

HOPF BIFURCATION FROM A TURNING POINT

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1. Introduction

In the usual formulations of the Hopf Bifurcation Theorem (see [3] for references) one considers a family of evolution equations depending smoothly on a real parameter and possessing a branch of steady states, which is parameterized by the parameter for the respective equation. In this case, there is no loss of generality assuming that this branch is given by the zero solution in the space attached to any parameter value. If, however, it is not possible to parameterize the given branch that way, a zero eigenvalue comes into the problem. Most authors exclude an eigenvalue of zero when dealing with Hopf Bifurcation. In the presence of such an eigenvalue there is no general existence theorem available; in fact, counterexamples may be found in [11]. In [11] we treated the case where the stationary solutions form a degenerate stationary pitchfork. In the present paper, we mainly discuss the bifurcation of periodic solutions from a turning point. The precise hypotheses will be given in Section 3. An intuitive description of the underlying situation is provided by Figure 1, where S is the given set of equilibria.

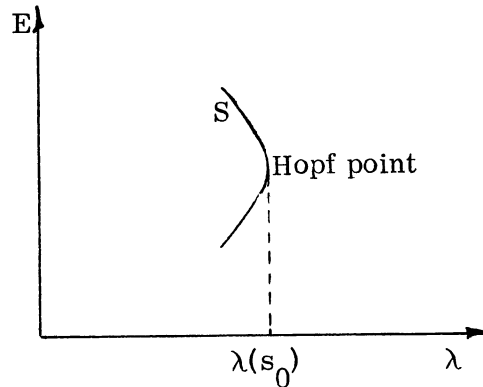


Figure 1

We shall discuss a special equation representing the generic case for Fig. 1. This will be done in Section 5. Section 3 contains hypotheses and results. We discuss the case when the stationary solutions form a straight line, too. In this case we assume that for a certain parameter value the linearization of the vector

field has eigenvalues on the imaginary axis. We show that exclusion of bifurcation of steady states together with a change of sign of the real parts of a pair of complex conjugate eigenvalues leads to bifurcation of periodic solutions. The proof relies on the principle of reduced stability (PRS), which has been introduced in [10] and will be extended in Section 2.

Recently several authors have investigated the effects of higher order bifurcation. See, for example, Chapter XIII in Chow and Hale [3] and the references at the end of this paper, as well as the references in [11]. In most of these studies one tries to obtain a complete description of the flow near the bifurcation point. This can be done only if the problem has a low codimension. In such a case one uses the center manifold theorem to deduce a differential equation on a low dimensional space which contains all relevant information on the change of the flow. The codimension of the original problem gives the number of parameters in this equation. In static bifurcation theory symmetries are used to decrease the codimension of a bifurcation problem [5].

2. The Principle of Reduced Stability

Let D, E be two real Banach spaces with $D \subset E$ continuously and densely embedded. We consider a nonlinear operator

$$F : D \times \mathbb{R} \rightarrow E \tag{2.1}$$

having the following properties:

- H1) F is real analytic on $U \times \Lambda$, where U is a neighborhood of zero in D , the topology on D is assumed to be induced by the graph norm of the linear operator $D_u F(0, 0)$ and Λ is an interval containing zero.
- H2) There is a curve $S = \{(u(s), \lambda(s)) \mid s \in [0, s_0], s_0 > 0\} \subset U \times \Lambda$ with $F(u(s), \lambda(s)) = 0$ for all $s \in [0, s_0]$, $u(0) = 0$, $\lambda(0) = 0$.
- H3) $T(s) \stackrel{\text{def}}{=} D_u F(u(s), \lambda(s))$ is a family of closed linear operators $E \rightarrow E$, with domain of definition $D(F(s)) = D$ for all $s \in [0, s_0]$.

H4) The Fréchet derivative $D_u F(0,0)$ has a semisimple isolated eigenvalue zero of multiplicity $m \geq 1$. (In our context semisimplicity of an eigenvalue means that the generalized eigenspace corresponding to this eigenvalue has a basis of eigenvectors.)

Let $P_0 : E \rightarrow E$ be the eigenprojector associated with this eigenvalue, i. e. ,

$$P_0 = -\frac{1}{2\pi i} \int_G R(z, D_u F(0,0)) dz , \quad (2.2)$$

where $R(z, T)$ denotes the resolvent of the operator T at $z \in \mathbb{C}$, and G is a closed curve in \mathbb{C} , surrounding zero and no other spectral point of $D_u F(0,0)$ lies within the region cut out by G .

Let Q_0 denote the projector $Q_0 = I - P_0$, where I is the identity on E . Applying these operators to (2.1) we obtain the following splitting

$$P_0 F(v + w, \lambda) = 0 \quad (2.3a)$$

$$Q_0 F(v + w, \lambda) = 0 \quad (2.3b)$$

where $P_0 u = v$, and $w = u - v$. The linear operator

$$D_w F(0,0) : Q_0 D \rightarrow Q_0 E$$

is a toplinear isomorphism. Thus, the implicit function theorem yields a solution (2.3b) near zero, i. e. , a map

$$w : V \times \Lambda' \rightarrow Q_0 D \quad (2.4)$$

such that for any pair $(v, \lambda) \in V \times \Lambda'$, equation (2.3b) is satisfied. Inserting the solution $w = w(v, \lambda)$ into (2.3a) completes the so-called Ljapunov-Schmidt reduction, yielding the bifurcation equation

$$\phi(v, \lambda) \stackrel{\text{def}}{=} P_0 F(v + w(v, \lambda), \lambda) = 0 . \quad (2.5)$$

In general, ϕ is a nonlinear real analytic map

$$\phi : W \times \Lambda' \rightarrow \mathbb{R}^m \quad (2.6)$$

where W is a neighborhood of zero in \mathbb{R}^m and Λ' is a subinterval of Λ . Corresponding to the solution curve S of

$$F(u, \lambda) = 0 \quad (2.7)$$

we get a curve $S' \subset \mathbb{R}^m \times \Lambda'$ given by

$$S' = \left\{ (v(s), \lambda(s)) \mid (v(s) + w(v(s)), \lambda(s)) \in S, 0 \leq s \leq s'_0 \right\}.$$

