ. Because the focus of this paper was on identifying sensemaking opportunities during instruction, the different pedagogical strategies provided by the instructors were not discussed. Future analyses will examine these strategies and the co-occurrence with sensemaking for a larger sample of teachers to determine if different pedagogical strategies are associated with the availability of different sensemaking opportunities for students.

4 Exploring Students' Descriptions of Mutation from A Cognitive Perspective Suggests How to Modify Instructional Approaches

4.1 Introduction

In biological organisms, mutations (changes in DNA sequence) are the source of variation. Variation is a key concept in the biological sciences which impacts students' understanding of two core areas, evolution and genetics (American Association for the Advancement of Science, 2011; Shaw, Van Horne, Zhang, & Boughman, 2008). However, few studies have been conducted on students' understanding of mutation. Existing studies have investigated students' misconceptions of mutation in the context of natural selection (Abraham et al., 2009; Brumby, 1984), genetics (Shaw et al., 2008; Smith & Knight, 2012), and more broadly as a source of variation (Bray Speth et al., 2014; Prevost et al., 2013).(Abraham et al., 2009; Brumby, 1984; Gregory, 2009; Mills Shaw et al., 2008; Nehm & Ridgway, 2011; Smith & Knight, 2012) This study investigates students' understanding of the origin of mutations using a cognitive approach to examine how students describe mutation in their written responses to a short answer question. The results provide insight into students' perspective on mutation that might be underlying their expression of noncanonical ideas in multiple contexts.

4.1.1 Mutation is a Fundamental Concept that is Difficult for Students to Understand

The idea of mutation is fundamental in understanding biology. Mutation can be defined as a "technical term for some change in the individual gene" (Carlson, 2011, p.1). The process of mutation can involve chromosomal rearrangements or the generation of small changes at the DNA level through errors in the action of DNA polymerase when replicating DNA. These errors can either occur spontaneously or can be induced by changes in the DNA structure caused by chemicals or X-rays. Mutation provides the source of variation essential for evolution to occur and is responsible for the cumulative changes that result in cancer. The field of genetics studies how mutations are passed from parents to offspring. A scientific understanding of mutation is therefore vital for biologists to understand evolution and genetics, and also for the public to make sense of issues such as exposure to mutagens and genetically modified foods.

Research has shown that mutation is a difficult concept for students in multiple contexts. In genetics education, where an understanding of mutation is critical to understanding genetic diseases, existing research has shown that students often hold incorrect ideas about the nature and consequences of mutations (Shaw et al., 2008; Smith & Knight, 2012). For example, on the Genetics Concept Assessment, students selected answers stating that “A change is a mutation only if it will produce a change in the amino acid sequence" and that "A frameshift mutation cannot lead to an early stop of translation" (Smith & Knight, 2012). Similarly, Shaw et al. (2008) found that students hold noncanonical ideas about the effect of a mutation on an organism. In essays responding to open ended questions about genetics, students frequently wrote that one mutation always causes one disease, a deterministic idea that is not consistent with the expert perspective that is more probabilistic about the outcomes of mutations. Holding deterministic views about mutation may explain why some students have difficulty with concepts such as incomplete dominance or the idea that inheriting a mutation in a cancer-associated gene is not a death sentence but only increases the probability of eventually developing cancer. While instruction that aligns with promising practices in science education has been shown to decrease the frequency of students selecting the most common incorrect answers for questions on the nature or effects of mutations, the incorrect response rate is still over twenty-five percent (Smith & Knight, 2012).

Mutation is also fundamental to understanding evolution, a core concept for biology literacy, because mutation provides the variation that is acted upon by evolutionary processes (AAAS, 2011). Multiple studies have found that students have difficulty understanding the role of mutation in the process of evolution, either not recognizing the role of mutation as the source of organismal variation or believing mutation is caused by environmental change (Abraham, et al., 2009; Bray Speth et al., 2014; Brumby, 1984; Gregory, 2009; Nehm & Ridgway, 2011; Prevost, et al., 2013). Even after explicit instruction, one-third of 182 students did not include mutation in their conceptual models for the origin of variation in the context of evolution, suggesting that failing to recognize the role of mutation in evolution is relatively robust to instruction (Bray Speth et al., 2014). When students do recognize the role of mutation in producing variation, they often state that mutations will suddenly appear in response to environmental changes (Abraham, et al., 2009; Brumby, 1976; Gregory, 2009; Nehm & Ridgway, 2011). Even after instruction designed to target this noncanonical idea, forty percent of students chose a response indicating that mutations were caused by a change in the environment (Clarke-Midura, Pope, Maruca, Abraham, & Meir, 2018).

Many of these studies explored students' understanding of the role or effects of mutation within the context of genetics or evolution (Bray Speth et al., 2014; Gregory, 2009; Shaw et al., 2008; Smith & Knight, 2012). Only rarely has students’ understanding about the mechanism of mutation been investigated. One exception is Prevost et al.'s

(2013) study which asked students to describe in writing how new alleles arise in a population. This lexical analysis revealed that students tended to describe types of mutations instead of the mechanism producing mutations. One limitation of this study was that in a lexical analysis, the removal of context from the analysis may have obscured the extent to which students understand the mutation mechanism.

Prevost et al. (2013) did not suggest an explanation for their results. Using theory derived from cognitive science, we hypothesize that students were treating mutations as an entity that existed independently of the process that produced it. We conducted an in-depth qualitative analysis of students' written responses to investigate whether students have a tendency to think of mutation only as an entity and not as the process/mechanism that produces changes in DNA. Students' focus on mutation as an entity may cause attributes associated with processes (e.g., steps, time, interactions between entities) to be ignored accounting for some of the catalogued noncanonical ideas about mutation. In the following section, we explore the background of entity and process thinking from cognitive psychology.

4.1.2 The Cognitive Perspective of Viewing Scientific Phenomena as an Entity or a Process

Bringing a cognitive lens to discipline based education research permits movement from descriptions of difficulties (misconceptions) to investigating patterns in learners' process of learning. The cognitive perspective emphasizes the nature of knowing and learning, which includes how concepts and theories are understood and descriptions of organized patterns in cognitive activities (Greeno, Collins, & Resnick, 1996). The cognitive perspective has been used as a framework to explain students' difficulties in understanding biological phenomena (Coley & Tanner, 2012; Gregory, 2009). The results of some studies have suggested that students' incorrect ideas about scientific phenomena may be arising from cognitive explanatory biases, such as teleological thinking (the phenomenon occurs to fulfill a goal) or anthropocentric thinking (organisms act as conscious agents) (Coley & Tanner, 2015; Nehm & Ridgway, 2011; Opfer, Nehm, & Ha, 2012). Opfer et al. (2012) proposed that students might resort to intuitive ideas if they lack a "scientifically normative explanation", i.e. mechanistic causes for the phenomena. However, others have shown that students' tendency to select cognitive construals is influenced by the context (Gouvea & Simon, 2018).

Other cognitive theorists have suggested that students' difficulty in explaining scientific phenomena stems not from their tendency to ascribe an incorrect mechanistic cause but from their tendency to misclassify the phenomenon as an entity instead of a process (Chi, Slotta, & de Leeuw, 1994). Entities are matter or objects, while processes are events or activities. According to this theory, the attributes for "Entities" and "Processes" are mutually exclusive (Chi et al., 1994; Chi, 2013). Entities "can be contained" (Chi, 2013), while processes cannot; processes are associated with the attribute of "occurring over time", while entities are not.

In physics, several researchers have shown across multiple contexts that students have a tendency to misattribute processes to the entity category (Chi et al., 1994; Eshach, Lin, & Tsai, 2018; Lancor, 2014; Reiner, Slotta, Chi, & Resnick, 2000; Slotta & Chi, 2006). This misattribution has been associated with what have been labeled robust misconceptions, noncanonical ideas that are very difficult to correct (Chi et al., 1994; Eshach et al., 2018; Lancor, 2014; Reiner et al., 2000; Slotta & Chi, 2006). For example, in physics, students tend to incorrectly describe heat as a collection of objects instead of a process involving the movement of molecules. Thus, they will incorrectly talk about "hot molecules" (Chi, 2013, p. 57). According to Chi (2013), this type of miscategorization is problematic because entities and processes do not share common attributes. Processes such as heating, electricity flowing through a circuit, or baking bread have an element of time associated with them. Entities such as the electrons contained in the circuit, the bread or the incorrect "hot molecules" do not occur over time (although they may continue to be in existence). Similarly, no process can have physical properties such as color, volume, or location (Chi, 2013). When students do miscategorize processes as entities, Chi (2013) claims that this is when robust misconceptions arise that are resistant to instruction because it is not enough to point out the misconception. Instead, students have to change the way they categorize. For example, students tend to struggle with understanding heat transfer, and they often express the incorrect idea that "coldness from the ice flows into the water" (Chi, 2013, p. 51). Achieving long-lasting change requires more than simply changing this one idea. It requires a fundamental shift in the idea that heat transfer is something that occurs as the result of a transfer of a thing, "coldness" that contains "cold", to a process-based concept that includes the movement and collision of molecules over time. The coding schemes to distinguish processes and entities employed by many of the researchers using this cognitive framework tend to be specific to the scientific phenomenon. Therefore, it is helpful to look elsewhere for a more general characterization of what constitutes evidence of process-based reasoning by students.

4.1.3 Characterizing Processes and Entities Using the Theory of Mechanism

In the science education literature, researchers have characterized students' explanations of scientific phenomenon in terms of their understanding of scientific mechanism (Russ, Scherr, Hammer & Mikeska, 2008). As reviewed in Russ et al. (2008), scientific mechanism is often equated with scientific process. While disagreement exists on the nuances, most definitions of scientific mechanism agree that a scientific mechanism describes how entities within a phenomenon interact to produce outcomes associated with the phenomenon (Bechtel & Abrahamsen, 2005; Glennan, 2005; Illiari & Williamson, 2012; Machamer, Darden, & Craver, 2000). Relating this definition back to the idea of categorization of entities and processes, the entities would be defined as the objects in the phenomenon. In the example of heat transfer when hot and cold liquids are mixed, the entities would be the molecules in the hot and cold liquids. The mechanism is the heat transfer process which occurs through the interaction of molecules spatially and temporally. As the molecules move and bump into one another, they affect the motion of other molecules, either speeding them up or slowing them down. Students who refer to hotness as a thing to be transferred are missing the temporal and spatial interactions that are essential for understanding how one activity leads to the next to produce the end product (medium liquid temperature) from the starting conditions (hot and cold liquid).

