ON A CLASS OF PERTURBED CONSERVATION LAWS

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Abstract. We investigate behaviour of the support and decay for large times of nonnegative entropy solutions of the Cauchy problem:

$$\begin{aligned} &\partial_t u + \frac{1}{m}\partial_x(u^m) = -u^p && \text{in } (0,\infty)\times \mathbf{R} \\ &u = u_0 && \text{in } \{0\}\times \mathbf{R} \ . \end{aligned}$$

Here $m > 1$, $p > 1$ and u_0 has compact support. Sharp estimates are given studying the Riemann problem and using comparison results.

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1. Introduction

We study the Cauchy problem

$$\text{(1.1)} \qquad \begin{cases} \partial_t u + \frac{1}{m}\partial_x(u^m) = -u^p & \text{in } (0,\infty)\times \mathbf{R} \\ u = u_0 & \text{in } \{0\}\times \mathbf{R} \ , \end{cases}$$

where $m > 1$, $p > 1$ and u_0 is a bounded nonnegative function with compact support.

Several situations of practical interest (see e.g. [BE], [Mu2]) can be described by the equation

$$\text{(1.2)} \qquad \partial_t u + \partial_x[\varphi(u)] + \psi(u) = 0 \ .$$

The asymptotic behaviour of the solution of the relative Cauchy problem as $t \to \infty$ was investigated (also in a more general framework) in [DV], [Mu1], [Ry]. In particular, the appearance of interesting qualitative phenomena (among others, localization of the support) was established under suitable assumptions on φ, ψ.

In this note we derive sharp estimates concerning the support of the solution of problem (1.1) and its intermediate asymptotics as $t \to \infty$ (a particular case of (1.1) was investigated in [Da]).

Let supp $u_0 = [a, b]$. Since u is nonnegative (see Section 2), we have

$$\operatorname{supp} u(t,\cdot) = [a, s(t)]$$

where $b \le s(t) < \infty$ for any $t \ge 0$. It will be seen that $t \to s(t)$ is an increasing function in $(0, \infty)$ with $s(0) = b$.

The following result will be proved.

THEOREM 1.1. (i) *Let* $1 < p < m$. *Then there exists* $x^* > 0$ *such that*

$$b \le s(t) \le b + x^* \qquad \textit{for any } t \ge 0 \ .$$

(ii) *Let* $1 < m \le p$. *Then there exist* $A, B > 0$ *such that for large* t

$$
\begin{aligned}
&A \ln t \le s(t) \le B \ln t && \text{if } p = m \ , \\
&A t^{\frac{p-m}{p-1}} \le s(t) \le B t^{\frac{p-m}{p-1}} && \text{if } m < p < m+1 \ , \\
&A\, t^{\frac{1}{m}} (\ln t)^{-\frac{m-1}{m}} \le s(t) \le B\, t^{\frac{1}{m}} (\ln t)^{-\frac{m-1}{m}} && \text{if } p = m+1 \ , \\
&A t^{\frac{1}{m}} \le s(t) \le B t^{\frac{1}{m}} && \text{if } m+1 < p \ .
\end{aligned}
$$

The constants x^*, A, B depend in general on $b-a$, $\|u_0\|_\infty$, m and p.

Case (i) above will be referred to as *localization*, case (ii) as *positivity* (the same terminology is used for models of nonlinear diffusion; see in particular [Ke], [BKP]).

Concerning the behaviour for large times of the solution of (1.1), we shall prove the following result.

THEOREM 1.2. (i) *Let* $1 < p < m$. *Then there exists* $c_0 > 0$ *such that*

$$
t \left[\frac{1}{p-1} - t \|u(t,\cdot)\|_\infty^{p-1} \right] \to c_0 \quad \text{as} \quad t \to \infty \ .
$$

(ii) *Let* $1 < m \le p$. *Then there exist* c_1, c_2, $c_3 > 0$ *such that as* $t \to \infty$

$$
\begin{aligned}
&t^{\frac{1}{p-1}} \|u(t,\cdot)\|_\infty \to c_1 && \text{if } m \le p < m+1 \ , \\
&c_2 \le (t \ln t)^{\frac{1}{m}} \|u(t,\cdot)\|_\infty \le c_3 && \text{if } p = m+1 \ , \\
&c_2 \le t^{\frac{1}{m}} \|u(t,\cdot)\|_\infty \le c_3 && \text{if } m+1 < p \ .
\end{aligned}
$$

The constants c_i in general depend on $b-a$, $\|u_0\|_\infty$, m and p.

Theorem 1.2 states that if $p \ne m+1$, for large times

$$
\|u(t,\cdot)\|_\infty = 0(t^{-\alpha})
$$

where $\alpha = \alpha(p) := \max\left\{\frac{1}{p-1}, \frac{1}{m}\right\}$ (see Fig. 1).

The above behaviour of $\alpha = \alpha(p)$ was derived in [Mul] using formal developments (see also [SNT]).

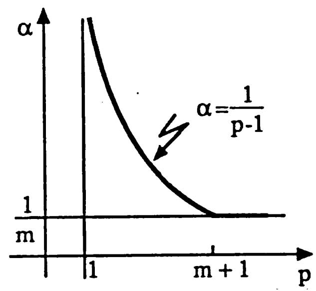

Fig. 1

Some comments are in order. Consider the following problems:

$$(1.3)\qquad \begin{cases} \dot{w} = -w^p & \text{in } (0,\infty) \\ w(0) = \|u_0\|_\infty & , \end{cases}$$

$$(1.4)\qquad \begin{cases} \partial_t v + \frac{1}{m}\partial_x(v^m) = 0 & \text{in } (0,\infty)\times \mathbf{R} \\ v = u_0 & \text{in } \{0\}\times \mathbf{R} \end{cases}$$

where u_0 is the Cauchy data of problem (1.1).

Both the solution of (1.3) and the (unique nonnegative) entropy solution of (1.4) are entropy supersolutions of (1.1) (see Definition 2.2). Then by Theorem 2.1 we have

$$(1.5)\qquad 0 \le u \le \min\{v, w\} \qquad \text{a.e. in } (0,\infty)\times \mathbf{R}\ .$$

Clearly,

$$(1.6)\qquad w(t) \sim 0(t^{-\frac{1}{p-1}}) \qquad \text{for large } t\ ;$$

it is also known (see e.g. [Sm]) that

$$(1.7)\qquad \|v(t,\cdot)\|_\infty \sim 0(t^{-\frac{1}{m}}) \qquad \text{for large } t\ .$$

Hence by (1.5)-(1.7)

$$\|u(t,\cdot)\|_\infty \le \text{const } t^{-\alpha}\ ,$$

with $\alpha = \alpha(p)$ as above.