Let $R(s)$ denote the Jacobian of ϕ along S' . The Principle of Reduced Stability states that the stability of the points on S (near 0) considered as stationary solutions of the evolution equation

$$\frac{du}{dt} = F(u, \lambda) \quad (2.8)$$

is given by the stability of the corresponding points on S' viewed as equilibria of the ordinary differential equation

$$\frac{dv}{dt} = \phi(v, \lambda). \quad (2.9)$$

This principle has been introduced in [10]. There it has been shown that if $R(s) = s^k R_k + O(s^{k+1})$ for $s \rightarrow 0$ and R_k has an eigenvalue zero of multiplicity not greater than one, then the principle is true; moreover, the Puiseux series for the critical eigenvalues of $T(s)$ and the eigenvalues of $R(s)$ have the same first nonvanishing term. Consequently, all but at most one eigenvalue converge to zero with the same order in s . In the appendix of [11] we proved that the same qualitative and quantitative results hold true if $R(s)$ is diagonal. In this case the different eigenvalues may converge to zero with an arbitrary order in s , but there is a rather special form required for the matrix $R(s)$. In this section we give a condition allowing a more general form for $R(s)$ and also different orders for the eigenvalues.

If one is only interested in the qualitative result, namely the stability properties, one does not have to study all eigenvalues but only compare the sign of $\max\{\operatorname{Re} \tau(s) \mid \tau(s) \in \sigma(R(s))\}$ with the sign of $\max\{\operatorname{Re} \tau(s) \mid \tau(s) \in \sigma(T(s))\}$, $\lim \tau(s) = 0$ for $s \rightarrow 0$. This question has been studied by Vanderbauwhede [13].

His result says that if R_k , defined as above, has an eigenvalue with a positive real part, then the points on S sufficiently close to zero are unstable, while they are stable if all eigenvalues of $R(s)$ are in the left half-plane. This result follows easily from [10]. In the second case the hypotheses of theorem 2 in [10] are trivially satisfied, while the first case follows from the proof of theorem 1 in [10]. Besides the stability of the bifurcating solutions, the quantitative result also gives the dimension of the unstable manifold and last but not least, allows us to prove bifurcation results.

An alternate approach to performing the Ljapunov-Schmidt reduction for bifurcation problems consists of the use of the center manifold. Using that method we had no difficulties in assigning the stability properties of an equilibrium by just considering its stability on the center manifold. But we had to look for periodic solutions of an ODE instead of looking for zeros of a map. The dimension of the state space of this ODE would be the number of eigenvalues on the imaginary axis (counted with multiplicities). This center manifold approach does not allow us to restrict attention to just a few eigenvalues on the imaginary axis. On the other hand, it gives insight into the change of structure of the flow near critical points. This cannot be achieved using the Ljapunov-Schmidt method.

In order to formulate our result, we need a spectral property of operators depending on parameters.

Definition: Let us say that the eigenvalue τ_0 of $T(\tilde{s})$ satisfies property (S) if

- (i) it is isolated and semisimple
- (ii) let $P(s)$ be the total projector for the τ_0 -group (the τ_0 -group for $T(s)$ consists of all eigenvalues of $T(s)$ which converge to τ_0 as $s \rightarrow \tilde{s}$) and let $\{V_j(s)\}_{j=1, \dots, p}$ be the irreducible subspaces of $T(s)|_{P(s)E}$. Assume that the $V_j(s)$ vary continuously with s , and that the limits $\lim_{s \rightarrow \tilde{s}} V_j(s)$ exist and $P(\tilde{s})E = \bigoplus_{j=1}^p V_j(\tilde{s})$. Let $\tau_j(s)$ be the eigenvalue of $T(s)|_{V_j(s)}$ and $D_j(s) = (T(s) - \tau_j(s))|_{V_j(s)}$ be the eigennilpotent of $T(s)$ on $V_j(s)$. If $\tau_j(s) = a_j(s - \tilde{s})^{k(j)} + o((s - \tilde{s})^{k(j)})$ we require $D_j(s) = o((s - \tilde{s})^{k(j)})$.

Remark: We say "the reduction principle for critical eigenvalues" (RPE) is true if the eigenvalues of $T(s)$ near zero are given in lowest nonvanishing order by the eigenvalues of $R(s)$.

Theorem 1: Assume the eigenvalue 0 of $R(0)$ satisfies property (S). Then the RPE is true.

Proof: By property (S) there exists a family of invertible matrices $X(s)$, varying continuously with s , such that $X(s)$ and $X^{-1}(s)$ are bounded for $s \rightarrow 0$ and continuous in $s = 0$, and

$$Q(s) = X(s)R(s)X^{-1}(s) \quad (2.10)$$

has the form

$$Q(s) = \begin{bmatrix} \tilde{\tau}_1(s) & g_1(s) & 0 & \dots & 0 \\ 0 & \tilde{\tau}_2(s) & g_2(s) & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & \dots & 0 & \dots & \tilde{\tau}_n(s) \end{bmatrix}$$

where $\tilde{\tau}_j(s)$ are the eigenvalues of $R(s)$ and the $g_j(s)$ vanish at least as fast as $\tilde{\tau}_j(s)$ at $s = 0$ (or the g_j vanish identically). In [10] it has been shown that the eigenvalues τ_j of T satisfy

$$\det \left[R(s) - \tau I - \sum_{\nu=1}^{\infty} \tau^{\nu} B_{\nu}(s) \right] = 0 \quad (2.11)$$

where $B_{\nu}(s) = O(s^2)$ as $s \rightarrow 0$, while the $\tilde{\tau}_j(s)$ satisfy

$$\det[R(s) - \tau I] = 0. \quad (2.12)$$

We have

$$X(s) \left[R(s) - \tau I - \sum_{\nu=1}^{\infty} \tau^{\nu} B_{\nu}(s) \right] X^{-1}(s) = Q(s) - \tau I - \sum_{\nu=1}^{\infty} \tau^{\nu} \tilde{B}_{\nu}(s) \quad (2.13)$$

with $\tilde{B}_\nu(s) = O(s^2)$, $s \rightarrow 0$. Thus the eigenvalues $\tau_j(s)$ satisfy

$$\det \left[Q(s) - \tau I - \sum_{\nu=1}^{\infty} t^\nu \tilde{B}_\nu(s) \right] = 0 . \quad (2.14)$$

