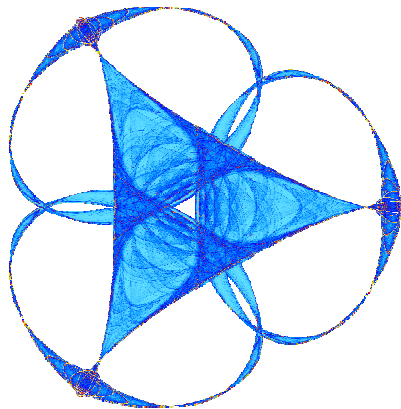ON A STOCHASTIC LERAY-α MODEL OF EULER EQUATIONS

By

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IMA Preprint Series #2412

(October 2012)

On a Stochastic Leray-α model of Euler equations

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October 25, 2012

Abstract

We deal with the 3D inviscid Leray-α model. The well posedness for this problem is not known; by adding a random perturbation we prove that there exists a unique (in law) global solution. The random forcing term formally preserves conservation of energy. The result holds for initial velocity of finite energy and the solution has finite energy a.s.. These results are easily extended to the 2D case.

MSC2010: 35Q31, 60H15, 35Q35.
Keywords: Inviscid Leray-α models, Euler equations, Multiplicative noise, Uniqueness in law, Stratonovitch integral, Girsanov formula.

1 Introduction

The motion of incompressible fluids is described by the Navier-Stokes equations

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla)v + \nabla p = f \\ \operatorname{div} v = 0 \end{cases} \tag{1}$$

for viscous fluids, or by the Euler equations

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p = f \\ \operatorname{div} v = 0 \end{cases} \tag{2}$$

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for inviscid fluids.

The unknown are the velocity field $v = v(t, x)$ and the pressure field $p = p(t, x)$; f is a given external force and $\nu > 0$ is the viscosity that corresponds to the inverse of the Reynolds number Re. When the fluid moves in a bounded domain, suitable boundary conditions are associated to these equations, respectively, the no-slip and slip conditions.

The above two systems have a quite different behavior; for instance, when $f = 0$ system (1) is dissipative while system (2) is conservative. It is well known, since the seminal work of Leray, that for initial velocity of finite energy the 3D Navier-Stokes system (1) has a global weak solution but its uniqueness is still an open problem. However, for the 3D Euler system (2) neither the global existence nor the uniqueness of global solutions are known when the initial velocity is of finite energy (we refer to the review paper [4] on this topic).

We recall that to prove the existence of solutions to (1) in $\mathbb{R}^d$, $d = 2, 3$, Leray [23] considered the following regularization for $\alpha > 0$

$$
\begin{cases}
\dfrac{\partial v^\alpha}{\partial t} - \nu \Delta v^\alpha + (u^\alpha \cdot \nabla) v^\alpha + \nabla p = f \\
u^\alpha = G_\alpha * v^\alpha \\
\operatorname{div} v^\alpha = 0
\end{cases}
\tag{3}
$$

where G_α is a smoothing kernel such that $u^\alpha \longrightarrow v^0$, in some sense, as $\alpha \to 0$. In particular, system (3) converges to the Navier-Stokes system (1) as $\alpha \to 0$.

In [8], a special smoothing kernel was considered, namely, the Green function associated with the operator $1 - \alpha\Delta$,

$$
u^\alpha = G_\alpha * v^\alpha = (1 - \alpha\Delta)^{-1} v^\alpha
\tag{4}
$$

for $\alpha > 0$. This kernel works as a kind of filter with width α and the parameter α reflects a sub-grid length scale in the model. This model was inspired by the Navier-Stokes α model (also known as the Camassa-Holm system or Lagrangian averaged Navier-Stokes α equations) of turbulence, see [6, 7, 18] and the references therein. Moreover, it has been demonstrated analytically and computationally that the Navier-Stokes α model is a powerful tool in the study of turbulence, see [18] and the reference therein. Along the same lines, it is worth mentioning that other α models, such as the Clark-α model [5] and the Navier-Stokes Voigt equations [20], have been used as a sub-grid scale models of turbulence.

In this paper, we are interested in a stochastic version of system (3) with $\nu = 0$ and regularization given by (4), that is the following stochastic Leray-α

model of Euler equations

$$\begin{cases} dv^\alpha + [(u^\alpha \cdot \nabla)v^\alpha + \nabla p] \, dt = ((\sigma \circ dW) \cdot \nabla)v^\alpha \\ v^\alpha = (1 - \alpha \triangle)u^\alpha \\ \text{div } v^\alpha = 0 \end{cases} \qquad (5)$$

Here W is a Brownian motion (in time) and $\circ dW$ referes to the Stratonovitch differential; the parameter α is positive. When $\alpha = 0$ the first equation of (5) reduces to the stochastic Euler equation.

The well posedness of weak solutions of the deterministic system (5) ($\sigma = 0$) is not known. In particular, when the initial velocity has finite energy, existence of global weak solutions can be proven if $\alpha > 0$, see the Appendix. However, the uniqueness is not known for both $d = 2$ and $d = 3$. If $\alpha = 0$ and $\sigma = 0$, global existence of solutions are known for initial velocity of finite energy and enstrophy for $d = 2$ while it is open for $d = 3$. Their uniqueness is an open problem for $d = 2$ and $d = 3$.

It is worth to point out that the analysis of the deterministic Leray-α Euler equations in 3D, that is system (5) with $\sigma = 0$, is more difficult than for the other approximations models for 3D inviscid fluids (i.e., Camassa-Holm, Clark and Voigt models). Indeed, for these other models there is formal conservation of the sum $\int |v^\alpha|^2 dx + \alpha \int |\nabla v^\alpha|^2 dx$ ($\alpha > 0$), whereas the model we are interested in has only formal conservation of $\int |v^\alpha|^2 dx$. From this point of view the deterministic Leray-α model for 3D Euler equations considered in this paper is closer to the 3D Euler equations than the Voigt, Camassa-Holm and Clark models which regularize much more the original Euler equations.

When adding an appropriate stochastic perturbation, we will prove that system (5) has a unique global solution (in law) when the initial condition is of finite energy. We will prove existence and uniqueness (in law) of solutions by means of Girsanov formula. The multiplicative noise in (5) formally preserves the conservation of energy (see Section 3 for the details). It is crucial to choose the random perturbation in (5) to be written in the Stratonovitch form in such a way that formally the energy of the vector field v^α is conserved.

All our results are stated for a three dimensional spatial domain (a box $[0, 2\pi]^3$, assuming periodic boundary conditions), but our proofs can be easily adapted to the two dimensional case. It would be interesting to study the behavior of the process v^α when α and/or σ converge to zero. This is the subject of future research.

In the past few years, there has been a huge effort to tackle the problem of using a similar noise in order to improve the qualitative properties of some non linear equations. In particular uniqueness of the stochastic equation

has been proved either when uniqueness is not known in the deterministic setting or with weaker assumptions than in the deterministic setting; for these results, see [1, 2, 3, 11, 13, 14, 15] and the references therein. We refer to [9, 16] for the 2D Euler equations, and to [10, 17] for some analysis on the 3D Navier-Stokes equations. For an overview of the problems and methods, we refer to [12].

As far as the content of this paper, in Section 2 we will introduce our functional setting and spaces. Section 3 will be devoted to the description of the stochastic Leray-α model in Fourier components. We will write the model in both the Stratonovitch and Itô forms. Section 4 will focus on the linear model: global existence and uniqueness of strong (in the probabilistic sense) solutions will be proved. The uniqueness proof is based on the study of a new linear problem constructed by means of the covariance matrix A_k. The nonlinear model will be studied in Section 5, where we prove the existence and uniqueness of solutions (in law) by means of a Girsanov formula. In Section 6, our results will be stated for the stochastic partial differential equation (5). To make our paper self-contained, in the Appendix we will give the proof of global existence of weak solutions for the deterministic Leray-α model of Euler equations.