Theorem 1.2 proves that the above estimate is sharp. It can be said that absorption prevails over convection if $p < m+1$, while the opposite holds if $p > m+1$. Similar remarks are familiar in the above referred parabolic case.

The main tool of the proof are comparison results. In fact, we shall prove Theorems 1.1, 1.2 only for the Riemann problem – namely for problem (1.1) with Cauchy data

$$u_0 = \sigma \chi_{[0,l]} \qquad (\sigma, l > 0) \ . \tag{1.8}$$

This can be done by elementary methods. Then the general case follows easily by Theorem 2.1; we omit the details.

Let us mention that estimates like those of Theorem 1.2 can be proved in L^p-norms $(1 \le p < \infty)$ by similar methods.

2. Preliminaries

By Σ_T we denote the region $\{(t,x) |\ t \in (0,T], x \in \mathbf{R}\}$ $(T > 0)$. Let $u_0 \in L^\infty(\mathbf{R})$, $u_0 \ge 0$ a.e. in $\mathbf{R}$. Following [Kr] we make the following definition.

DEFINITION 2.1. *By an entropy solution of problem (1.1) in* Σ_T *we mean a nonnegative function* $u \in L^\infty(\Sigma_T)$ *such that:*

(i) for any $k \in \mathbf{R}$ *and any* $\varphi \in C_0^\infty(\Sigma_T)$, $\varphi \ge 0$,

$$\iint_{\Sigma_T} \left\{ |u-k|\varphi_t + sgn(u-k) \left[\frac{u^m - k^m}{m} \varphi_x - u^p \varphi \right] \right\} \mathrm{d}t\, \mathrm{d}x \ge 0 \ ; \tag{2.1}$$

(ii) there exists a set E *of zero measure in* $[0,T]$ *such that for* $t \in [0,T] \backslash E$ *the function is defined almost everywhere for* $x \in \mathbf{R}$

and for any real interval (a_1, a_2)

$$\lim_{\substack{t\to 0\\ t\notin E}} \int_{a_1}^{a_2} |u(t,x) - u_0(x)|\,\mathrm{d}\,x = 0 \quad . \tag{2.2}$$

The following definition is also of interest ([NT]; see also [LP]).

DEFINITION 2.2. *An entropy subsolution (respectively supersolution) of problem (1.1) in* Σ_T *is any nonnegative* $u^- \in L^\infty(\Sigma_T)$ *(respectively* $u^+ \in L^\infty(\Sigma_T)$*), which satisfies the requirements of Definition 2.1 with (2.1), (2.2) replaced by*

$$\iint_{\Sigma_T} \Big\{ [u^\mp - k]_\pm \varphi_t + H_\pm(u^\mp - k) \cdot \\ \cdot \left[\frac{(u^\mp)^m - k^m}{m} \varphi_x - (u^\mp)^p \varphi \right] \Big\} \mathrm{d}\,t\,\mathrm{d}\,x \geq 0 \tag{2.3}$$

and respectively

$$\lim_{\substack{t\to 0\\ t\notin E}} \int_{a_1}^{a_2} [u^\mp(t,x) - u_0(x)]_\pm \mathrm{d}\,x \leq 0 \quad . \tag{2.4}$$

Here $[r]_\pm := \frac{1}{2}(|r| \pm r)$, $H_\pm(r) := \frac{1}{2}(sgn r \pm 1)$; either upper or lower signs hold in (2.3), (2.4).

It is known [Kr] that problem (1.1) has a unique entropy solution in Σ_T for any $T > 0$. Uniqueness can be regarded as a consequence of the following comparison result ([Kr]; see also [NT]).

For any $M, R > 0$ a family of cones of propagation for problem (1.1) is introduced as follows:

$$K := \{(t,x) |\ |x| \leq R - M^{m-1} t \ \ , \ \ 0 \leq t \leq \min\{T \, , \, M^{1-m} R\}\} \ .$$

We also set

$$S_\tau := K \cap \{t = \tau\} \qquad (0 \leq \tau \leq \min\{T, M^{1-m} R\}) \ .$$

THEOREM 2.1. *Let* u^-, u^+ *be an entropy subsolution, respectively supersolution of problem (1.1). Assume that*

$$|u^{\mp}(t,x)| \leq M \qquad \text{for any} \quad (t,x) \in [0,T] \times [-R,R] \ .$$

Then for almost every $t \in [0, \min\{T, M^{1-m}R\}]$

$$\int_{S_t} |u^-(t,x) - u^+(t,x)|_+ \mathrm{d}\,x \leq \int_{S_0} |u^-(0,x) - u^+(0,x)|_+ \mathrm{d}\,x \ . \tag{2.5}$$

Let us observe an immediate consequence of the above theorem. It is easily seen that both the solution of (1.3) and the (unique, nonnegative entropy) solution of (1.4) are entropy supersolutions of (1.1). Hence we have the following

PROPOSITION 2.1. *Let* $u_0 \in L^\infty(\mathrm{R})$, $u_0 \geq 0$. *Then the following inequalities hold almost everywhere in* Σ_T, *for any* $T > 0$*:*

$$u(t,x) \leq \frac{\|u_0\|_\infty}{[1 + (p-1)\|u_0\|_\infty^{p-1} t]^{\frac{1}{p-1}}} \ ; \tag{2.8}$$

$$u(t,x) \leq v(t,x) \ , \tag{2.9}$$

where v *denotes the unique (nonnegative) entropy solution of problem (1.4).*

3. Similarity solutions

When discussing the asymptotic form of the solution of (1.1) for large times, we shall encounter the following family of similarity solutions:

$$u(t,x) = t^{-\frac{1}{p-1}} f(\eta) \quad , \quad \eta := x t^{\frac{m-p}{p-1}} \quad . \tag{3.1}$$

Here f is the solution of the problem:

$$\begin{cases} \left(f^{m-1} + \frac{m-p}{p-1}\eta\right) f' - f\left(\frac{1}{p-1} - f^{p-1}\right) = 0 \quad \text{in} \quad (0,\overline{\eta}) \\ f \geq 0 \\ \frac{f^{m-1}}{\eta} \to 1 \quad \text{as} \quad \eta \to 0 \quad , \end{cases} \tag{3.2}$$

where

$$\overline{\eta} := \begin{cases} \infty & \text{if} \quad p \leq m \\ \frac{(p-1)^{\frac{p-m}{p-1}}}{p-m} & \text{if} \quad p > m \quad . \end{cases}$$

Clearly,

$$f = 0 \quad , \quad f = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} =: f^*$$

are stationary solutions of the differential equation in (3.2).

