THE DOUBLE POROSITY MODEL FOR SINGLE PHASE FLOW IN NATURALLY FRACTURED RESERVOIRS

BY

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THE DOUBLE POROSITY MODEL FOR SINGLE PHASE FLOW IN NATURALLY FRACTURED RESERVOIRS

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1. Introduction

As early as 1953, Pirson [6] gave a qualitative description of a fractured reservoir as a reservoir with two porous structures–matrix (porous rock) and fracture. It was not until the early 1960's, however, that a quantitative description of this double porosity concept appeared [3], [9]. Briefly, a fractured reservoir is a region of space $\Omega \subset \mathbb{R}^d$ (d=2 or 3) which is partitioned by the thin fractures into disjoint, simply connected matrix blocks $\Omega_i \subset \Omega$. The idea is to consider at each point of space Ω not one but two porosities (actually, two sets of reservoir properties), one associated with the matrix, and the other associated with the fracture system. The matrix porosity, denoted $\phi_i(\underline{x})$ for $\underline{x} \in \Omega_i$, is the macroscopic matrix pore void space per unit bulk volume. The fracture porosity, $\phi(\underline{x})$ for $\underline{x} \in \Omega$, is the macroscopic fracture void space per unit bulk volume.

Some of the fluid flows in the matrix, while the rest flows in the fracture system. The matrix flow is certainly of Darcy type. We also assume a macroscopic Darcy flow over the reservoir Ω for the fracture system flow, even though it is literally constrained to the physical fractures. This is a reasonable assumption for reservoirs with thin, interconnected fractures and low fluid velocities. Consequently, at each point of space we have tensors for both the matrix permeability $\mathbf{k}_i(\underline{x})$, $\underline{x} \in \Omega_i$, and the fracture permeability $\mathbf{k}(\underline{x})$, $\underline{x} \in \Omega$.

Several authors have used the double porosity concept to model single phase flow in a fractured reservoir, most notably Barenblatt, et al. [3], Warren and Root [9], kazemi [5], and de Swaan O. [8]. Many other authors have considered double porosity models for multi-phase, multi-component systems (for examples, see [7] and its references).

The most important and difficult thing to model is the exchange of fluid between the matrix and the fracture system. Generally, this exchange is assumed to be in a quasi-steady state; that is, an ad hoc transfer function is introduced which does not

depend explicitly on time but rather depends on the difference between the matrix fluid pressure and the fracture system fluid pressure. This assumption effectively ignores the pattern of flow in an individual matrix block. Fluid exchange actually takes place at the surfaces of the blocks. The rate of exchange depends upon the fluid pressures, the geometry of the matrix blocks, and the past history of the matrix flow.

Two exceptions to the quasi-steady state approach are to be found in the papers of Kazemi [5] and de Swaan O. [8]. In both of these papers, the fluid exchange is related to the flow through the matrix blocks' surfaces. Unfortunatly, both papers also constrain the fracture flow to the physical fractures, rather than considering it macroscopically spread out over the entire reservoir. This places restrictions on the geometry of the reservoir and otherwise severely complicates any field-scale numerical solution procedure. These two models are not genuinely of the double porosity type, at least not as it has been described above, as at each point of space there is only a single porosity.

While the quasi-steady state assumption is often reasonable for single phase flow when the matrix blocks are small [3], [5], [9], it is still valuable to properly model the unsteady state nature of the fluid exchange without unnecessarily restricting the geometry nor complicating the approximation process. Obviously, such a model is needed for simulations where accuracy is quite important or where the quasi-steady state assumption is invalid; moreover, this model should give us greater understanding of the quasi-steady state assumption itself, and it should provide a stepping-stone to properly modeling multi-phase, multi-component fractured reservoir flow.

Such a model has been described and analyzed by the author in [1]. The approach taken there is being used by the author in modeling a completely miscible displacement [2], and by Douglas, et al. [4] in modeling an immiscible waterflood. Here, one of the assumptions of the modeling process in [1] shall be relaxed, as described below.