"Mechanistic reasoning involves more than noting which causes are associated with which effects" (Russ et al. 2008, p. 506). Russ et al. (2008) gives the example of a student that responds to a teacher's prompt by stating that "Gravity's pulling the book down before the paper" (p. 518). According to the framework for mechanistic reasoning that they developed, this student has named an activity "pulling down" that is engaged in by gravity; the gravity is the cause for the effect of "pulling down". In this phrasing, the mechanistic process of gravity is a thing that does the pulling. Without additional reasoning about either the chaining of events (the steps in a mechanism) or the interactions occurring between the entities involved in the mechanism of gravity (e.g., the paper and the Earth), this statement shows a lack of mechanistic understanding of the scientific phenomenon of objects falling down.

Slotta, Chi, and Joram (1995) make a similar argument for the importance of context when they discuss how to categorize "wind" in two very similar example statements. They state that on example statement "That sound you hear is just the wind moving through the canyon" (p. 385) should be categorized as an entity statement for "wind". In contrast, Slotta et al. (1995) state that another example statement about wind, "The wind in the canyon is just the air moving" (p. 385), as evidence of process thinking about wind. They explain that the different contextual aspects associated with students' descriptions of what is moving results in the two different categorizations, but don't expand in a way that is generalizable to other phenomena.

However, by applying Russ et al.'s (2008) framework, it is possible to develop more generalizable parameters for distinguishing between entity and process statements. In the first statement ("wind moving through the canyon") Russ et al. (2008) would probably have said that the student had named a cause (wind) for the effect (sound) but had not described a mechanism. On the other hand, the second statement which was classified as a process by Slotta et al. (1995) contains indicators of two categories of Russ et al.'s (2008) framework that are associated with mechanistic descriptions: "Identifying Organization of Entities" and "Chaining". The second statement specifies how two entities (air and the canyon) are organized with respect to one another: the wind in the canyon is caused by the air *moving [in* the canyon]. There is also an indication of chaining because the statement implies that the air has to move in the canyon for wind to occur. These two features (Identifying Organization of Entities and Chaining) are absent for the process of wind from the entity-classified statement “the wind moving through the canyon”, which merely states that the wind is an entity that is moving through the canyon [although one could make the case that this is a process statement for sound].

4.1.3.1 The dual nature of mutation as an entity and a process

Unlike many of the physical phenomena that have been studied which can only be seen as either an entity (e.g. temperature), or a process (e.g. heating), the word “mutation” describes both an entity (the change in the DNA) and a process (how that change is produced). (Another example is adaptation, which can be defined both as a feature (an entity) that becomes more frequent in a population as a result of a selective process, and as a process by which populations change over time in response to specific environmental pressures.) Both the entity-based perception and process-based perception of mutation have affordances and limitations depending on context. For example, an entity-based perception of mutations facilitates understanding inheritance because it is necessary to know that mutations/alleles have fixed locations on chromosomes to understand recombination and the effect of the chromosome movements on sorting alleles in to gametes. On the other hand, the perception of mutation as a dynamic process with attributes such as time, steps and interactions is critical to understanding other biological processes. The process-based perspective of mutation is required to understand why the development of cancer is a multistep process and why a change in the environment cannot directly cause a change in the DNA in evolutionary processes.

Experts switch with ease between entity and process-based associations (Chi, 2013; Gupta, Elby, & Conlin, 2014; Gupta, Hammer, & Redish, 2010). However, based on prior research in physics (Eshach et al., 2018; Lancor, 2014; Reiner et al., 2000; Slotta & Chi, 2006), it is likely that students tend to perceive mutations primarily as entities, subjugating the process that produced them.

Adopting a cognitive lens and investigating how students perceive mutation rather than what students know about mutation, will permit a broader instructional approach that will elevate students' scientific understanding across multiple contexts. Moreover, such research has the potential to add to cognitive understanding because the entity-based perception of mutation is not a misconceived idea as in prior studies (Brookes, Horton, Van Heuvelen, & Etkina, 2005; Brookes & Etkina, 2015; Gupta et al., 2014, 2010; Slotta & Chi, 2006). This study uses a free-response question to elicit the ideas that are accessible to students about mutation, what students can access from their memory in that context (Tulving & Pearlstone, 1966). The prompt question was adapted from Prevost et al. (2013) and was intentionally left vague to elicit entity/process responses rather than to assess what students know about mutation or the mechanisms for mutation. We used qualitative analyses to answer the research questions:

In written responses to a prompt asking students to explain how mutations originate in populations of organisms:

1) What topics do students discuss in their answers with respect to mutation?

2) To what extent do students describe mutation as an entity or as a process?

3) What types of noncanonical ideas are expressed?

4) How do students' noncanonical ideas about mutation expressed in their writing relate to their entity/process focused treatment of mutation?

4.2 Methods

4.2.1 Context

Participants in this study were students enrolled in the first semester of an introductory course for biology majors at a large Midwestern research university in the United States of America. Sixty-four students (*N* = 64) completed the assessment questions used in this study, 90% of whom were freshmen. The course was co-taught by two experienced instructors with an average teaching experience of over 15 years who have been trained in active learning techniques.

Analysis of the textbook (Brooker, Widmaier, Graham, & Stiling, 2015) used by students in this study showed that in the chapter that covers mutation, the seventeen figures depict the types and effects of germ-line versus somatic cell mutations, the phenotypes for genetic disease caused by point mutations, and types of structural changes in the DNA caused by mutagens. All of these figures treated mutation as an entity and none depicted the process of mutation.

4.2.2 Data Collection

The free response question was administered during the 11th week of a fifteen-week long course, after students had received instruction in mutation and DNA replication. This method was chosen in order to reveal what perspectives are accessible to (readily retrieved in the moment) students versus what might be available (previously memorized information which can be elicited if the right cue is provided) (Tulving &

Pearlstone, 1966). While multiple choice questions tend to capture whether students know the correct answers in response to explicit or implicit cues in the prompts and distractors, free response questions can elicit a more complex thinking or a higher-level cognitive processing (Martinez, 1999). Compared to the interview format, the use of written responses to questions avoids potential cueing by the interviewers in interviews and is less resource intensive for larger samples (National Research Council, 2001; Wiggins, 1993).

The two questions in Table 4.1 were adapted from Prevost et al. (2013). Students randomly received either one of the questions. We used two organisms here because research has shown that assessment item features and contexts (such as organism type) can affect students' thinking (Heredia, Furtak, & Morrison, 2012; Nehm & Ha, 2011). Because research has shown that students tend to describe processes as entities (Eshach et al., 2018; Lancor, 2014; Slotta & Chi, 2006), this question was phrased to theoretically bias towards a process-based description of mechanisms underlying mutation. The word "how" was used as to prompt for descriptions of a process. Prior studies have shown that 42% of primary and secondary students answered “how” questions about a scientific phenomenon with causal or mechanistic explanations and another 43% provided a reason for the phenomenon (Abrams & Southerland, 2001). The word "originate" was used because theoretically this word should lead students into describing the molecular mechanisms *producing* mutations. Because we were also interested in whether students would have a dual perspective on mutations in their writing, the process cues were balanced by the use of the word “mutations” as a noun, an entity cue.

Table 4.1
Free response questions used to reveal student perspectives on mutation

Questions	
Version 1	Explain how mutations originate in populations of lizards. (You can illustrate your answer with drawings.)
Version 2	Explain how mutations originate in populations of bacteria. (You can illustrate your answer with drawings.)

4.2.3 Data Analysis

All sixty-four participants responded to the assessment questions. Thirty students responded to the question on lizards. Thirty-four responded to the question on bacteria. Student responses were analyzed in three phases. Phase 1 catalogued the topics contained in students' responses. Phase 2 analyzed whether the responses reflected an entity- or a process-focused treatment of mutation. Phase 3 characterized noncanonical ideas about mutation contained in students' responses.

4.2.3.1 Phase 1: Coding for topics in students' descriptions

The first phase of coding broadly characterized the topics students mentioned with respect to mutation. The goal of this phase of coding was to determine whether students mentioned a cause for the origin of mutations.

Six codes were developed from phrases in students' responses using inductive coding (Glaser & Strauss, 2017): "DNA Replication", "Induction", "Other Cause", "Definition/Types", "Natural Selection", and "Reproduction". Examples of these codes are shown in Table 4.2. A single student response could be assigned multiple codes because students could mention multiple topics with respect to mutation in their answer and therefore, the unit of coding for Phase 1 is a phrase within the entire response. Six responses included drawings along with the written answers. The drawings were analyzed using these six codes to capture the ideas the drawings conveyed.

Thirty responses (n=30) were coded by the first author who has a master's degree in botany and is pursuing a doctoral degree in science education and independently by the second author who has a doctoral degree in biology and in science education. The average Cohen's kappa was 0.94, indicating an almost perfect reliability (McHugh, 2012).

The six codes reflecting topics mentioned by students in their answers were grouped in to three topic categories (Table 4.2): Cause, Description, and Propagation. The Cause category contains codes that mention a cause for mutation whether correct (i.e., DNA replication or Induction) or incorrect (Other Cause, e.g., change in the environment). The Description category contains the Definition/Types Code. In this category, students describe what a mutation is or the types of mutations. The Propagation Category contains codes that refer to how a mutation will spread through a population (Natural Selection) or be propagated from parent to offspring (Reproduction).

Table 4.2

Descriptions and examples of six codes developed in the process of coding for response topics (The correspondent phrase was bolded in student response.)

