

HOMOGENIZATION LIMITS OF THE EQUATIONS
OF ELASTICITY IN THIN DOMAINS

BY

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Homogenization limits of the equations
of elasticity in thin domains

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Introduction

We study pure bending of a flat, linearly elastic three dimensional plate with rapidly varying composition. A uniform coercivity-condition and a uniform boundedness-condition are placed on the constitutive elastic law, but we require no special structure in composition, such as periodicity or quasi-periodicity. It is shown that all limiting vertical displacements, coming from the equations of 3-d elasticity with plate-thickness approaching 0, must necessarily satisfy a fourth order equation of the form

$$(1) \quad \partial_{\alpha\beta} (M_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} w) = F$$

on the (plate-) midplane Ω . The existence of a limiting equation of the form (1) is well known for plates with slowly varying composition [3,14,16]; recently Caillerie has studied plates with rapidly varying periodic composition [2]. The analysis presented here extends those results to plates with arbitrary variation in composition. For the periodic case there is an (essentially) unique limiting rigidity tensor $M_{\alpha\beta\gamma\delta}$. In addition to the particular form of the local variation it only depends on the limiting ratio of the thickness and the length scale of variation. The tensor $M_{\alpha\beta\gamma\delta}$ may be expressed in terms of energies of certain periodic cell problems.

Without structure assumptions about the local composition it is no longer possible to give explicit formulas for the $M_{\alpha\beta\gamma\delta}$. We believe that in interesting cases it may be possible to find upper and lower bounds for its eigenvalues much in the same way as has been (at least partially) done for certain composites [5,12,13,19].

The problem studied here is also closely related to the problem of plates with rapidly varying periodic thickness considered in [7-10]. If voids are thought of as occupied by a material of zero strength, then a plate with

rapidly varying thickness may in principle be thought of as a flat plate with rapidly varying composition. Due to our coercivity-condition, voids are not permitted in the flat plates considered here; they represent an added difficulty which we, at this point, technically do not know how to handle without extra assumptions about the local structure of the composition. Some of the ideas of Γ -convergence may be relevant to this problem, since they have successfully been applied to study limits of non-coercive functionals in other circumstances [1].

Plates with rapidly varying composition are of interest in structural optimization; in certain design contexts they are known to be stronger than plates with only slow variation in composition. We refer the reader to [10] for a more detailed discussion of the relation between an optimal design problem and plates with rapidly varying composition (in that particular discussion the variation in composition has the form of a rapidly varying thickness).

The approach taken here is a variation of the method of H -convergence introduced by Murat and Tartar [15,17]. One major difference is that in this case both the dimension of the domain in which the limiting equation is satisfied as well as the order of the limiting equation differs from that associated to the equations with rapid variations. In the analysis this difference is probably most apparent in the construction of the isomorphism from $H^{-2}(\Omega)$ to $H^2(\Omega)$, which is the candidate for the resolvent of our limiting operator.

The organization of this paper is as follows. In the first section we briefly provide some preliminaries concerning the equations of 3-d elasticity, and in addition we give a precise statement of the convergence result to be proven later. It is very convenient to rescale the thickness variable of the plate to the interval $(-1,1)$; the three dimensional equations are now all in

the same domain - but they become singularly perturbed as the old thickness parameter approaches zero. In section 2 we apply Korn's and Poincaré's inequalities to the solutions of the rescaled equations, and this leads directly to estimates of various expressions and then to statements about weakly convergent subsequences and the structure of their limits. A major part of any convergence argument, using the method of H-convergence, is to construct the isomorphism which is the candidate for the resolvent of the limit operator. In this case such an operator must necessarily map $H^{-2}(\Omega)$ onto $H^2(\Omega)$ and it turns out that it may be constructed from the three dimensional equations by restricting attention to external loads that are uniform throughout the thickness. Section 4 contains the final step of the convergence argument, namely the verification of the right constitutive relationship between curvature and bending moments. This is accomplished by integration by parts of the energy bilinear form. The trial-functions are the solutions to the three dimensional elastic problem with the prescribed loads, and the test functions are picked so that they satisfy the three dimensional elastic equations with an external load which is uniform throughout the thickness, and so that they furthermore have constant curvatures in the limit as the thickness approaches zero. It is possible to select such test functions because of the aforementioned isomorphism. This last part of the proof is a variation of Tartar's div-curl lemma, which again is a special case of the so-called method of compensated compactness [18].

1. Preliminaries and statement of the main result

We shall write $\underline{x} = (x_1, x_2, x_3)$ for vectors in \mathbb{R}^3 and $\underline{x} = (x_1, x_2)$ for vectors in \mathbb{R}^2 . Latin indices will usually range from 1 to 3, and Greek ones from 1 to 2; the summation convention applies whenever indices are repeated. We write $\partial_i = \partial/\partial x_i$ and $\partial_{ij} = \partial^2/\partial x_i \partial x_j$. The three dimensional flat plate

of thickness 2ε is given by

$$R(\varepsilon) = \{\underline{x} : \underline{x} \in \Omega, |x_3| < \varepsilon\}$$

where Ω is a smooth bounded domain in \mathbb{R}^2 and ε denotes a small parameter, with say $0 < \varepsilon < 1$. We shall denote by $\partial_+ R(\varepsilon)$ and $\partial_- R(\varepsilon)$ the upper and lower faces of the plate

$$\partial_{\pm} R(\varepsilon) = \{\underline{x} : \underline{x} \in \Omega, x_3 = \pm \varepsilon\}$$

and by $\partial_0 R(\varepsilon)$ we denote the outer edge of the plate

$$\partial_0 R(\varepsilon) = \{\underline{x} : \underline{x} \in \partial\Omega, |x_3| < \varepsilon\}.$$

Associated with any displacement $\underline{u} = (u_1, u_2, u_3)$ of \mathbb{R}^3 is its strain tensor

$$e_{ij}(\underline{u}) = \frac{1}{2} (\partial_i u_j + \partial_j u_i)$$

and the corresponding stress tensor

$$\sigma_{ij}(\underline{u}) = b_{ijkl} e_{kl}(\underline{u}).$$

We are concerned with spatially inhomogeneous materials and so the components b_{ijkl} of the elastic tensor will be bounded measurable functions in \underline{x} . Furthermore, these functions are permitted to depend on the thickness parameter ε and (except for certain symmetries) they will only be restricted by the following two requirements:

$$(2) \quad b_{ijkl}^\varepsilon(\underline{x}) e_{ij} e_{kl} > c_1 \sum_{i,j} |e_{ij}|^2$$

$$(3) \quad \left(\sum_{i,j} |b_{ijkl}^\varepsilon(\underline{x}) e_{kl}|^2 \right)^{1/2} < c_2 \left(\sum_{i,j} |e_{ij}|^2 \right)^{1/2}$$

a.e. in $R(\varepsilon)$ for any symmetric 2×2 tensor e , with constants $c_1 > 0$ and c_2

that are independent of ε . Note: throughout this paper c_1 and C_2 always refer to these same constants whereas the letters c and C will denote generic positive constants (independent of ε).

Remark 1

One simple example of rapid variation in composition, which falls within the framework of our study and which has received quite a bit of attention, is the case

$$b_{ijkl}^\varepsilon(\underline{x}) = b_{ijkl}^1(\underline{x}/\varepsilon) ,$$

where $b_{ijkl}^1(\underline{y})$ is periodic in \underline{y} and satisfies (2), (3) (cf. [2]). We also mention the work in [7-10] about plates with rapidly varying periodic thickness:

$$\tilde{R}(\varepsilon) = \{ \underline{x} : \underline{x} \in \Omega, |x_3| < \varepsilon h(\underline{x}/\varepsilon^a) \} , \quad 0 < a < \infty .$$

If we denote $h_{\max} = \max h$ then formally, at least, such a plate corresponds to an inhomogeneous material

$$b_{ijkl}^\varepsilon(\underline{x}) = b_{ijkl}^1(\underline{x}/\varepsilon^a, x_3/\varepsilon)$$

with

$$b_{ijkl}^1(\underline{y}) = \begin{cases} c_{ijkl} & |y_3| < h(\underline{y}) \\ 0 & |y_3| > h(\underline{y}) \end{cases}$$

occupying the flat domain

$$\{ \underline{x} : \underline{x} \in \Omega, |x_3| < \varepsilon h_{\max} \} .$$

The possibility that b_{ijkl}^ε may vanish on $R(\varepsilon)$ is technically a significant extra difficulty which we shall not include in our analysis of arbitrary, rapidly varying composition. □

We always assume that the elastic tensor obeys the symmetries

$$(4) \quad b_{ijkl}^\epsilon = b_{jikl}^\epsilon = b_{ijlk}^\epsilon = b_{klij}^\epsilon$$

and that the horizontal planes are planes of elastic symmetry, which means (cf. [11])

$$(5) \quad b_{\alpha\beta\gamma 3}^\epsilon = 0 \quad , \quad b_{\alpha 333}^\epsilon = 0 \quad .$$

Finally we assume that b^ϵ is even with respect to x_3 :

$$(6) \quad b_{ijkl}^\epsilon(\underline{x}, x_3) = b_{ijkl}^\epsilon(\underline{x}, -x_3) \quad .$$