Pressure must be a continuous variable; hence, the matrix pressure at the surface of each block and the pressure of the fluid in the physical fractures around the block must be equal. On each block, this fact gives rise to a boundary condition for the matrix flow. Let $p(\underline{x},t)$ denote the fracture system pressure. The pressure in a physical fracture is not necessarily the value of p evaluated at the fracture location, as p is a macroscopic variable with respect to the system of fractures. A reasonable value for the pressure in a physical fracture is given by taking some local average of p. It is appropriate to take the physical fracture pressure around $\partial\Omega_i$ as some average in the vicinity of the entire block, since the diameters of the matrix blocks are small compared to the smallest dimension of the reservoir itself (i.e., the reservoir is *naturally* fractured).

In [1], it was assumed that the variation of p over each Ω_i was so small as to be negligible. A simple local average of p was taken, giving a constant (in space) pressure (and density) over $\partial\Omega_i$. Consequently, fluid could only be absorbed or emitted uniformly by each matrix block; fluid could not flow from one side to another through

any block.

In the model to be presented here, the variation of p near each Ω_i will influence the matrix flow. At each time, the physical fracture fluid density around $\partial\Omega_i$ shall be assumed to be the restriction of a globally linear function. More generally, one could take a higher degree polynomial; the model will generalize. However, if the spatial variation over the blocks is so large as to require this generality, one can hardly claim to have a double porosity reservoir.

An outline of the paper follows. In section 2 we will derive our model from physical considerations by applying the double porosity concept to a naturally fractured reservoir. The matrix/fracture fluid exchange will influence the matrix flow through a boundary condition on each block as described above, and it will influence the fracture system flow through a macroscopically distributed source. In the last three sections, we will extend most of the results of [1] to the present model. We will again make the (unphysical) assumption that f_e is in $L^2(\Omega\times J)=L^2(J;L^2(\Omega))$; that is, f_e will not be concentrated at points. This will ensure a smooth solution to the differential problem. We will analyze the model in section 3 by showing that it is mathematically well posed. In section 4, we will describe a finite element method that is easy to implement in field-scale simulation, and we will prove its convergence at the optimal rate in section 5.

2. Derivation of the Model

We will now apply the double porosity concept to a single phase fluid in a naturally fractured reservoir. We will assume that the fluid is an ideal liquid; that is, a fluid of constant viscosity $\mu>0$ and compressibility $c>0$:

$$\rho^{-1}d\rho = c\,dp, \qquad \sigma^{-1}d\sigma = c\,dq, \tag{2.1}$$

where $\rho(\underline{x},t)$ and $p(\underline{x},t)$ are the fracture system fluid density and pressure, respectively, and $\sigma(\underline{x},t)$ and $q(\underline{x},t)$ are the corresponding quantities for the matrix fluid.

First consider fluid flow in the matrix blocks. Since the blocks are small and typically the fracture system has a higher flow capacity (permeability) than the matrix, we make the following two assumptions:

1. An individual matrix block will interact only with the fractures surrounding it; hence, matrix blocks do not interact directly with each other, nor with external sources or sinks. (Kazemi's numerical results [5] indicate that this is a good assumption.)
2. At any given time, the spatial variation of the fluid density in the physical fractures around a block is sufficiently well taken into account by a linear function.

Over the i *th* matrix block, assumption 1 and conservation of mass combined with Darcy's law and (2.1) give us

$$\phi_i\sigma_t - \underline{\nabla}\cdot(\boldsymbol{\kappa}_i\underline{\nabla}\sigma - \sigma^2\underline{\Gamma}_i) = 0, \qquad (\underline{x},t)\epsilon\Omega_i\times J, \tag{2.2}$$

where the t subscript denotes partial differentiation in time, $\boldsymbol{\kappa}_i(\underline{x})=\mathbf{k}_i(\underline{x})/\mu c$ (a tensor), $\underline{\Gamma}_i(\underline{x})$ is the gravitational constant times $\mathbf{k}_i(\underline{x})/\mu$ applied to the gradient of the vertical coordinate, and $J=(0,T]$ is the time interval of interest. It is sufficient in reservoir simulation to linearize the quadratic term; hence, let $\sigma_0(\underline{x})$ be some reference density function and approximate

$$\sigma^2 = [(\sigma - \sigma_0) + \sigma_0]^2 \approx 2\sigma_0\sigma - \sigma_0^2. \tag{2.3}$$