Category	Code	Description	Student response
Cause	DNA replication	The student includes replication of the hereditary material of the organism as a way that mutation originates.	**Mutations originate *via* errors in replication of DNA,** or through horizontal transfer of DNA among individuals. (Jennifer)
	Induction	The student claims that mutation is caused by agents in the environment, which include chemicals, or radiation. The student can use "outside factors", "induced", when talking about mutagens.	Mutation can occur in a variety of ways. Sometimes mutations occur spontaneously during DNA replication. If the lizard population is **exposed to mutagens such as UV radiation or certain chemicals**, this could also cause mutation. (Alan)
	Other Cause	The student includes causes other than DNA replication, induction for how mutations arise, which may include transcription, transduction, change in the environment.	There are different ways that mutations originate in bacteria, although I can really only think of a few. **The insertion of viral DNA can cause mutations** by changing the DNA sequence. Other ways to alter the DNA sequence, which causes mutations, is the addition or deletion of base pairs within DNA that can create a whole new amino acid sequence by changing one or more of the amino acids. (Sandy)
Description	Definition/ Types	Student talks about what a mutation is, e.g. a change in the sequence or student lists types of mutation (e.g., insertions, deletion).	A lot of things can cause mutations. **It could be a frameshift, missense, nonsense mutation. These can be insertions, deletions, or frameshift** that can either change or not affect the amino acid sequence that will be transcribed and translated. These are then passed down from the parents' gametes. (Holly)
Propagation	Inheritance	Student includes part of or the whole process of reproduction to describe how mutation gets passed down and preserved in the population.	**It begins with a mutation in one organism and when it spreads either it survives in the environment or it dies**. (Anne)
	Natural Selection	The student talks about part or the whole process of how the mutation benefits or harms the organisms as related to natural selection. Or the student talks about how the environment interacts with the presence of certain mutation.	Mutations originate slowly. Usually neutral mutations will arise that do not do anything for the individual's fitness. However, **some lizards may accumulate an advantageous mutation which increases its fitness, allowing it to reproduce more. Because of this, there will be more lizards in the population with this mutation.** (Kelly)

4.2.3.2 Phase 2: Coding for Defining Characteristic, Steps, Interactions

The second phase of coding was to determine whether within the entire response, students were treating mutation as an entity or a process. Student responses that did not contain a Cause code (DNA Replication, Induction, Other Cause), were coded as treating mutation as an entity. These responses did not even name an activity that could result in the production of mutation. Thus, they did not even make a cause and effect association (Russ et al, 2008).

However, for the responses that were coded as Cause, we needed to determine whether students were simply identifying a cause /effect association or whether they were providing a process-based description. Therefore, this phase of coding moved beyond coding which words were mentioned in students' responses to a coding scheme that captured context (Slotta et al., 1995). Drawing on the distinctions between entity and process descriptions provided by Slotta et al. (1995) as well as the Framework for Mechanistic Reasoning (Russ et al., 2008), we developed a coding scheme that applied to this context but may have broader applications as well. Levels 6 (Organization of Entities) and 7 (Chaining: Backward and Forward) of the Framework for Mechanistic Reasoning (Russ et al., 2008) indicate that students have moved beyond naming a cause or activity and therefore, merely showing a knowledge of input-output relations (Russ et al., 2008; Slotta et al., 1995). Levels 6 and 7 indicate an acknowledgement of the mechanism/process that underlies this relationship: how an output was dynamically produced from the initial state. The Framework for Mechanistic Reasoning was developed to characterize the discourse of young children in conversation with one another. However, this context involves written descriptions by undergraduate students in response to a prompt. In this static context, it can be difficult to detect evidence of chaining. Therefore, we chose to more specifically define chaining as “Steps”; the response shows evidence of sequential steps. The Level 6 code, Organization of Entities is vague and could describe spatial organization between entities that were not associated with one another, although that does not seem to be the way it was applied (Russ et al., 2008). Therefore, this code was more specifically defined for this context as “Interactions”, evidence that two or more entities interacted with one another in some way. These codes of “Steps” and “Interactions” also better encapsulate the mutation mechanism which is a series of activities (binding, copying, repairing) that is carried out by more than one entity that interacts with one another (Machamer et al., 2000). If the response containing a named causal activity did not show evidence of steps or interactions, then the response was coded as naming a “Defining Characteristic”. In these cases, the named activities are attributes (albeit causal) of the entity, but don’t demonstrate a knowledge of a process or mechanism (Russ et al., 2008).

The three quotes in Table 4.3 will be used to demonstrate how this coding scheme was applied. All three of the responses are presented in their entirety. Jennifer stated that “Mutations originate via errors in replication of DNA, or through horizontal transfer of DNA among individuals. She has named two activities, one partially correct (“DNA replication”) and one incorrect (“horizontal transfer”) that according to her cause mutations (“mutations originate via”). Mutations were treated as the effect of these causal activities, the end product. Moreover, mutations were equated with “errors in replication of DNA” another entity (errors) that result from an activity (DNA replication). There is no evidence in this response of the dynamic mutational process that involves interactions between DNA and DNA polymerase resulting in errors that may then be repaired by the same enzyme or a different enzyme. Attributes of the entity have been defined (caused by x and y), but not attributes of the process (entities interacting in a spatiotemporal sequence to convent an initial state in to an end state).

Table 4.3
Descriptions and examples of three codes used to code for Entity- or Process-focused phrases (The Entity-focused phrase is italicized, and the Process-focused phrase is double-lined.)

Code	**Description**	**Student response**
Defining Characteristic (Entity focused)	The student describes the process as a definition statement and provides no explanation of the process.	Mutations originate *via errors in replication of DNA, or through horizontal transfer of DNA among individuals*. (Jennifer)
Steps (Process focused)	The student attempts to describe steps in the process from a starting point to an ending point.	In the process of DNA replication/transcription, there are very rare events in which an enzyme either makes a mistake when replicating DNA by changing a nucleotide in a sequence or an RNA polymerase encoding RNA makes a transcript that slightly differs from the desired product. Any mistakes, such as these, that are not corrected by the cell permanently alter the genotype of that cell as well as any subsequent offspring made with genetic info from altered organism. (Brendan)
Interaction (Process focused)	The student attempts to describe the interaction among distinct objects or between objects and the outside factors to explain the dynamics in the process.	During DNA replication in a single gamete, a mutation may occur. Mutations arise from DNA polymerase attaching an incorrect base pair at any point during the replication process. If this mutated gamete is able to grow into maturity with this mutation and reproduce, the mutation will begin to spread through the population. If the mutation happens to provide an advantage in fitness, the mutation will continue to spread via higher rate of reproduction of the more fit mutated lizards. (Victor)

Contrast Jennifer's response to Brendan's response. Interestingly, Brendan did not mention the word mutation. Instead, he described a series of events involving interactions between more than one entity. First, he stated that there were "rare events in which an enzyme either makes a mistake when replicating DNA by changing a nucleotide in a sequence"[1]. Even this initial statement differs in a key way from Jennifer's statement that "mutations originate via errors in replication of DNA". While Jennifer named mutations as the output of an activity, Brendan is describing a process that occured over time when he proceeded "Any mistakes, such as these [referring to the output of the first activity], that are not corrected…." Thus, he named a second activity "correction" that occurs after the first, describing a mutation process that has two sequential steps and therefore must occur over time – a key attribute of processes, but not entities.

The third response, by Victor, also differs from Jennifer's response (albeit more subtly). Victor began by saying that "During DNA replication in a single gamete, a mutation may occur." If Victor had stopped there, then this would have been equivalent to Jennifer's naming of an activity that causes an output, a thing called "mutation". However, Victor continued "Mutations arise from DNA polymerase attaching an incorrect base pair at any point during the replication process." In this part of the description, Victor described one entity (DNA polymerase) doing something (attaching) with another entity (base pair); describing an interaction between the two entities (DNA polymerase and base pair).

Using this coding scheme, all responses that named a causal activity for mutation whether correct or incorrect (Cause category) were coded for Defining Characteristic,

[1] Brendan also incorrectly states that the initial event could be RNA polymerase making a mistake during transcription. We are not coding for correctness of answers, but indicators of steps or interactions.

Steps, and Interactions by the first author and independently by the second author. The average Cohen's kappa was 0.93, indicating an almost perfect reliability (McHugh, 2012).

4.2.3.2.1 Categorizing responses as an entity, process or dual focused treatment of mutation

Any causal response that was coded as Steps or Interactions was categorized as process-focused treatment of mutation because we argue that these codes show evidence within the written description of attributes associated with processes. Any causal response that was coded as Defining Characteristic used the activity to define/characterize mutation as an entity that result from a particular activity and was thus classified as Entity-focused. Because some responses were coded in Phase 1 as Cause and either Propagation and/or Description, some responses showed evidence of both Entity-focused treatment of mutation (from the Propagation or Description) and Process-focused treatment of mutation (if the Cause was coded as containing Steps and Interactions). These responses were classified as Dual-focused.

4.2.3.2.2 Coding for the entity/process treatment of Propagation of mutations

To determine if students could apply a Process-focused characterization to another mechanism, we re-examined responses that were coded as naming activities associated with Propagation of mutation using the same coding scheme used for Cause of mutations (described in Table 4.3). Therefore, all phrases of Propagation of mutation were coded to see if students described the steps or interactions associated with reproduction (the process of how mutations are passed down from parents to the next generation) or evolution (the process of how mutations are spread in the population under natural selection), or simply provided defining characteristics. These responses will be described in detail in the Results section. All of the Propagation phrases (n=33) were coded by the first author and independently by the second author. The average Cohen's kappa was 0.82, indicating a strong reliability (McHugh, 2012).