The equations of elastostatic equilibrium for the clamped, vertically loaded, three dimensional plate $R(\epsilon)$ are

$$(7) \quad \begin{aligned} -\partial_j [\sigma_{ij}(\underline{u}^\epsilon)] &= \begin{cases} 0 & i=1,2 \\ \epsilon^2 F^\epsilon & i=3 \end{cases} & \text{in } R(\epsilon) \\ \sigma_{ij}(\underline{u}^\epsilon) \nu_j &= \begin{cases} 0 & i=1,2 \\ \epsilon^3 f_\pm^\epsilon & i=3 \end{cases} & \text{on } \partial_\pm R(\epsilon) \\ \underline{u}^\epsilon &= 0 & \text{on } \partial_0 R(\epsilon) \end{aligned}$$

where $\underline{\nu} = (0, 0, \pm 1)$ denotes the outward normal to $\partial_\pm R(\epsilon)$. The loads are scaled in order to insure that \underline{u}^ϵ stays bounded as ϵ goes to zero. For convenience we shall assume that F^ϵ is even in x_3 and that $f_+^\epsilon(\underline{x}) = f_-^\epsilon(\underline{x})$; the common boundary load is denoted $f^\epsilon(\underline{x})$. Because of the linearity of the problem, this represents no loss of generality in the limit $\epsilon \rightarrow 0$, as the energy corresponding to odd loading is negligible compared to that of even loading for an elastic law with the symmetries (5) and (6) (cf. [8]). Notice that from the assumptions about the loads and (5), (6) it follows that

$$u_1^\epsilon, u_2^\epsilon \text{ are odd, } u_3^\epsilon \text{ is even}$$

$$\sigma_{\alpha\beta}^\epsilon, \sigma_{33}^\epsilon \text{ are odd, } \sigma_{\alpha 3}^\epsilon \text{ is even}$$

with respect to x_3 ; X_ϵ will denote the space of all admissible displacements that obey these symmetries:

$$X_\epsilon = \{ \underline{u} \in H^1(R(\epsilon)) : \underline{u} |_{\partial_0 R(\epsilon)} = 0, u_1, u_2 \text{ are odd and } u_3 \text{ is even in } x_3 \}$$

where $H^1(R(\epsilon))$ is the space of (vector valued) functions with square integrable first derivatives. The problem (7) now has the following variational formulation

$$\underline{u}^\epsilon \in X_\epsilon \text{ and}$$

$$(8) \quad \int_{R(\epsilon)} \sigma_{ij}(\underline{u}^\epsilon) e_{ij}(\underline{v}) d\mathbf{x} = \epsilon^2 \int_{R(\epsilon)} F^\epsilon v_3 d\mathbf{x} + 2\epsilon^3 \int_{\partial_+ R(\epsilon)} f^\epsilon v_3 d\tilde{\mathbf{x}}$$

$$\text{for any } \underline{v} \in X_\epsilon .$$

Unless explicitly stated otherwise we shall always assume that

$$(9) \quad F^\epsilon \in L^2(R(\epsilon)), f^\epsilon \in L^2(\Omega)$$

with

$$(10) \quad F^\epsilon(\underline{x}, \epsilon y) \rightarrow F^0(\underline{x}, y) \text{ in } L^2(\Omega \times (-1, 1)), \text{ and}$$

$$f^\epsilon(\underline{x}) \rightarrow f^0(\underline{x}) \text{ in } L^2(\Omega)$$

as $\epsilon \rightarrow 0$ (the regularity assumptions on the loads can be somewhat relaxed - see Remark 4 - but we do not feel that this serves any purpose in the present context.)

Since we are not imposing any requirements on the structure of the rapid variation in the b_{ijkl}^ϵ , we will not in general obtain convergence of \underline{u}^ϵ

as ε approaches zero. Instead our main result concerns convergent subsequences (which corresponds to the compactness property in the theory of Γ -convergence, cf. [4]).

Theorem

Let $\{\varepsilon_k\}_{k=1}^{\infty}$ be any given sequence converging to zero. There exist a subsequence $\{\varepsilon_{k(\ell)}\}_{\ell=1}^{\infty}$ - for simplicity denoted $\{\varepsilon_{\ell}\}_{\ell=1}^{\infty}$ - and a tensor-valued function $M_{\alpha\beta\gamma\delta}(\underline{x})$ such that

i)
$$M_{\alpha\beta\gamma\delta} = M_{\beta\alpha\gamma\delta} = M_{\alpha\beta\delta\gamma} = M_{\gamma\delta\alpha\beta}$$

ii) $M_{\alpha\beta\gamma\delta}$ lies in $L^{\infty}(\Omega)$ with

$$M_{\alpha\beta\gamma\delta}(\underline{x}) t_{\alpha\beta} t_{\gamma\delta} > \frac{2}{3} c_1 \sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \text{ and}$$

$$\left(\sum_{\alpha,\beta} |M_{\alpha\beta\gamma\delta}(\underline{x}) t_{\gamma\delta}|^2 \right)^{1/2} < \frac{2}{3} c_2 \left(\sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \right)^{1/2}$$

a.e. in Ω , for any symmetric 2x2 tensor t .

iii) For any even external load $F^{\varepsilon_{\ell}}$ and any boundary loads $f_{+}^{\varepsilon_{\ell}} = f_{-}^{\varepsilon_{\ell}} = f^{\varepsilon_{\ell}}$, satisfying (9) and (10), the solution to the problem (7), $\underline{u}^{\varepsilon_{\ell}}$, as ε_{ℓ} approaches zero converges to

$$(-x_3 \partial_1 w(\underline{x}), -x_3 \partial_2 w(\underline{x}), w(\underline{x})),$$

where $w \in H^2(\Omega)$ solves the problem

$$\partial_{\alpha\beta} (M_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} w) = F^0 \text{ in } \Omega$$

$$w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega.$$

The effective load $F^0(\underline{x})$ is given by

$$F^0(\underline{x}) = \int_{-1}^1 F^0(\underline{x}, y) dy + 2f^0(\underline{x}) ,$$

where F^0 and f^0 are the limits from (10). The convergence is in the weak topologies:

$$u_3^{\varepsilon_\ell}(\underline{x}, \varepsilon_\ell y) \rightharpoonup w(\underline{x}) \text{ in } H^1(\Omega \times (-1, 1))$$

$$\frac{1}{\varepsilon_\ell} u_\alpha^{\varepsilon_\ell}(\underline{x}, \varepsilon_\ell y) \rightharpoonup -y \partial_\alpha w(\underline{x}) \text{ in } H^1(\Omega \times (-1, 1)) .$$

Remark 2

It is possible to prove a similar theorem without the symmetry requirement that $b_{ijkl}^\varepsilon = b_{klij}^\varepsilon$ (i.e. without assuming that b_{ijkl}^ε is a symmetric operator on 2×2 tensors). Of course, the resulting tensor $M_{\alpha\beta\gamma\delta}$ will not possess this symmetry either. The coercivity estimate stays the same, but the operator norm estimate for $M_{\alpha\beta\gamma\delta}$, $2/3 C_2$, is replaced by $\frac{2}{3} \frac{(C_2)^2}{c_1}$. We consider the symmetric case, since it is the only physically interesting in the context of elastostatics. □

Remark 3

It follows immediately from the statement of our theorem that

$$\frac{1}{2\varepsilon_\ell} \int_{-\varepsilon_\ell}^{\varepsilon_\ell} u_3^{\varepsilon_\ell} dx_3 \rightharpoonup w(\underline{x}) \text{ in } H^1(\Omega) ;$$

it is actually shown in the proof of the theorem that $\frac{1}{2\varepsilon_\ell} \int_{-\varepsilon_\ell}^{\varepsilon_\ell} u_3^{\varepsilon_\ell} dx_3$ converges strongly towards w in $H^1(\Omega)$. □

Remark 4

As stated earlier, the assumptions (10) on the loads are not the weakest

possible. For the solution of (7) (or rather of (8)) to make sense it is necessary and sufficient that the functional

$$v \rightarrow \epsilon^2 \int_{R(\epsilon)} F^\epsilon v \, d\underline{x} + 2\epsilon^3 \int_{\partial_+ R(\epsilon)} f^\epsilon v \, d\underline{x}$$

is in the dual of $\{v \in H^1(R(\epsilon)) : v|_{\partial_0 R(\epsilon)} = 0, v \text{ is even in } x_3\}$, with the integrals representing appropriate duality pairings.

Define a rescaled functional F^ϵ as follows:

$$\langle F^\epsilon, v \rangle = \epsilon^{-1} \int_{R(\epsilon)} F^\epsilon v(\underline{x}, x_3/\epsilon) \, d\underline{x} + 2 \int_{\partial_+ R(\epsilon)} f^\epsilon v(\underline{x}, x_3/\epsilon) \, d\underline{x}.$$

F^ϵ is then an element of the dual of $\{v \in H^1(R(1)) : v|_{\partial_0 R(1)} = 0, v \text{ is even in } x_3\}$ and our theorem still holds provided F^ϵ converge strongly in this dual space. The effective load F^0 is given by the functional

$$\langle F^0, w \rangle = \langle \lim_{\epsilon \rightarrow 0} F^\epsilon, w \rangle,$$

where the function $w(\underline{x})$ in the second expression is interpreted as a function of all three variables $(x_1, x_2, x_3) = (\underline{x}, x_3)$ (only independent of x_3). We note that it is not possible to obtain all elements of the dual of $H^2(\Omega)$ as effective loads by this construction. The solution to the limit problem

$$\begin{aligned} \partial_{\alpha\beta} (M_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} w) &= F^0 \text{ in } \Omega \\ w = \frac{\partial w}{\partial n} &= 0 \text{ on } \partial\Omega \end{aligned},$$

may be defined variationally for any F^0 in the dual of $H^2(\Omega)$, but for certain (very unsmooth) F^0 , w is not related to solutions of the 3d equations of elasticity through the limiting process discussed in this paper. □