The i *th* matrix block flow equation becomes

$$\phi_i\sigma_t - \underline{\nabla}\cdot[\boldsymbol{\kappa}_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i] = 0, \qquad (\underline{x},t)\epsilon\Omega_i\times J, \tag{2.4}$$

where $\underline{\gamma}_i(\underline{x})=\sigma_0(\underline{x})\underline{\Gamma}_i(\underline{x})$.

By assumption 1, the boundary condition for (2.4) is given entirely by continuity of pressure (equivalently, of density, by (2.1)) with the fracture flow. As explained in the introduction, we should locally average the macroscopic fracture density to obtain the fluid density in the physical fractures. For simplicity, let us assume that the fractures are infinitely thin so that $\overline{\Omega}=\cup_i\overline{\Omega}_i$. Now, to make this local averaging process explicit, let $\{\chi_i(\underline{x})\}$ be some partition of unity over Ω such that each χ_i is or is approximately the characteristic function of Ω_i; that is, $\sum_i\chi_i\equiv 1$, $\chi_i\geq 0$, the support of $\chi_i\approx\Omega_i$, and $\int\chi_i dx=|\Omega_i|=$ the measure of Ω_i. Later, the χ_i will be described further.

There are many ways to linearly approximate ρ near Ω_i. Perhaps the best way to approximate ρ by its local averages is to express it in terms of an orthonormal expansion. Let $\{1,\Lambda_{i,1}(\underline{x}),\ldots,\Lambda_{i,d}(\underline{x})\}$ be an orthonormal basis for the linear functions with respect to the inner product given by integration against the weight $|\Omega_i|^{-1}\chi_i$. (For example, perform Gram-Schmidt orthogonalization and normalization to $\{1, x_1,\ldots,x_d\}$. If Ω_i is a rectangular parallelepiped and χ_i is its characteristic function, then the $\Lambda_{i,j}$ are just scaled Legendre polynomials.) Let $\underline{\Lambda}_i(\underline{x})=(\Lambda_{i,1}(\underline{x}),\ldots,\Lambda_{i,d}(\underline{x}))$, and let $(\cdot,\cdot)$ be the $L^2(\Omega)$ or $(L^2(\Omega))^d$ inner product. Then we have that

$$\rho = \frac{1}{|\Omega_i|}[(\rho,\chi_i) + (\rho,\underline{\Lambda}_i\chi_i)\cdot\underline{\Lambda}_i] + O([\mathrm{diam}(\Omega_i)]^2), \qquad (\underline{x},t)\epsilon\Omega_i\times J, \tag{2.5}$$

and (by assumption 2) our boundary condition shall be

$$\sigma = \frac{1}{|\Omega_i|}[(\rho,\chi_i) + (\rho,\underline{\Lambda}_i\chi_i)\cdot\underline{\Lambda}_i], \qquad (\underline{x},t)\epsilon\partial\Omega_i\times J. \tag{2.6}$$

The initial condition for (2.4) must be given:

$$\sigma(\underline{x},0) = \sigma^0(\underline{x}), \qquad \underline{x}\in\Omega_i. \tag{2.7}$$

The fracture flow is governed by an equation analogous to (2.4), except that two source terms appear. The effect of external sources (sinks) $f_e(\underline{x},t)$ has been reserved for the fracture system (assumption 1). In addition, the fluid produced through the matrix blocks' surfaces is another source. Denote by $f_i(\underline{x},t)$ the source from the i *th* block. Then the entire matrix source is $\sum_i f_i$. Hence,

$$\phi\rho_t - \underline{\nabla}\cdot[\,\mathbf{k}\underline{\nabla}\rho - (2\rho - \rho_0)\underline{\gamma}\,] = f_e + \sum_i f_i, \qquad (\underline{x},t)\in\Omega\times J, \tag{2.8}$$

where $\mathbf{k}(\underline{x})=\mathbf{k}(\underline{x})/\mu c$ (a tensor), $\rho_0(\underline{x})$ is the fracture system's reference density function, $\underline{\gamma}(\underline{x})=\rho_0(\underline{x})\underline{\Gamma}(\underline{x})$, and $\underline{\Gamma}(\underline{x})$ is the gravitational constant times $\mathbf{k}(\underline{x})/\mu$ applied to the gradient of the vertical coordinate.