2   Functional setting

Let the spatial domain be a torus $\mathbb{T}$, i.e. $x \in \mathbb{R}^3$ and periodic boundary conditions on the cube $[0, 2\pi]^3$ are assumed. Notice that if v is a solution, then also $v + c$ ($c \in \mathbb{R}$) is a solution. Therefore, we consider mean zero velocity vectors, i.e. $\int_{[0,2\pi]^3} v(t, x) \, dx = 0$.

We fix notations. Given a complex number $z \in \mathbb{C}$, we denote, respectively, by $\Re z$ and $\Im z$ its real and imaginary part; hence, $\overline{z} = \Re z - i\Im z$ and the product of two complex numbers z and w is $wz = (\Re w \Re z - \Im w \Im z) + i(\Re w \Im z + \Im w \Re z)$.

Morever, let $x \in \mathbb{C}^3$ be represented as $x = (x^{(1)}, x^{(2)}, x^{(3)})$. For $x, y \in \mathbb{C}^3$, we set $\langle x, y \rangle = \sum_{j=1}^{3} x^{(j)} \overline{y}^{(j)}$ and $\|x\|^2 = \langle x, x \rangle$. This defines, as a particular case, also the scalar product and the norm in $\mathbb{R}^3$.

For each $k \in \mathbb{Z}^3$, let $e_k(x) := e^{i\langle x, k \rangle}$, $x \in \mathbb{R}^3$. The family $\{e_k\}_{k \in \mathbb{Z}^3}$ is a complete orthogonal basis for the space $L^2(\mathbb{T}, \mathbb{C})$.

In this Section, we write the deterministic Leray-α model of the Euler

4

system (5)

$$\begin{cases} \dfrac{\partial v^\alpha}{\partial t} + (u^\alpha \cdot \nabla)v^\alpha + \nabla p = 0 \\ v^\alpha = (1 - \alpha\triangle)u^\alpha \\ \text{div } v^\alpha = 0, \end{cases} \qquad (6)$$

in Fourier components; this will be given in (8). In the next Section, the stochastic forcing term will be introduced.

Assume $v(t, \cdot)$ and $u(t, \cdot)$ are in $(L^2(\mathbb{T}, \mathbb{C}))^3$; then

$$v(t, x) = \sum_{k \in \mathbb{Z}^3} v_k(t)e_k(x) \qquad \text{and} \qquad u(t, x) = \sum_{k \in \mathbb{Z}^3} u_k(t)e_k(x)$$

We have $v_0 = 0$ and $u_0 = 0$, since v has mean value zero. Moreover, since v and u are real valued and $e_{-k}(x) = \overline{e_k(x)}$, we have

$$v_{-k}(t) = \overline{v_k}(t), \qquad u_{-k}(t) = \overline{u_k}(t).$$

We set $\|v(t, \cdot)\|_{l^2}^2 = \sum_k \|v_k(t)\|^2$.

We substitute in $(6)_2$ and get

$$\sum_k v_k(t)e_k(x) = (1 - \alpha\triangle) \sum_k u_k(t)e_k(x).$$

From now on, we will drop the index α in the unknowns for simplicity.

Since $\triangle e_k(x) = -\|k\|^2 e_k(x)$ then $v_k(t) = (1 + \alpha\|k\|^2)\, u_k(t)$ and

$$\sum_k v_k(t)e_k(x) = \sum_k \left(1 + \alpha\|k\|^2\right) u_k(t)e_k(x).$$

From equation $(6)_3$ we get the incompressibility condition

$$\langle v_k(t), k \rangle = 0 \qquad \forall k \in \mathbb{Z}^3, \ t \in \mathbb{R}.$$

Finally, using $(6)_1$ we obtain

$$\frac{d}{dt} \sum_k v_k(t)e_k(x)$$

$$= -\left(\sum_h u_h(t)e_h(x) \cdot \nabla\right) \sum_{k'} e_{k'}(x)v_{k'}(t) - \nabla p(t, x)$$

$$= -\sum_{h,k'} e_h(x) \left(u_h^{(1)}(t)\tfrac{\partial}{\partial x^{(1)}} + u_h^{(2)}(t)\tfrac{\partial}{\partial x^{(2)}} + u_h^{(3)}(t)\tfrac{\partial}{\partial x^{(3)}}\right) e_{k'}(x)v_{k'}(t) - \nabla p(t, x)$$

$$= -\sum_{h,k'} e_h(x) \left(u_h^{(1)}(t)ik'^{(1)} + u_h^{(2)}(t)ik'^{(2)} + u_h^{(3)}(t)ik'^{(3)}\right) e_{k'}(x)v_{k'}(t) - \nabla p(t, x)$$

$$= -i\sum_{h,k'} e_{h+k'}(x)\langle u_h(t), k' \rangle v_{k'}(t) - \nabla p(t, x),$$

5

that is

$$\frac{d}{dt}\sum_k v_k(t)e_k(x) = -i\sum_{h,k'}\frac{\langle v_h(t), k'\rangle}{1+\alpha\|h\|^2}P_k(v_{k'}(t))e_{h+k'}(x)$$

where

$$P_k(v) := v - \frac{\langle v, k\rangle}{\langle k, k\rangle}\,k \tag{7}$$

is the projection onto the space orthogonal to k.

Summing up, since $\langle v_h, k\rangle = \langle v_h, k-h\rangle$, we obtain the system (6) written in Fourier components

$$\begin{cases} \dfrac{dv_k}{dt}(t) = -i\sum_{h\in\mathbb{Z}^3}\dfrac{\langle v_h(t), k\rangle}{1+\alpha\|h\|^2}P_k(v_{k-h}(t)) \\ \langle v_k(t), k\rangle = 0 \\ v_{-k}(t) = \overline{v_k}(t) \end{cases} \tag{8}$$

for any k.

We notice that, given $\alpha \geq 0$, if $\sum_k \|v_k(t)\|^2 < \infty$, then the series in the r.h.s. of $(8)_1$ is convergent and for any k we have

$$\left\|\frac{dv_k}{dt}(t)\right\| \leq \|v(t)\|_{l^2}^2\|k\|.$$

System (8) enjoys an important property: formally the energy $E(t) := \frac{1}{2}\sum_k \|v_k(t)\|^2$ is conserved under the dynamics given by (8). Indeed,

$$\frac{d}{dt}\|v_k(t)\|^2 = \sum_h 2\Re\left\{-i\frac{\langle v_h(t), k\rangle}{1+\alpha\|h\|^2}\langle P_k(v_{k-h}(t)), v_k(t)\rangle\right\}$$

Summing over all components, we formally obtain

$$\frac{dE}{dt}(t) = \sum_k\sum_h \Im\left\{\frac{\langle v_h(t), k\rangle}{1+\alpha\|h\|^2}\langle v_{k-h}(t), v_k(t)\rangle\right\}$$

which vanishes, since the sum contains terms which cancel each other according to the following equality

$$\frac{\langle v_{h'}, k'\rangle}{1+\alpha\|h'\|^2}\langle v_{k'-h'}, v_{k'}\rangle = \overline{\frac{\langle v_h, k\rangle}{1+\alpha\|h\|^2}\langle v_{k-h}, v_k\rangle} \tag{9}$$

for $h' = -h$ and $k' = k-h$.

Let us finally notice that conservation of energy formally holds also for $\alpha = 0$.