PROPOSITION 3.1. *Every solution of problem (3.2) takes values in the interval* $[0, f^*]$.

Proof. Let f be a nonconstant solution of (3.2). Since

$$f' = f^{2-m} \frac{\frac{1}{p-1} - f^{p-1}}{1 + \frac{m-p}{p-1}\frac{\eta}{f^{m-1}}} =: F_0(\eta, f) \quad , \tag{3.3}$$

it follows that

$$\frac{f'}{\eta^{\frac{2-m}{m-1}}} \to \frac{1}{m-1} \quad \text{as} \quad \eta \to 0 \quad .$$

In general we get easily from (3.3)

$$f^{(k)} \sim \eta^{\frac{1}{m-1} - k} \quad \text{as} \quad \eta \to 0 \quad (k = 1, 2, \ldots) \quad .$$

Hence

$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0 \quad , \quad f^{(k)}(0) = \begin{cases} \infty & \text{if } m > (k+1)/k \\ > 0 & \text{if } m = (k+1)/k \\ 0 & \text{if } m < (k+1)/k \end{cases} .$$

Then there exists $\eta_0 > 0$ such that

$$f(\eta) \in (0, f^*) \quad \text{for any} \quad \eta \in (0, \eta_0) \ .$$

On the other hand, it is easily seen that F_0 is C^1 in a neighbourhood of $(\eta, 0)$, respectively (η, f^*), for any $\eta > 0$. Hence by uniqueness $\eta_0 = \overline{\eta}$. ∎

Remark 3.1. It is easy to check that F_0 is not C^1 in any neighbourhood of $(0,0)$.

PROPOSITION 3.2. *Every solution of problem (3.2) is increasing in* $(0, \overline{\eta})$.

Proof. The conclusion is obvious by (3.2) if $p \leq m$. If $p > m$, observe that

$$1 - \frac{p-m}{p-1} \frac{\eta}{f^{m-1}} \to \frac{m-1}{p-1} > 0 \quad \text{as} \quad \eta \to 0 \ .$$

Set

$$\eta_1 := \sup \left\{ \eta \in (0, \overline{\eta}) | \quad \frac{\eta}{f^{m-1}} < \frac{p-1}{p-m} \right\} \ ;$$

let us prove that $\eta_1 = \overline{\eta}$.

Observe that by Proposition 3.1 $f' > 0$ in $(0, \eta_1)$. If $\eta_1 < \overline{\eta}$, we would have

$$f' \to \infty \quad \text{as} \quad \eta \to \eta_1^- \ .$$

On the other hand, deriving (3.3) gives easily

$$f'' = \frac{f'}{f^{m-1} - \frac{p-m}{p-1}\eta} \left\{ -(m-1) f^{m-2} f' - p f^{p-1} + \frac{p-m+1}{p-1} \right\} \ ,$$

whence

$$f'' \to -\infty \quad \text{as} \quad \eta \to \eta_1^- \ .$$

The contradiction proves the result. ∎

If $p = m$, problem (3.2) can be integrated explicitly. If $p \neq m$, the differential equation

$$\left(f^{m-1} + \frac{m-p}{p-1}\eta\right) \mathrm{d}f - f\left(\frac{1}{p-1} - f^{p-1}\right) \mathrm{d}\eta = 0$$

has the integrating factor

$$\mu(f) = -f^{p-m-1}\left(\frac{1}{p-1} - f^{p-1}\right)^{\frac{m+1-2p}{p-1}} \ ,$$

as is easily checked. Then we get the following result.

THEOREM 3.1. *There exists a unique nontrivial solution of problem (3.2). If $p = m$, the solution is*

$$f(x) = \left[\frac{1 - e^{-(m-1)x}}{m-1}\right]^{\frac{1}{m-1}} \quad (x \geq 0) \ . \tag{3.4}$$

If $p \neq m$, the solution is implicitly defined by the equality

$$\left[\frac{1}{p-1} - f^{p-1}\right]^{\frac{m-p}{p-1}} \left[\frac{\eta}{f^{m-p}} + \frac{1}{m-p}\right] = \frac{1}{m-p}\left(\frac{1}{p-1}\right)^{\frac{m-p}{p-1}} \ . \tag{3.5}$$

Remark 3.2. The constant in the right-hand side of (3.5) is $(-\overline{\eta})$, if $p > m$.

COROLLARY 3.1. *Let f denote the nontrivial solution of problem (3.2). Then $f \to f^*$ as $\eta \to \overline{\eta}$.*

Proof. The conclusion is obvious from (3.4) if $p = m$. If $p \neq m$, observe that by Propositions 3.1, 3.2 f has a limit as $\eta \to \overline{\eta}$. Were this limit strictly less than f^*, equality (3.5) couldn't be satisfied for any $\eta \in (0, \overline{\eta})$. Hence the conclusion. ∎

We note for further reference the following result; the elementary proof is omitted.

COROLLARY 3.2. *Let f be the unique nontrivial solution of problem (3.2). Then*

$$\frac{f^{m-1}}{\eta} = 1 - \frac{m-1}{2}\eta^{\frac{p-1}{m-1}} + o(\eta^{\frac{p-1}{m-1}}) \quad as \quad \eta \to 0 \ . \tag{3.6}$$

4. Solution of the Riemann problem

In this section we construct the solution of problem (1.1) with Cauchy data given by (1.7).