The eigenvalues of $R(s)$ are precisely the zeros of

$$\det [Q(s) - \tau I] = 0 . \quad (2.15)$$

In [11] we showed that the solutions of (2.14) and (2.15) have the same first non-vanishing term in the Puiseux series if $Q(s)$ is diagonal. In the more general case we just observe that the argument given in [11] still provides a proof.

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Theorem 2: If the eigenvalue zero of $D_u F(0,0)$ satisfies property (S) and if for $s \neq 0$ the 0-group does not contain zero, then the RPE is true.

Proof: There exists a family of linear maps (see Vanderbauwhede [13])

$$\tilde{\phi}(s) : \ker D_u F(0,0) \rightarrow \ker D_u F(0,0)$$

such that

- a) $\tilde{\phi}(s)$ is similar to $T(s)|_{P(s)E}$ (for small s)
- b) $\tilde{\phi}(s) = \eta(s)R(s)$, with $\eta(s) = I + O(s^2)$ for $s \rightarrow 0$.

We may assume that $\tilde{\phi}(s)$ is upper triangular and that the diagonal does not contain zero. Now a straightforward application of Newton's diagram yields the result.

3. Hypotheses and Statement of Results

We consider an evolution equation

$$\frac{du}{dt} + G(u, \lambda) = 0$$

in a real Hilbert space E , depending on a real parameter λ varying in an interval $\Lambda \subset \mathbb{R}$. The domain of definition $D(G(\cdot, \lambda))$ of the nonlinear operator $G(\cdot, \lambda)$ is a

dense subspace $D \subset E$ and is independent of $\lambda \in \Lambda$. We assume the existence of a curve

$$S = \{(u(s), \lambda(s)), s \in [-a, b], a, b > 0\} \quad (3.2)$$

of stationary solutions of (3.1), which is analytic in s . Let us assume $\lambda(0) = 0$, $u(0) = 0$. We write $T(s) = D_u G(u(s), \lambda(s))$ and $A = T(0)$. We collect our hypotheses on the spectrum of $T(s)$, on the analytical properties, and on the geometry of S :

- S1) Zero is a semisimple eigenvalue of multiplicity $\rho_0 \geq 0$ of A ;
- S2) A has N pairs of complex conjugate eigenvalues $\pm i\omega_\nu$, $\omega_\nu > 0$, $\nu = 1, \dots, N$; these are semisimple and have finite multiplicity ρ_ν ;
- S3) there exists a $j_0 \in \{1, \dots, N\}$ with $\rho_{j_0} = 1$;
- S4) the greatest common divisor of the elements of $K = \{k \in \mathbb{N} - \{1\} \mid ik\omega_{j_0} \in \sigma(A)\}$ is greater than one;
- S5) $T(s)$ has no purely imaginary eigenvalues for small $s \neq 0$;
- S6) all eigenvalues $\{ik\omega_{j_0} \mid k \in K_0\}$ satisfy property (S), $K_0 = K \cup \{0\}$;
- S7) there is a curve $\gamma(s)$ of eigenvalues of $T(s)$ passing through $i\omega_{j_0}$ with $(d^{\ell_0}/ds^{\ell_0})\text{Re } \gamma(0) \neq 0$ and ℓ_0 is minimal and odd.
- A1) $-A$ is sectorial and for $\mu \notin \sigma(A)$, $(A - \mu I)$ has compact resolvent.
- A2) $T(s)$ is holomorphic of type (A) (that means: there exists a holomorphic family $T(\mathcal{K})$ of type (A), such that $T(s)$ is obtained from $T(\mathcal{K})$ by restricting \mathcal{K} to real values, see Kato [7] for the notion of a holomorphic family of type (A));
- A3) $G : D \times \Lambda \rightarrow E$ is real analytic, the topology on D being defined by the graph norm of A ;
- G1) there exists a ball B of radius $\mathcal{E} > 0$ around zero in D and an interval $\Delta = (-\delta, \delta)$ such that $\{(u, \lambda) \in B \times \Delta \mid G(u, \lambda) = 0\} = S \cap (B \times \Delta)$;
- G2) there exist positive numbers $\underline{a}, \underline{b} \leq \min(a, b)$, such that $S_1 = \{(u(s), \lambda(s)) \mid s \in [-\underline{a}, \underline{b}]\} = S \cap (B \times \Delta)$ and $\lambda'(s) \neq 0$ for $s \in [-\underline{a}, 0) \cup (0, \underline{b}]$;

Since $\lambda(s)$ is an analytic function of s and $\lambda'(s)$ does not vanish identically, we know that there exists a number $m \in \mathbb{N}$, $m \geq 1$ with

$$\lambda'(0) = \dots = \frac{d^{m-1}\lambda(0)}{ds^{m-1}} = 0, \quad \frac{d^m\lambda(0)}{ds^m} \neq 0. \quad (3.3)$$

We distinguish two cases:

- A) m is even
- B) m is odd.

In the case A the curve S has a turning point at $s = 0$, and the number ρ_0 defined in S1) is positive. We shall state our results for both cases but we provide complete proofs only for the more difficult case A. However, most parts of our exposition will be general enough to cover both cases. Our assumptions G1), G2) imply that we can choose the constants ε and δ such that

$$\lambda(-a) = \lambda(b) = -\delta \text{ in case A} \quad (3.4A)$$

$$\lambda(-a) = -\delta, \quad \lambda(b) = \delta, \text{ in case B.} \quad (3.4B)$$

We shall always assume that we have chosen our constants such that (3.4A) or (3.4B) hold.