3 The stochastic Leray-α model in Fourier components

We are interested in a stochastic equation obtained from (8) by adding a random forcing term in such a way that energy is formally conserved. To this end we consider the system of Stratonovich equations

$$dY_k(t) = -i \sum_{h \in \mathbb{Z}^3} P_k(Y_{k-h}(t)) \frac{\langle Y_h(t)dt + \sigma \circ dW_h(t), k \rangle}{1 + \alpha \|h\|^2}, \qquad k \in \mathbb{Z}^3, k \neq \vec{0} \tag{10}$$

where $\{W_h\}_{h \in \mathbb{Z}^3}$ is a family of independent $\mathbb{C}^3$-valued Brownian motions on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P)$, except for $\langle W_h(t), h \rangle = 0, W_{-h}(t) = \overline{W_h}(t)$. According to the properties of Stratonovich integral (see [19]) we have formally that $dE(t) = 0$. The computations are similar to the previous ones, using (9) and

$$\langle Y_{k-h}, Y_k \rangle \langle \sigma \circ dW_{-h}, k \rangle = \overline{\langle Y_k, Y_{k-h} \rangle \langle \sigma \circ dW_h, k \rangle}.$$

Let us make precise the Stratonovich formulation of system (10) in order to write it in terms of Itô integrals. Indeed, the Stratonovitch formulation gives insights on the behaviour of the system, but computations will be done on the Itô formulation.

Set

$$J := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1 > 0 \text{ or } (k_1 = 0, k_2 > 0) \text{ or } (k_1 = 0, k_2 = 0, k_3 > 0)\}.$$

Let $\{W'_h\}_{h \in J}$ be a family of independent $\mathbb{C}^3$-valued standard Brownian motions. Define for any $h \in J$

$$W_h := P_h(W'_h), \qquad W_{-h} := \overline{W}_h \tag{11}$$

Therefore $\langle W_h, h \rangle = 0, W_{-h} = \overline{W}_h$ for any $h \in \mathbb{Z}^3, h \neq \vec{0}$. Hence it is enough to give the family $\{W'_h\}_{h \in J}$ in order to define the stochastic part of system (10).

Next, define

$$\widetilde{\widetilde{W}}_{h,k}(t) := \langle W_h(t), k \rangle \tag{12}$$

then each $\widetilde{\widetilde{W}}_{h,k}$ is a $\mathbb{C}$-valued Brownian motion, whose real and imaginary part are independent. Since $\widetilde{\widetilde{W}}_{h,k}(t) = \langle W_h(t), k \rangle = \langle P_h(W'_h(t)), k \rangle = \langle W'_h(t), P_h(k) \rangle$, then

$$Var[\Re \widetilde{\widetilde{W}}_{h,k}(1)] = Var[\Im \widetilde{\widetilde{W}}_{h,k}(1)] = \|P_h(k)\|^2 = \|k\|^2 \sin^2(\theta)$$

where θ is the angle between h and k. Now, setting

$$\widetilde{W}_{h,k}\|P_h(k)\| := \widetilde{\widetilde{W}}_{h,k} \tag{13}$$

we have defined standard real Brownian motions $\Re\widetilde{W}_{h,k}$ and $\Im\widetilde{W}_{h,k}$.

Set

$$\sigma_h := \frac{\sigma}{1 + \alpha\|h\|^2}. \tag{14}$$

Therefore (10) is

$$\begin{aligned}
dY_k(t) = &- i\sum_{h\in\mathbb{Z}^3} \frac{\langle Y_h(t), k\rangle}{1 + \alpha\|h\|^2} P_k(Y_{k-h}(t))dt \\
&- i\sum_{h\in\mathbb{Z}^3} \sigma_h\|P_h(k)\|P_k(Y_{k-h}(t)) \circ d\widetilde{W}_{h,k}(t)
\end{aligned} \tag{15}$$

Indeed, for the Stratonovich integral we have

$$\int_0^t P_k(Y_{k-h}(s)) \langle \sigma_h \circ dW_h(s), k\rangle$$

$$= \sigma_h P_k \int_0^t Y_{k-h}(s) \circ d\widetilde{\widetilde{W}}_{h,k}(s)$$

$$= \sigma_h\|P_h(k)\|P_k \int_0^t Y_{k-h}(s) \circ d\widetilde{W}_{h,k}(s)$$

$$= \sigma_h\|P_h(k)\|P_k \int_0^t \Re Y_{k-h}(s) \circ d\Re\widetilde{W}_{h,k}(s) - \sigma_h\|P_h(k)\|P_k \int_0^t \Im Y_{k-h}(s) \circ d\Im\widetilde{W}_{h,k}(s)$$

$$+ i\sigma_h\|P_h(k)\|P_k \int_0^t \Re Y_{k-h}(s) \circ d\Im\widetilde{W}_{h,k}(s) + i\sigma_h\|P_h(k)\|P_k \int_0^t \Im Y_{k-h}(s) \circ d\Re\widetilde{W}_{h,k}(s). \tag{16}$$

We have the corresponding Itô formulation.

Theorem 1. *Let $\{Y_k\}_{k\in\mathbb{Z}^3, k\neq\vec{0}}$ be a sequence of continuous and adapted processes defined on a given filtered probability space such that $\sum_k \|Y_k(t)\|^2 < \infty$ a.s.. If the sequence solves the following system*

$$\begin{aligned}
dY_k(t) = &- i\sum_{h\in\mathbb{Z}^3} \frac{\langle Y_h(t), k\rangle}{1 + \alpha\|h\|^2} P_k(Y_{k-h}(t))dt \\
&- i\sum_{h\in\mathbb{Z}^3} \sigma_h\|P_h(k)\|P_k(Y_{k-h}(t))d\widetilde{W}_{h,k}(t) \\
&- \sum_{h\in\mathbb{Z}^3} \sigma_h^2\|P_h(k)\|^2 P_k(P_{k-h}(Y_k(t)))dt, \qquad \forall k
\end{aligned} \tag{17}$$

then it solves system (15).

Proof. We are going to prove that when the Itô integral in the r.h.s. is written as a Stratonovich integral, we get (15). The corrective term appearing in this transformation comes from the quadratic variation $[\sigma_h \| P_h(k) \| P_k(Y_{k-h}), \widetilde{W}_{h,k}]$ (see [19]).

Let us work on the real and imaginary part; this makes the proof long but clear. Let $k' = k - h$; from (15) we have

$$\sigma_h \| P_h(k) \| P_k(\Re Y_{k'}(t)) = \sigma_h \| P_h(k) \| P_k(\Re Y_{k'}(0)) + \int_0^t \ldots \ldots ds$$

$$+ \sum_{h' \in \mathbb{Z}^3} \int_0^t \sigma_h \| P_h(k) \| P_k \left(\sigma_{h'} \| P_{h'}(k') \| P_{k'}(\Re Y_{k'-h'}(s)) \right) \circ d\Im \widetilde{W}_{h',k'}(s)$$

$$+ \sum_{h' \in \mathbb{Z}^3} \int_0^t \sigma_h \| P_h(k) \| P_k \left(\sigma_{h'} \| P_{h'}(k') \| P_{k'}(\Im Y_{k'-h'}(s)) \right) \circ d\Re \widetilde{W}_{h',k'}(s).$$

$$\sigma_h \| P_h(k) \| P_k(\Im Y_{k'}(t)) = \sigma_h \| P_h(k) \| P_k(\Im Y_{k'}(0)) + \int_0^t \ldots \ldots ds$$

$$- \sum_{h' \in \mathbb{Z}^3} \int_0^t \sigma_h \| P_h(k) \| P_k \left(\sigma_{h'} \| P_{h'}(k') \| P_{k'}(\Re Y_{k'-h'}(s)) \right) \circ d\Re \widetilde{W}_{h',k'}(s)$$

$$+ \sum_{h' \in \mathbb{Z}^3} \int_0^t \sigma_h \| P_h(k) \| P_k \left(\sigma_{h'} \| P_{h'}(k') \| P_{k'}(\Im Y_{k'-h'}(s)) \right) \circ d\Im \widetilde{W}_{h',k'}(s).$$