4.2.3.3 Phase 3: Coding for noncanonical ideas

All responses were analyzed for the presence of noncanonical ideas about mutations. Noncanonical ideas that appeared at least twice in students' responses were captured and grouped in to categories. Three categories were identified: Incorrect Cause, Confound Mechanisms, Determinism. For example, determinism denotes when a student elaborates on the effect of mutation, the student conveys the idea that one mutation will lead to a change in trait, or the student states that only beneficial mutations or non-harmful mutations can be passed on (e.g. "If this mutation is silent it won't impact anything, if it is detrimental to the survival, that bacteria will be less fit, if it is beneficial, it could be passed on to gametes proceeding it.") The description of these codes and examples are listed in Table 4.4. The first author and a second coder with a doctoral degree in molecular biology coded all responses (n=64) independently, and the average Cohen's kappa was 0.85, indicating a strong reliability (McHugh, 2012).

Table 4.4
Descriptions and examples of codes used for identifying noncanonical ideas (the noncanonical idea is highlighted using wavy lines)

Code	**Description**	**Student response**
Incorrect Cause	The student indicates that mutations originate because of environment change or evolutionary mechanisms.	Mutations can occur from random chance, by gene flow, genetic drift**…**
Confound Mechanisms	The student describes a mechanism that produces mutation, and the student uses transcription or translation instead of DNA replication.	In the process of DNA replication/transcription, there are very rare events in which an enzyme either makes a mistake when replicating DNA by changing a nucleotide in a sequence or an RNA polymerase encoding RNA makes a transcript that slightly differs from the desired product. Any mistakes, such as these, that are not corrected by the cell permanently alter the genotype of that cell as well as any subsequent offspring made with genetic info from altered organism.
Determinism	When the student elaborates on the effect of mutation, the student conveys the idea that one mutation will lead to a change in trait. Or the student states that only beneficial mutations or non-harmful mutations can be passed on.	Mutations can be random accidents that arose during transcription during binary fission, or if phages or other bacteria inject their DNA into the bacteria. They can also occur due to mutagens. If this mutation is silent it won't impact anything, if it is detrimental to the survival, that bacteria will be less fit, if it is beneficial, it could be passed on to gametes proceeding it.

4.3 Results

The first section of the results will discuss how students treat mutation in their responses: as an entity or as a process. We will then present the types of noncanonical ideas that were observed and the association with the treatment of mutation. All example responses are presented in their entirety. Responses from both question contexts, the lizards and bacteria, were collapsed because no significant differences in the patterns described below were identified according to the organism type (Table 4.5).

Table 4.5
The distribution by question context of coded topic phrases, and the treatment of mutation

Context	Coded topic phrase			Treatment of mutation		
	Cause	Description	Propagation	Process-focused only	Entity-focused only	Dual-focused
Lizard (N=30)	23	16	18	2	23	5
Bacteria (N=34)	25	16	15	0	32	2
Total	48	32	33	2	55	7

Note. There was no significant difference between lizards and bacteria for coded topic phases ($\chi^2 = 0.35$, $p = .84$), or treatment of mutation (Fisher exact test, $p = .10$).

4.3.1 Phase 1: Most Students Named A Cause for Mutation

Out of 64 student responses, 48 mentioned a cause for mutation, suggesting that 75% of students recognized that the question was asking for a causal response. Twenty-five percent of student responses (16) only mentioned Description or Propagation without listing or describing any cause. Fifty percent of the 64 responses included the topic of Description indicating that they contained a definition of mutation as a change in DNA or described the different types of mutations. Fifty-two percent of the 64 responses contained the topic of Propagation stating how mutations were passed down to the next generation or spread in the population. On average, each student response included 1.8 topic categories (i.e. Description or Propagation or Cause).

4.3.2 Phase 2: Few Responses Reflected A Process-focused Treatment of Mutation

Of the 64 student responses we analyzed, nine contained Process-focused phrases with respect to mutation – phrases that described either steps or interactions in the mechanism of mutation (Table 4.3). Out of these nine responses, two contained only

Process-focused phrases and were classified as Process-focused only responses (Table 4.6). Seven contained both Process-focused and Entity-focused phrases and were classified as Dual-focused responses (Table 4.6).

Table 4.6
Distribution of student responses under Process-focused, Entity-focused, and Dual-focused categories (N=64)

Treatment of Mutation	Number of students	Percentage of students
Process-focused only	2	3.1%
Entity-focused only	55	86%
Dual-focused	7	10.9%

4.3.2.1 A Process-focused only treatment of mutation

Only two out of 64 responses provided Process-focused treatment of mutation without also containing an Entity-focused description of mutation. Ethan’s response to the prompt is:

> Mutations in a lizard population can arise spontaneously but this is less common due to proofreading and other counter measures. Additionally, if a lizard is exposed to chemicals, UV radiation, or any number of other mutagens, the number of mistakes in DNA transcription/translation increases and therefore there is less of a chance that all these get caught and either fixed or destroyed by the body.

His response provided evidence for a process-focused treatment of mutation, because multiple steps (which occur over time) were included in his description. The steps were 1) exposure to mutagens (“if a lizard is exposed to chemicals, UV radiation, or any number of other mutagens" 2) mistakes in DNA occur ("the number of mistakes in DNA transcription/translation increases"), 3) mistakes get caught ("therefore there is less of a chance that all these get caught"), and 4) mistakes are "either fixed or destroyed". Although Ethan confounded DNA replication with DNA transcription/ translation, we are not coding for correctness of answers, but for indicators of steps or interactions.

4.3.2.2 A Dual-focused treatment of mutation

Seven out of 64 responses included both phrases that indicated a focus on process and ones that indicated a focus on describing mutation as an entity. Mary stated:

> Mutations originate in a population of bacteria when the polymerase molecule makes a mistake when extending the second strand of complementary DNA. This mistake can be inserting the wrong base pair. This might happen because there are 4 nitrogenous base pairs, 2 pyrimidines and 2 primines; Adenine pairs with Thymine while Guanine pairs with Cytosine. The polymerase could accidentally pair the other pyrimidine with the primine. This may be harmful during translation of the mRNA of the gene. This substitution mutation can be silent in that the codon still codes for the same amino acid. It could code for another amino acid which may not have a huge effect on the protein other than changing the protein shape slightly making it more difficult for the protein to function properly. Or it could code for a stop codon which would terminate translation early and completely throw off protein function.

Mary's response was classified as Process-focused because it was coded as showing evidence of Interactions. Mary included two sections that were indicative of polymerase interacting with base pairs (indicated by the underlined sections). The most explicit statement occurs when she stated that “The polymerase could accidentally pair the other pyrimidine with the primine.” Although not coded as explicitly showing evidence of Steps, there is evidence that this response contains a suggestion of time associated with the formation of mutation with the use of the phrase “when polymerase molecule makes a mistake when extending…”. Mary also included descriptions of one type of mutation (a “substitution mutation”) which depicted mutation as an entity to be categorized and was coded as Description. Therefore, Mary’s response can be categorized as a Dual-focused treatment of mutation.

Three of these seven students whose responses reflected a Dual treatment of mutation described the propagation of mutation through the population after describing the mechanism of mutation.

4.3.3 The Majority of Student Responses Reflected an Entity-focused Treatment of Mutation

Student responses that only contained Entity-focused phrases were much more common that those that contained both. Eighty-six percent of (55) students treated mutation only as an entity in their responses. Table 4.7 shows the distribution of coded topic phrases that were mentioned under each of the response categories. For example, 55 out of 64 student responses treated mutation as an entity, either by describing the Cause of mutation (DNA Replication or Induction or Other Cause) as a defining statement, or by only including the Description or Propagation of mutations. As summarized in Table 4.7, out of 55 Entity-focused responses, 39 student responses named a causal activity that was coded as describing a Defining Characteristic of mutation Cause and thus classified as Entity-focused. The remaining 16 responses did not include any Cause phrases. Twelve of these responses were coded as providing a Description (categorizing or defining mutation) and eleven were coded as Propagation (discussing how mutations are inherited).

Table 4.7
The distribution of coded topic phrases under response categories of Entity-focused, Process-focused and Dual-focused treatments of mutation

Entity/Process Focused (Phase 2 codes)	Topics Mentioned (Phase 1 codes)		
Response Category (64 student responses)	Cause	Description	Propagation
Entity-focused (55)	Cause as a Defining Characteristic (39)	Yes (16)	Yes (10)
			None (6)
		None (23)	Yes (9)
			None (14)
	No Cause (16)	Yes (12)	Yes (7)
			None (5)
		None (4)	Yes (4)
			None (0)
Process-focused (2)	Cause with Steps or Interactions (2)	Yes (0)	-
		None (2)	Yes (0)
			None (2)
Dual-focused (7)	Cause with Steps or Interactions (7)	Yes (4)	Yes (0)
			None (4)
		None (3)	Yes (3)
			None (0)
Phrases total	48	32	33

4.3.3.1 Entity-focused responses that mentioned a cause for mutation

Out of 48 student responses that mentioned Cause, 39 responses treated the cause as a defining characteristic and mutation is treated as an entity – something to be defined. In the response below, Emily defined a mutation as an entity that can be “caused by something (e.g. UV radiation)", noting the cause (UV radiation) and the effect (mutation) without delineating the dynamics of the process (the Steps or Interactions. Emily proceeded to characterize the types of mutations ("addition of a base, deletion of a base") and the possible effects, treating mutation as the end product to be categorized. Because Emily's response provided no evidence of the steps or interactions (the dynamic activity) associated with a process even though she named a causal activity, her response was coded as Entity-focused.

> Mutations can either be random or can be caused by something (e.g. UV radiation). There are a few types of mutations that may occur such as addition of a base, deletion of a base, or it may switch the base (not complimentary). It can have different effects on expression depending on the type of mutation and the location of the mutation, for example a mutation in the 3rd spot of an amino acid group will most likely not change the expression of the gene.

4.3.3.2 Entity-focused responses that didn't mention a cause for mutation

Because the question asked students to explain how mutations originate, not how they are propagated or for a definition, it was expected that all students would mention a cause for mutation. However, sixteen students did not even mention a cause for mutation (Table 4.7). Instead, these students defined a mutation and/or listed types of mutations (Description phrases) and/or wrote about how mutations were transferred from parents to offspring or propagated in a population (Propagation phrases). All of these responses were classified as Entity-focused treatment of mutation. For example, Kelly wrote:

> Mutations originate slowly. Usually neutral mutations will arise that do not do anything for the individual's fitness. However, some lizards may accumulate an advantageous mutation which increases its fitness, allowing it to reproduce more. Because of this, there will be more lizards in the population with this mutation.