We will now define the matrix source term. We must be careful to model the fluid exchange from the point of view of the fracture system in a manner that is consistent with the boundary condition (2.6).

At each point $\underline{x}\in\partial\Omega_i$, the i *th* block loses (or gains) an amount of fluid equal to

$$-[\,\mathbf{k}_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i\,]\cdot\underline{\nu}_i, \tag{2.9}$$

where $\underline{\nu}_i(\underline{x})$ is the outer unit normal to $\partial\Omega_i$. This fluid should be macroscopically spread out as f_i over the domain Ω near Ω_i in such a way that f_i has the same macroscopic effect. Since the block only detects the linear variation of the fracture flow, f_i and (2.9) should have the same effect on a linear fracture flow.

Since (2.9) is a distribution (supported on $\partial\Omega_i$), its effect is determined by considering its action on a test function $\varphi\in C^\infty(\Omega)$:

$$\begin{aligned}
&\int_{\partial\Omega_i} [\,\mathbf{k}_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i\,]\cdot\underline{\nu}_i\,\varphi\,ds \\
&\quad = \int_{\Omega_i} \underline{\nabla}\cdot\{[\,\mathbf{k}_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i\,]\varphi\}\,dx \\
&\quad = (\underline{\nabla}\cdot[\,\mathbf{k}_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i\,],\,\varphi)_i + (\mathbf{k}_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i,\,\underline{\nabla}\varphi)_i \\
&\quad = (\phi_i\sigma_t,\,\varphi)_i + (\mathbf{k}_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i,\,\underline{\nabla}\varphi)_i,
\end{aligned} \tag{2.10}$$

by the divergence theorem and (2.4). (Here, $(\cdot,\cdot)_i$ is the inner product on $L^2(\Omega_i)$ or $(L^2(\Omega_i))^d$.) If we now restrict φ to lie in the set of linear functions, then

$$\varphi = \frac{1}{|\Omega_i|}[(\varphi,\,\chi_i) + (\varphi,\,\Delta_i\chi_i)\cdot\Delta_i] \tag{2.11}$$

and

$$\int_{\partial\Omega_i} [\kappa_i \underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i]\cdot\underline{\nu}_i \varphi \, ds$$

$$= \frac{1}{|\Omega_i|}\Big[(\phi_i\sigma_t, 1)_i (\varphi, \chi_i) + (\phi_i\sigma_t, \Delta_i)_i\cdot(\varphi, \Delta_i\chi_i)$$

$$+ (\kappa_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\Delta_i)_i\cdot(\varphi, \Delta_i\chi_i)\Big]$$

$$= \left[\frac{1}{|\Omega_i|}\Big\{(\phi_i\sigma_t, 1)_i + [(\phi_i\sigma_t, \Delta_i)_i + (\kappa_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\Delta_i)_i]\cdot\Delta_i\Big\}\chi_i, \varphi\right]. \tag{2.12}$$

(Note that $\underline{\nabla}\Delta_i$ is a tensor in the above expression.) Consequently,

$$f_i = -\frac{1}{|\Omega_i|}\Big\{(\phi_i\sigma_t, 1)_i + [(\phi_i\sigma_t, \Delta_i)_i + (\kappa_i\underline{\nabla}\sigma - (2\sigma - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\Delta_i)_i]\cdot\Delta_i\Big\}\chi_i, \quad (\underline{x},t)\in\Omega\times J. \tag{2.13}$$

Higher order variations in φ were ignored in defining f_i, just as higher order variations in ρ were ignored in defining the boundary condition (2.6). To put it another way, the function f_i is the best localized (by χ_i) linear approximation to the distribution (2.9). Mass is conserved by f_i. Moreover, f_i takes into account the spatial variation of the source (2.9), up to the linear order.