Theorem 3: We assume the hypotheses given in Section 3, namely S1) - S7), A1) - A3), G1) and G2). Then in both cases A and B there exists a continuum C of nontrivial periodic solutions of (3.1) emanating at $(u, \lambda) = (0, 0)$ from the curve S given by (3.2). If the constants δ , are chosen sufficiently small, then C contains either

- a) a solution (u, λ) of (3.1) where u is a nontrivial periodic function of t and $|\lambda| = \delta_j$, or
- b) a solution (u, λ) of (3.1) where $|\lambda| \leq \delta$ and u is nontrivial periodic and $\sup_{t \in \mathbb{R}^+} \|u(t)\|_E = \varepsilon$.

We obtain these solutions as stationary solutions of an equation between spaces of 2π -periodic functions, which are defined below.

Remark 1: For case A, Theorem 3 seems to be new, even in the most simple case when the operator A has only one pair of simple complex conjugate eigenvalues and a simple eigenvalue zero. This case is not a consequence of any kind of Hopf Bifurcation Theorem, because the geometry of S implies the existence of the zero eigenvalue.

Remark 2: If $\rho_0 = 0$, i. e., if no eigenvalue zero occurs, case B is a consequence of the work of Alexander and Yorke [1]. Without any condition on the algebraic structure of the critical eigenvalues they have shown, that odd parity (see [1]) leads to bifurcation of periodic solutions. The proof in [1] relies on algebraic topology. Using the Fuller index, Chow et al. [2] proved a similar result for nonzero parity. The proofs in [1] and [2] apply only to the finite dimensional case. For evolution equations in Hilbert spaces, the same conclusion as in [2] is a consequence of Fiedler's work [4].

Remark 3: If all eigenvalues $ik\omega_{j_0}$ are simple, then hypothesis S6) is satisfied.

Remark 4: If $K = \emptyset$, $\rho_0 = \rho_{j_0} = 1$, Theorem 3 case B follows from Theorem A in [11].

Remark 5: Hypothesis S6) may be superfluous. The proof below indicates that it is only technical. There is just one conclusion drawn from it, namely, that $\tilde{\gamma}$ (see (4.23) below) gives in lowest order $\text{Re } \gamma$ (see S7)). This should be true in general.

4. The Proof

We divide the proof into several steps:

Step 1: A reformulation in a space of 2π -periodic functions

Step 2: The Ljapunov-Schmidt reduction

Step 3: A further reduction

Step 4: The conclusion.

Before carrying out this program let us refer to Kielhöfer [8] and Lauterbach [11]. In the first paper one may find all details of Step 1 which are left out here, while the second paper gives a rather detailed description of the more transparent situation for ODE's, case B with $K = \emptyset$, $\rho_0 = \rho_{j_0} = 1$.
Step 1: Let (\cdot, \cdot) denote the inner product in E , $\|\cdot\| = (\cdot, \cdot)^{1/2}$. Define the space

$$H_0 = L_2([0, 2\pi], E) .$$

This space is a Hilbert space, the inner product being given by

$$(\cdot, \cdot)_0 = \int_0^{2\pi} (\cdot, \cdot) dt .$$

The functions in H_0 can be differentiated in the sense of distributions (see Lions and Magenes [12, p. 6]). Let \mathcal{D} denote the operator of differentiation. Since the subspace of those functions with $\mathcal{D}u \in H_0$ is embedded into $C([0, 2\pi], E)$ [12, p. 19] the following definition makes sense:

$$H_2 = \text{cl}_{\|\cdot\|_2} \left\{ u \in H_0 \mid \mathcal{D}u \in H_0, u \in L_2([0, 2\pi], D), u(0) = u(2\pi) \right\}$$

where $\|\cdot\|_2 = (\cdot, \cdot)_2^{1/2}$ and $(\cdot, \cdot)_2 = \int_0^{2\pi} ((\mathcal{D}\cdot, \mathcal{D}\cdot) + (A\cdot, A\cdot)) dt$.

Let us consider the operator $J_0 = \omega \mathcal{D} + A$, for $\omega \in \mathbb{R}^+$, defined on H_0 with $D(J_0) = H_2$. The functions in $\ker J_0$ may be characterized as follows:

- (i) if $u \in \ker J_0$ and u is a constant function of time, then $u \in \ker A$;
- (ii) if $u \in \ker J_0$ and u is time-periodic, then $u(t) = Z(t/\omega)u_0$, where $Z(t)$ denotes the holomorphic semigroup generated by $-A$.

By periodicity $u_0 \in E_0$, E_0 being the subspace of E corresponding to the spectral parts of A on the imaginary axis. Since A is sectorial and has compact resolvent, this part of the spectrum consists of finitely many eigenvalues of finite multiplicity. Thus we have

$$u(t) = v^{(1)} + \sum_{k=1}^N \sum_{j=1}^{\rho_k} \left(\alpha_{k,j} \exp(-i \frac{t}{\omega} \omega_k) \phi_{k,j} + \overline{\alpha_{k,j}} \exp(i \frac{t}{\omega} \omega_k) \overline{\phi_{k,j}} \right) .$$

The $\alpha_{k,j}$ are complex numbers, $v^{(1)} \in \ker A$, and the $\phi_{k,j}$ are eigenvectors of A to the eigenvalue $i\omega_k$ satisfying