2. The rescaled problem - a priori estimates and limit behaviour

In this paragraph we study a rescaling of the problem (7) to the fixed domain $R(1) = \Omega \times (-1,1)$. Independent variables in $R(1)$ will be denoted (\underline{x}, y) and we define the new dependent variables as follows

$$\begin{aligned} U_{\alpha}^{\varepsilon}(\underline{x}, y) &= \frac{1}{\varepsilon} u_{\alpha}^{\varepsilon}(\underline{x}, \varepsilon y) \\ U_3^{\varepsilon}(\underline{x}, y) &= u_3^{\varepsilon}(\underline{x}, \varepsilon y) \\ E_{ij}^{\varepsilon}(\underline{x}, y) &= \frac{1}{\varepsilon} e_{ij}(\underline{u}^{\varepsilon})(\underline{x}, \varepsilon y) \\ \sum_{ij}^{\varepsilon}(\underline{x}, y) &= \frac{1}{\varepsilon} \sigma_{ij}(\underline{u}^{\varepsilon})(\underline{x}, \varepsilon y) . \end{aligned}$$

The strain tensor of $\underline{U}^{\varepsilon}$ respective to the variables (\underline{x}, y) is given by

$$(11) \quad \left(\begin{array}{c|c} E_{\alpha\beta}^{\varepsilon} & \varepsilon E_{\alpha 3}^{\varepsilon} \\ \hline \varepsilon E_{3\beta}^{\varepsilon} & \varepsilon^2 E_{33}^{\varepsilon} \end{array} \right) ,$$

and from an application of Korn's inequality in the domain $R(1)$ (cf. [6]), it now follows that

$$(12) \quad \|\underline{U}^{\varepsilon}\|_{H^1(R(1))} \leq C \left(\sum_{\alpha, \beta} \|E_{\alpha\beta}^{\varepsilon}\|_{L^2(R(1))}^2 + \varepsilon^2 \sum_{\alpha} \|E_{\alpha 3}^{\varepsilon}\|_{L^2(R(1))}^2 + \varepsilon^4 \|E_{33}^{\varepsilon}\|_{L^2(R(1))}^2 \right)^{1/2}$$

(remember $\underline{U}^{\varepsilon}$ vanishes on $\partial_0 R(1)$). By rescaling the system (7) we see that

$$(13) \quad \begin{aligned} -\partial_{\beta} \sum_{\alpha\beta}^{\varepsilon} - \frac{1}{\varepsilon} \frac{\partial}{\partial y} \sum_{\alpha 3}^{\varepsilon} &= 0 \\ -\frac{1}{\varepsilon} \partial_{\beta} \sum_{3\beta}^{\varepsilon} - \frac{1}{2} \frac{\partial}{\partial y} \sum_{33}^{\varepsilon} &= F^{\varepsilon}(\underline{x}, \varepsilon y) \quad \text{in } R(1) \\ \sum_{\alpha 3}^{\varepsilon} &= 0, \quad \frac{1}{\varepsilon} \sum_{33}^{\varepsilon} = \pm f^{\varepsilon}(\underline{x}) \quad \text{on } \partial_{\pm} R(1), \end{aligned}$$

or in a variational form

$$(14) \quad \int_{R(1)} \left[\sum_{\alpha\beta}^{\epsilon} e_{\alpha\beta}(\underline{v}) + \frac{2}{\epsilon} \sum_{\alpha 3}^{\epsilon} e_{\alpha 3}(\underline{v}) + \frac{1}{2} \sum_{33}^{\epsilon} e_{33}(\underline{v}) \right] dx dy$$

$$= \int_{R(1)} F^{\epsilon}(\underline{x}, \epsilon y) v_3 dx dy + 2 \int_{\partial_+ R(1)} f^{\epsilon} v_3 dx$$

for any $\underline{v} \in X_1 = \{ \underline{v} \in H^1(R(1)) : \underline{v}|_{\partial_0 R(1)} = 0, v_1, v_2 \text{ are odd and } v_3 \text{ is even in } y \}$. Here we have used the notation $e(\underline{v})$ for the strain of the displacement field $\underline{v}(\underline{x}, y)$ relative to the variables (\underline{x}, y) .

Lemma 1

The norms

$$\| \underline{u}^{\epsilon} \|_{H^1(R(1))}, \| E_{ij}^{\epsilon} \|_{L^2(R(1))}, \text{ and}$$

$$\| \sum_{ij}^{\epsilon} \|_{L^2(R(1))} \quad \text{are all bounded by}$$

$$C \left(\| F^{\epsilon}(\underline{x}, \epsilon y) \|_{L^2(R(1))} + \| f^{\epsilon} \|_{L^2(\Omega)} \right)$$

Proof:

Inserting $\underline{v} = \underline{u}^{\epsilon}$ into (14) and using the formula (11) for $e(\underline{u}^{\epsilon})$ we get

$$(15) \quad \int_{R(1)} \sum_{ij}^{\epsilon} E_{ij}^{\epsilon} dx dy = \int_{R(1)} F^{\epsilon}(\underline{x}, \epsilon y) u_3^{\epsilon} dx dy + 2 \int_{\partial_+ R(1)} f^{\epsilon} u_3^{\epsilon} dx$$

The coercivity of the elastic tensor and the estimate (12) now leads to

$$c_1 \sum_{i,j} \| E_{ij}^{\epsilon} \|_{L^2(R(1))}^2 < C \left(\| F^{\epsilon}(\underline{x}, \epsilon y) \|_{L^2(R(1))} + \| f^{\epsilon} \|_{L^2(\Omega)} \right) \| \underline{u}^{\epsilon} \|_{H^1(R(1))}$$

$$< C \left(\| F^{\epsilon}(\underline{x}, \epsilon y) \|_{L^2(R(1))} + \| f^{\epsilon} \|_{L^2(\Omega)} \right) \left(\sum_{i,j} \| E_{ij}^{\epsilon} \|_{L^2(R(1))}^2 \right)^{1/2}$$

so that

$$(16) \quad \left(\sum_{i,j} \|E_{ij}^\varepsilon\|^2 \right)^{1/2} \leq C \left(\|F^\varepsilon(\underline{x}, \varepsilon y)\|_{L^2(\mathbb{R}(1))} + \|f^\varepsilon\|_{L^2(\Omega)} \right).$$

The desired estimates follow directly from (16) in combination with (3) and (12). □

Remark 5

Based on Lemma 1 and the formulas for $E_{\alpha 3}^\varepsilon$ and E_{33}^ε we get that

$$(17) \quad \left\| \frac{\partial}{\partial y} U_\alpha^\varepsilon + \partial_\alpha U_3^\varepsilon \right\|_{L^2(\mathbb{R}(1))} \leq C\varepsilon \left(\|F^\varepsilon(\underline{x}, \varepsilon y)\|_{L^2(\mathbb{R}(1))} + \|f^\varepsilon\|_{L^2(\Omega)} \right)$$

$$(18) \quad \left\| \frac{\partial}{\partial y} U_3^\varepsilon \right\|_{L^2(\mathbb{R}(1))} \leq C\varepsilon^2 \left(\|F^\varepsilon(\underline{x}, \varepsilon y)\|_{L^2(\mathbb{R}(1))} + \|f^\varepsilon\|_{L^2(\Omega)} \right).$$
□

According to our assumption (10) $F^\varepsilon(\underline{x}, \varepsilon y)$ converges in $L^2(\mathbb{R}(1))$ and f^ε in $L^2(\Omega)$; Lemma 1 now gives that $\|\underline{U}^\varepsilon\|_{H^1(\mathbb{R}(1))}$, $\|E_{ij}^\varepsilon\|_{L^2(\mathbb{R}(1))}$ and $\|\sum_{ij}^\varepsilon\|_{L^2(\mathbb{R}(1))}$ are bounded independently of ε . From any sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to zero it is thus possible to extract a subsequence $\{\varepsilon_\ell\}_{\ell=1}^\infty$ such that

$$(19) \quad \underline{U}^{\varepsilon_\ell} \rightharpoonup \underline{U}^0 \text{ in } H^1(\mathbb{R}(1))$$

$$(20) \quad E_{ij}^{\varepsilon_\ell} \rightharpoonup E_{ij}^0 \text{ in } L^2(\mathbb{R}(1))$$

$$(21) \quad \sum_{ij}^{\varepsilon_\ell} \rightharpoonup \sum_{ij}^0 \text{ in } L^2(\mathbb{R}(1))$$

as ε_ℓ approaches zero.

Lemma 2

The third component of \underline{U}^0 , U_3^0 , is independent of y and belongs to

$H^2(\Omega)$. Furthermore

$$U_\alpha^0 = -y \partial_\alpha U_3^0 \quad \text{and}$$

$$E_{\alpha\beta}^0 = -y \partial_{\alpha\beta} U_3^0 .$$

Proof:

From (18) we get that $\frac{\partial}{\partial y} U_3^0 = 0$, U_3^0 is therefore independent of y . From (17)

we get $\frac{\partial}{\partial y} U_\alpha^0 = -\partial_\alpha U_3^0$ and since U_α^0 is odd with respect to y , $U_\alpha^0 = -y \partial_\alpha U_3^0$.