Bearing in mind that

$$\widetilde{W}_{h',k-h} \quad \text{is independent of} \quad \widetilde{W}_{h,k} \qquad \text{if } h' \neq -h \text{ and } h' \neq h$$

and

$$\widetilde{W}_{h',k-h} = \overline{\widetilde{W}_{h,k}} \qquad \text{if } h' = -h.$$

$$\widetilde{W}_{h',k-h} = \widetilde{W}_{h,k} \qquad \text{if } h' = h$$

we are reduced to take into account only the terms with $h' = -h$ and $h' = h$. Moreover, we use that $P_h(k) = P_{-h}(k-h) = P_h(k-h)$. Therefore the corrective term for $\sigma_h \| P_h(k) \| P_k(\Re Y_{k-h}(s)) \circ d\Im \widetilde{W}_{h,k}(s)$ is

$$-\frac{1}{2} \sigma_h^2 \| P_h(k) \|^2 P_k \left(P_{k-h} \left(\Re Y_k(s) \right) \right) + \frac{1}{2} \sigma_h^2 \| P_h(k) \|^2 P_k \left(P_{k-h} \left(\Re Y_{k-2h}(s) \right) \right) \ ;$$

that for $\sigma_h \| P_h(k) \| P_k(\Im Y_{k-h}(s)) \circ d\Re \widetilde{W}_{h,k}(s)$ is

$$-\frac{1}{2} \sigma_h^2 \| P_h(k) \|^2 P_k \left(P_{k-h} \left(\Re Y_k(s) \right) \right) - \frac{1}{2} \sigma_h^2 \| P_h(k) \|^2 P_k \left(P_{k-h} \left(\Re Y_{k-2h}(s) \right) \right) \ ;$$

9

that for $-\sigma_h \|P_h(k)\| P_k(\Re Y_{k-h}(s)) \circ d\Re \widetilde{W}_{h,k}(s)$ is

$$-\frac{1}{2}\sigma_h^2 \|P_h(k)\|^2 P_k\left(P_{k-h}\left(\Im Y_k(s)\right)\right) - \frac{1}{2}\sigma_h^2 \|P_h(k)\|^2 P_k\left(P_{k-h}\left(\Im Y_{k-2h}(s)\right)\right) \; ;$$

that for $\sigma_h \|P_h(k)\| P_k(\Im Y_{k-h}(s)) \circ d\Im \widetilde{W}_{h,k}(s)$ is

$$-\frac{1}{2}\sigma_h^2 \|P_h(k)\|^2 P_k\left(P_{k-h}\left(\Im Y_k(s)\right)\right) + \frac{1}{2}\sigma_h^2 \|P_h(k)\|^2 P_k\left(P_{k-h}\left(\Im Y_{k-2h}(s)\right)\right) \; .$$

Summing up all the contributions, we get the expression given in the Proposition. $\square$

The aim of this paper is to study the stochastic system (17) with initial data of finite energy.

4 The linear model

Let us consider the linear system obtained by neglecting the nonlinear terms in (17):

$$
\begin{cases}
dY_k(t) & = -i\sum_{h\in\mathbb{Z}^3} \sigma_h \|P_h(k)\| P_k(Y_{k-h}(t)) d\widetilde{B}_{h,k}(t) \\
& \quad - \sum_{h\in\mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 P_k(P_{k-h}(Y_k(t))) dt \\[2mm]
\langle Y_k(t), k \rangle & = 0 \\[2mm]
Y_{-k}(t) & = \overline{Y_k(t)} \\[2mm]
Y_k(0) & = y_k
\end{cases}
\tag{18}
$$

for each $k \neq \overrightarrow{0}$. Here, $\{\widetilde{B}_{h,k}\}$ is a family of $\mathbb{C}$-valued Brownian motions obtained from a family of independent $\mathbb{C}^3$-valued standard Brownian motions $\{B'_h\}_{h\in J}$ defined on a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, Q)$ with the same procedure presented in (11)-(13).

In the next section, we will see how Girsanov transform allows to pass from the linear to the nonlinear system.

Notice that if $\mathbb{E}^Q \int_0^T \|Y(t)\|_{l^2} dt < \infty$, then the terms in the r.h.s. of $(18)_1$

are well defined. Indeed, for the Itô integrals we use that

$$\mathbb{E}^Q \| \sum_{h \in \mathbb{Z}^3} \sigma_h \|P_h(k)\| \int_0^t P_k(Y_{k-h}(s)d\widetilde{B}_{h,k}(s)\|^2$$

$$= \mathbb{E}^Q \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 \int_0^t \|P_k(Y_{k-h}(s))\|^2 ds$$

$$\leq \sigma^2 \|k\|^2 \mathbb{E}^Q \int_0^t \sum_{h \in \mathbb{Z}^3} \|Y_{k-h}(s)\|^2 ds$$

$$= \sigma^2 \|k\|^2 \mathbb{E}^Q \int_0^t \|Y(s)\|_{l^2}^2 ds$$

and for the deterministic integrals (Q-a.s.)

$$\| \int_0^t \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 P_k(P_{k-h}(Y_k(s))) \ ds\|$$

$$\leq \int_0^t \| \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 P_k(P_{k-h}(Y_k(s)))\| ds$$

$$\leq (\sum_{h \in \mathbb{Z}^3} \sigma_h^2) \|k\|^2 \int_0^t \|Y_k(s)\| ds$$

with $\sum_{h \in \mathbb{Z}^3} \sigma_h^2 = \sum_{h \in \mathbb{Z}^3} \frac{\sigma^2}{(1+\alpha\|h\|^2)^2} < +\infty$.

We are interested in the stochastic system (18) with deterministic initial data $y = \{y_k\}_k$ of finite energy. We shall deal with strong solution in the probabilistic sense, that is the filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, Q)$ and the Brownian motions $\{B'_h\}_{h \in J}$ are given a priori. We shall prove existence and uniqueness of solutions of the following type.

Definition 1. *Given $y \in l^2$, an energy controlled strong solution for system (18) is a family of continuous and adapted $\mathbb{C}^3$-valued stochastic processes $\{Y_k\}_{k \in \mathbb{Z}^3}$ such that for all $t \geq 0$*

$$\begin{cases} Y_k(t) & = & y_k - i \sum_{h \in \mathbb{Z}^3} \sigma_h \|P_h(k)\| \int_0^t P_k(Y_{k-h}(s))d\widetilde{B}_{h,k}(s) \\ & & - \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 \int_0^t P_k(P_{k-h}(Y_k(s)))ds \\ \langle Y_k(t), k \rangle & = & 0 \\ Y_{-k}(t) & = & \overline{Y_k(t)} \end{cases}$$

11

Q-a.s. for all k, and for any $t > 0$

$$\sum_{k \in \mathbb{Z}^3} \|Y_k(t)\|^2 \le \sum_{k \in \mathbb{Z}^3} \|y_k\|^2 \qquad Q - a.s.$$

4.1 Existence of a strong solution

Theorem 2. *For any initial data of finite energy, there exists an energy controlled strong solution to system* (18).