- (i) $\{\phi_{k,1}, \dots, \phi_{k,\rho_k}\}$ is a basis for the eigenspace corresponding to the eigenvalue $i\omega_k$, $k = 0, 1, \dots, N$;
- (ii) if $(\omega_k/\omega) \in \mathbb{Z}$ then there exists a basis $\phi_{k,j}(s)$ of the total eigenspace of $T(s)$ for the $i\omega_k$ -group consisting of eigenvectors or generalized eigenvectors, such that

$$\phi_{k,j}(s) \rightarrow \phi_{k,j} \quad \text{for } s \rightarrow 0 .$$

Due to the periodicity with period 2π , ω_k/ω must be an integer. Therefore only those eigenvalues which are integer-multiple of ω are relevant for $\ker J_0$. As on page 59 in [8] we can show that zero is a semisimple eigenvalue of J_0 . Define $\omega = \omega_{j_0}$ and J_0 as above with this ω . We expect to find periodic solutions with period $\frac{2\pi}{\omega+\mu}$ for small μ . Thus we reformulate our problem substituting t by $t/(\omega+\mu)$ as an equation in H_0 :

$$\mu \mathcal{D}u + J_0 u = N(u, \lambda), \quad u \in H_2 \tag{4.1}$$

where $N(\cdot, \cdot)$ contains the terms not written down explicitly. Considering equation (4.1) instead of (3.1) we admit solutions possessing less regularity. By Theorem 1.6 in [8] we know, however, that solutions of (4.1) having a small norm are strict solutions of (3.1).

Step 2: Lemma 1.2 in [8] implies that zero is isolated in $\sigma(J_0)$ and therefore the range $R(J_0)$ is closed. Thus we find projectors $P, Q : H_0 \rightarrow H_0$ commuting with J_0 such that

$$J_0 P = 0, \quad Q J_0 = J_0 \tag{4.2}$$

Define $Pu = v$, $Qu = w$ for $u \in H_0$. Again by Lemma 1.2 in [8]

$$J_0|_{\mathbb{QH}_2} : \mathbb{QH}_2 \rightarrow \mathbb{QH}_0$$

is a toplinear isomorphism. Therefore we may apply the Ljapunov-Schmidt reduction as described in Section 2. First of all we get the splitting

$$\mu \mathcal{D}v = \text{PN}(v+w, \lambda) \tag{4.3a}$$

$$\mu \mathcal{D}w + J_0 w = \text{QN}(v+w, \lambda) \tag{4.3b}$$

The Fréchet derivative of (4.3b) at $(w, v, \lambda, \mu) = (0, 0, 0, 0)$ is given by the operator $J_0|_{\mathbb{QH}_2}$, which is, according to the remark above, an isomorphism. Therefore the implicit function theorem yields a local solution

$$w : \mathbb{PH}_0 \times \Lambda \times \mathbb{R} \rightarrow \mathbb{QH}_2, \quad w(0, \lambda, \mu) = 0 \tag{4.4}$$

of (4.3b). Inserting w in (4.3a) yields the bifurcation equation

$$\mu \mathcal{D}v = \text{PN}(v+w(v, \lambda, \mu), \lambda), \quad v \in \ker J_0 \tag{4.5}$$

Step 3: Our main problem is to eliminate μ from equation (4.5). Our plan of the proof is to use the RPE in calculating the Brouwer index of the solutions on the curve S for equation (4.5). The homotopy invariance of the Brouwer degree will give the result. We begin with the observation that there is a natural splitting of $\ker J_0$ given by

$$\ker J_0 = \ker A \oplus V_2, \tag{4.6}$$

where V_2 is a vector space containing nontrivial time-periodic functions with mean value zero. This decomposition is orthogonal with respect to $(\cdot, \cdot)_0$, thus the operator P may be written as the sum $P = P_1 + P_2$. Note that these projection operators commute with \mathcal{D} . Moreover, we have $P_1 \mathcal{D} = 0$. Writing $v^{(1)} = P_1 v$ and $v^{(2)} = P_2 v$, we get a further splitting of (4.5) into

$$P_1 N(v^{(1)} + v^{(2)} + w(v^{(1)} + v^{(2)}, \lambda, \mu), \lambda) = 0 \tag{4.7a}$$

$$\mu \mathcal{D}v^{(2)} = P_2 N(v^{(1)} + v^{(2)} + w(v^{(1)} + v^{(2)}, \lambda, \mu), \lambda) \tag{4.7b}$$

This system has a special feature: if $v^{(2)} = 0$, then (4.7b) is satisfied. Therefore the stationary solutions of (3.1) are related in a 1-1 manner to the solutions of (4.7a) with $v^{(2)} = 0$. Let

$$S' = \left\{ (P_1 u(s), \lambda(s)) \mid s \in [-\underline{a}, \underline{b}] \right\} . \quad (4.8)$$

(We assume that we have chosen the constant ε, δ and $\underline{a}, \underline{b}$ such that the bifurcation function is defined on $(B \cap \ker J_0) \times \Delta$.)

Let us now investigate (4.7b). We start with an orthogonal decomposition of V_2 . Let V_3 be the subspace of V_2 spanned by

$$\psi_1 = \operatorname{Re}(\phi_{j_0, 1} e^{-it}) , \quad \psi_2 = \operatorname{Im}(\phi_{j_0, 1} e^{-it}) .$$

Let

$$\psi_1^* = \operatorname{Re}(\phi_{j_0, 1}^* e^{-it}) , \quad \psi_2^* = \operatorname{Im}(\phi_{j_0, 1}^* e^{-it}) ,$$

where $\phi_{j_0}^*$ is the eigenvector of A^* to the eigenvalue $i\omega_{j_0}$ such that

$(\phi_{j_0}, \phi_{j_0}^*)_0 = 1/2\pi$. Let r_1, r_2 be defined by

$$r_\ell = (v, \psi_\ell^*)_0 \quad \text{for } \ell = 1, 2 . \quad (4.9)$$

We write $v^{(2)} \in V_2$ as $v^{(2)} = r_1 \psi_1 + r_2 \psi_2 + \tilde{v}$. We claim that for any multilinear map $B : E \times E \dots \times E \rightarrow E$

$$(B(\tilde{v}, \dots, \tilde{v}), \psi_\ell^*)_0 = 0 \quad \text{for } \ell = 1, 2 . \quad (4.10)$$

In order to prove the claim we express \tilde{v} as a finite sum

$$\tilde{v} = \sum_{k \in K} \sum_{j=1}^{\rho_k} (\alpha_{k,j} e^{-ikt} \phi_{k,j} + \bar{\alpha}_{k,j} e^{ikt} \bar{\phi}_{k,j}) .$$