Note that χ_i defines and weights the region of space over which the i*th* block influences the fracture flow. Effectively, χ_i has related the scale of the matrix macroscopic averaging (which is on the order of the size of the rock grains) to the scale of the fracture system macroscopic averaging (which is on the order of the size of the blocks).

As a boundary condition for (2.8), let us simply take the "no flow" Neumann condition:

$$[\kappa\underline{\nabla}\rho - (2\rho - \rho_0)\underline{\gamma}]\cdot\underline{\nu} = 0, \qquad (\underline{x},t)\in\partial\Omega\times J, \tag{2.14}$$

where $\underline{\nu}(\underline{x})$ is the outer unit normal to $\partial\Omega$. Finally, give the initial condition as

$$\rho(\underline{x},0) = \rho^0(\underline{x}), \qquad \underline{x}\in\Omega. \tag{2.15}$$

In summary, (2.4), (2.6), and (2.7) define the flow in the matrix blocks, while (2.8) with (2.13), (2.14), and (2.15) define the fracture system flow.

3. Analysis of the Model

For the analysis below, we shall tacitly assume the following hypotheses. Ω is a simply connected, bounded domain with smooth boundary, and each Ω_i is a convex domain. Each of the functions ϕ, ϕ_i, κ, κ_i, $\underline{\Gamma}$, $\underline{\Gamma}_i$, and χ_i is bounded and sufficiently smooth. Additionally, ϕ and the ϕ_i are bounded below by positive constants, and the tensors κ and κ_i, for all i, are symmetric and uniformly positive-definite.

Let $W^{r,p}(\Omega)$ denote the usual Sobolev space of $r \geq 0$ times differential functions in $L^p(\Omega)$, $1 \leq p \leq \infty$. Let $H^r(\Omega)=W^{r,2}(\Omega)$, and let $H^{-r}(\Omega)$ be its dual. $H^1{}_0(\Omega)=\{u \epsilon H^1(\Omega) | u=0 \text{ on } \partial\Omega\}$. $H^s(J;H^r(\Omega))$, s a nonnegative integer, is the usual space having the norm

$$\|u\|_{H^s(J;H^r(\Omega))} = \left\{ \int_J \sum_{\ell=0}^{s} \left\| \frac{\partial^\ell u}{\partial t^\ell} \right\|^2_{H^r(\Omega)} dt \right\}^{1/2}.$$

Similarily, we have the space $L^\infty(J;H^r(\Omega))$. It will be convenient to define the space

$$H^1{}_\Lambda(\Omega) = \left\{ v \epsilon H^1(\Omega) \,\middle|\, \begin{array}{l} \text{the trace of } v \text{ on } \partial\Omega \text{ is identical to the} \\ \text{restriction of a linear function to } \partial\Omega \end{array} \right\}.$$

Of course, in each of these spaces, the domain Ω may be changed to Ω_i, $\partial\Omega$, $\partial\Omega_i$, or even $\Omega_m = \cup_i \Omega_i$.

C and ϵ will always denote generic positive constants.

Following [1], we will use the method of continuity to show that the model is well posed. We will present the argument in a more direct form here. This form of the argument is completely analogous to that used in the error analysis below.

For each $\lambda \epsilon [0,1]$, consider the problem defined by

$$\phi u_t - \underline{\nabla}\cdot(\kappa \underline{\nabla} u - 2\underline{\gamma} u) = G(\underline{x},t) + \lambda \sum_i F_i, \qquad (\underline{x},t) \epsilon \Omega \times J, \tag{3.1a}$$

$$(\kappa \underline{\nabla} u - 2\underline{\gamma} u)\cdot \underline{\nu} = u_0(\underline{x}), \qquad (\underline{x},t) \epsilon \partial\Omega \times J, \tag{3.1b}$$

$$u(\underline{x},0) = u^0(\underline{x}), \qquad \underline{x} \epsilon \Omega, \tag{3.1c}$$