Proof. We consider the finite dimensional system associated to (18); for each integer $N > 0$ this is obtained by neglecting the components of index higher than N in such a way that the energy is conserved. Set $\Gamma_N^k = \{h \in \mathbb{Z}^3 : 0 < \|h\| < N, 0 < \|h - k\| < N\}$. Then the linear Galerkin system is

$$\begin{cases} dY_k(t) & = & -i \sum_{h \in \Gamma_N^k} \sigma_h \|P_h(k)\| P_k(Y_{k-h}(t)) d\widetilde{B}_{h,k}(t) \\ & & - \sum_{h \in \Gamma_N^k} \sigma_h^2 \|P_h(k)\|^2 P_k(P_{k-h}(Y_k(t))) dt \\ \\ \langle Y_k(t), k \rangle & = & 0 \\ \\ Y_{-k}(t) & = & \overline{Y(t)_k} \\ \\ Y_k(0) & = & y_k \end{cases} \qquad (19)$$

for each $k \in \mathbb{Z}^3, 0 < \|k\| < N$. We consider inital data $Y^N(0) = y^N$ obtained from the inital data y of the full system (18) by putting to 0 the components y_k with $\|k\| \ge N$.

By linearity the Galerkin system has a unique global strong solution $Y^N = \{Y_k^N\}_{0 < \|k\| < N}$. Each component is a continuous and adapted process. Moreover, energy is conserved, that is for any $t > 0$

$$d \sum_{\|k\| < N} \|Y_k^N(t)\|^2 = 0 \qquad Q - \text{a.s.}$$

To prove it, again we use the properties of the family $\{\widetilde{B}_{h,k}\}$ of Brownian motions and the projector defined in (7).

Therefore, for an inital data of finite energy we have for any $t > 0$

$$\|Y^N(t)\|_{l^2} = \|y^N\|_{l^2} \le \|y\|_{l^2} \quad Q - a.s. \qquad (20)$$

This implies that for any $p \in [1, \infty)$ we have

$$\mathbb{E}^Q \int_0^T \|Y^N(t)\|_{l^2}^p dt = \|y^N\|_{l^2}^p \le \|y\|_{l^2}^p \qquad \forall N. \qquad (21)$$

12

Therefore the sequence $\{Y^N\}_N$ is a bounded sequence in $L^p(\Omega \times [0,T]; l^2)$ for any $1 \le p \le \infty$. This implies that there exists a sequence $\{Y^{N_i}\}_{i=1}^{\infty}$ and a process $Y \in L^\infty(\Omega \times [0,T]; l^2)$ such that

$$\lim_{i \to \infty} Y^{N_i} = Y \quad \text{weakly in } L^p(\Omega \times [0,T]; l^2) \qquad \text{for } p < \infty$$

and

$$\lim_{i \to \infty} Y^{N_i} = Y \quad \star\text{-weakly in } L^\infty(\Omega \times [0,T]; l^2).$$

In particular

$$\lim_{i \to \infty} Y_k^{N_i} = Y_k \text{ weakly in } L^2(\Omega \times [0,T]).$$

Now we consider the convergence of the integrals in the r.h.s. of $(19)_1$. The Itô integral, considered as a linear operator, is strongly continuous in $L^2(\Omega \times [0,T])$; hence it is weakly continuous (see .e.g. [22, 24]). This implies that each stochastic integral converges weakly:

$$\lim_{N \to \infty} \int_0^{\cdot} P_k(Y_{k-h}^N(s)) d\tilde{B}_{h,k}(s) = \int_0^{\cdot} P_k(Y_{k-h}(s)) d\tilde{B}_{h,k}(s) \qquad \text{weakly in } L^2(\Omega \times [0,T])$$

On the other side, using the independence of the Itô integrals and the Itô isometry we have

$$\mathbb{E}^Q \| \sum_{\|h\| < N} \sigma_h \|P_h(k)\| \int_0^T P_k(Y_{k-h}^N(s)) d\widetilde{B}_{h,k}(s) \|^2$$

$$= \sum_{\|h\| < N} \sigma_h^2 \|P_h(k)\|^2 \mathbb{E}^Q \| \int_0^T P_k(Y_{k-h}^N(s)) d\widetilde{B}_{h,k}(s) \|^2$$

$$= \sum_{\|h\| < N} \sigma_h^2 \|P_h(k)\|^2 \mathbb{E}^Q \int_0^T \|P_k(Y_{k-h}^N(s))\|^2 ds$$

$$\le \sigma^2 \|k\|^2 \sum_{\|h\| < N} \mathbb{E}^Q \int_0^T \|Y_{k-h}^N(s)\|^2 ds$$

$$\le \sigma^2 \|k\|^2 \|y\|_{l^2}^2 \quad \text{by (21)}.$$

Hence

$$\lim_{N \to \infty} \sum_{\|h\| < N} \int_0^{\cdot} \sigma_h \|P_h(k)\| P_k(Y_{k-h}^N(s)) d\tilde{B}_{h,k}(s) = \sum_h \int_0^{\cdot} \sigma_h \|P_h(k)\| P_k(Y_{k-h}(s)) d\tilde{B}_{h,k}(s)$$

weakly in $L^2(\Omega \times [0,T])$.

For the deterministic integral, it is an easy computation to identify the limit:

$$\int_0^\cdot P_k(P_{k-h}(Y_k^N(s)))ds \rightarrow \int_0^\cdot P_k(P_{k-h}(Y_k(s)))ds \qquad \text{weakly in } L^2(\Omega \times [0,T]).$$

For the limit Y, we have for any $t \geq 0$

$$\|Y(t)\|_{l^2} \leq \|y\|_{l^2} \qquad Q - a.s.$$

Therefore Y is an energy controlled strong solution.

Moreover, using again the estimate

$$\mathbb{E}^Q \int_0^T \|Y(t)\|_{l^2}^2 dt < \infty$$

in a classical way we obtain that the process given by the infinite sum of Itô integrals $\sum_h \sigma_h \|P_h(k)\| \int_0^t P_k(Y_{k-h}(s))d\tilde{B}_{h,k}(s)$ has a continuous modification. Hence, we conclude that the process Y has a continuous modification.
□

4.2 The covariance matrices

In Theorem 2 we have proved existence of energy controlled solutions $Y = \{Y_k\}_{k \in \mathbb{Z}^3}$ of (18); now we want to show their uniqueness. The idea is to study the time evolution of the covariance matrices $\{A_k\}_{k \in \mathbb{Z}^3}$, defined as follows:

$$A_k^{j_1,j_2}(t) = \mathbb{E}^Q \left[\Re Y_k^{(j_1)}(t) \Re Y_k^{(j_2)}(t) + \Im Y_k^{(j_1)}(t) \Im Y_k^{(j_2)}(t) \right] \qquad j_1, j_2 = 1, 2, 3$$

We collect the properties of A_k. Since $\langle Y_k(t), k \rangle = 0$ for any t and k, then k is an eigenvector for $A_k(t)$ corresponding to the 0 eigenvalue. $A_k(t)$ is a symmetric and semi-positive definite matrix; therefore the trace of $A_k(t)$ is non negative. Moreover, we have

$$\sum_{k \in \mathbb{Z}^3} \text{Tr}(A_k(t)) \leq \|y\|_{l^2}^2 \tag{22}$$

for any $t \geq 0$. Finally, $P_k A_k(t) P_k = A_k(t)$, where P_k is the real matrix previously defined in (7), which is symmetric semi-positive definite; P_k has the 0 eigenvalue with eigenvector k and the eigenvalue 1 of double multiplicity.

Bearing in mind (18) and the properties of the Brownian motions $\tilde{B}_{h,k}$, with some long but easy computations we get that each A_k fulfils a linear equation.