Now the left-hand side in (4.10) is a sum of expressions of the form

$$\left(B(\pm \phi_{k_1, \ell_1}, \dots, \pm \phi_{k_p, \ell_p}) \exp[(\pm k_1 \pm k_2 \pm \dots \pm k_p)it], \psi_\ell^* \right)_0 \beta, \quad \beta \in \mathbb{R}$$

where the k_ℓ are elements of K . Since by S5) the greatest common divisor of the elements in K is greater than one, we have

$$\pm k_1 \pm k_2 \pm \dots \pm k_p \neq \pm 1 . \quad (4.11)$$

Since the inner product in (4.10) involves an integral over the interval $[0, 2\pi]$, the claim is proved. Since $\mathcal{D}\psi_1 = -\psi_2$ and $\mathcal{D}\psi_2 = \psi_1$ the part of equation (4.7b) on V_3 has the following form:

$$\mu r_2 = g_1(r_1, r_2)H(v, \lambda, \mu) \quad (4.12)$$

$$-\mu r_1 = g_2(r_1, r_2)L(v, \lambda, \mu) \quad (4.13)$$

with $g_\ell(0,0) = 0$, for $\ell = 1, 2$. Setting $r_2 = 0$ we fix the phase of the periodic solution being sought. Since $g_2(0,0) = 0$ we may now divide the second equation through r_1 . In contrast to a similar step, which has been performed by Kielhöfer [9, p. 506] we may lose solutions when dividing through r_1 , because there may exist solutions of (3.1) with $r_1 = r_2 = 0$. This apparently cannot happen in the case of Hopf Bifurcation.

Differentiation of (4.3b) after inserting $w(v, \lambda, \mu)$ with respect to v yields $w_v(0, 0, \mu) = 0$. This implies

$$\left. \frac{\partial}{\partial \mu} \frac{1}{r_1} g_2(r_1, 0)L(v, \lambda, \mu) \right|_{(r_1, \tilde{v}, v^{(1)}, \lambda, \mu) = (0, 0, 0, 0, 0)} = 0 .$$

Therefore we can solve the equation

$$-\mu = \frac{1}{r_1} g_2(r_1, 0)L(v^{(1)} + v, \lambda, \mu) + r_1 \psi_1 \quad (4.14)$$

using the implicit function theorem. This solution has the form

$$\mu = \mu(r_1, v^{(1)}, v, \lambda), \quad \mu(0, 0, 0, 0) = 0 \quad (4.15)$$

This completes Step 3.

Step 4: For any $v \in \ker J_0$ define

$$\mu(v, \lambda) = \mu(r_1, v^{(1)}, \tilde{v}, \lambda)$$

where $r_1, v^{(1)}$, and \tilde{v} denote the respective components of v . Observe that

$\mu(v, \lambda)$ is real analytic. Put $\mu(v, \lambda)$ into (4.1) and consider the one-parameter bifurcation problem

$$\mu(v, \lambda) \mathcal{Q} u + J_0 u = N(u, \lambda) \quad , \quad u \in H_2 \quad . \quad (4.16)$$

Now, apply the Ljapunov-Schmidt reduction to (4.16) using the projection operators defined in (4.2). Since our hypotheses and Theorem 2 imply that the RPE is true, we know that the eigenvalues of the linear operators

$$T_1(s) = \mu(Pu(s), \lambda(s)) \mathcal{Q} + J_0 - N_u(u(s), \lambda(s)) \quad (4.17)$$

and

$$R(s) = \mu(v(s), \lambda(s)) \mathcal{Q} - \frac{\partial}{\partial v} PN(v(s) + w(s), \lambda(s)) \quad (4.18)$$

have the same first nonvanishing terms, where

$$w(s) = w(v(s), \lambda(s), \mu(v(s), \lambda(s))) \quad .$$

We claim that $R(s)$ has block diagonal form, i. e. ,

$$R(s) = \begin{pmatrix} R_1(s) & 0 & 0 \\ 0 & \Gamma(s) & 0 \\ 0 & 0 & R_2(s) \end{pmatrix}$$

where R_1 and R_2 are the parts of $R(s)$ on $\ker A$ and on the complement \tilde{V} of V_3 in V_2 respectively, while $\Gamma(s)$ is a diagonal operator on V_3 . We begin by considering the part on $\ker A$. We show

$$\frac{\partial}{\partial v^{(2)}} \left(-P_1 N(v^{(1)} + v^{(2)} + w(v^{(1)} + v^{(2)}, \lambda, \mu), \lambda) \right) = 0 \quad (4.20)$$

where the derivative is taken at a point with $v^{(2)} = 0$ and solving (4.7a). Equation (4.20) is equivalent with

$$\frac{\partial}{\partial v^{(2)}} \left(-N(\cdot), \phi_{0, \ell} \right)_0 = 0, \quad \ell = 1, \dots, \rho_0$$

where the argument is the same as in (4.20) and the derivative is calculated at the same point.

The map

$$N(v + w(v, \lambda, \mu), \lambda)$$

is real analytic on $\ker J_0 \times \mathbb{R}^2$ and may be written as

$$N(v + w(v, \lambda, \mu), \lambda) = \sum \lambda^j \mu^\ell B_{j\ell k}^{(1)}(v, \dots, v)$$

where $B_{j\ell k}^{(1)}$ is a k -linear map $\ker J_0 \rightarrow E$.