$E_{\alpha\beta}^{\varepsilon_\ell} = e_{\alpha\beta}(U^{\varepsilon_\ell}) = \frac{1}{2} (\partial_\alpha U_\beta^{\varepsilon_\ell} + \partial_\beta U_\alpha^{\varepsilon_\ell})$ converges as a distribution towards

$\frac{1}{2} (\partial_\alpha U_\beta^0 + \partial_\beta U_\alpha^0) = -y \partial_{\alpha\beta} U_3^0$; on the other hand it also converges towards

$E_{\alpha\beta}^0$, and consequently $E_{\alpha\beta}^0 = -y \partial_{\alpha\beta} U_3^0$. We already know that $U_3^0 \in H^1(\Omega)$

with $U_3^0 = 0$ on $\partial\Omega$. From the fact that $-y \partial_\alpha U_3^0 = U_\alpha^0 \in H^1(R(1))$ we conclude

that $U_3^0 \in H^2(\Omega) \cap H^1(\Omega)$. It only remains to prove that $\frac{\partial}{\partial n} U_3^0 = 0$ on $\partial\Omega$,

where $\partial/\partial n$ is the outward normal derivative.

Since $U_\alpha^{\varepsilon_\ell} = 0$ on $\partial_0 R(1) = \partial\Omega \times (-1,1)$ and $U_\alpha^{\varepsilon_\ell}$ converges weakly towards U_α^0 in $H^1(R(1))$, it follows that $-y \partial_\alpha U_3^0 = U_\alpha^0 = 0$ on $\partial\Omega \times (-1,1)$. This necessarily implies that $\partial_\alpha U_3^0 = 0$ on $\partial\Omega$, and so $\frac{\partial}{\partial n} U_3^0 = n_\alpha \partial_\alpha U_3^0 = 0$ on $\partial\Omega$. □

If $v(\underline{x}, y)$ is a function on $R(1)$ then we define

$$\bar{v}(\underline{x}) = \frac{1}{2} \int_{-1}^1 v(\underline{x}, y) dy .$$

From (19) and the fact that U_3^0 is independent of y it follows that $\bar{U}_3^{\varepsilon_\ell} \rightarrow U_3^0$ in $H^1(\Omega)$; this result may be slightly improved:

Lemma 3

$\bar{U}_3^{\varepsilon_\ell}$ converges strongly towards U_3^0 in $H^1(\Omega)$ as ε_ℓ approaches zero.

Proof:

We may write

$$\begin{aligned} \partial_{\alpha} \bar{U}_3^{\epsilon}(\underline{x}) &= \frac{1}{2} \int_{-1}^1 \partial_{\alpha} U_3^{\epsilon}(\underline{x}, y) dy \\ &= \frac{1}{2} \int_{-1}^1 (\epsilon E_{\alpha 3}^{\epsilon}(\underline{x}, y) - \frac{\partial}{\partial y} U_{\alpha}^{\epsilon}(\underline{x}, y)) dy \\ &= \epsilon \bar{E}_{\alpha 3}^{\epsilon}(\underline{x}) - \frac{1}{2} (U_{\alpha}^{\epsilon}(\underline{x}, 1) - U_{\alpha}^{\epsilon}(\underline{x}, -1)) . \end{aligned}$$

From the estimate of $\|E_{\alpha 3}^{\epsilon}\|_{L^2(\mathbb{R}(1))}$ in Lemma 1 it therefore follows that

$$(22) \quad \partial_{\alpha} \bar{U}_3^{\epsilon}(\underline{x}) + \frac{1}{2} (U_{\alpha}^{\epsilon}(\underline{x}, 1) - U_{\alpha}^{\epsilon}(\underline{x}, -1)) \rightarrow 0 \text{ in } L^2(\Omega) \text{ as } \epsilon \rightarrow 0.$$

We also know that $U_{\alpha}^{\epsilon_{\ell}} \rightharpoonup U_{\alpha}^0$ in $H^1(\mathbb{R}(1))$, and this implies that $U_{\alpha}^{\epsilon_{\ell}}(\underline{x}, \pm 1)$ converge weakly in $H^{1/2}(\Omega)$, and thus strongly in $L^2(\Omega)$, towards

$$U_{\alpha}^0(\underline{x}, \pm 1) = \mp \partial_{\alpha} U_3^0(\underline{x}) . \text{ It follows immediately from (22) that}$$

$$\partial_{\alpha} (\bar{U}_3^{\epsilon_{\ell}} - U_3^0) \rightarrow 0 \text{ in } L^2(\Omega) ,$$

which shows that $\bar{U}_3^{\epsilon_{\ell}}$ converges strongly towards U_3^0 in $H^1(\Omega)$, as ϵ_{ℓ} approaches zero. □

Integration in y on both sides of (21) gives $\int_{-1}^1 \sum_{\alpha 3}^{\epsilon_{\ell}} dy \rightarrow \int_{-1}^1 \sum_{\alpha 3}^0 dy$ in $L^2(\Omega)$. It turns out that $\int_{-1}^1 \sum_{\alpha 3}^0 dy = 0$, and furthermore that it is possible to find the weak limit of $\frac{1}{\epsilon_{\ell}} \int_{-1}^1 \sum_{\alpha 3}^{\epsilon_{\ell}} dy$ in $H^{-1}(\Omega)$ (our notation for the dual of $H^1(\Omega)$).

Lemma 4

$\frac{1}{\epsilon_{\ell}} \int_{-1}^1 \sum_{\alpha 3}^{\epsilon_{\ell}} dy$ converges weakly in $H^{-1}(\Omega)$ towards $\partial_{\beta} \int_{-1}^1 y \sum_{\alpha \beta}^0 dy$ as ϵ_{ℓ} approaches zero.

Proof:

Performing an integration by parts and using the first and third equations in (13) we obtain

$$\begin{aligned}
 \frac{1}{\varepsilon} \int_{-1}^1 \sum_{\alpha 3}^{\varepsilon} dy &= -\frac{1}{\varepsilon} \int_{-1}^1 y \frac{\partial}{\partial y} \sum_{\alpha 3}^{\varepsilon} dy \\
 (23) \qquad \qquad \qquad &= \int_{-1}^1 y \partial_{\beta} \sum_{\alpha \beta}^{\varepsilon} dy \\
 &= \partial_{\beta} \int_{-1}^1 y \sum_{\alpha \beta}^{\varepsilon} dy .
 \end{aligned}$$

Since $\sum_{\alpha \beta}^{\varepsilon \ell} \rightarrow \sum_{\alpha \beta}^0$ in $L^2(\mathbb{R}(1))$ we know that $\int_{-1}^1 y \sum_{\alpha \beta}^{\varepsilon \ell} dy \rightarrow \int_{-1}^1 y \sum_{\alpha \beta}^0 dy$ in $L^2(\Omega)$ and therefore $\frac{1}{\varepsilon \ell} \int_{-1}^1 \sum_{\alpha 3}^{\varepsilon \ell} dy \rightarrow \partial_{\beta} \int_{-1}^1 y \sum_{\alpha \beta}^0 dy$ in $H^{-1}(\Omega)$. \square

The tensor $-\int_{-1}^1 y \sum_{\alpha \beta}^0 dy$ will play a significant role in the construction of the tensor $M_{\alpha \beta \gamma \delta}$; it eventually turns out that $-\int_{-1}^1 y \sum_{\alpha \beta}^0 dy = M_{\alpha \beta \gamma \delta} \partial_{\gamma \delta} U_3^0$. At this point we only observe that

$$(24) \qquad -\partial_{\alpha \beta} \int_{-1}^1 y \sum_{\alpha \beta}^0 dy = \int_{-1}^1 F^0(\underline{x}, y) dy + 2f^0(\underline{x}) \quad \text{in } \Omega,$$

where F^0 and f^0 are the limits of $F^{\varepsilon}(\underline{x}, \varepsilon y)$ and f^{ε} respectively (cf. (10)). To get (24) one uses (23) and the second and fourth identities in (13) to write

$$\begin{aligned}
 -\partial_{\alpha \beta} \int_{-1}^1 y \sum_{\alpha \beta}^{\varepsilon} dy &= -\frac{1}{\varepsilon} \partial_{\alpha} \int_{-1}^1 \sum_{\alpha 3}^{\varepsilon} dy \\
 (25) \qquad \qquad \qquad &= \frac{1}{\varepsilon^2} \int_{-1}^1 \frac{\partial}{\partial y} \sum_{33}^{\varepsilon} dy + \int_{-1}^1 F^{\varepsilon}(\underline{x}, \varepsilon y) dy \\
 &= 2f^{\varepsilon}(\underline{x}) + \int_{-1}^1 F^{\varepsilon}(\underline{x}, \varepsilon y) dy.
 \end{aligned}$$

Passing to the limits in this identity as ϵ approaches zero along the sequence $\{\epsilon_\ell\}_{\ell=1}^\infty$ we are led to (24).

Remark 6

So far we have obtained a number of convergence results for subsequences $\underline{U}^\epsilon, E_{ij}^\epsilon$ and \sum_{ij}^ϵ corresponding to a specific set of loads (F^ϵ, f^ϵ) converging to (F^0, f^0) . Since the appropriate norms of the differences between the U^ϵ 's, the E^ϵ 's and the \sum^ϵ 's corresponding to different loads (F^ϵ, f^ϵ) and (G^ϵ, g^ϵ) are bounded by $C(\|(F^\epsilon - G^\epsilon)(\underline{x}, \epsilon y)\|_{L^2(R(1))} + \|f^\epsilon - g^\epsilon\|_{L^2(\Omega)})$ (Lemma 1), it follows that we may pick the same index sequence $\{\epsilon_\ell\}_{\ell=1}^\infty$ for any loads (F^ϵ, f^ϵ) that converge to this (F^0, f^0) in the sense of (10).