This response described one type of mutation, "neutral mutation" and the Propagation of mutation “some lizards may accumulate an advantageous mutation which increases its fitness, allowing it to reproduced more”. Both of these phrases treat mutations as a thing that can be classified into bins or can be propagated in the population. There is some evidence of time associated with the phrase “Mutations originate slowly” but even in this phrase mutations are treated as something that appears over time, not a process which is causing something to appear. Therefore, Kelly’s response is characterized as Entity-focused.

4.3.4 The Majority of Student Responses Included a Process-focused Treatment of Propagation of Mutation

Surprisingly, considering that the prompt asked about origination of mutations, not propagation, over half (33 of 64) of students' responses included Propagation phrases that characterized how mutations were transferred from parent to offspring either in a single mating or in populations (Table 4.7). Although mutation was treated as an entity (something that can be passed) in the Propagation phrases, the mechanism of Propagation has the potential to be described as a process or an entity. We wanted to know whether students could adopt a process-focus when describing a biological phenomenon, as opposed to whether they did when writing about mutation. Therefore, in the 33 responses that contained Propagation phrases, student writing about propagation was coded using the same three codes used to distinguish between an Entity and Process-focused treatment of mutation (Defining Characteristic, Steps, Interactions).

Holly wrote:

> A lot of things can cause mutations. It could be a frameshift, missense, or sense mutation. These can be insertions, deletions, or frameshift that can either change or not affect the amino acid sequence that will be transcribed and translated. These are then passed down from the parents' gametes.

Holly included the topic of Propagation, stating that mutations are passed down in the process of reproduction. This reflected an Entity-focused treatment of Propagation, because the phrase provided a naming activity "passed down" without any dynamics (Interaction or Steps) in the process of Propagation. On the other hand, Ali's response shows evidence of a Process-focused treatment of Propagation. He wrote:

> If it was a helpful mutation, then natural selection will benefit this lizard+ it will be more fit than the other lizards + will have more offspring. These offspring will have that mutation + will also be more fit, so they will have also have more offspring + pass the mutation onto them. Eventually after generations, the mutation will be prominent in the population.

This response included the sequence of activities as steps for the process of Propagation, starting from the lizard with a helpful mutation, then the lizard will become more fit, then it will have more offspring, then these offspring will have more offspring, until the termination condition "be prominent in the population". Ali described multiple steps in the process for how mutations can be propagated in a population, thus this response was categorized as a Process-focused treatment of Propagation.

As shown in Figure 4.1, Propagation phrases were mentioned in 33 student responses, and Cause phrases were mentioned in 48 student responses. The first bar shows the proportion of Entity-focused responses (21%, grey) and Process-focused responses (79%, black) for Propagation of mutation; the second bar shows the proportion of Entity-focused responses (83%, grey) and Process-focused responses (17%, black) for Cause of mutation. The results show that the majority of responses described Propagation of mutations as a process (black portion in first bar in Figure 4.1), while the majority of responses described Cause for mutations as an entity (grey portion in second bar in Figure 4.1).

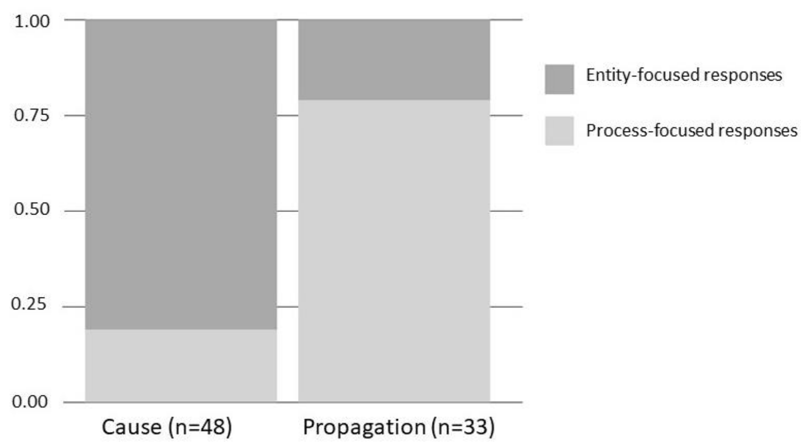


Figure 4.1. Percentage of Entity- and Process-Focused responses in responses that described Cause of mutation versus Propagation of mutation.

Alex's response shown below included an Entity-focused treatment of mutation and a Process-focused treatment of Propagation of mutations.

> Due to either mistakes in gene replication or a mutagen substance in germlines or an embryo, a new allele arises. Rarely that allele is beneficial

and few individuals that have it can reproduce more frequently and are more fit than others to survive. Natural selection acts on the new allele and select for that and against the older one. After a long time and if conditions have remained stable the new allele is dominant in the population.

The Cause phrase defined "mistakes in gene replication or a mutagen substance" (something) as the cause for mutation (bold and italicized phrase). He has named an activity and ascribed a cause ("mistakes in gene replication or a mutagen substance in germlines or an embryo") to an effect ("a new allele arises") but has not provided a mechanistic characterization of mutation that could be coded as Interactions or Steps. In contrast, the depiction of Propagation includes multiple steps: 1) "Allele is beneficial", 2) "Few individuals that have it can reproduce more frequently", 3) "Natural selection acts on the new allele and select for that and against the older one", and 4) "After a long time and if conditions have remained stable the new allele is dominant in the population." The phrase of "after a long time" (while not coded) is also indicative of a Process-focused treatment of Propagation because time is a key attribute of processes.

4.3.5 Noncanonical Ideas Occur in Both Process- and Entity-Focused Responses

Because it has been proposed that students' process/entity perspective is associated with noncanonical thinking (Chi, 2013), responses were also coded for the presence of noncanonical ideas. Three main types of noncanonical ideas were identified after coding the 64 student responses: Incorrect Cause (including a cause other than DNA replication or Induction to explain how mutations arise), Confound Mechanism (confounding transcription or translation with DNA replication), Determinism (mutation must lead to change in protein or phenotype; only beneficial or non-harmful mutations can be passed down). Twenty-eight students had responses with noncanonical ideas that fit in to these three categories (only two students had more than one noncanonical idea). Table 4.8 shows the association between the process/entity treatment of mutation in the response and the type of noncanonical idea.

Table 4.8
Noncanonical ideas about mutation organized by process/entity treatment of mutation

Entity/Process Treatment of Mutation	Noncanonical ideas		
	Confound Mechanism	Determinism	Incorrect Cause
Dual-focused responses (N=7)	2	1	
Process-focused responses (N=2)	1		
Entity-focused responses (N=55)	3	5	19

Most of the students (19 out of 28) with a noncanonical idea mentioned Incorrect Cause for mutation, with or without mentioning the correct causes for mutation of DNA replication or induction. Four students included the idea that mutations occur to adapt to the environment. Four students who answered the bacteria question included transduction as the cause for mutation. Eleven students each mentioned a random cause such as "mishap in the cell formation", "random cellular metabolic processes", "crossing over", "gene flow", "horizontal transfer" etc. All of these nineteen responses provided an Entity-focused description of mutation; no step or interaction was mentioned to explain the origin of mutation.

Six responses confounded transcription/translation with DNA replication. These were categorized separately from Incorrect Cause because it was difficult to ascertain whether this was a naming difficulty or whether students really did not understand how mutations are generated. Out of these six responses, three responses were Entity-focused descriptions, one was a Process-focused description, and two reflected a Dual treatment of mutation. For example, Luke wrote "Most mutations occur through *a random set of ‘mistakes’ that occur through DNA replication, translation and/or transcription*", using the correct cause "DNA replication" and incorrect cause "translation and/or transcription" to define mutation. Brendan's response reflecting a Dual treatment of mutation stated:

> In the process of DNA replication/transcription, there are very rare events in which an enzyme either makes a mistake when replicating DNA by changing a nucleotide in a sequence or an RNA polymerase encoding RNA makes a transcript that slightly differs from the desired product. Any mistakes, such as these, that are not corrected by the cell permanently alter the genotype of that cell as well as any subsequent offspring made with genetic info from altered organism.

Brendan included the steps that 1) an enzyme makes a mistake, and 2) mistakes are not corrected, indicating a Process-focused treatment of mutation (double lined phrases). He also described that the mutation was spread to "subsequent offspring", treating mutation as an entity that can be passed down to offspring. Therefore, this whole response was coded as a Dual treatment of mutation. However, the interaction of RNA polymerase with RNA was incorrectly listed as being involved in mutation (wavy lined phrase), which is a noncanonical idea confounding transcription with DNA replication.

Six responses, five of which are Entity-focused descriptions of mutation, included a form of Determinism for two types of ideas. Four out of those six responses wrote that only beneficial or non-harmful mutations can be passed on. For example, Sally included the phrases that "If this mutation is silent it won't impact anything, if it is detrimental to the survival, that bacteria will be less fit, if it is beneficial, it could be passed on to gametes proceeding it." Sally described three effects of mutation, silent, detrimental and beneficial, but indicated that only beneficial mutations can be passed down, which is a deterministic idea. Two responses included the idea that a mutation would definitely cause a change in phenotype, as Kay wrote:

> Mutations originate in bacteria when part of the DNA sequence is changed. This can be the addition, removal or replacement of nucleotides. When the nucleotides are changed, the DNA can not code for what is was originally intended for. If it codes something different, for example, spots on an animal's fur, this is considered a mutation. In bacteria, these mutations are then expressed in it, such as the function or form of the bacteria.

The definition and types of mutations were listed ("the addition, removal or replacement of nucleotides"), while Kay mentioned that any mutation will lead to a change in phenotypes (wavy lined phrase) which is a deterministic idea.