By a rearrangement of the sum we may assume that

$$N(v^{(1)} + v^{(2)} + w(v^{(1)} + v^{(2)}), \lambda, \mu), \lambda) = \sum \lambda^j \mu^\ell B_{j\ell k}^{(1)}(v^{(1)}) B_{j\ell k}^{(2)}(v^{(2)})$$

where $B_{j\ell k}^{(1)}$ is an analytic mapping into the space of k -linear forms $\ker A \rightarrow E$ and $B_{j\ell k}^{(2)}$ is k -linear. In order to show (4.20) we need only observe that

$$\left(B_{j\ell 1}^{(1)}(v^{(1)}) B_{j\ell 1}^{(2)}(v^{(2)}), \phi_{0,p} \right)_0 = 0, \quad p = 1, \dots, \rho_0$$

for all $j, \ell \geq 0$. But this inner product involves an intergral over a full period of some trigonometric polynomial with mean value zero. Therefore we have established (4.20). Similarly, one proves the appearance of the other zeros in (4.19).

Now we claim that $\Gamma(s)$ is a diagonal matrix. We consider the part of equation (4.7b) on V_3 which is linear in r_1, r_2 at $v^{(2)} = 0$. This part is equivariant with respect to the group action

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \rightarrow S_\theta \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \quad \text{with } S_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (4.21)$$

As a result of this equivariance this part of the equation has the form

$$h(v^{(1)}, \lambda, \mu) \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + q(v^{(1)}, \lambda, \mu) \begin{pmatrix} -r_2 \\ r_1 \end{pmatrix}. \quad (4.22)$$

This follows from the fact that any matrix commuting with all S_θ has the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Since the second equation is satisfied by $\mu(v, \lambda)$ after setting $r_2 = 0$, we find $q(v^{(1)}, \lambda, \mu(v^{(1)}, \lambda)) = 0$ identically.

Now we see

$$\Gamma(s) = \begin{pmatrix} h(v^{(1)}(s), \lambda(s), \mu(s)) & 0 \\ 0 & h(v^{(1)}(s), \lambda(s), \mu(s)) \end{pmatrix}$$

where $\mu(s) = \mu(v(s), \lambda(s))$. This proves the claim. Moreover, one finds that $\Gamma(s)$ has a double eigenvalue. The most important step in the course of our proof is the following formula for the double eigenvalue $\tilde{\gamma}(s)$ of $\Gamma(s)$:

$$\tilde{\gamma}(s) = \frac{1}{\ell_0!} \frac{d^{\ell_0}}{ds^{\ell_0}} \operatorname{Re} \gamma(0) s^{\ell_0} + O(s^{\ell_0+1}) . \quad (4.23)$$

(The number ℓ_0 was defined in S7).)

The proof of this formula proceeds as follows:

- a) $\operatorname{Re} \gamma(s)$ is a double eigenvalue of $T_1(s)$ (this is obvious):
- b) The set of lowest order terms of eigenvalues of $T_1(s)$ near zero is exactly the same set as the set given by the lowest order terms of eigenvalues of $R(s)$ (this follows from the RPE).

c) The curve $\tilde{\gamma}(s)$ of eigenvalues of $R(s)$ corresponds to the curve $\gamma(s)$ of eigenvalues of $T(s)$ passing through $i\omega$. This fact is intuitively quite obvious, but we give a proof. Perform the Ljapunov-Schmidt reduction for

$$Au - N(u, \lambda) = 0 . \quad (4.24)$$

Doing so, we end up with equation (4.7a) with $v^{(2)} = 0$. Since the is true for this reduction we find the eigenvalues of $T(s)$ near zero as eigenvalues of $R_1(s)$ (in lowest order). These eigenvalues of $T(s)$ are also eigenvalues of $T_1(s)$. Therefore none of the eigenvalues of $R_1(s)$ corresponds to $\gamma(s)$. Similarly, one find the first nonvanishing terms of the eigenvalues of $R_2(s)$ by the respective reduction for the equations

$$\mu Q u + J_{0k} u = N(u, \lambda), \quad u \in H_2 \quad (3.25)$$

where $J_{0k} = \omega_k Q + A$, $k \in K$. For these reductions we do not obtain the same equation as the respective part of (4.7b), but still we have the same linear parts at $v^{(2)} = 0$. The RPE applied to (4.25) proves the assertion c) and hence (4.23).

By our choice of μ , equation (4.13) holds identically if $r_2 = 0$. Therefore we fix the phase of the unknown periodic solution setting $r_2 = 0$. Now there is no loss of information rewriting equations (4.7a, b) without equation (4.13). For this new equation we still have a set S'' of solutions corresponding to the set S and having the form

$$S'' = \left\{ \left(v^{(1)}(s), r_1(s), \tilde{v}(s), \lambda(s) \right) \mid s \in [-\underline{a}, \underline{b}] \right\} \quad (4.26)$$

corresponding to the stationary solutions of (3.1). Next we show that the equation obtained by the procedure described above, call it

$$M(v^{(1)}, r_1, \tilde{v}, \lambda) = 0, \quad (4.27)$$

has solutions near $(0, 0, 0, 0)$, which do not lie on S'' . The linearization of M along S'' is given by

$$\begin{pmatrix} R_1(s) & 0 & \dots & 0 \\ 0 & \tilde{\gamma}(s) & & 0 \\ 0 & & & R_2(s) \end{pmatrix}. \quad (4.28)$$

We assume that (4.27) has no solution lying off S'' . We calculate the Brouwer degree $d(M(\cdot, \cdot, \cdot, \lambda), B', 0)$, where B' is an open neighborhood of zero in $\ker A \times \mathbb{R} \times \tilde{V}$ for two values of λ having opposite sign. Assume we are in case A. Then $-\delta = \lambda(-\underline{a}) = \lambda(\underline{b})$ and we may choose all the constants and the neighborhood B' such that $(v^{(1)}(s), r_1(s), \tilde{v}(s)) \in B'$. Then we get

$$d(M(\cdot, \cdot, \cdot, -\delta), B', 0) = \text{sgn det } R(-\underline{a}) + \text{sgn det } R(\underline{b}) \quad (4.29)$$

By the shape of S'' we know that

$$\text{sgn det } R_1(-\underline{a}) = -\text{sgn det } R_1(\underline{b}). \quad (4.30)$$

Moreover, from S7) and the RPE it follows that $\tilde{\gamma}(s)$ changes sign at $s = 0$. We claim

$$\text{sgn det} R_2(s) = 1 \quad \text{for all small } s \neq 0 . \quad (4.31)$$