Proposition 3. *For each $k \neq \vec{0}$, A_k fulfils the differential equation*

$$\frac{dA_k}{dt}(t) = - \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 P_k P_{k-h} A_k(t)$$
$$- \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 A_k(t) P_{k-h} P_k \qquad (23)$$
$$+ 2 \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 P_k A_{k-h}(t) P_k$$

This Proposition shows the non trivial fact that the covariance matrices satisfy a closed differential system.

4.3 Pathwise uniqueness

We prove pathwise uniqueness for system (18).

Theorem 4. *There exists at most one energy controlled strong solution to system (18), that is given two energy controlled strong solutions $Y_{[1]}$ and $Y_{[2]}$ to system (18) defined on the same probability space $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, Q)$ and with the same initial data $y \in l^2$ and Brownian motions, we have for any $t \geq 0$*

$$Y_{[1]}(t) = Y_{[2]}(t) \qquad Q - a.s.$$

Proof. Define
$$Y := Y_{[1]} - Y_{[2]}.$$

The idea of the proof is to take the difference $Y = Y_{[1]} - Y_{[2]}$; by linearity Y solves (18) but with initial data $Y(0) = 0$. Let $\{A_k\}_{k \in \mathbb{Z}^3}$ be the covariance matrices of Y; these matrices satisfy (23) with zero initial condition and regularity (24). Thus, the uniqueness problem for (18) is transformed in the easier uniqueness problem for the deterministic system (23). Indeed, in order to show that for any $t > 0$ we have $Y(t) = 0$ Q-a.s. it is enough to prove that system (23) with zero initial condition has the unique solution which vanishes, i.e. for any $k \neq \vec{0}$, given $A_k(0) = 0$ we have $A_k(t) = 0$ for all $t > 0$.

For any $T > 0$ define

$$B_k := \int_0^T A_k(t)dt.$$

Each tensor B_k enjoys the same properties of A_k. Since

$$\sum_{k \in \mathbb{Z}^3} \text{Tr}(A_k(t)) \leq 4\|y\|_{l^2}^2, \qquad (24)$$

15

then

$$\sum_{k \in \mathbb{Z}^3} \mathrm{Tr}(B_k) \leq 4\|y\|_{l^2}^2 T;$$

thus

$$\lim_{k \to \infty} \mathrm{Tr}(B_k) = 0. \tag{25}$$

Writing equation (23) in the integral form we have

$$A_k(T) = \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 \big[2P_k B_{k-h} P_k - P_k P_{k-h} B_k - B_k P_{k-h} P_k\big]. \tag{26}$$

If $B_h = 0$ for all h, then $A_k(T) = 0$ and the proof is completed.

We proceed by contradiction. Suppose that there exists k_1 such that B_{k_1} does not vanish; then the maximal eigenvalue λ_1^* of B_{k_1} is positive. Starting from B_{k_1} and λ_1^* we construct a sequence of B_{k_n} and λ_n^*'s (with λ_n^* the maximal eigenvalue of B_{k_n}) such that $\{\lambda_n^*\}_{n \in \mathbb{N}}$ is a stricly increasing sequence. Therefore $\lim_{n \to \infty} \lambda_n^* > 0$. On the other hand, each B_k is semipositive definite and therefore $\mathrm{Tr}(B_{k_n}) \geq \lambda_n^*$. It follows that

$$\lim_{n \to \infty} \mathrm{Tr}(B_{k_n}) > 0$$

which is impossible because of (25).

Therefore we are left to prove that given $k_n \in \mathbb{N}$ such that B_{k_n} has maximal eigenvalue $\lambda_n^* > 0$, then there exists $k_{n+1} \in \mathbb{N}$ such that $B_{k_{n+1}}$ has maximal eigenvalue $\lambda_{n+1}^* > \lambda_n^* > 0$.

Let ϕ_n be the eigenvector corresponding to λ_n^*. Therefore

$$B_{k_n} \phi_n = \lambda_n^* \phi_n \tag{27}$$

Moreover ϕ_n is orthogonal to k_n, since they are eigenvectors corresponding to different eigenvalues; therefore

$$P_{k_n} \phi_n = \phi_n \tag{28}$$

From (26) we have

$$0 \leq \langle \phi_n, A_{k_n}(T) \phi_n \rangle = \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 (2\langle \phi_n, P_{k_n} B_{k_n-h} P_{k_n} \phi_n \rangle$$
$$- \langle \phi_n, P_{k_n} P_{k_n-h} B_{k_n} \phi_n \rangle$$
$$- \langle \phi_n, B_{k_n} P_{k_n-h} P_{k_n} \phi_n \rangle)$$

Using that each B_k is symmetric and (27)-(28), we get

$$0 \leq 2 \sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 \big[\langle \phi_n, B_{k_n-h} \phi_n \rangle - \lambda_n^* \langle \phi_n, P_{k_n-h} \phi_n \rangle\big]$$

16

For fixed n, we have that $\langle \phi_n, B_{k_n-h}\phi_n \rangle$ tends to 0 as $\|h\| \to \infty$, because of (25). Therefore, some addends in the sum are negative; if the sum must be non negative there must exist at least one addend positive, i.e.

$$\exists \tilde{h}: \ \langle \phi_n, B_{k_n-\tilde{h}}\phi_n \rangle - \lambda_n^* \langle \phi_n, P_{k_n-\tilde{h}}\phi_n \rangle > 0$$

Set $k_{n+1} = k_n - \tilde{h}$. Then

$$\langle \phi_n, B_{k_{n+1}}\phi_n \rangle - \lambda_n^* \langle \phi_n, P_{k_{n+1}}\phi_n \rangle > 0$$

Setting $\psi_n = P_{k_{n+1}}\phi_n$ and using that $B_{k_{n+1}} = P_{k_{n+1}} B_{k_{n+1}} P_{k_{n+1}}$ we get

$$\langle \psi_n, B_{k_{n+1}}\psi_n \rangle > \lambda_n^* \langle \psi_n, \psi_n \rangle$$

which implies that the maximal eigenvalue λ_{n+1}^* of $B_{k_{n+1}}$ is larger than λ_n^*. $\square$

5 The nonlinear model

Consider the nonlinear system in the Itô form

$$
\begin{cases}
dY_k(t) = -i \displaystyle\sum_{h \in \mathbb{Z}^3} \frac{\langle Y_h(t), k \rangle}{1 + \alpha \|h\|^2} P_k(Y_{k-h}(t)) dt \\[4mm]
\qquad - i \displaystyle\sum_{h \in \mathbb{Z}^3} \sigma_h \|P_h(k)\| P_k(Y_{k-h}(t)) d\widetilde{B}_{h,k}(t) \\[4mm]
\qquad - \displaystyle\sum_{h \in \mathbb{Z}^3} \sigma_h^2 \|P_h(k)\|^2 P_k(P_{k-h}(Y_k(t))) dt \\[4mm]
\langle Y_k(t), k \rangle = 0 \\[2mm]
Y_{-k}(t) = \overline{Y_k}(t) \\[2mm]
Y_k(0) = y_k
\end{cases}
\tag{29}
$$

Starting from the solution of the linear system (18) we construct a solution to this nonlinear system by means of Girsanov transform. We shall deal with solutions on any fixed finite time interval $[0, T]$.