Let us denote by $x = x(t,y)$ the characteristic of (1.1) issued at $x = y$; set also

$$U = U(t,y) := u(t,x(t,y)) \ .$$

Then the couple (x,U) satisfies the system

$$\begin{cases} \dot{x} = U^{m-1} & \\ \dot{U} = -U^p & (t>0) \\ x = y \ , \quad U = U_0 & (t=0) \ , \end{cases} \tag{4.1}$$

where $U_0 := u_0(y)$. By integration we get easily:

$$U(t,y) = U_0[1 + (p-1)U_0^{p-1}t]^{-\frac{1}{p-1}} \ , \tag{4.2}$$

$$x(t,y) = \begin{cases} y + \frac{U_0^{m-p}}{m-p}\{1 - [1 + (p-1)U_0^{p-1}t]^{-\frac{m-p}{p-1}}\} & \text{if } p \neq m \\ y + \frac{1}{m-1}\log[1 + (m-1)U_0^{m-1}t] & \text{if } p = m \ . \end{cases} \tag{4.3}$$

We shall denote by $x = c(t)$ the characteristic issued at $x = 0$. Then we have:

$$c(t) = \begin{cases} \frac{\sigma^{m-p}}{m-p}\{1 - [1 + (p-1)\sigma^{p-1}t]^{-\frac{m-p}{p-1}}\} & \text{if } p \neq m \\ \frac{1}{m-1}\log[1 + (m-1)\sigma^{m-1}t] & \text{if } p = m \ . \end{cases} \tag{4.4}$$

The elementary proof of the following result is omitted.

LEMMA 4.1. *The map* $\eta = C(t) := c(t)t^{\frac{m-p}{p-1}}$ *is strictly increasing in* $(0,\infty)$. *Moreover: (i)* $C(t) \to 0$ *as* $t \to 0$, *(ii)* $C(t) \to \overline{\eta}$ *as* $t \to \infty$.

Let us denote by $x = s_0(t)$ the shock issued at $x = l$. The Rankine-Hugoniot jump condition reads in this case:

$$\text{(4.5)} \qquad \begin{cases} \dot{s}_0 = \frac{1}{m}u^{m-1}(t, s_0(t) - 0) \quad (t > 0) \\ s_0(0) = l \ . \end{cases}$$

As long as the shock s_0 doesn't intersect the characteristic c, we have

$$u(t, s_0(t) - 0) = U(t, s_0(t)) = \sigma[1 + (p-1)\sigma^{p-1}t]^{-\frac{1}{p-1}}$$

(observe that by (4.2) U is independent from x).

Then integrating (4.5) we get:

$$\text{(4.6)} \quad s_0(t) = \begin{cases} l + \frac{\sigma^{m-p}}{m(m-p)}\{1 - [1 + (p-1)\sigma^{p-1}t]^{-\frac{m-p}{p-1}}\} & \text{if } p \neq m \\ l + \frac{1}{m(m-1)}\log[1 + (m-1)\sigma^{m-1}t] & \text{if } p = m \ . \end{cases}$$

Since $m > 1$, it is clear from (4.1), (4.5) that s_0 is slower than c. A necessary and sufficient condition for their intersection is given in the following proposition.

PROPOSITION 4.1. (i) *Let* $1 < p < m$. *Then the shock* s_0 *intersects the characteristic* c *if and only if*

$$\text{(4.7)} \qquad l < \frac{m-1}{m(m-p)}\sigma^{m-p} \ .$$

If (4.7) holds, the intersection is unique.

(ii) *Let* $1 < m \leq p$. *Then there is a unique intersection between* s_0 *and* c.

Proof. If follows from (4.4), (4.6) that any intersection between s_0 and c must have abscissa

$$\text{(4.8)} \qquad \overline{x} = \frac{m}{m-1}l \ .$$

Hence an intersection exists if and only if for some $\bar{t} > 0$

$$c(\bar{t}) = \bar{x} \ .$$

Since $c(0) = 0$, c is strictly increasing and

$$\lim_{t\to\infty} c(t) = \begin{cases} \frac{\sigma^{m-p}}{m-p} & \text{if } p < m \\ \infty & \text{if } p \geq m \ , \end{cases}$$

the conclusion follows. ∎

Remark 4.1. It should be remarked that in the case $p < m$ both c and s_0 tend to a finite limit as $t \to \infty$. This explains heuristically the localization of the solution which occurs in this case (see Section 5).

Let us define

$$\bar{t} := \begin{cases} c^{-1}(\bar{x}) & \text{if } c \text{ and } s_0 \text{ intersect} \\ \infty & \text{otherwise} \ . \end{cases}$$

If there is intersection, consider the shock $x = s_1(t)$ issued at $(\bar{t}, \bar{x})$. The Rankine-Hugoniot condition gives

$$\begin{cases} \dot{s}_1 = \frac{1}{m} u^{m-1}(s_1(t) - 0) & (t > \bar{t}) \\ s_1(\bar{t}) = \bar{x} \ . \end{cases} \tag{4.9}$$

In such case $u(s_1(t) - 0)$ is given by (3.1), namely

$$u(s_1(t) - 0) = t^{-\frac{1}{p-1}} f(s_1(t) t^{\frac{m-p}{p-1}}) \ . \tag{4.10}$$

In any case we set

$$s(t) := \begin{cases} s_0(t) & t \in [0, \bar{t}) \\ s_1(t) & t \in [\bar{t}, \infty) \ . \end{cases}$$

Then the half-plane $[0, \infty) \times \mathbf{R}$ is the union of the following regions:

$$I_0 := \{(t, x) | t \geq 0 \ , \ x < 0\}$$

$$I_1 := \{(t, x) | t \geq 0 \ , \ 0 \leq x \leq \min\{c(t), s(t)\}\}$$

$$I_2 := \{(t, x) | t \geq 0 \ , \ \min\{c(t), s(t)\} < x \leq s(t)\}$$

$$I_3 := \{(t, x) | t \geq 0 \ , \ s(t) < x\} \ .$$

The situation is depicted in Fig. 2.

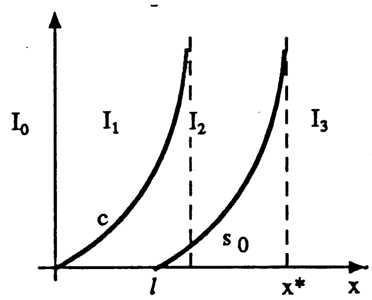

Fig. 2a $1 < p < m$; (4.7) doesn't hold.

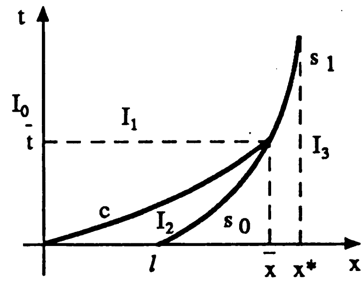

Fig. 2b $1 < p < m$; (4.7) holds.

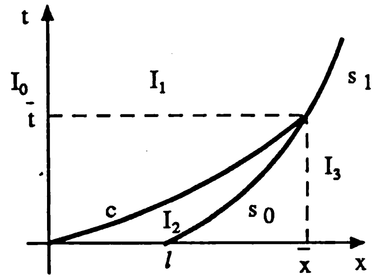

Fig. 2c $1 < m \leq p$.