and, for each i,

$$\phi_i v_t - \underline{\nabla}\cdot(\kappa_i \underline{\nabla} v - 2\underline{\gamma}_i v) = G_i(\underline{x},t), \qquad (\underline{x},t) \epsilon \Omega_i \times J, \tag{3.2a}$$

$$v = \frac{1}{|\Omega_i|}[(u, \chi_i) + (u, \Delta_i \chi_i)\cdot \Delta_i], \qquad (\underline{x},t) \epsilon \partial\Omega_i \times J, \tag{3.2b}$$

$$v(\underline{x},0) = v^0(\underline{x}), \qquad \underline{x} \epsilon \Omega_i, \tag{3.2c}$$

and

$$F_i = -\frac{1}{|\Omega_i|}\Big\{(\phi_i v_t, 1)_i + [(\phi_i v_t, \Delta_i)_i + (\kappa_i \underline{\nabla} v - 2\underline{\gamma}_i v, \underline{\nabla}\Delta_i)_i]\cdot\Delta_i\Big\}\chi_i, \qquad (\underline{x},t)\in\Omega\times J. \tag{3.3}$$

Note that u_0 is a function of $\underline{x}$ only.

The weak form of (3.2a) for a test function $\psi\in H^1_\Lambda(\Omega_i)$ (where ψ is identical to the linear function $\overline{\psi}_i$ on $\partial\Omega_i$) is,

$$\begin{aligned}(\phi_i v_t, \psi)_i &+ (\kappa_i \underline{\nabla} v, \underline{\nabla}\psi)_i - (2\underline{\gamma}_i v, \underline{\nabla}\psi)_i \\ &= (G_i, \psi)_i + \int_{\partial\Omega_i} (\kappa_i \underline{\nabla} v - 2\underline{\gamma}_i v)\cdot\underline{\nu}_i \overline{\psi}_i \, ds \\ &= (G_i, \psi - \overline{\psi}_i)_i - (F_i, \overline{\psi}_i).\end{aligned} \tag{3.4}$$

A similar weak form of (3.1a,b) for a test function $\varphi\in H^1(\Omega)$ can be found. If we add this weak form to λ times the sum on i of (3.4), we get

$$\begin{aligned}(\phi u_t, \varphi) &+ \lambda\sum_i(\phi_i v_t, \psi)_i + (\kappa\underline{\nabla} u, \underline{\nabla}\varphi) + \lambda\sum_i(\kappa_i\underline{\nabla} v, \underline{\nabla}\psi)_i \\ &= (2\underline{\gamma} u, \underline{\nabla}\varphi) + \lambda\sum_i(2\underline{\gamma}_i v, \underline{\nabla}\psi)_i + (G, \varphi) + \lambda\sum_i(G_i, \psi - \overline{\psi}_i)_i \\ &\quad + \int_{\partial\Omega} u_0\varphi \, ds + \lambda\sum_i(F_i, \varphi - \overline{\psi}_i).\end{aligned} \tag{3.5}$$

A priori energy estimates can be derived from (3.5). It is necessary to take $\overline{\psi}_i = |\Omega_i|^{-1}[(\varphi, \chi_i) + (\varphi, \Delta_i\chi_i)\cdot\Delta_i]$ so that the last term above vanishes. This condition is met for the usual choices $\varphi=u$, $\psi=v$ and $\varphi=u_t$, $\psi=v_t$ because of the boundary condition (3.2b). We obtain

$$\begin{aligned}\|u_t\|^2_{L^2(J;L^2(\Omega))} &+ \|u\|^2_{L^\infty(J;H^1(\Omega))} + \lambda\{\|v_t\|^2_{L^2(J;L^2(\Omega_m))} + \|v\|^2_{L^\infty(J;H^1(\Omega_m))}\} \\ &\le C\{\|G\|^2_{L^2(J;L^2(\Omega))} + \|u_0\|^2_{H^{1/2}(\partial\Omega)} + \|u^0\|^2_{H^1(\Omega)} \\ &\qquad + \lambda[\sum_i\|G_i\|^2_{L^2(J;L^2(\Omega_i))} + \|v^0\|^2_{H^1(\Omega_m)}]\} \\ &\quad + \epsilon\{\|u\|^2_{L^2(J;H^2(\Omega))} + \lambda\|v\|^2_{L^2(J;H^2(\Omega_m))}\},\end{aligned} \tag{3.6}$$