$L^2(R(1)) \times L^2(\Omega)$ is a separable Hilbert space; let $\{(F_N^0, f_N^0)\}_{N=1}^\infty$ be a basis. Following the previous argument we may for each N find a subsequence

$$\{\epsilon_\ell^N\}_{\ell=1}^\infty \subseteq \{\epsilon_\ell^{N-1}\}_{\ell=1}^\infty \subseteq \dots \subseteq \{\epsilon_\ell^1\}_{\ell=1}^\infty \subseteq \{\epsilon_k\}_{k=1}^\infty$$

so that the convergence results listed above hold for solutions to the problem (7) for any (F^ϵ, f^ϵ) converging to (F_N^0, f_N^0) . By taking the diagonal subsequence of all the $\{\epsilon_\ell^N\}_{\ell=1}^\infty$ we obtain a subsequence $\{\epsilon_\ell\}_{\ell=1}^\infty$ for which the convergence results hold simultaneously for all N . Since linear combinations of the $\{(F_N^0, g_N^0)\}_{N=1}^\infty$ are dense in $L^2(R(1)) \times L^2(\Omega)$, and since appropriate norms of the differences between the U^{ϵ_ℓ} 's, the E^{ϵ_ℓ} 's and the \sum^{ϵ_ℓ} 's, in the limit, are bounded by the norm of the difference between the limits of the loads in $L^2(R(1)) \times L^2(\Omega)$ (Lemma 1), it follows that the convergence results of this section hold for the fixed subsequence $\{\epsilon_\ell\}_{\ell=1}^\infty$ for any loads (F^ϵ, f^ϵ) that converge in the sense of (10).

3. An auxiliary isomorphism

If the homogenized limit operator is to have the form $\partial_{\alpha\beta}(M_{\alpha\beta\gamma\delta}\partial_{\gamma\delta})$ then it must necessarily be an isomorphism between $H^0_2(\Omega)$ and $H^{-2}(\Omega)$ (our notation for the dual of $H^0_2(\Omega)$). We shall now study in more detail a particular case of the boundary value problem (7), where the exterior load is independent of x_3 and ϵ and where the boundary loads vanish. We show that, in the limit as ϵ_ℓ approaches 0, this naturally leads to an isomorphism between $H^0_2(\Omega)$ and $H^{-2}(\Omega)$. We owe the initial suggestion, that it might be easier to obtain an isomorphism using vanishing boundary loads, to L. Tartar. For the remainder of this paper $\{\epsilon_\ell\}_{\ell=1}^\infty$ always refers to the "universal" subsequence of $\{\epsilon_k\}_{k=1}^\infty$ selected by the diagonalization process discussed in Remark 6.

Let G be in $L^2(\Omega)$ and let $\underline{v}^\epsilon \in X_\epsilon$ be the solution to

$$-\partial_j[\sigma_{ij}(\underline{v}^\epsilon)] = \begin{cases} 0 & i=1,2 \\ \epsilon^2 G(\underline{x}) & i=3 \end{cases} \quad \text{in } R(\epsilon)$$

$$(26) \quad \sigma_{ij}(\underline{v}^\epsilon)v_j = 0 \quad i=1,2,3 \quad \text{on } \partial_\pm R(\epsilon)$$

$$\underline{v}^\epsilon = 0 \quad \text{on } \partial_0 R(\epsilon) .$$

As before we introduce rescaled variables:

$$v_\alpha^\epsilon(\underline{x}, y) = \frac{1}{\epsilon} v_\alpha^\epsilon(\underline{x}, \epsilon y)$$

$$v_3^\epsilon(\underline{x}, y) = v_3^\epsilon(\underline{x}, \epsilon y)$$

$$\tilde{E}_{ij}^\epsilon = \frac{1}{\epsilon} e_{ij}(\underline{v}^\epsilon)(\underline{x}, \epsilon y)$$

$$\tilde{L}_{ij}^\epsilon = \frac{1}{\epsilon} \sigma_{ij}(\underline{v}^\epsilon)(\underline{x}, \epsilon y)$$

From the analysis in the previous section we know among other things that

$$\bar{v}_3^\varepsilon \rightarrow v_3^0 \text{ in } H^1(\Omega)$$

as ε_ℓ approaches zero. We furthermore know that $v_3^0 \in H^2(\Omega)$.

Lemma 5

For any $G \in L^2(\Omega)$

$$\begin{aligned} \|v_3^0\|_{H^2(\Omega)}^2 &< C \int_{\Omega} G v_3^0 dx && \text{and} \\ \|G\|_{H^{-2}(\Omega)}^2 &< C \int_{\Omega} G v_3^0 dx . \end{aligned}$$

Proof

The identity corresponding to (15) in this case ($F^\varepsilon(\underline{x}, \varepsilon y) = G(\underline{x})$, $f^\varepsilon = 0$) reads

$$\int_{R(1)} \sum_{ij}^{\varepsilon} \tilde{E}_{ij}^\varepsilon dx dy = \int_{R(1)} G v_3^\varepsilon dx dy ;$$

because of the coercivity assumption (2) and the fact that G is independent of y it follows that

$$(27) \quad c_1 \sum_{\alpha, \beta} \|\tilde{E}_{\alpha\beta}^\varepsilon\|_{L^2(R(1))}^2 < c_1 \sum_{i,j} \|\tilde{E}_{ij}^\varepsilon\|_{L^2(R(1))}^2 < 2 \int_{\Omega} G \bar{v}_3^\varepsilon dx .$$

From Lemma 2 we know

$$\tilde{E}_{\alpha\beta}^\varepsilon \rightharpoonup -y \partial_{\alpha\beta} v_3^0 \text{ in } L^2(R(1)) .$$

Passing to the limit in (27) along the sequence $\{\varepsilon_\ell\}$, and using the weak lower

semicontinuity of the norm, we thus obtain

$$\frac{2}{3} c_1 \sum_{\alpha, \beta} \|\partial_{\alpha\beta} v_3^0\|_{L^2(\Omega)}^2 = c_1 \sum_{\alpha, \beta} \|y \partial_{\alpha\beta} v_3^0\|_{L^2(\mathbb{R}(1))}^2 < 2 \int_{\Omega} G v_3^0 dx \quad .$$

This proves the first inequality of our statement, since $(\sum_{\alpha, \beta} \|\partial_{\alpha\beta} \cdot\|_{L^2(\Omega)}^2)^{1/2}$ is one of the equivalent norms on $H^2(\Omega)$.

The identity corresponding to (25) in the present situation reads

$$\partial_{\alpha\beta} \int_{-1}^1 y \tilde{\sum}_{\alpha\beta}^{\epsilon} dy = -2G(\underline{x}) \quad .$$

Consequently

$$(28) \quad \|G\|_{H^{-2}(\Omega)}^2 < C \sum_{\alpha, \beta} \left\| \int_{-1}^1 y \tilde{\sum}_{\alpha\beta}^{\epsilon} dy \right\|_{L^2(\Omega)}^2 < C \sum_{\alpha, \beta} \|\tilde{\sum}_{\alpha\beta}^{\epsilon}\|_{L^2(\mathbb{R}(1))}^2 \quad .$$

The $\tilde{E}_{ij}^{\epsilon}$ and $\tilde{\sum}_{ij}^{\epsilon}$ are related by

$$\tilde{\sum}_{ij}^{\epsilon} = b_{ijkl}^{\epsilon}(\underline{x}, \epsilon y) \tilde{E}_{kl}^{\epsilon} \quad ,$$

and this in combination with (3) and (28) leads to

$$\|G\|_{H^{-2}(\Omega)}^2 < C \sum_{i, j} \|\tilde{E}_{ij}^{\epsilon}\|_{L^2(\mathbb{R}(1))}^2 \quad .$$

The last inequality of (27) then gives

$$\|G\|_{H^{-2}(\Omega)}^2 < C \int_{\Omega} G \bar{v}_3^{\epsilon} dx \quad ,$$

which in the limit as ϵ approaches zero along the sequence $\{\epsilon_{\ell}\}_{\ell=1}^{\infty}$ yields the desired second inequality. □

We define an operator from $L^2(\Omega)$ into $H^2(\Omega)$ by $: G \rightarrow v_3^0$. It follows directly from the two inequalities in Lemma 5 that

$$c \|G\|_{H^{-2}(\Omega)} < \|v_3^0\|_{H^2(\Omega)} < C \|G\|_{H^{-2}(\Omega)} \quad ,$$

i.e., the above operator may be extended as an injective and bounded linear operator $S : H^{-2}(\Omega) \rightarrow H^0(\Omega)$. From the second inequality in Lemma 5 it now follows, using the Lax-Milgram Lemma, that S maps $H^{-2}(\Omega)$ onto $H^0(\Omega)$.

In summary:

The operator $G \rightarrow v_3^0$ may be extended as

(29) an isomorphism S between $H^{-2}(\Omega)$ and $H^0(\Omega)$.

Let G_i , $i=1,2$, be two elements of $L^2(\Omega)$, and let $\underline{v}_{(i)}^\epsilon$ denote the solution of (26), corresponding to $G = G_i$, $i=1,2$. According to the variational formulation (8) and the symmetry of the elastic tensor b_{ijkl}^ϵ

$$\begin{aligned} \epsilon^2 \int_{R(\epsilon)} G_1 v_{(2),3}^\epsilon dx &= \int_{R(\epsilon)} \sigma_{ij}(\underline{v}_{(1)}^\epsilon) e_{ij}(\underline{v}_{(2)}^\epsilon) dx \\ &= \int_{R(\epsilon)} \sigma_{ij}(\underline{v}_{(2)}^\epsilon) e_{ij}(\underline{v}_{(1)}^\epsilon) dx \\ &= \epsilon^2 \int_{R(\epsilon)} G_2 v_{(1),3}^\epsilon dx . \end{aligned}$$

From this we immediately conclude that

$$\int_{\Omega} G_1 \bar{v}_{(2),3}^\epsilon dx = \int_{\Omega} G_2 \bar{v}_{(1),3}^\epsilon dx ,$$

and thus in the limit, as ϵ approaches zero along the sequence $\{\epsilon_\ell\}_{\ell=1}^\infty$, we obtain

(30)
$$\int_{\Omega} G_1 S(G_2) dx = \int_{\Omega} G_2 S(G_1) dx .$$

By continuity the identity (30) is satisfied for any $G_1 \in H^{-2}(\Omega)$, i.e., we have shown that S is selfadjoint.