Now define $u : [0,\infty) \times \mathbf{R} \to [0,\infty)$ as follows:

$$(4.11) \qquad u(t,x) := \begin{cases} t^{-\frac{1}{p-1}} f(xt^{\frac{m-p}{p-1}}) & ((t,x) \in I_1) \\ \sigma[1+(p-1)\sigma^{p-1}t]^{-\frac{1}{p-1}} & ((t,x) \in I_2) \\ 0 & ((t,x) \in I_0 \cup I_3) \ . \end{cases}$$

Let us prove the following result.

THEOREM 4.1. *The function u defined by (4.11) is the unique nonnegative entropy solution of the Riemann problem (1.1), (1.7) in Σ_T, for any $T > 0$.*

Proof. The function u is piecewise continuous; moreover, it is continuous on the axis $x = 0$ (since $f(0) = 0$) and satisfies the Rankine-Hugoniot condition at $x = s(t)$ $(t > 0)$. Hence the conclusion follows if we prove that u is continuous on the curve $(t, c(t))$ for any $t \in (0, \bar{t})$.

This is an immediate consequence of (3.4), (4.4) if $p = m$. In the case $p \neq m$ define

$$(4.12) \qquad \begin{aligned} F(t) &:= t^{\frac{1}{p-1}} u(t, c(t)+0) \\ &= \sigma t^{\frac{1}{p-1}}[1+(p-1)\sigma^{p-1}t]^{-\frac{1}{p-1}} \quad (t \in (0,\bar{t})) \ . \end{aligned}$$

It is easily seen that

$$\dot{F} = \frac{F}{t}\left(\frac{1}{p-1} - F^{p-1}\right) \quad (t \in (0,\bar{t})) \;\; ;$$

thus $\dot{F} > 0$ in $(0,\bar{t})$.

It can be easily checked that the function $\eta = C(t)$ has derivative

$$\dot{C} = \frac{1}{t}\left(F^{m-1} + \frac{m-p}{p-1}C\right) \;\; ; \tag{4.14}$$

here use of (4.1), (4.12) has been made.

By Lemma 4.1 and the above remarks, the map $\eta \to F(C^{-1}(\eta))$ is defined in $(0,\bar{\eta})$ and satisfies the differential equation in (3.2). Moreover, by Lemma 4.1 we have:

$$\begin{aligned}\frac{F^{m-1}(C^{-1}(\eta))}{\eta} &= \frac{[C^{-1}(\eta)]^{\frac{m-1}{p-1}}}{\eta}\sigma^{m-1}[1+(p-1)\sigma^{p-1}C^{-1}(\eta)]^{-\frac{m-1}{p-1}} \\ &= \frac{C^{-1}(\eta)}{c(C^{-1}(\eta))}\sigma^{m-1}[1+(p-1)\sigma^{p-1}C^{-1}(\eta)]^{-\frac{m-1}{p-1}} \to 1 \text{ as } \eta \to 0 \;.\end{aligned}$$

It follows that $F(C^{-1}(\eta)) = f(\eta)$, where f is the unique (nontrivial) solution of problem (3.2). By definition (4.11), (4.12) this implies

$$u(t,c(t)-0) = u(t,c(t)+0) \text{ for any } t \in (0,\bar{t}) \;\; ; \tag{4.15}$$

hence the conclusion. ∎

Remark 4.2. Equality (4.15) also holds at $t = \bar{t}$ if $\bar{t} < \infty$.

We can summarize the previous discussion as follows. The solution of the Riemann problem (1.1), (1.8)

(i) for any fixed $t \in (0,\bar{t})$ increases in the region I_2 from zero to the value $t^{-\frac{1}{p-1}}f(C(t))$ (see Proposition 3.2) and is constant in the region I_3;

(ii) for any $t > \bar{t}$ increases in the region I_2 from zero to the value $t^{-\frac{1}{p-1}}f(S(t))$, where

$$S(t) := s(t)t^{\frac{m-p}{p-1}} \quad . \tag{4.16}$$

Hence we have for any $t > 0$

$$\|u(t,\cdot)\|_\infty = t^{-\frac{1}{p-1}} f(S(t)) \quad . \tag{4.17}$$

Equality (4.17) is the link between asymptotic estimates of the support and those of the L^∞-norm of the solution.

5. The Riemann problem: support properties

Let us first observe a regularity property of the right interface $s(\cdot)$.

PROPOSITION 5.1. $s \in C^1(0,\infty)$.

Proof. If there is no intersection between s_0 and c, the result is obvious from (4.5). In the case of intersection we have

$$\dot{s}_0(\bar{t}) = \dot{s}_1(\bar{t}) = \dot{s}(\bar{t}) \quad .$$

by Remark 4.2. Hence the conclusion. ∎

The following preliminary estimate of s will be of use.

LEMMA 5.1. *For any* $t \geq 0$

$$s(t) \leq \begin{cases} l + \frac{\sigma^{m-p}}{m(m-p)}\{1 - [1 + (p-1)\sigma^{p-1}t]^{-\frac{m-p}{p-1}}\} & \text{if } p \neq m \\ l + \frac{1}{m(m-1)} \log[1 + (m-1)\sigma^{m-1}t] & \text{if } p = m \quad . \end{cases} \tag{5.1}$$

Proof. From Proposition 2.1 and Theorem 4.1 we get

$$u(t,x) \leq \sigma[1 + (p-1)\sigma^{p-1}t]^{-\frac{1}{p-1}} \quad \text{in } (0,\infty) \times \mathbf{R} \quad . \tag{5.2}$$

Hence by (4.5), (4.9) we get

$$\begin{cases} \dot{s} \leq \frac{\sigma^{m-1}}{m}[1 + (p-1)\sigma^{p-1}t]^{-\frac{m-1}{p-1}} & (t > 0) \\ s(0) = l \quad . \end{cases} \tag{5.3}$$

Then the conclusion follows. ∎

Now we can prove the following result.

THEOREM 5.1. *(Localization). Let $1 < p < m$. Then there exists $x^* = x^*(\sigma) > 0$ such that*

$$l \leq s(t) \leq l + x^* \quad \text{for any} \quad t \geq 0 \ . \tag{5.4}$$

Proof. Define

$$x^* := \frac{\sigma^{m-p}}{m(m-p)} \ ; \tag{5.5}$$

then the bound on the right follows from (5.1). Since s is increasing, the conclusion follows. ∎

The situation is different for $p \geq m$, as the following result shows.