Definition 2. *Given $y \in l^2$, a weak solution of equation (29) in l^2 is a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$, a sequence of independent $\mathbb{C}^3$-valued Brownian motions $W' = \{W_h'\}_{h \in J}$ on $(\Omega, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$ and an l^2-valued stochastic process $Y := (Y_k)_{k \in \mathbb{Z}^3}$ on $(\Omega, \{\mathcal{F}_t\}_{t \in [0,T]}, P)$, with continuous*

and adapted components Y_k such that for all $t \in [0, T]$

$$
\begin{cases}
Y_k(t) = y_k - i \sum_{h \in \mathbb{Z}^3} \int_0^t \frac{\langle Y_h(s), k \rangle}{1 + \alpha \|h\|^2} P_k(Y_{k-h}(s)) ds \\
\qquad - i \sum_{h \in \mathbb{Z}^3} \frac{\sigma}{1 + \alpha \|h\|^2} \|P_h(k)\| \int_0^t P_k(Y_{k-h}(s)) d\widetilde{W}_{h,k}(s) \\
\qquad - \sum_{h \in \mathbb{Z}^3} \left(\frac{\sigma}{1 + \alpha \|h\|^2} \right)^2 \|P_h(k)\|^2 \int_0^t P_k(P_{k-h}(Y_k(s))) ds \\
\\
\langle Y_k(t), k \rangle = 0 \\
Y_{-k}(t) = \overline{Y_k(t)}
\end{cases}
$$

P-a.s. for each $k \in \mathbb{Z}^3$. We denote this solution by $((\Omega, \{\mathcal{F}_t\}_{t \in [0,T]}, P), Y, W')$.

Moreover, it is called an energy controlled weak solution if for all $t \in [0, T]$ this solution satisfies

$$
\sum_{k \in \mathbb{Z}^3} \|Y_k(t)\|^2 \le \sum_{k \in \mathbb{Z}^3} \|y_k\|^2 \qquad P - a.s.
$$

As usual, the Brownian motions $\widetilde{W}_{h,k}$ are constructed by the family $\{W'_h\}_{h \in J}$ (see (11)-(13)).

First we present the Girsanov result. Let $Y = \{Y_h\}_h$ be the strong energy controlled solution of (18). Define

$$
L(t) = \frac{1}{\sigma} \sum_{h \in J} \int_0^t Y_h(s) dB_h(s)
$$

Then L is a martingale and its quadratic variation $[L, L]$ is well defined and given by

$$
[L, L](t) = \frac{1}{\sigma^2} \int_0^t \sum_{h \in J} Y_h(s)^2 ds \tag{30}
$$

We have

Proposition 5. *Let Y be the strong solution of system (18) with the family of independent $\mathbb{C}^3$-valued standard Brownian motions $\{B'_h\}_{h \in J}$ on $(\Omega, \{\mathcal{F}_t\}_{t \ge 0}, Q)$. Then*

$$
W'_h(t) = B'_h(t) - \frac{1}{\sigma} \int_0^t Y_h(s) ds, \qquad h \in J, \ 0 \le t \le T, \tag{31}
$$

defines a family of independent $\mathbb{C}^3$-valued standard Brownian motions on $(\Omega, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ with the measure P, defined on $(\Omega, \mathcal{F}_T)$, which is absolutely continuous with respect to the measure Q and

$$\frac{dP}{dQ} = e^{L(T) - \frac{1}{2}[L,L](T)}$$

Proof. Since $\|Y(t)\|_{l^2} \leq \|y\|_{l^2}$ Q-a.s., Novikov condition

$$\mathbb{E}^Q \left[e^{\frac{1}{\sigma^2} \int_0^T \sum_{h \in J} \|Y_h(s)\|^2 ds} \right] < \infty \tag{32}$$

holds true. Therefore, Girsanov transform now applies in a classical way (see, e.g. [19, 21]). $\qquad\square$

We point out that $\frac{dP}{dQ} > 0$, Q-a.s.; therefore also $\frac{dQ}{dP}$ is well defined. Thus the measures P and Q are equivalent (i.e. absolutely continuous to each other).

Our main result is

Theorem 6. *For any initial data of finite energy, system (29) has an energy controlled weak solution. Moreover, this solution is unique in law.*

Proof. As far as the existence is concerned, we have that (18) has a unique strong solution Y defined on $(\Omega, \{\mathcal{F}_t\}, Q)$ and satisfying (18). Using the Girsanov transform (31), we get that $((\Omega, \{\mathcal{F}_t\}, P), Y, W')$ is a weak solution of (29). Moreover, the measure P is equivalent to the measure Q; then $\|Y(t)\|_{l^2} \leq \|y\|_{l^2}$ P- and Q-a.s. This means that this weak solution is an energy controlled solution.

As far as the uniqueness is concerned, if there were two different weak solutions of the nonlinear system (29), then each of them would give rise to a weak solution of the linear system (18); these are obtained starting from $((\Omega, \{\mathcal{F}_t\}, P), Y, W')$ and getting $((\Omega, \{\mathcal{F}_t\}, Q), Y, B')$ via Girsanov transform. On the other side, the pathwise uniqueness for the linear system (18) implies the weak uniqueness; this comes from Yamada-Watanabe theorem, which is usually known for finite dimensional systems but whose validity holds also in the infinite dimensional setting as soon as the Itô stochastic integrals are well defined. Now, using the absolute continuity of P and Q, we deduce that the nonlinear system (29) has a unique solution (in law). $\qquad\square$

Remark 7. *i) The proof shows that our technique can be applied for any $\alpha > 0$ to more general models, that is we can deal with a noise defined by means of $\sigma_h = \frac{\sigma}{(1 + \alpha \|h\|^2)^p}$ and with the smoothing term given by $u^\alpha = (1 - \alpha \Delta)^{-p} v^\alpha$, for any $p > 4/3$.*
ii) In the 2-dimensional case, all our computations can be extended for $p > 1/2$.

6 The formulation in SPDE

The stochastic model considered so far in Fourier components can be written as a stochastic partial differential equation, as follows

$$\begin{cases} dv + (u \cdot \nabla)v \ dt \ + \nabla p \ dt \ = \sum_{h \in \mathbb{Z}^3} \sum_{j=1}^3 \sigma_h e_h \frac{\partial v}{\partial x^{(j)}} \circ dW_h^{(j)} \\ v = (1 - \alpha \triangle)u \\ \text{div } v = 0 \\ v(0) = v_0 \end{cases} \tag{33}$$

For simplicilty we have dropped out the index α in the unknowns.

The first equation of system (33) can be written in a more compact form as

$$dv + \sum_{j=1}^3 \frac{\partial v}{\partial x^{(j)}} \ u^{(j)} \ dt \ + \nabla p \ dt \ = \sum_{j=1}^3 \frac{\partial v}{\partial x^{(j)}} \circ dW^{(j)}. \tag{34}$$

Here the random field W is given as $W(t,x) := \sum_{h \in \mathbb{Z}^3} \sigma_h e_h(x) W_h(t)$.

More precisely, the Wiener process W has the following form

$$W(t,x) = 2\sigma \sum_{h \in J} \frac{\cos(\langle h, x \rangle)\Re W_h(t) - \sin(\langle h, x \rangle)\Im W_h(t)}{1 + \alpha \|h\|^2}, \tag{35}$$

where $\{W_h\}_{h \in J}$ is a family of independent $\mathbb{C}^3$-valued Brownian motions on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, P)$, such that $\langle W_h(t), h \rangle = 0$.

Let us denote by H and V the subspaces of $(L^2(\mathbb{T}))^3$ and $(H^1(\mathbb{T}))^3$ respectively, given by vectors fields divergence free and periodic:

$$H = \{v \in \left(L^2(\mathbb{T})\right)^3, \quad \nabla \cdot v = 0, \quad \int_{\mathbb{T}} v(x)dx = 0, v \cdot n \text{ periodic on } \mathbb{T}\}$$

$$V = \{v \in H : v \in [H^1(\mathbb{T})]^3, \quad u \text{ periodic on } \mathbb{T}\}$$

where n is the unit normal to the boundary of the spatial domain.