Finally let us consider the operator that takes $G \in L^2(\Omega)$ to the tensor $-\int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^0 dy \in L^2(\Omega)$. The \tilde{E}_{ij}^ϵ and $\tilde{\Sigma}_{ij}^\epsilon$ are related by

$$\tilde{\Sigma}_{ij}^\epsilon = b_{ijkl}^\epsilon(\underline{x}, \epsilon y) \tilde{E}_{kl}^\epsilon ,$$

and using (3) and the last inequality in (27), we thus get

$$\begin{aligned} \sum_{\alpha, \beta} \left\| \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^\epsilon dy \right\|_{L^2(\Omega)}^2 &< C \sum_{\alpha, \beta} \left\| \tilde{\Sigma}_{\alpha\beta}^\epsilon \right\|_{L^2(R(1))}^2 \\ (31) \qquad \qquad \qquad &< C \sum_{i, j} \left\| \tilde{E}_{ij}^\epsilon \right\|_{L^2(R(1))}^2 \\ &< C \int_{\Omega} G \bar{V}_3^\epsilon dx . \end{aligned}$$

Because of the weak lower semicontinuity of the norm it follows, by passing to the limit in (31) along the sequence $\{\epsilon_\ell\}_{\ell=1}^\infty$, that

$$(32) \qquad \sum_{\alpha, \beta} \left\| \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^0 dy \right\|_{L^2(\Omega)}^2 < C \int_{\Omega} G V_3^0 dx .$$

We just proved that $\|V_3^0\|_{H^2(\Omega)} < C \|G\|_{H^{-2}(\Omega)}$ and from (32) we therefore get

$$\sum_{\alpha, \beta} \left\| \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^0 dy \right\|_{L^2(\Omega)}^2 < C \|G\|_{H^{-2}(\Omega)}^2 ,$$

or

$$(33) \qquad \text{The operator } G \rightarrow -\int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^0 dy \text{ may be extended as a bounded linear operator from } H^{-2}(\Omega) \text{ into } L^2(\Omega) .$$

We shall refer to this extension as $T_{\alpha\beta}$.

4. The Proof of the Main Result

We already proved that the rescaled displacements

$$\underline{U}^\varepsilon = \left(\frac{1}{\varepsilon} u_1^\varepsilon(\underline{x}, \varepsilon_\ell y), \frac{1}{\varepsilon} u_2^\varepsilon(\underline{x}, \varepsilon_\ell y), u_3^\varepsilon(\underline{x}, \varepsilon_\ell y) \right)$$

converge weakly towards

$$\left(-y \partial_1 U_3^0(\underline{x}), -y \partial_2 U_3^0(\underline{x}), U_3^0(\underline{x}) \right)$$

in $H^1(R(1))$, with $U_3^0 \in H^2(\Omega)$.

In this section we verify that there exists a tensor $M_{\alpha\beta\gamma\delta}$ (independent of the loads $F^\varepsilon, f^\varepsilon$), with the properties i) and ii) listed in our theorem, for which

$$(34) \quad - \int_{-1}^1 y \sum_{\alpha\beta}^o dy = M_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} U_3^o .$$

From (24) we know that

$$-\partial_{\alpha\beta} \int_{-1}^1 y \sum_{\alpha\beta}^o dy = \int_{-1}^1 F^o(\underline{x}, y) dy + 2f^o(\underline{x}) ,$$

and by combining with (34) we therefore get that U_3^o satisfies

$$\partial_{\alpha\beta} (M_{\alpha\beta\gamma\delta} \partial_{\gamma\delta} U_3^o) = \int_{-1}^1 F^o(\underline{x}, y) dy + 2f^o(\underline{x}) \text{ in } \Omega$$

$$\text{with } U_3^o = \frac{\partial}{\partial n} U_3^o = 0 \text{ on } \partial\Omega .$$

Except for a change of notation (replace U_3^o by the simpler w) this will complete the proof of our theorem.

Our verification of the existence of the tensor $M_{\alpha\beta\gamma\delta}$ proceeds by the method of compensated compactness (cf. [18]); specifically we adapt the so-called div-curl lemma of L. Tartar to the present problem. $\underline{u}^\varepsilon$ as previously denotes the solution of (7) (or(8)) with loads $F^\varepsilon \in L^2(R(\varepsilon))$ and $f_+^\varepsilon = f_-^\varepsilon = f^\varepsilon \in L^2(\Omega)$, and $\underline{v}^\varepsilon$ denotes the solution of (26) with $G \in L^2(\Omega)$.

$\underline{U}^\varepsilon$, E_{ij}^ε , \sum_{ij}^ε and $\underline{V}^\varepsilon$, $\tilde{E}_{ij}^\varepsilon$, $\tilde{\sum}_{ij}^\varepsilon$ denote the rescaled variables corresponding to $\underline{u}^\varepsilon$ and $\underline{v}^\varepsilon$ respectively. Let ϕ be an arbitrary but fixed function in $\mathcal{D}(\Omega)$, we shall then compute the limit of

$$\int_{R(1)} \sum_{ij}^\varepsilon \tilde{E}_{ij}^\varepsilon \phi \, d\tilde{x}d\tilde{y} = \int_{R(1)} \tilde{\sum}_{ij}^\varepsilon E_{ij}^\varepsilon \phi \, d\tilde{x}d\tilde{y}$$

in two different ways as ε goes to zero along the sequence $\{\varepsilon_\ell\}_{\ell=1}^\infty$.

Inserting the test field $\phi \underline{V}^\varepsilon$ into (14), and using the fact that $e(\underline{V}^\varepsilon)$ has the form (11), with E^ε replaced by \tilde{E}^ε , we get

$$(35) \quad \int_{R(1)} \sum_{ij}^\varepsilon \tilde{E}_{ij}^\varepsilon \phi \, d\tilde{x}d\tilde{y} = \int_{R(1)} F^\varepsilon(\underline{x}, \varepsilon y) V_3^\varepsilon \phi \, d\tilde{x}d\tilde{y} + 2 \int_{\partial_+ R(1)} f^\varepsilon V_3^\varepsilon \phi \, d\tilde{x} \\ - \int_{R(1)} \sum_{\gamma\delta}^\varepsilon V_{\gamma}^\varepsilon \partial_\delta \phi \, d\tilde{x}d\tilde{y} - \frac{1}{\varepsilon} \int_{R(1)} \sum_{\gamma 3}^\varepsilon V_3^\varepsilon \partial_\gamma \phi \, d\tilde{x}d\tilde{y}$$

(other terms vanish because ϕ is independent of y). From (19), compactness and Lemma 2 we get that

$$V_3^{\varepsilon_\ell} \phi \rightarrow V_3^0 \phi \text{ in } L^2(R(1)) \\ V_3^{\varepsilon_\ell}(\underline{x}, 1) \phi \rightarrow V_3^0 \phi \text{ in } L^2(\Omega) \\ V_{\gamma}^{\varepsilon_\ell} \partial_\delta \phi \rightarrow -y \partial_\gamma V_3^0 \partial_\delta \phi \text{ in } L^2(R(1));$$

at the same time F^{ε_ℓ} , f^{ε_ℓ} and $\sum_{\alpha\beta}^{\varepsilon_\ell}$ are all weakly convergent in L^2 (indeed the first two converge strongly). We therefore conclude that as ε approaches 0 along the sequence $\{\varepsilon_\ell\}_{\ell=1}^\infty$, the first three terms on the right hand side of (35) approach

$$(36) \quad \int_{\Omega} \left(\int_{-1}^1 F^0(\underline{x}, y) dy + 2f^0(\underline{x}) \right) V_3^0 \phi \, d\tilde{x} + \int_{\Omega} \int_{-1}^1 y \sum_{\gamma\delta}^0 dy \partial_\gamma V_3^0 \partial_\delta \phi \, d\tilde{x} \, .$$

The last term $-\frac{1}{\varepsilon} \int_{R(1)} \sum_{\gamma 3}^{\varepsilon} v_3^{\varepsilon} \partial_{\gamma} \phi \, d\tilde{x} \, dy$ requires special attention. Using Poincaré's inequality on vertical lines and the fact that

$$\frac{\partial}{\partial y} v_3^{\varepsilon} = \varepsilon^2 \tilde{E}_{33}^{\varepsilon} \quad (\text{cf. (11)}) \quad \text{we get}$$

$$\begin{aligned} & \left| \frac{1}{\varepsilon} \int_{R(1)} \sum_{\gamma 3}^{\varepsilon} (v_3^{\varepsilon} - \bar{v}_3^{\varepsilon}) \partial_{\gamma} \phi \, d\tilde{x} \, dy \right| \\ & \leq C \frac{1}{\varepsilon} \left\| \sum_{\gamma 3}^{\varepsilon} \right\|_{L^2(R(1))} \left\| v_3^{\varepsilon} - \bar{v}_3^{\varepsilon} \right\|_{L^2(R(1))} \\ & \leq C \frac{1}{\varepsilon} \left\| \sum_{\gamma 3}^{\varepsilon} \right\|_{L^2(R(1))} \left\| \frac{\partial}{\partial y} v_3^{\varepsilon} \right\|_{L^2(R(1))} \\ & = C\varepsilon \left\| \sum_{\gamma 3}^{\varepsilon} \right\|_{L^2(R(1))} \left\| \tilde{E}_{33}^{\varepsilon} \right\|_{L^2(R(1))} \quad , \end{aligned}$$

and since both $\left\| \sum_{\gamma 3}^{\varepsilon} \right\|_{L^2(R(1))}$ and $\left\| \tilde{E}_{33}^{\varepsilon} \right\|_{L^2(R(1))}$ are bounded this last term is of order ε . It thus suffices to study the limiting behavior of

$$-\frac{1}{\varepsilon} \int_{R(1)} \sum_{\gamma 3}^{\varepsilon} \bar{v}_3^{\varepsilon} \partial_{\gamma} \phi \, d\tilde{x} \, dy = \int_{\Omega} -\frac{1}{\varepsilon} \int_{-1}^1 \sum_{\gamma 3}^{\varepsilon} dy \, \bar{v}_3^{\varepsilon} \partial_{\gamma} \phi \, d\tilde{x} \quad .$$