THEOREM 5.2. *(Positivity). Let $1 < m \leq p$. Then for any $t \geq \bar{t}$*

$$s(t) = \frac{1}{m-1} \log \left\{ 1 + [e^{(m-1)\bar{x}} - 1] \left(\frac{t}{\bar{t}} \right)^{\frac{1}{m}} \right\} \quad \text{if} \ p = m \ , \tag{5.6}$$

$$s(t) \geq \bar{x} \left(\frac{t}{\bar{t}} \right)^{\frac{p-m}{m(p-1)}} \quad \text{if} \ p > m \ . \tag{5.7}$$

Proof. Recall that an intersection $(\bar{t}, \bar{x})$ between s_0 and c_0 exists if $p \geq m$ (Proposition 4.1). Problem (4.9) can be rewritten:

$$\begin{cases} \dot{s} = \frac{1}{m} t^{-\frac{m-1}{p-1}} f^{m-1}(s(t) t^{\frac{m-p}{p-1}}) & (t > \bar{t}) \\ s(\bar{t}) = \bar{x} \ . \end{cases} \tag{5.8}$$

If $p = m$, this can be integrated to give (5.6). If $p > m$, Propositions 3.1, 3.2 imply that

$$f^{m-1}(s(t) t^{\frac{m-p}{p-1}}) \geq \frac{p-m}{p-1} s(t) t^{\frac{m-p}{p-1}} \ .$$

Substituting into (5.8) gives

$$\begin{cases} \dot{s} \geq \frac{p-m}{m(p-1)} \frac{s}{t} & (t > \bar{t}) \\ s(\bar{t}) = \bar{x} \ , \end{cases}$$

whence (5.7) follows. ∎

Remark 5.1. Observe that by (4.8), (4.4) $\overline{x} = \overline{x}(l, m)$, $\overline{t} = \overline{t}(l, \sigma, m, p)$.

When $p \geq m$, to get sharp estimates of the behaviour of s as $t \to \infty$ it is convenient to use the coordinates (t, η). Therefore we shall investigate the properties of the map $\eta = S(t)$ defined in (4.16).

Using (5.8) it is easily checked that S satisfies the differential equation

$$\dot{S} = \frac{1}{t}\left[\frac{f^{m-1}(S)}{m} - \frac{p-m}{p-1}S\right] \quad \text{for any } t > \overline{t} \; . \tag{5.9}$$

This can be rewritten as an autonomous equation in the variable $\tau := \log t$, namely:

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} = \frac{f^{m-1}(S)}{m} - \frac{p-m}{p-1}S \quad \text{for any } \tau > \overline{\tau} \; , \tag{5.10}$$

where $\overline{\tau} := \log \overline{t}$.

Concerning stationary solutions of (5.10) the following results can be proved (the case $p < m$ is mentioned for completeness).

PROPOSITION 5.2. (i) *Let* $1 < p \leq m$ *or* $p \geq m+1$. *Then the unique stationary solution of (5.10) is* $S = 0$.
(ii) *Let* $1 < m < p < m+1$. *Then equation (5.10) has exactly two stationary solutions,* $S = 0$ *and* $S = S_1 > 0$. *Moreover,* $S_1 \to \infty$ *as* $p \to m^+$, $S_1 \to 0$ *as* $p \to (m+1)^-$.

The situation is qualitatively depicted in Fig. 3.

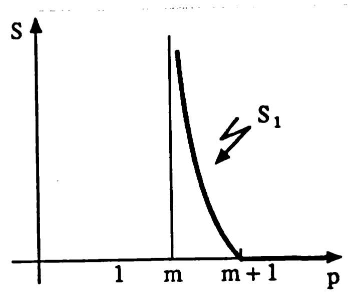

Fig. 3

PROPOSITION 5.3. *Let* $1 < m \leq p$. *Then:*

(i) *the trivial solution is completely unstable if* $m \leq p < m+1$, *or globally attractive if* $m+1 \leq p$;

(ii) *the stationary solution* $S = S_1$ *is globally attractive whenever it exists (namely, if* $m < p < m+1$*).*

Remark 5.2. It will be apparent that the trivial solution of (5.10) is completely unstable even if $1 < p < m$. In such case by (5.7)

$$S(t) \geq \text{ const. } t^{1/m} \quad \text{as} \quad t \to \infty \ .$$

Let us turn to the proof of Proposition 5.2 (suppose $p > m$, otherwise the proof is trivial). Due to the results of Section 3 (see in particular Proposition 3.2 and equality (3.5)), the stationary solutions $\eta = S(t) \equiv S$ of (5.10) are in one-to-one correspondence with the solutions of the following system:

$$\text{(5.11)} \qquad \begin{cases} \frac{f^{m-1}}{m} = \frac{p-m}{p-1}\eta \\ \eta f^{p-m} - \frac{1}{p-m} = -\overline{\eta}[(f^*)^{p-1} - f^{p-1}]^{\frac{p-m}{p-1}} \end{cases} .$$

Let us prove the following result.

LEMMA 5.2. *(i) Let* $1 < m < p < m+1$. *Then system (5.11) has exactly two solutions,* $(0,0)$ *and* (η_1, f_1). *Here* $\eta_1 \in (0,\infty)$, $f_1 \in (0, f^*)$; *moreover:*

$$(\eta_1, f_1) \to (\infty, f^*) \quad \text{as} \quad p \to m^+ \ ,$$

$$(\eta_1, f_1) \to (0,0) \quad \text{as} \quad p \to (m+1)^- \ .$$

(ii) Let $m+1 \leq p \, (m>1)$. *Then (5.11) has only the trivial solution* $(0,0)$.