Moreover, identifying H with its dual H' we get the Gelfand triple (V, H, V')

$$V \subset H \simeq H' \subset V'.$$

The norms are inheritaed from the spaces $(L^2(\mathbb{T}))^3$ and $(H^1(\mathbb{T}))^3$.

Definition 3. *Given $v_0 \in H$, a weak solution of (33) in H is a filtered probability space $(\Omega, \{\mathcal{F}_t\}, P)$, a sequence of independent $\mathbb{C}^3$-valued Brownian motions $\{W_h\}_h$ on $(\Omega, \{\mathcal{F}_t\}, P)$ and an H-valued continuous and adapted*

stochastic process v on $(\Omega, \{\mathcal{F}_t\}, P)$, such that

$$\int_\mathbb{T} \langle v(t,x), \phi(x)\rangle dx - \int_0^t \int_\mathbb{T} \langle (u(s,x)\cdot\nabla)\phi(x), v(s,x)\rangle ds\ dx$$

$$= \int_\mathbb{T} \langle v_0(x), \phi(x)\rangle dx$$

$$- \sum_{h\in\mathbb{Z}^3} \sigma_h \sum_{j=1}^3 \int_0^t \left(\int_\mathbb{T} e_h(x)\langle \frac{\partial\phi}{\partial x^{(j)}}(x), v(s,x)\rangle dx\right) \circ dW_h^{(j)}(s),\ \ P-a.s.$$

$$(36)$$

for each $t \in [0,T]$ and for all test functions $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, periodic on $\mathbb{T}$, divergence free and of class C^1. We denote this solution by $((\Omega, \{\mathcal{F}_t\}, P), v, W)$. Moreover, it is called an energy controlled weak solution if for all $t \geq 0$ this solution satisfies

$$\|v(t,\cdot)\|_H \leq \|v_0\|_H \qquad P-a.s.$$

This weak formulation corresponds to the stochastic equation (34). Indeed, for a more regular solution v, by integration by parts in (36) one gets (34). This is a classical result for the Euler equation and the stochastic part uses the properties of the Brownian motions. Therefore we have the following result.

Proposition 8. *The equality $v(t,x) = \sum_{h\in\mathbb{Z}^3} Y_h(t)e_h(x)$ relates the solutions of the stochastic PDE (36) and of the stochastic Fourier system (15) (or (17)).*

Finally, we can reformulate our result for the SPDE:

Theorem 9. *For any initial velocity of finite energy, equation (34) has an energy controlled weak solution. Moreover, this solution is unique in law.*

7   Appendix

In this section, we present a proof of global existence of a weak solution for the deterministic system (6) with initial velocity of finite energy; here weak has to be understood in the sense of PDE's. No written proof has been found in the published literature, whereas there are results of local (in time) existence and uniqueness for very regular initial velocity; however, uniqueness of weak solutions is an open problem. Anyway, Edriss Titi has presented a proof of global existence of weak solutions, as a private communication.

For simplicity, let us drop the index α.

Theorem 10. *(Due to E. S. Titi) Let $v_0 \in H$ and $T > 0$. Then there exists a global weak solution v to the system (6) such that*

$$v \in L^\infty([0,T]; H) \cap C([0,T]; V')$$

and

$$\int_{\mathbb{T}} \langle v(t,x), \phi(x) \rangle dx - \int_0^t \int_{\mathbb{T}} \langle (u(s,x) \cdot \nabla)\phi(x), v(s,x) \rangle ds \, dx \tag{37}$$
$$= \int_{\mathbb{T}} \langle v_0(x), \phi(x) \rangle dx$$

for each $t \in [0,T]$ and for all test functions $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, periodic with period box $\mathbb{T}$, divergence free and of class C^1.

Proof. Let P_N be the finite dimensional projector that is $v^N = P_N v$ means $v^N(x) = \sum_{\|h\| < N} (\int_{\mathbb{T}} v(x) e_h(x) dx) e_h(x)$. Then, we get the following finite-dimensional system system

$$\begin{cases} \dfrac{dv^N}{dt} + P_N[(u^N \cdot \nabla)v^N] = 0 \\ u^N + \alpha \Delta u^N = v^N \end{cases} \tag{38}$$

From $(38)_1$, we get $\frac{d}{dt}\|v^N(t)\|_H^2 = 0$ for any N and t; then $\|v^N(t)\|_H^2 \leq \|v_0\|_H^2$. Thus

$$\sup_N \|v^N\|_{L^\infty(0,T;H)} < \infty; \tag{39}$$

hence using $(38)_2$

$$\sup_N \|u^N\|_{L^\infty(0,T;(H^2(\mathbb{T}))^3)} < \infty. \tag{40}$$

Using again equation $(38)_1$

$$\|\frac{dv^N(t)}{dt}\|_{V'} = \sup_{\|\phi\|_V = 1} |\langle (u^N(t) \cdot \nabla)v^N(t), P_N\phi \rangle|$$
$$= \sup_{\|\phi\|_V = 1} |\langle (u^N(t) \cdot \nabla)P_N\phi, v^N(t) \rangle|$$
$$\leq \|u^N(t)\|_{(L^\infty(\mathbb{T}))^3} \|P_N\phi\|_V \|v^N(t)\|_H.$$

Using again (40) and the embedding theorem $H^2(\mathbb{T}) \subset L^\infty(\mathbb{T})$, we get that

$$\sup_N \|\frac{dv^N}{dt}(t)\|_{L^\infty(0,T;V')} < \infty. \tag{41}$$

The estimate (41) means that $\{v^N\}$ is uniformly Lipschitz in V'. On the other side using the estimate (39), $\{v^N(t)\}$ is inside a bounded ball of H. Hence, the set $\{v^N(t), \quad \forall t\}$ is a compact subset of V'. Using the Ascoli-Arzelà theorem, we can extract a subsequence called again $v^N(t)$ such that

$$v^N \longrightarrow v \quad \text{in} \quad C([0,T];V')$$

and $v \in C([0,T];V')$. Moreover, using the estimate (41), the limit v is $Lip([0,T];V')$.

Using (41) and $(38)_2$ we get that

$$\sup_N \|\frac{du^N}{dt}(t)\|_{L^\infty(0,T;V)} < \infty; \tag{42}$$

this result and (40) allow to use the compactness theorem (Aubin-Lions). Therefore we can extract a subsequence, again called u^N such that

$$u^N \longrightarrow u = (1 - \alpha\Delta)^{-1}v \quad \text{in} \quad L^p([0,T];H^{2-\epsilon}),$$

for some arbitrary $\epsilon > 0$ and p finite. We shall take $\epsilon < \frac{1}{2}$ in order to use that $H^{2-\epsilon}(\mathbb{T}) \subset L^\infty(\mathbb{T})$. Now, we have all the ingredients to pass to the limit in the system (38) that we are going to write in the weak form:

$$\int_{\mathbb{T}} \langle v^N(t,x) - v^N(s,x), \phi(x)\rangle dx = \int_s^t \int_{\mathbb{T}} \langle (u^N(r,x) \cdot \nabla)P_N\phi(x), v^N(r)\rangle dx \, dr.$$

It is easy to pass to the limit on the l.h.s. of the above equality. Let us focus of the r.h.s. of above equality: the non linear term $\langle (u^N \cdot \nabla)P_N\phi, v^N\rangle$ converges weakly in $L^1(0,T;V')$, since u^N converges strongly in $L^2(0,T;[L^\infty(\mathbb{T})]^3)$ and v^N converges weakly in $L^2(0,T;H)$ (due to (39), considering possibly a new subsequence). $\qquad\square$

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