4.4 Discussion

This study identified three ways students treated mutation in response to the prompt: “Explain how mutations originate in populations of lizards/bacteria”: Entity only, Process only, and Dual (both Entity and Process). In contrast to previous studies on how surface features (such as organism type) affect students' thinking (Heredia, Furtak, & Morrison, 2012; Nehm & Ha, 2011; Prevost et al., 2013), students in our study tended to treat mutation as an entity for both organism types. There were slightly greater proportion of responses for the lizard that were Process-focused or Dual-focused (7/30 for lizards versus 2/34 for bacteria. However, this tendency did not rise to the level of significance. Given the paucity of responses containing a process perspective, a larger sample would be necessary to draw firm conclusions about the effect of organism type on students' responses.

We found that even after instruction in DNA replication and mutation, the majority of students treated mutation as an entity in response to a prompt that calls for an explanation of the process of mutation in both the lizard and bacterial populations. It is worth noting that 48 students (75%) interpreted the question as intended, at least naming a cause for an effect. However, this study shows that most (39/48) of these students do not go beyond naming an activity that produces an effect which is not sufficient to show the mechanistic reasoning (Russ et al, Machamer, Craver and Darden) that we argue is associated with process-focused descriptions based on parallels with Slotta et al. (1995). This extends prior studies that found that students tend to describe physical phenomena as entities (Eshach et al., 2018; Gupta et al., 2014; Lancor, 2014; Reiner et al., 2000; Slotta & Chi, 2006; Wiser & Amin, 2001). In these studies, the description of physical phenomena as entities instead of processes was often incorrect (Reiner et al., 2000; Wiser & Amin, 2001). Our findings show that even when an entity and process focused characterization of the scientific phenomenon are both valid, students tend to provide an entity-focused description.

These results suggest that theoretical discussions of why students use an entity perspective to explain scientific phenomenon needs to move beyond targeting incorrect categorization of phenomena. It has been proposed that incorrect categorization of scientific phenomena as entities instead of processes may be associated with students' noncanonical ideas about the phenomenon (Chi, 2013). Therefore, this study also examined the association between entity and process focused categorization of mutation and noncanonical ideas about mutation. We found that noncanonical ideas were associated with approximately half of Entity-focused responses and approximately half of Process-focused responses. Therefore, with respect to phenomena where either an Entity-focused or Process-focused treatment is correct, there does not seem to be an association between a particular perspective and the presence of noncanonical ideas.

4.4.1 Consequences of an Entity-Focused versus Process-Focused Perspective of Mutation

Although students tend to describe mutation as an entity regardless of the context, experts in biology maintain dual perspectives and switch fluidly between the two depending on the context, in the same way that experts in physics switch between Entity-focused metaphors and Process-focused explanations (Brookes & Etkina, 2015; Brookes et al., 2005; Gupta et al., 2014, 2010). In physics, straddling these two mindsets is believed to be productive and essential to understanding and communicating concepts such as energy and gravity (Gupta et al., 2014, 2010). Gupta et al. (2014, 2010) have suggested that the Entity-focused descriptions for non-material concepts can be used to better communicate with peers when discussing Process-focused concepts (Gupta et al., 2010), and this Entity-focused view can help learners focus on the sense making process rather than on using the "right" words (Gupta et al., 2014). Chi and colleagues have argued against the use of entity perspectives of processes even as analogy or metaphor because this treatment could lead to noncanonical ideas that will be hard to displace (Slotta, 2011; Slotta & Chi, 2006). The data presented here suggest that noncanonical ideas are associated with responses containing only Entity-focused perspectives and with responses containing Dual-focused and Process-focused perspectives. Moreover, the majority of students with a Process-focused perspective are capable of switching between an Entity and Process-focused perspective within the same short response. Furthermore, as we discuss below, there are affordances to students using both perspectives when considering mutation. We suggest therefore, that it may be appropriate for instructors to leverage students' existing Entity-focused perceptions of a phenomenon, particularly in cases where both perspectives are valid with the goal of switching perspectives to suit the context. Additional research needs to be done to determine the affordances and constraints of such an approach when the Entity-focused perspective is not legitimate.

In biology, the word "mutation" has both Entity-focused attributes (fixed location) and Process-focused attributes (steps and interactions) depending on the context. An Entity-focused view of mutation (e.g. with a fixed location on the chromosome) may help learners focus on sense making of the process of inheritance of mutation, while the Process-focused view of mutation initiates a mechanistic thinking and the sense making of the effects that mutation can have in the processes of natural selection and the development of genetic diseases or cancer. If students stick to an Entity-focused view of mutation, they may struggle with concepts requiring an understanding of Process-focused attributes of mutation. For example, students may have difficulty associating the idea that it takes time for mutations to form and accumulate in cancer (accounting for the benefits of early detection). Additionally, they may have difficulty recognizing that an interaction between the environment and the DNA would be necessary in order for an organism to mutate in response to the environment, accounting for the persistence of this often-heard noncanonical idea (Abraham, et al., 2009; Bray

Speth et al., 2014; Brumby, 1984; Gregory, 2009; Nehm & Ridgway, 2011). Others have attributed the persistence of this idea to students' difficulty in understanding the randomness of mutation (Abraham et al., 2009; Bray Speth et al., 2014) or to the presence of cognitive biases, such as intentionality or teleology (Brumby, 1984; Gregory, 2009; Nehm & Ridgway, 2011). However, an alternative explanation is that students may not be recognizing that mutation is a process that has attributes of time, steps and interactions. Therefore, they would not consider that most environmental contexts would not physically be able to cause changes in the DNA. They would also be unlikely to associate the idea of randomness with mutation because they would not be considering mutation as a process that has this attribute. Instead, they would be thinking of mutation as an entity that already exists.

4.4.1.1 Noncanonical ideas appeared in both Process-focused and Entity-focused treatment of mutation

Chi et al. (1994) have argued that misconceiving processes as entities can lead to robust misconceptions, because the attributes associated with these two categories are mutually exclusive. Thus, students would incorrectly assign inappropriate Entity-focused attributes to the process. In the contexts studied by Chi and others, the fundamental error is the incorrect assignment of a process to an entity category (Chi, 2013; Slotta & Chi, 2006). Therefore, it has been argued that correcting students' categorization will correct these robust misconceptions (Chi, 2013).

This study, however, found that students who had process descriptions and those who had entity descriptions both expressed noncanonical ideas. For example, the noncanonical idea of describing errors in transcription as responsible for mutations was present in all categories of descriptions (dual, process only, entity only treatment of mutation). It could be that this noncanonical idea is of the type Chi (2013) calls "false beliefs" or "flawed mental models" and may not be the robust type that is associated with confounding entities and processes. This type of noncanonical idea can be corrected relatively easily by explicit or implicit refutation, e.g. showing students the correct belief or model and asking students to compare and contrast it with their noncanonical ideas (Chi, 2013).

4.4.4.2 Effect of an absence of a Process-focused perspective on students' understanding of phenomena

The most common non-canonical idea in this study is ascribing an Incorrect cause (19 responses out of 64). All 19 responses show an Entity-focused treatment of mutation. The wide variety of incorrect causes students specified as generating mutations aligns with previous studies that showed students had difficulty in understanding mutation (Bray Speth et al., 2014; Prevost et al., 2013). Because only 9 Process-focused descriptions were identified in our study as opposed to 55 entity-focused descriptions, future research on a sample with more Process-focused responses is needed to determine if a Process-focused view of mutation inoculates students against noncanonical ideas associated with incorrect causes.

4.4.2 Students are Capable of Process-focused Explanations of Scientific Phenomenon

To our knowledge, this is the only scientific phenomenon with a dual nature where student treatment of the phenomenon has been investigated using the cognitive theory of entity/process classification. In previous studies, the Entity-focused perspective of investigated scientific phenomenon is often an incorrect understanding (Eshach et al., 2018; Gupta et al., 2014; Lancor, 2014; Reiner et al., 2000; Slotta & Chi, 2006; Wiser & Amin, 2001). While in this study, an entity-focused perspective is a choice that students make which reflects how they access their knowledge about mutation. Our results showed that in response to a question that should have elicited a description of the process of mutation, most students defaulted to an entity-focused description of mutation, but one that is not incorrect. Their responses include phrases that define mutation as altered DNA or errors in DNA replication, or describe types of mutations, or show the propagation of mutations in the population. Most students do not include phrases that describe the processes of mutation. They don't describe the interactions and steps in the process of mutation even though they are answering this prompt after instruction in DNA replication and mutation. This indicates that mutation as a process is not accessible to them when they answer this prompt, even though in other contexts, they may show that this information is available to them (Tulving & Pearlstone, 1966).

Many students were able to access information about interactions and steps of another scientific phenomenon in their descriptions. Taking advantage of students' spontaneous and unexpected production of descriptions of propagation of mutations (either from parent to offspring or within populations), we investigated whether students produced Entity or Process-focused descriptions of propagation. Many students who produce an Entity-focused response about mutation are capable of process descriptions of propagation. Almost twice as many descriptions of the process of propagating a mutation in a population are Process-focused as opposed to Entity-focused. This implies that students are not inherently “process thinkers” or “entity thinkers”. Instead, it is necessary to look at the instructional context surrounding mutation as opposed to propagation.

4.4.2.1 Instructional context favors entity-focused treatment of mutation

Students' tendency to describe mutation as an entity could be the result of the tradition in biology instruction of a treatment of mutation that emphasizes the entity-focused perspective. The textbook used in this class indicates bias towards depicting mutations as entities to be characterized instead of processes with steps and interactions of objects. A similar pattern was observed with another commonly used textbook for college biology majors, Biological Sciences (Freeman et al., 2017). Similarly, in physics, physical phenomena tend to be treated as an entity in textbooks, as over 80% of the clauses containing "heat" in two of three mainstream textbooks for introductory physics are describing "heat" as an entity, suggesting that instruction may play a role in students' entity-focused view of physical phenomena as well (Brookes et al., 2005).