By the RPE the eigenvalues of $R_2(s)$ coincide in lowest nonvanishing order with the real parts of the eigenvalues of $D_u G(u(s), \lambda(s))$ near $\pm i k \omega$ for $k \in K$. Thus we find always two eigenvalues of $R_2(s)$ having the same sign. This proves the claim. By (4.29), (4.30), (4.31) and the change of sign for $\gamma(s)$ we obtain

$$d(M(\cdot, \cdot, \cdot, -\delta), B', 0) = \pm 2 . \quad (4.32)$$

From the shape of S'' and our assumption that there are no solutions of (4.18) not lying on S'' , we find

$$d(M(\cdot, \cdot, \cdot, \delta), B', 0) = 0 . \quad (4.33)$$

Equations (4.32) and (4.33) yield a contradiction to the homotopy invariance of the Brouwer degree. Thus there must exist a continuum C of solutions for (4.27) emanating from zero lying off S'' . The solutions on C correspond to periodic solutions of (3.1) and the proof of Theorem 3 is complete.

5. A Special Equation

In this section we consider a special equation representing a generic situation for the bifurcation problem (3.1) case A under the hypotheses given in Section 3. We are going to discuss the bifurcation of periodic solutions together with their stability properties. By the RPE the stability assignments given below reflect the stability properties of the corresponding solution of (3.1) in a correct way. The Floquet theory for evolution equations is developed in [6]. We consider the bifurcation equation for (3.1) with $K = \emptyset$, $\rho_0 = 1$, i. e.,

$$\begin{aligned} g_1(r, z, \lambda) &= Mrz + \lambda r + Kr^3 + o(r(z + \lambda + r^2)) \\ g_2(r, z, \lambda) &= \lambda + Hz^2 + Lr^2 + o(\lambda + z^2 + r^2) \end{aligned} \quad (5.1)$$

for positive H . In (5.1) we have normalized the coefficients of λ and λr to one. The stationary solutions are given by

$$\lambda = -Hz^2 + O(z^3) . \quad (5.2)$$

The eigenvalues of the Jacobian of g along this solution determine the (linearized) stability. The following table gives the number of eigenvalues of $T(s)$ with positive real parts. (As usual in the stability analysis of bifurcating solutions we assume that the remaining part of the spectrum of A lies in the right half-plane and is bounded away from the imaginary axis.)

	$M > 0$	$M < 0$
$z > 0$	3	1
$z < 0$	0	2

In order to find periodic solutions of (3.1) we look for solutions of (5.1) with $r \neq 0$, i.e.,

$$\lambda = -Mz - Kr^2 + O(z^2 + r^4) \quad (5.3)$$

From $g_2(r, z, \lambda) = 0$ one finds

$$\lambda = -Hz^2 - Lr^2 + O(|z|^3 + r^3) \quad (5.4)$$

giving

$$Mz + Kr^2 + Hz^2 + Lr^2 + O(z^2 + r^3) = 0 . \quad (5.5)$$

In lowest order we get

$$z = \frac{L-K}{M} r^2 + O(r^3) . \quad (5.6)$$

From (5.4) we find the bifurcation direction by

$$\lambda = -Lr^2 + O(r^3) . \quad (5.7)$$

Thus the sign of L determines the direction of bifurcation. Let us now pursue the question of stability for this branch of periodic solutions. We calculate the Jacobian of g

$$\begin{pmatrix} Mz + \lambda + 3Kr^2 & Mr \\ 2Lr & 2Hz \end{pmatrix} \quad (5.8)$$

The determinant of (5.8) along the branch of periodic solutions is given by

$$-2LMr^2 + O(r^4) . \quad (5.9)$$

Thus, for $LM < 0$, the two eigenvalues have the same sign, if $LM > 0$ they have opposite signs. Therefore, in the second case, the periodic solution is unstable. Independent of the direction of bifurcation, we find that the dimension of the unstable manifold is equal to one. In the first case, i.e., $LM < 0$, we calculate the trace of (5.8) along the periodic branch. It is given by

$$\frac{L-K}{M} (M+2H) - Lr^2 + 3Kr^2 + O(r^3) . \quad (5.10)$$

This leads us to consider

$$\text{sgn} \left[K + \frac{H}{M} (L-K) \right] . \quad (5.11)$$

If this expression is negative then the bifurcating periodic orbits are stable. In the other case they are unstable, the dimension of the unstable manifold being two. We collect these results in Figure 2. Stationary solutions are indicated by a solid line, periodic solutions by broken lines. The numbers give the dimension of the unstable manifold. The alternative in b) and c) depends on the sign in (5.11).

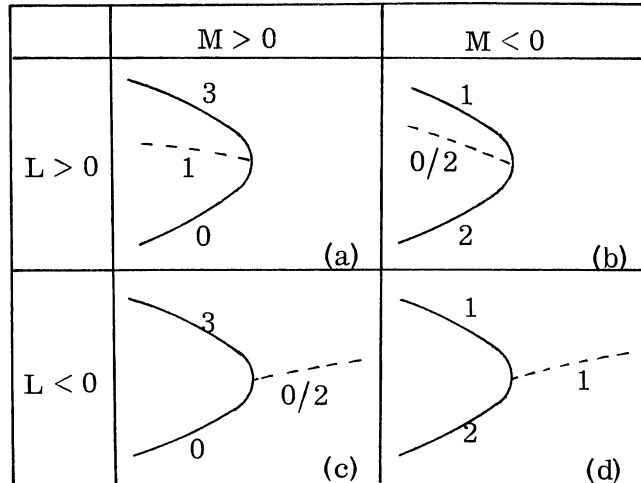


Figure 2.

We want to add one more remark. If one considers one or another of the above constants M , H , L or K as a parameter, one obtains paths connecting the different pictures above. One should expect to find subharmonic bifurcation or bifurcation into invariant tori at values of these parameters where the stability of the periodic orbit changes.

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