According to Lemma 3 $\bar{v}_3^{\varepsilon_\ell}$ converges strongly in $H^1(\Omega)$ and according to Lemma 4

$\frac{1}{\varepsilon_\ell} \int_{-1}^1 \sum_{\gamma 3}^{\varepsilon_\ell} dy$ converges weakly in $H^{-1}(\Omega)$ towards $\partial_\delta \int_{-1}^1 y \sum_{\gamma \delta}^0 dy$,

hence $\int_{\Omega} -\frac{1}{\varepsilon_\ell} \int_{-1}^1 \sum_{\gamma 3}^{\varepsilon_\ell} dy \, \bar{v}_3^{\varepsilon_\ell} \partial_{\gamma} \phi \, d\tilde{x}$ converges to

$$(37) \quad - \int_{\Omega} \partial_\delta \int_{-1}^1 y \sum_{\gamma \delta}^0 dy \, v_3^0 \partial_{\gamma} \phi \, d\tilde{x}$$

as ε_ℓ approaches 0. Collecting the terms in (36) and (37) we get

$$\lim_{\varepsilon_\ell \rightarrow 0} \int_{R(1)} \sum_{ij}^{\varepsilon_\ell} \tilde{E}_{ij}^{\varepsilon_\ell} \phi \, d\tilde{x} \, dy$$

$$\begin{aligned}
&= \int_{\Omega} F^0 v_3^0 \phi \, d\tilde{x} + \int_{\Omega} \int_{-1}^1 y \sum_{\gamma\delta}^0 dy \, \partial_{\gamma} v_3^0 \partial_{\delta} \phi \, d\tilde{x} \\
&\quad - \int_{\Omega} \partial_{\delta} \int_{-1}^1 y \sum_{\gamma\delta}^0 dy \, v_3^0 \partial_{\gamma} \phi \, d\tilde{x}
\end{aligned}$$

We integrate the last two terms in the right hand side by parts to remove derivatives from ϕ . Using the fact that ϕ vanishes on $\partial\Omega$ and the identity (24) we thus obtain

$$\begin{aligned}
(38) \quad &\lim_{\varepsilon_{\ell} \rightarrow 0} \int_{R(1)} \sum_{ij}^{\varepsilon_{\ell}} \tilde{E}_{ij}^{\varepsilon_{\ell}} \phi \, d\tilde{x} \, dy \\
&= - \int_{\Omega} \int_{-1}^1 y \sum_{\gamma\delta}^0 dy \, \partial_{\gamma\delta} v_3^0 \phi \, d\tilde{x}
\end{aligned}$$

Exchanging the roles of $\underline{U}^{\varepsilon}$ and $\underline{V}^{\varepsilon}$ in the above argument we would similarly obtain

$$\begin{aligned}
(39) \quad &\lim_{\varepsilon_{\ell} \rightarrow 0} \int_{R(1)} \sum_{ij}^{\varepsilon_{\ell}} \tilde{E}_{ij}^{\varepsilon_{\ell}} \phi \, d\tilde{x} \, dy \\
&= - \int_{\Omega} \int_{-1}^1 y \sum_{\gamma\delta}^0 dy \, \partial_{\gamma\delta} U_3^0 \phi \, d\tilde{x} \, dy .
\end{aligned}$$

Because of the symmetry of the elastic law:

$$\int_{R(1)} \sum_{ij}^{\varepsilon_{\ell}} \tilde{E}_{ij}^{\varepsilon_{\ell}} \phi \, d\tilde{x} \, dy = \int_{R(1)} \sum_{ij}^{\varepsilon_{\ell}} E_{ij}^{\varepsilon_{\ell}} \phi \, d\tilde{x} \, dy ,$$

and it then follows from (38) and (39) that

$$\begin{aligned}
&\int_{\Omega} - \int_{-1}^1 y \sum_{\gamma\delta}^0 dy \, \partial_{\gamma\delta} v_3^0 \phi \, d\tilde{x} \\
&= \int_{\Omega} - \int_{-1}^1 y \sum_{\gamma\delta}^0 dy \, \partial_{\gamma\delta} U_3^0 \phi \, d\tilde{x} \, dy .
\end{aligned}$$

In terms of the operators S and T defined in the previous section, this may be stated

$$(40) \quad \int_{\Omega} - \int_{-1}^1 y \sum_{\gamma\delta}^0 dy \partial_{\gamma\delta} S(G) \phi d\tilde{x} \\ = \int_{\Omega} T_{\gamma\delta}(G) \partial_{\gamma\delta} U_3^0 \phi d\tilde{x}$$

The identity (40) has so far only been verified for G in $L^2(\Omega)$ (and any $\phi \in \mathcal{D}(\Omega)$), but because of the continuity of the operators S and $T_{\gamma\delta}$ (cf. (29),(33)) it follows immediately that (40) holds for any G in $H^{-2}(\Omega)$. Let $\Omega' \subset\subset \Omega$ and pick $\psi \in \mathcal{D}(\Omega)$ with $\psi \equiv 1$ in Ω' . S is an isomorphism between $H^{-2}(\Omega)$ and $H^2(\Omega)$ (cf. (29)); insertion of $G = S^{-1}(1/2 x_{\alpha} x_{\beta} \psi)$ into (40) yields

$$(41) \quad \int_{\Omega} - \int_{-1}^1 y \sum_{\alpha\beta}^0 dy \phi d\tilde{x} = \int_{\Omega} T_{\gamma\delta}(S^{-1}(1/2 x_{\alpha} x_{\beta} \psi)) \partial_{\gamma\delta} U_3^0 \phi d\tilde{x}$$

for all ϕ in $\mathcal{D}(\Omega')$ (here we use that $\psi \equiv 1$ on $\text{supp}(\phi)$, $\phi \in \mathcal{D}(\Omega')$). Since $\Omega' \subset\subset \Omega$ is arbitrary we conclude from (41) that there exists $M_{\alpha\beta\gamma\delta}(\tilde{x})$ with

$$- \int_{-1}^1 y \sum_{\alpha\beta}^0 dy = M_{\alpha\beta\gamma\delta}(\tilde{x}) \partial_{\gamma\delta} U_3^0,$$

as stated in (34). $M_{\alpha\beta\gamma\delta}$ is given by

$$(42) \quad M_{\alpha\beta\gamma\delta} = T_{\gamma\delta}(S^{-1}(\frac{1}{2} x_{\alpha} x_{\beta} \psi)) \text{ in } \Omega',$$

where ψ is any element of $\mathcal{D}(\Omega)$: $\psi \equiv 1$ in Ω' . It is clear from the formula (42) for $M_{\alpha\beta\gamma\delta}$ that it obeys the symmetries

$$M_{\alpha\beta\gamma\delta} = M_{\beta\alpha\gamma\delta} = M_{\alpha\beta\delta\gamma}.$$

By taking the U_3^0 (and $\sum_{\gamma\delta}^0$), that correspond to loads $F^\varepsilon = F(\underline{x}) \in L^2(\Omega)$, $f^\varepsilon=0$ and inserting in (40) we obtain

$$(43) \quad \int_{\Omega} T_{\gamma\delta}(F) \partial_{\gamma\delta} S(G)\phi \, dx \sim \int_{\Omega} T_{\gamma\delta}(G) \partial_{\gamma\delta} S(F)\phi \, dx$$

Because of continuity (43) holds for all $F, G \in H^{-2}(\Omega)$. Pick

$$F = S^{-1}\left(\frac{1}{2} x_\alpha x_\beta \psi\right) \quad \text{and} \quad G = S^{-1}\left(\frac{1}{2} x_\rho x_\sigma \psi\right) \quad \text{with} \quad \psi \equiv 1 \quad \text{in} \quad \Omega',$$

it then follows from (42) and (43) that

$$\begin{aligned} M_{\alpha\beta\rho\sigma} &= T_{\rho\sigma}\left(S^{-1}\left(\frac{1}{2} x_\alpha x_\beta \psi\right)\right) \\ &= T_{\alpha\beta}\left(S^{-1}\left(\frac{1}{2} x_\rho x_\sigma \psi\right)\right) \\ &= M_{\rho\sigma\alpha\beta} \end{aligned}$$

in Ω' . Since Ω' is arbitrary this verifies the last symmetry of $M_{\alpha\beta\gamma\delta}$.