Proof. Solving the first equation of (5.11) with respect to η and substituting into the second gives

$$\frac{1}{p-m} H\left(\frac{f}{f^*}\right) = 0$$

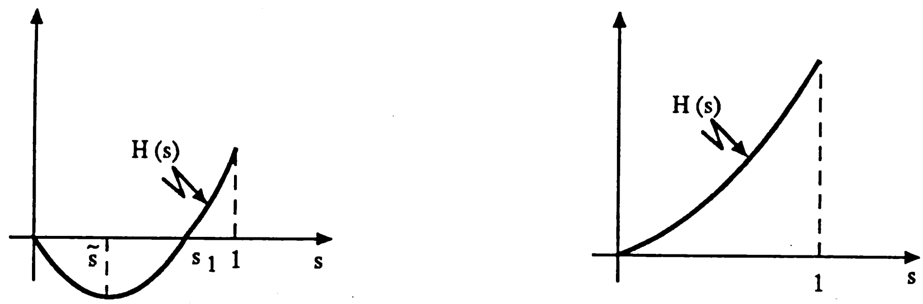

Fig. 4a $m < p < m+1$. **Fig. 4b** $p \geq m+1$.

where

(5.12) $$H(s) := 1 - \frac{s^{p-1}}{m} - (1 - s^{p-1})^{\frac{p-m}{p-1}} \quad (s \in [0,1]) \ .$$

It follows that

$$H(0) = 0 \quad , \quad H(1) = 1 - \frac{1}{m} > 0 \ ;$$

moreover,

(5.13) $$H'(s) = s^{p-2}\left[-\frac{p-1}{m} + (p-m)(1-s^{p-1})^{-\frac{m-1}{p-1}}\right] \ .$$

Hence the equation $H'(s) = 0$ has a root $\tilde{s} > 0$ if and only if

(5.14) $$\frac{m(p-m)}{p-1} < 1 \quad \Leftrightarrow \quad p < m+1 \ .$$

It is also easy to see that in case (i)

$$H'(0) = \cdots = H^{(k-1)}(0) = 0 \quad , \quad H^{(k)}(0) < 0 \ ,$$

where $k = [p]$, whereas in case (ii)

$$H'(0) = 0 \quad , \quad H'(s) > 0 \ \text{ for any } \ s \in (0,1) \ .$$

It follows easily that (see Fig. 4)

- in case (i) there exists a unique $s_1 \in (\tilde{s}, 1]$ such that $H(s_1) = 0$;
- in case (ii) $H(s) = 0 \Rightarrow s = 0$.

It is clear from (5.13) that

$$\tilde{s} \to 1 \Rightarrow s_1 \to 1 \quad \text{as} \quad p \to m^+ \ ,$$

$$\tilde{s} \to 0 \quad \text{as} \quad p \to (m+1)^- \ .$$

Hence in the latter case

$$H(\tilde{s}) = 1 - \frac{p-1}{m(p-m)} + \frac{m-1}{p-m}\tilde{s}^{p-1} \to 0 \Rightarrow s_1 \to 0 \quad \text{as} \quad p \to (m+1)^-$$

by the continuity of H.

Finally, define in case (i)

$$f_1 := s_1 f^* \quad , \quad \eta_1 := \frac{p-1}{m(p-m)} f_1^{m-1} \quad ;$$

then the conclusion follows easily. ∎

The proof of Proposition 5.2 follows immediately from Lemma 5.2 if $p > m$. We omit the details.

The proof of Proposition 5.3 will follow from several lemmas.

LEMMA 5.3. *Let $1 < m < p$. Then every solution of equation (5.10) is bounded.*

Proof. It follows from Lemma 5.1 that

$$S(t) \le t^{-\frac{p-m}{p-1}} \left[l + \frac{1}{m(p-m)\sigma^{p-m}} \{[1 + (p-1)\sigma^{p-1} t]^{\frac{p-m}{p-1}} - 1\} \right] \to$$
$$\to \frac{\overline{\eta}}{m} =: \tilde{\eta} \quad \text{as} \quad t \to \infty \ .$$

Hence the conclusion follows. ∎

Let us denote

$$F_1(\eta) := \frac{f^{m-1}(\eta)}{m} - \frac{p-m}{p-1}\eta \ .$$

It is easily seen that $F_1 \in C^1([0, \tilde{\eta}])$ whenever $1 < p < m$. Let us prove the following result.

LEMMA 5.4. *Let $1 < m \le p$. Then the trivial solution of equation (5.10) is completely unstable if $m \le p < m+1$, or asymptotically stable if $m + 1 \le p$.*

Proof. According to Corollary 3.2 we can rewrite

$$
\begin{aligned}
(5.15)\quad F_1(\eta) &= \eta\left\{\frac{1}{m}\left[1-\frac{m-1}{2}\ \eta^{\frac{p-1}{m-1}}+o\left(\eta^{\frac{p-1}{m-1}}\right)\right]-\frac{p-m}{p-1}\right\}\\
&= \left(\frac{1}{m}-\frac{p-m}{p-1}\right)\eta-\frac{m-1}{2m}\ \eta^{\frac{p+m-2}{m-1}}+o\left(\eta^{\frac{p+m-2}{m-1}}\right)\ .
\end{aligned}
$$

Then by (5.10), (5.15) there exists $\delta > 0$ such that for any $S_0 \in (0,\delta)$ and any $\tau > 0$

$$
(5.16)\quad
\begin{aligned}
&S(\tau) \geq \delta\ e^{\frac{1}{2}\left(\frac{1}{m}-\frac{p-m}{p-1}\right)\tau} && \text{if } m \leq p < m+1\ ,\\
&S(\tau) \leq \delta\left(1+\frac{\delta^{\frac{m-1}{m}}}{4}\tau\right)^{-\frac{m-1}{m}} && \text{if } p = m+1\ ,\\
&S(\tau) \leq \delta\ e^{\left(\frac{1}{m}-\frac{p-m}{p-1}\right)\tau} && \text{if } p > m+1\ .
\end{aligned}
$$

Hence the conclusion follows. ∎

Let us also note the following result.

LEMMA 5.5. *Let* $1 < m < p$. *Then every solution of equation (5.10) is monotone.*

Proof. Let $S = S(\tau)$ be a nonconstant solution of (5.10). If $\tau_1 > 0$ exists such that $\frac{dS}{d\tau}(\tau_1) = 0$, then either $S(\tau_1) = 0$ or $S(\tau_1) = S_1$ by Proposition 5.2. This is impossible by the uniqueness theorem since $F_1 \in C^1([0,\tilde{\eta}])$. Hence the result. ∎

Now we can prove Proposition 5.3.

Proof of Proposition 5.3. Let $p > m$. By Lemmas 5.3, 5.5 every nonconstant solution of (5.10) tends to a finite limit as $\tau \to \infty$. It is easily seen that such a limit is a stationary solution. Then the conclusion follows by Proposition 5.2 and Lemma 5.4. ∎

Let us prove Theorem 1.1 for the Riemann problem (1.1), (1.7).