Exposure to the entity-focused perspective of scientific phenomenon may be reinforced by the coverage of science within the classroom. In the classroom in this study, a review of notes and classroom artifacts showed that the instructors spent much more time covering mutation as an entity (either embedded within a process) or describing the types of mutations than they did on explaining the process. This may reflect instructors' prior training (e.g., evolution has a more Entity-focused treatment of mutation than molecular biology) and/or a traditional view of how mutations should be taught. The instruction of mutation reported in previous studies also focused on the treatment of mutation as an entity (Clarke-Midura et al., 2018; Smith & Knight, 2012). On the other hand, instructors may perceive that the Process-focused perspective of mutation should be taught in an advanced biology class rather than an introductory class. Bray Speth et al. (2014) mentioned that in an introductory biology class mutation was taught as the origin of variation after instruction of evolution, because "a detailed overview of different molecular types of mutation and of DNA repair mechanisms was beyond the scope of the course" (p. 531).

More broadly, students may also have been conditioned by prior science instruction to view science as a collection of facts rather than a set of explanations of how and why things happen (Danielak, Gupta, & Elby, 2014; McComas, Clough, & Almazroa, 1998; Russ, 2018; Sandoval, 2003). For example, researchers have shown that educators tend to reward students' knowledge of the correct terminology, rather than reward their efforts in making sense of the phenomenon (Russ, 2018; Russ, Coffey, Hammer, & Hutchison, 2009; Talanquer, Bolger, & Tomanek, 2015). This conditioning may explain why others have found that students struggle with constructing causal explanations for scientific phenomena (Abrams, Southerland & Cummins, 2001; Haskel-Ittah & Yarden, 2018).

4.5 Implications for Instruction

This study shows that students tend to describe mutation as an entity, not including the Process-focused attributes of mutation, when they were answering how mutations originate. Because a Process-focused perspective can benefit students' sense making of other biological phenomena, we suggest that the instruction should emphasize the Process-focused attributes of mutation, e.g. the steps and interactions, while minimizing the instruction of categorization or definition of mutations. In addition, our findings suggest that instructors need to be aware that the characteristics of mutation that are accessible to novices may not include process-focused attributes. Therefore, instructors should clarify when they use the word "mutation" whether they are referring to the process or the entity. Such steps may help to improve communication and minimize the development of noncanonical ideas.

4.6 Future Directions

This study was designed to explore how students described mutation, not what they could understand about the mechanism of mutation. Moreover, the study was designed to explore what ideas about mutation were accessible to students not what was available to them. Thus, it remains unclear from this work whether students' perspective is influenced by their understanding of the mechanism of mutation and whether or not students *can* associate process attributes to mutation (versus whether they *do*). While we chose the approach taken because we wanted to understand what students might be accessing when they hear or see the word "mutation", additional studies are needed to explore these questions. However, our results show that when instructors use the word "mutation" in class in a process context (as in our written prompt), it is likely that students are not associating process attributes with the concept which may be contributing to noncanonical ideas that appear in diverse contexts.

5 Conclusion

5.1 Key Findings across Three Studies

The three studies in this dissertation focused on different aspects of sensemaking in science. The first two studies focused on sensemaking of mathematical equations in science, and study 3 investigated students' sensemaking of a scientific mechanism. These studies contributed to the understanding of the sensemaking provided by instructors in the science classroom and student sensemaking of mechanisms within scientific phenomena.

The Sci-Math Sensemaking Framework was developed in study 1 based on themes identified from an extensive literature review, which includes nine categories grouped into two dimensions of science and mathematics sensemaking about mathematical equations in science. This framework can efficiently move forward the theoretical discussion and analysis of the teaching and learning of mathematical equations in science. Because Sci-Math Sensemaking Framework is a consistent framework with the potential to capture various types of sensemaking in science classrooms, results from different studies on sensemaking of mathematical equations in science can be compared and synthesized with this shared framework. Communications from researchers across science education fields, and even researchers in mathematics education field who used this framework can generate insights into the connections among scientific disciplines and between mathematics and science in science education.

The Sci-Math Sensemaking Framework was then adopted to describe the sensemaking opportunities created by four instructors when equations about the same biology phenomenon were taught. Four instructors were found to provide various types of sensemaking opportunities in their classes and two of the four instructors explicitly linked the science and mathematics sensemaking. This study for the first time provides evidence for blending sensemaking of science and mathematics during instruction and proposes a new way of categorizing blended sensemaking opportunities, as coordinated sensemaking (within a field) or as blended (across fields). Identification of the sensemaking opportunities provided by different instructors in different scientific disciplines using this framework and coordinating with studies on students' problem solving has the potential to identify features of effective teaching approaches of equations in science.

Last, students' sensemaking of mutation, a difficult but important scientific phenomenon in biology, was investigated using students' written responses to open-ended questions. The word "mutation" can be correctly used to describe both an entity and a process, but the qualitative analysis showed that, despite a prompt asking for a mechanistic description, many students default to entity-based responses. Previous studies showed that students tend to describe a process as an entity incorrectly and attributed this to incorrect categorization of processes as entities (Chi, 2013; Chi et al., 1994). Because mutation can be correctly perceived as both a process and an entity depending on context, this study suggests that the explanation for students' tendency to revert to entity-based descriptions of scientific phenomenon needs to move beyond incorrect categorization of phenomenon. The absence of process-focused description of mutation in students' responses indicates that the instruction should be modified to emphasize the steps and interactions in the process.

This dissertation establishes a conceptual framework and provides evidence around the sensemaking around mathematical equations in science, and students' sensemaking of a scientific phenomenon, mutation. Three studies in this dissertation provide new perspectives in understanding the sensemaking of scientific phenomena. The resources embodied in mathematical equations are summarized based on an extensive literature review, which has the potential to guide more research on the teaching and learning of equations. Alternative explanations for students' struggle with making sense of important but difficult scientific ideas are proposed for researchers and instructors and the uniqueness of mutation with a dual nature provides new thoughts on how the existing theory can be used to explain students' learning. The theoretical frameworks of sensemaking in mathematics and science used in this work can bridge the existing findings from previous studies and new understandings gained from this dissertation. This will enable meaningful discussion and future research on key sensemaking features in educational practices and students' learning outcome.

5.2 Implications

These studies on sensemaking have implications for instruction of the equations and processes in science classrooms. The Sci-Math Sensemaking Framework provides nine categories of knowledge that can be used as resources during instruction to achieve students' sensemaking of both mathematics and science knowledge. Findings from this dissertation suggest that instruction should focuses on the steps and interactions during a scientific phenomenon/process, and if mathematical equations can be used to teach this process, instructors need to explicitly connect the mathematical knowledge to the scientific knowledge contained in the equations. Because students tend to memorize scientific facts and mathematical procedures with a lack of conceptual understanding (Bing & Redish, 2008; Schuchardt & Schunn, 2016; Tuminaro & Redish, 2007), the focus of instruction should move away from rote memorization or application without understanding. Instructors should provide sensemaking opportunities for students to use one type of knowledge to support and explain other types of knowledge, especially the connection between mathematics and science knowledge. The potential of using mathematical equations to focus students on the steps and interactions in a scientific phenomenon should be explored in science classes.

5.3 Future directions

The Sci-Math Sensemaking Framework needs to be applied to analysis of teaching of mathematical equations across different scientific disciplines. The categories in this framework are open for modification if additional categories are found based on this analysis. Research needs to be done to determine if the Sci-Math Sensemaking Framework can be used to investigate how students make sense of equations during class and in their problem-solving process, providing understanding of students' ways of sensemaking of equations in science. Another interesting avenue of research would be to how different factors (such as course level, student population, instructor's belief) affect instructors' choices of providing sensemaking space and how these choices impact students' ability to make sense of certain types of knowledge in equations.

Sensemaking of scientific mechanism has been treated as the key feature in science sensemaking (Kapton, 2016). Sci-Mechanism sensemaking is considered as the most advance type of sensemaking in the science dimension, but not all instructors have provided Sci-Mechanism for students. From students' description of mutation as an entity rather than as a process or mechanism, future work is needed that describes whether or how scientific mechanisms are taught in science classes. One potentially fruitful direction will be research on how scientific mechanism can be taught through mathematical equations and whether this approach will focus students' attention on the steps and interactions of the scientific process, rather than the definitions or terminologies in science.

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Standing on the shoulders of giants: Challenges and recommendations of literature search in information systems research. *Communications of the Association for Information Systems*, *37*, 9. https://doi.org/10.17705/1CAIS.03709

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7 Appendix: Themes in 63 Publications Retained for the Development of Sci-Math Sensemaking Framework