At this point we only know that $M_{\alpha\beta\gamma\delta} \in L^2_{loc}(\Omega)$; we now verify that the $M_{\alpha\beta\gamma\delta}$ are indeed L^∞ -functions. Consider the identity (38) corresponding to $F^\varepsilon = G(\underline{x})$ and $f^\varepsilon=0$, and replace ϕ by ϕ^2 :

$$(44) \quad \begin{aligned} &\lim_{\varepsilon \rightarrow 0} \int_{R(1)} \sum_{ij}^{\varepsilon} \tilde{E}_{ij}^{\varepsilon} \phi^2 \, d\underline{x} \, dy \\ &= - \int_{\Omega} \int_{-1}^1 y \sum_{\alpha\beta}^0 \tilde{\sum}_{\alpha\beta}^0 \, dy \, \partial_{\alpha\beta} V_3^0 \phi^2 \, d\underline{x} \, dy. \end{aligned}$$

It follows directly from (3) and the fact that b_{ijkl}^ε is symmetric that

$$C_2^{-1} \sum_{i,j} \left| \sum_{ij}^{\varepsilon} \right|^2 < \sum_{ij}^{\varepsilon} \tilde{E}_{ij}^{\varepsilon},$$

consequently

$$\begin{aligned}
& C_2^{-1} \int_{R(1)} \sum_{i,j} |\tilde{\Sigma}_{ij}^\varepsilon|^2 \phi^2 dx dy \\
& < \int_{R(1)} \sum_{i,j} \tilde{\Sigma}_{ij}^\varepsilon \tilde{E}_{ij}^\varepsilon \phi^2 dx dy .
\end{aligned}$$

Because of the weak lower semicontinuity of the norm, (44) now yields

$$\begin{aligned}
(45) \quad & C_2^{-1} \int_{R(1)} \sum_{i,j} |\tilde{\Sigma}_{ij}^o|^2 \phi^2 dx dy < \\
& - \int_{\Omega} \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^o dy \partial_{\alpha\beta} V_3^o \phi^2 dx .
\end{aligned}$$

Holder's inequality gives

$$\begin{aligned}
& \int_{\Omega} \sum_{\alpha,\beta} \left| \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^o dy \right|^2 \phi^2 dx \\
& < \frac{2}{3} \int_{R(1)} \sum_{\alpha,\beta} |\tilde{\Sigma}_{\alpha\beta}^o|^2 \phi^2 dx dy ,
\end{aligned}$$

and therefore in combination with (45) it gives

$$\begin{aligned}
& \frac{3}{2} C_2^{-1} \int_{\Omega} \sum_{\alpha,\beta} \left| \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^o dy \right|^2 \phi^2 dx \\
& < - \int_{\Omega} \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^o dy \partial_{\alpha\beta} V_3^o \phi^2 dx \\
& < \left(\int_{\Omega} \sum_{\alpha,\beta} \left| \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^o dy \right|^2 \phi^2 dx \right)^{1/2} \left(\int_{\Omega} \sum_{\alpha,\beta} |\partial_{\alpha\beta} V_3^o|^2 \phi^2 dx \right)^{1/2} .
\end{aligned}$$

From this we conclude that

$$\int_{\Omega} \sum_{\alpha,\beta} \left| \int_{-1}^1 y \tilde{\Sigma}_{\alpha\beta}^o dy \right|^2 \phi^2 dx < \left(\frac{2}{3} C_2 \right)^2 \int_{\Omega} \sum_{\alpha,\beta} |\partial_{\alpha\beta} V_3^o|^2 \phi^2 dx ,$$

or in terms of the operators S and T :

$$(46) \quad \int_{\Omega} \sum_{\alpha, \beta} |T_{\alpha\beta}(G)|^2 \phi^2 d\tilde{x} < \left(\frac{2}{3} C_2\right)^2 \int_{\Omega} \sum_{\alpha, \beta} |\partial_{\alpha\beta} S(G)|^2 \phi^2 d\tilde{x} .$$

If we pick $G = S^{-1}(\frac{1}{2} x_{\gamma} x_{\delta} t_{\gamma\delta} \psi)$ for some constant symmetric 2x2 tensor t and some $\psi \in \mathcal{D}(\Omega)$, $\psi \equiv 1$ in $\Omega' \subset\subset \Omega$, then

$$\begin{aligned} T_{\alpha\beta}(S^{-1}(\frac{1}{2} x_{\gamma} x_{\delta} t_{\gamma\delta} \psi)) &= T_{\alpha\beta}(S^{-1}(\frac{1}{2} x_{\gamma} x_{\delta} \psi)) t_{\gamma\delta} \\ &= M_{\alpha\beta\gamma\delta} t_{\gamma\delta} \end{aligned}$$

in Ω' . Inserting in (46) we get for any $\phi \in \mathcal{D}(\Omega')$

$$(47) \quad \int_{\Omega'} \sum_{\alpha, \beta} |M_{\alpha\beta\gamma\delta} t_{\gamma\delta}|^2 \phi^2 d\tilde{x} < \left(\frac{2}{3} C_2\right)^2 \int_{\Omega'} \sum_{\alpha, \beta} |t_{\alpha\beta}|^2 \phi^2 d\tilde{x}$$

(since $S(G) = \frac{1}{2} x_{\gamma} x_{\delta} t_{\gamma\delta} \psi$). (47) says that $(\sum_{\alpha, \beta} |M_{\alpha\beta\gamma\delta} t_{\gamma\delta}|^2)^{1/2}$

is an L^2 multiplier of norm $< \frac{2}{3} C_2 (\sum_{\alpha, \beta} |t_{\alpha\beta}|^2)^{1/2}$, consequently

$(\sum_{\alpha, \beta} |M_{\alpha\beta\gamma\delta} t_{\gamma\delta}|^2)^{1/2}$ is in $L^\infty(\Omega')$ and

$$(\sum_{\alpha, \beta} |M_{\alpha\beta\gamma\delta} t_{\gamma\delta}|^2)^{1/2} < \frac{2}{3} C_2 (\sum_{\alpha, \beta} |t_{\alpha\beta}|^2)^{1/2}$$

a.e. in Ω' . Since $\Omega' \subset\subset \Omega$ is arbitrary this proves that $M_{\alpha\beta\gamma\delta} \in L^\infty(\Omega)$

and it also verifies the second inequality in ii). It remains to show that

$M_{\alpha\beta\gamma\delta}$ is coercive. Due to the coercivity of $b_{ij}^{\epsilon}{}_{kl}$ (cf. (2))

$$c_1 \sum_{i,j} |\tilde{E}_{ij}^{\epsilon}|^2 < \sum_{ij}^{\epsilon} \tilde{E}_{ij}^{\epsilon} ,$$

so

$$c_1 \int_{R(1)} \sum_{i,j} |\tilde{E}_{ij}^\varepsilon|^2 \phi^2 dx dy < \int_{R(1)} \sum_{i,j} \tilde{E}_{ij}^\varepsilon \tilde{E}_{ij}^\varepsilon \phi^2 dx dy .$$

Passing to the limit in ε along the sequence $\{\varepsilon_\ell\}_{\ell=1}^\infty$, using the relation $\tilde{E}_{\alpha\beta}^0 = -y \partial_{\alpha\beta} V_3^0$ (Lemma 2), the identity (44) and the weak lower semicontinuity of the norm, we get

$$\begin{aligned} c_1 \int_{R(1)} y^2 \sum_{\alpha,\beta} |\partial_{\alpha\beta} V_3^0|^2 \phi^2 dx dy &< c_1 \int_{R(1)} \sum_{i,j} |\tilde{E}_{ij}^0|^2 \phi^2 dx dy \\ &< - \int_{\Omega} \int_{-1}^1 y \sum_{\alpha\beta}^0 dy \partial_{\alpha\beta} V_3^0 \phi^2 dx . \end{aligned}$$

Because of the constitutive relation (34) (which has already been verified) this yields

$$\begin{aligned} \frac{2}{3} c_1 \int_{\Omega} \sum_{\alpha,\beta} |\partial_{\alpha\beta} V_3^0|^2 \phi^2 dx \\ < \int_{\Omega} M_{\alpha\beta\gamma\delta} \partial_{\alpha\beta} V_3^0 \partial_{\gamma\delta} V_3^0 \phi^2 dx , \end{aligned}$$

or in terms of the operator S :

$$(48) \quad \frac{2}{3} c_1 \int_{\Omega} \sum_{\alpha,\beta} |\partial_{\alpha\beta} S(G)|^2 \phi^2 dx < \int_{\Omega} M_{\alpha\beta\gamma\delta} \partial_{\alpha\beta} S(G) \partial_{\gamma\delta} S(G) \phi^2 dx .$$

If we pick $G = S^{-1}(\frac{1}{2} x_\gamma x_\delta t_{\gamma\delta} \psi)$ for some constant symmetric 2x2 tensor t and some ψ in $\mathcal{D}(\Omega)$, $\psi \equiv 1$ in $\Omega' \subset \subset \Omega$ then

$$\partial_{\alpha\beta} S(G) = t_{\alpha\beta}$$

in Ω' . Inserting in (48) we get for any $\phi \in \mathcal{D}(\Omega')$

$$\frac{2}{3} c_1 \int_{\Omega'} \sum_{\alpha,\beta} |t_{\alpha\beta}|^2 \phi^2 dx < \int_{\Omega'} M_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta} \phi^2 dx ,$$

from which it immediately follows that

$$\frac{2}{3} c_1 \sum_{\alpha, \beta} |t_{\alpha\beta}|^2 \leq M_{\alpha\beta\gamma\delta} t_{\alpha\beta} t_{\gamma\delta}$$

a.e. in Ω' . Since $\Omega' \subset \subset \Omega$ is arbitrary this establishes the first inequality in ii). We have thus completed the proof of our theorem.

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