Proof of Theorem 1.1. (i) is the content of Theorem 5.1. (ii) If $p = m$ the result follows from (5.6). If $m < p < m+1$ it follows from

Proposition 5.3 and the definition (4.16), since in this case

$$S(t) \to S_1 > 0 \qquad \text{as } t \to \infty \ .$$

Let $p > m+1$. Rewrite (5.10) as follows:

$$\frac{\mathrm{d}\,S}{\mathrm{d}\,\tau} = -\omega S + \frac{1}{m}\left(\frac{f^{m-1}}{S} - 1\right) S \ , \tag{5.17}$$

where

$$-\omega := \frac{1}{m} - \frac{p-m}{p-1} < 0 \ .$$

By Proposition 5.3

$$S(\tau) \to 0 \qquad \text{as } \tau \to \infty \ .$$

Hence by Corollary 3.2 there exists $\tau_1 > 0$ such that

$$\frac{\mathrm{d}\,S}{\mathrm{d}\,\tau} \leq -\omega S \qquad \text{for any } \tau > \tau_1 \ ,$$

whence

$$S(\tau)e^{\omega\tau} \leq S(\tau_1)e^{\omega\tau_1} =: B \qquad (\tau > \tau_1) \ . \tag{5.18}$$

Let us integrate (5.17) on the interval $(\overline{\tau}, \tau)$, where $\overline{\tau} := \log \overline{t}$. We get

$$S(\tau)e^{\omega\tau} = S(\overline{\tau})e^{\omega\overline{\tau}} e^{\frac{1}{m}\int_{\overline{\tau}}^{\tau}\left[\frac{f^{m-1}(S(\zeta))}{S(\zeta)} - 1\right]\mathrm{d}\,\zeta} \ . \tag{5.19}$$

By (3.6) and (5.18) we also have:

$$\int_{\overline{\tau}}^{\tau}\left[\frac{f^{m-1}(S(\zeta))}{S(\zeta)} - 1\right]\mathrm{d}\,\zeta \geq \int_{\overline{\tau}}^{\tau_1}\left[\frac{f^{m-1}(S(\zeta))}{S(\zeta)} - 1\right]\mathrm{d}\,\zeta -$$
$$- (m-1)B^{\frac{p-1}{m-1}} \int_{\tau_1}^{\tau} e^{-\omega\cdot\frac{p-1}{m-1}\zeta}\mathrm{d}\,\zeta \qquad (\tau > \tau_1 > \overline{\tau}) \ .$$

Since the second integral in the right–hand side is convergent, we get

$$0 < A \leq S(\tau)e^{\omega\tau} \qquad (\tau > \tau_1)$$

with a suitable definition of the constant A. Since

$$S(\tau)e^{\omega\tau} = s(t)t^{-\frac{p-m}{p-1}}t^{-\frac{1}{m}+\frac{p-m}{p-1}} = s(t)t^{-\frac{1}{m}}$$

for $\tau = \log t$, the conclusion follows. The proof of the case $p = m + 1$ is similar (see inequality (5.16)). This completes the proof. ∎

6. The Riemann problem: intermediate asymptotics

Let us prove Theorem 1.2 for the Riemann problem (1.1), (1.8). Due to equality (4.17), the proof follows easily from the estimates of the support derived above.

Proof of Theorem 1.2. (i) If no intersection arises between s_0 and c, we have (see (4.11)):

$$\|u(t,\cdot)\|_\infty = \sigma[1 + (p-1)\sigma^{p-1}t]^{-\frac{1}{p-1}} \quad .$$

In such case

$$t\left[\frac{1}{p-1} - t\|u(t,\cdot)\|_\infty^{p-1}\right] \longrightarrow \frac{1}{(p-1)^2\sigma^{p-1}} \qquad \text{as } t \to \infty \ ,$$

which proves the claim.

If there is intersection, from the definition (4.11) of u and equality (3.5) we get

$$\text{(6.1)} \qquad \left\{t\left[\frac{1}{p-1} - t\|u(t,\cdot)\|_\infty^{p-1}\right]\right\}^{\frac{m-p}{p-1}} = \frac{1}{m-p}\left(\frac{1}{p-1}\right)^{\frac{m-p}{p-1}} \cdot \\ \cdot\left\{\frac{S(t)t^{-\frac{m-p}{p-1}}}{f^{m-p}(S(t))} + \frac{t^{-\frac{m-p}{p-1}}}{m-p}\right\} \quad .$$

In this case by Theorem 5.1

$$s(t) = S(t)t^{-\frac{m-p}{p-1}} \longrightarrow s_\infty \le x^* \qquad \text{as } t \to \infty \ ,$$

$$\Rightarrow \quad S(t) \to \infty \quad \Rightarrow \quad f(S(t)) \to \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$$

by Corollary 3.1. Then from (6.1) we get

$$t\left[\frac{1}{p-1} - t\|u(t,\cdot)\|_\infty^{p-1}\right] \longrightarrow \left(\frac{s_\infty}{m-p}\right)^{\frac{p-1}{m-p}}$$

as $t \to \infty$.

(ii) If $p = m$ we get from (3.4):

$$t^{\frac{1}{m-1}}\|u(t,\cdot)\|_\infty = f(s(t)) \to \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}} \qquad \text{as } t \to \infty \ ,$$

due to Theorem 1.1 (observe that $S = s$ for $p = m$).

If $m < p < m+1$, $S(t) \to S_1 > 0$ as $t \to \infty$ (see Proposition 5.3). Hence by continuity of f we have (see Lemma 5.2)

$$t^{\frac{1}{p-1}}\|u(t,\cdot)\|_\infty = f(S(t)) \to f_1 > 0 \qquad \text{as } t \to \infty \ .$$

If $p > m+1$, it is known from Theorem 1.1 that

$$S(t) \sim t^{\frac{1}{m} - \frac{p-m}{p-1}} \qquad \text{as } t \to \infty \ .$$

Then

$$t^{\frac{1}{p-1}}\|u(t,\cdot)\|_\infty = f(S(t)) \sim t^{\frac{1}{m-1}\left(\frac{1}{m} - \frac{p-m}{p-1}\right)} \qquad \text{as } t \to \infty \ .$$

Since

$$\frac{1}{p-1} - \frac{1}{m-1}\left(\frac{1}{m} - \frac{p-m}{p-1}\right) = \frac{1}{m} \ ,$$

the claim follows.

If $p = m + 1$, by Theorem 1.1 we have

$$f(S(t)) \sim (\ln t)^{-\frac{1}{m}} \quad ;$$

then the conclusion follows. This completes the proof. ∎

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