ID	Retained publication	Sci-Label	Sci-Description	Sci-Pattern	Sci-Mechanism	Math-Procedure	Math-Rule	Math-Structure	Math-Relation	Math-Concept
R 01	Pospiech, G. (2019). Framework of mathematization in physics from a teaching perspective. In Mathematics in Physics Education (pp. 1–33). Springer.		1	1		1		1	1	
R 02	Bain, K., Rodriguez, J. M. G., & Towns, M. H. (2019). Chemistry and Mathematics: Research and Frameworks To Explore Student Reasoning. Journal of Chemical Education, 96(10), 2086-2096.							1	1	
R 03	Schuchardt, A. (2016). Learning biology through connecting mathematics to scientific mechanisms: Student outcomes and teacher supports (Doctoral dissertation, University of Pittsburgh).	1			1	1		1	1	1
R 04	Odden, T. O. B., & Russ, R. S. (2019). Defining sensemaking: Bringing clarity to a fragmented theoretical construct. Science Education, 103(1), 187-205.									
R 05	Rittle-Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural Knowledge of mathematics. In Oxford handbook of numerical cognition (pp. 1118–1134). Oxford University Press					1				1
R 06	Bain, K., Rodriguez, J.-M. G., & Towns, M. H. (2019). Investigating student		1	1		1		1		

	understanding of rate constants: When is a constant "Constant"? Journal of Chemical Education, 96(8), 1571–1577. https://doi.org/10.1021/acs.jchemed.9b00005									
R 07	Baroody, A. J., Feil, Y., & Johnson, A. R. (2007). An Alternative Reconceptualization of procedural and conceptual knowledge. Journal for Research in Mathematics Education, 38(2), 115–131. https://doi.org/10.2307/30034952					1	1		1	1
R 08	Baxter, J. A., Ruzicka, A., Beghetto, R. A., & Livelybrooks, D. (2014). Professional development strategically connecting mathematics and science: The impact on teachers' confidence and practice. School Science and Mathematics, 114(3), 102–113. https://doi.org/10.1111/ssm.12060			1						
R 09	Becker, N. M., Rupp, C. A., & Brandriet, A. (2017). Engaging students in analyzing and interpreting data to construct mathematical models: An analysis of students' reasoning in a method of initial rates task. Chemistry Education Research and Practice, 18(4), 798–810. https://doi.org/10.1039/c6rp00205f	1		1		1			1	
R 10	Becker, N., & Towns, M. (2012). Students' understanding of mathematical expressions in physical chemistry contexts: An analysis using Sherin's	1		1				1	1	

	symbolic forms. Chemistry Education Research and Practice, 13(3), 209–220. https://doi.org/10.1039/C2RP00003B									
R 11	Bing, T. J., & Redish, E. F. (2007). The Cognitive Blending of Mathematics and Physics Knowledge. AIP Conference Proceedings, 883, 26–29. https://doi.org/10.1063/1.2508683	1				1	1	1		
R 12	Brahmia, S. W., Boudreaux, A., & Kanim, S. E. (2016). Developing Mathematization with Physics Invention Tasks. ArXiv Preprint, (February), arXiv:1602.02033.		1					1		
R 13	Cañadas, M. C., Molina, M., & del Río, A. (2018). Meanings given to algebraic symbolism in problem-posing. Educational Studies in Mathematics, 98(1), 19–37. https://doi.org/10.1007/s10649-017-9797-9							1	1	
R 14	Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for Research in Mathematics Education, 33(5), 352. https://doi.org/10.2307/4149958								1	
R 15	Case, J. M., & Gunstone, R. F. (2003). Approaches to learning in a second year chemical engineering course. International Journal of Science Education, 25(7), 801–819. https://doi.org/10.1080/09500690305033					1				

R 16	Dixon, J. A., Deets, J. K., & Bangert, A. (2001). The representations of the arithmetic operations include functional relationships. Memory and Cognition, 29(3), 462–477. https://doi.org/10.3758/BF03196397						1		1	
R 17	Etkina, E., Warren, A., & Gentile, M. (2006). The role of models in physics instruction. The Physics Teacher, 44(1), 34–39. https://doi.org/10.1119/1.2150757			1	1					
R 18	Even, R. (1990). Subject matter knowledge for teaching and the case of functions. Educational Studies in Mathematics, 21(6), 521–544. Retrieved from https://www.jstor.org/stable/3482390									1
R 19	Gupta, A., & Elby, A. (2011). Beyond epistemological deficits: Dynamic explanations of engineering students' difficulties with mathematical sense-making. International Journal of Science Education, 33(18), 2463-2488.			1		1				
R 20	Haapasalo, L., & Kadijevich, D. (2000). Two types of mathematical knowledge and their relation. Journal Für Mathematik-Didaktik, 21(2), 139–157. https://doi.org/10.1007/BF03338914					1	1			1
R 21	Hansson, L., Hansson, Ö., Juter, K., & Redfors, A. (2015). Reality–theoretical models–mathematics: A ternary perspective on physics lessons in upper-secondary school. Science & Education,	1				1	1			

	24(5–6), 615–644. https://doi.org/10.1007/s11191-015-9750-1									
R 22	Hestenes, D. (2010). Modeling theory for math and science education. In A. Lesh, R., Galbraith, P. L., Haines, C. R., & Hurford (Ed.), Modeling Students’ Mathematical Modeling Competencies (pp. 13–41). https://doi.org/10.1007/978-1-4419-0561-1_3			1	1			1	1	
R 23	Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In James Hiebert (Ed.), Conceptual and procedural knowledge: The case of mathematics (pp. 1–27). Lawrence Erlbaum Associates, Inc.					1	1	1		1
R 24	Hu, D., & Rebello, N. S. (2013). Using conceptual blending to describe how students use mathematical integrals in physics. Physical Review Special Topics - Physics Education Research, 9(2), 1–15. https://doi.org/10.1103/PhysRevSTPER.9.020118	1		1		1		1		1
R 25	Izsák, A. (2004). Students’ Coordination of Knowledge When Learning to Model Physical Situations. Cognition and Instruction, 22(1), 81–128. https://doi.org/10.1207/s1532690Xci2201_4	1						1	1	
R 26	Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children’s algebraic reasoning in elementary school. Journal					1		1	1	1

	for Research in Mathematics Education, 38(3), 258–288.									
R 27	Kuo, E., Hull, M. M., Gupta, A., & Elby, A. (2013). How students blend conceptual and formal mathematical reasoning in solving physics problems. Science Education, 97(1), 32–57. https://doi.org/10.1002/sce.21043	1		1		1		1	1	
R 28	Lazenby, K., & Becker, N. M. (2019). A modeling perspective on supporting students' reasoning with mathematics in chemistry. In M. H. Towns, K. Bain, & J.-M. G. Rodriguez (Eds.), It's Just Math: Research on Students' Understanding of Chemistry and Mathematics (Vol. 1316, pp. 9–24). https://doi.org/10.1021/bk-2019-1316.ch002				1				1	
R 29	Lehavi, Y., Bagno, E., Eylon, B.-S., Mualem, R., Pospiech, G., Böhm, U., … Karam, R. (2017). Classroom evidence of teachers' PCK of the interplay of physics and mathematics. In T. Greczyło & E. Dębowska (Eds.), Key Competences in Physics Teaching and Learning (pp. 95–104). https://doi.org/10.1007/978-3-319-44887-9	1	1			1			1	
R 30	Lehrer, R., & Schauble, L. (2010). What kind of explanation is a model? In M. K. Stein & L. Kucan (Eds.), Instructional Explanations in the Disciplines (pp. 9–22). https://doi.org/10.1007/978-1-4419-0594-9_2		1							

R 31	Levy, S. T., & Wilensky, U. (2009). Crossing levels and representations: The Connected Chemistry (CC1) curriculum. Journal of Science Education and Technology, 18(3), 224-242.								1	
R 32	Michelsen, C. (2015). Mathematical modeling is also physics - Interdisciplinary teaching between mathematics and physics in Danish upper secondary education. Physics Education, 50(4), 489–494. https://doi.org/10.1088/0031-9120/50/4/489			1						
R 33	Moss, J., & Case, R. (1999). Developing children’s understanding of the rational numbers: A new model and an experimental curriculum. Journal for Research in Mathematics Education, 30(2), 122–147.						1	1	1	1
R 34	Njini, P. (2016). Challenges faced by trainee teachers in the learning of the chain rule: a case study of a Midlands Teachers Training College (Doctoral dissertation, BUSE).									
R 35	Peled, I., & Segalis, B. (2005). It’s not too late to conceptualize: Constructing a generalized subtraction schema by abstracting and connecting procedures. Mathematical Thinking and Learning, 7(3), 207–230. https://doi.org/10.1207/s15327833mtl0703_2					1				1
R 36	Phuong, H. T. M. (2019). On the Procedural-Conceptual Based Taxonomy and Its Adaptation to the Multi-Dimensional Approach SPUR to Assess Students’									

	Understanding Mathematics. American Journal of Educational Research, 7(3), 212-218.									
R 37	Pietrocola, M. (2009). Mathematics As Structural Language of Physical Thought. In M. Vicentini & E. Sassi (Eds.), Connecting research in physics education with teacher education. International Commission on Physics Education.	1				1			1	
R 38	Quale, A. (2011). On the role of mathematics in physics. Science & Education, 20(3–4), 359–372. https://doi.org/10.1007/s11191-010-9278-3	1		1						
R 39	Radmehr, F., & Drake, M. (2019). Revised Bloom's taxonomy and major theories and frameworks that influence the teaching, learning, and assessment of mathematics: a comparison. International Journal of Mathematical Education in Science and Technology, 50(6), 895–920. https://doi.org/10.1080/0020739X.2018.1549336					1	1			1
R 40	Redish, E. F. (2017). Analysing the competency of mathematical modelling in physics. In T. Greczyło & E. Dębowska (Eds.), Key Competences in Physics Teaching and Learning (pp. 25–40). https://doi.org/10.1007/978-3-319-44887-9	1		1	1	1		1	1	
R 41	Redish, E. F., & Kuo, E. (2015). Language of physics, language of math: disciplinary culture and dynamic epistemology. Science &	1			1	1		1	1	

	Education, 24(5–6), 561–590. https://doi.org/10.1007/s11191-015-9749-7									
R 42	Rodriguez, J.-M. G., Bain, K., Hux, N. P., & Towns, M. H. (2019). Productive features of problem solving in chemical kinetics: more than just algorithmic manipulation of variables. Chemistry Education Research and Practice, 20(1), 175–186. https://doi.org/10.1039/C8RP00202A	1		1				1	1	
R 43	Rodriguez, J. M. G., Santos-Diaz, S., Bain, K., & Towns, M. H. (2018). Using symbolic and graphical forms to analyze students' mathematical reasoning in chemical kinetics [Research-article]. Journal of Chemical Education, 95(12), 2114–2125. https://doi.org/10.1021/acs.jchemed.8b00584							1	1	
R 44	Schuchardt, A. M., & Schunn, C. D. (2016). Modeling scientific processes with mathematics equations enhances student qualitative conceptual understanding and quantitative problem solving. Science Education, 100(2), 290–320. https://doi.org/10.1002/sce.21198	1			1	1				
R 45	Sherin, B. L. (2001). How Students Understand Physics Equations. Cognition and Instruction, 19(4), 479–541. https://doi.org/10.1207/S1532690XCI1904_3							1	1	
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