where C is independent of λ, and ϵ is as small as we like. The last two terms on the right side above arise from bounding the terms containing $\underline{\gamma}$ and $\underline{\gamma}_i$ in such a way that only the H^1-norms of these quantities appear (see [1]). Consequently, C depends on $\underline{\gamma}$ and $\underline{\gamma}_i$ only through their H^1-norms. This is important to notice, since $\underline{\gamma}$ and $\underline{\gamma}_i$ are proportional to ρ_0 and σ_0, respectively.

Energy estimates of (3.4) and (3.2b) with $\psi=v$ and with $\psi=v_t$ can be made directly.

Because v_t appears explicitly in F_i, an integration by parts in time argument is needed when $\psi=v$ (the same argument is needed to treat $(2\underline{\mathcal{X}}_i v, \underline{\nabla}\psi)_i$ when $\psi=v_t$). The final result shows that

$$\begin{aligned}
&\|v_t\|^2_{L^2(J;L^2(\Omega_m))} + \|v\|^2_{L^\infty(J;H^1(\Omega_m))} \\
&\quad \le C\{\sum_i \|G_i\|^2_{L^2(J;L^2(\Omega_i))} + \|v^0\|^2_{H^1(\Omega_m)} + \|u_t\|^2_{L^2(J;L^2(\Omega))} + \|u\|^2_{L^\infty(J;L^2(\Omega))}\} \\
&\qquad + \epsilon\|v\|^2_{L^2(J;H^2(\Omega_m))},
\end{aligned} \tag{3.7}$$

where C and ϵ are as above. Hence, we can omit the λ's in (3.6).

Elliptic regularity and direct estimates of (3.1a) and (3.2a) finally show that

$$\begin{aligned}
&\|u\|_{H^1(J;L^2(\Omega))} + \|u\|_{L^2(J;H^2(\Omega))} + \|v\|_{H^1(J;L^2(\Omega_m))} + \|v\|_{L^2(J;H^2(\Omega_m))} \\
&\quad \le C\{\|G\|_{L^2(J;L^2(\Omega))} + \|u_0\|_{H^{1/2}(\partial\Omega)} + \|u^0\|_{H^1(\Omega)} \\
&\qquad + \sum_i \|G_i\|_{L^2(J;L^2(\Omega_i))} + \|v^0\|_{H^1(\Omega_m)}\},
\end{aligned} \tag{3.8}$$

where C is as above.

This result and the method of continuity give us the

<u>Lemma:</u>

If $G\epsilon L^2(J;L^2(\Omega))$, $u_0\epsilon H^{1/2}(\partial\Omega)$, $u^0\epsilon H^1(\Omega)$, $G_i\epsilon L^2(J;L^2(\Omega_i))$, and $v^0\epsilon H^1(\Omega_i)$ (for each i) satisfy the compatibility relations

$$\frac{1}{|\Omega_i|}[(u^0, \chi_i) + (u^0, \underline{\Delta}_i\chi_i)\cdot\underline{\Delta}_i] = v^0, \qquad \underline{x}\epsilon\partial\Omega_i, \text{ for each } i, \tag{3.9}$$

then the problem (3.1)-(3.3) with $\lambda=1$ has a unique solution $u\epsilon H^1(J;L^2(\Omega))\cap L^2(J;H^2(\Omega))$ and $v\epsilon H^1(J;L^2(\Omega_m))\cap L^2(J;H^2(\Omega_m))$ that satisfies (3.8) with the constant C depending on $\underline{\mathcal{X}}$ and $\underline{\mathcal{X}}_i$ only through their H^1-norms.

We can now easily derive the theorem below as a corollary to the lemma. We need only set

$$\begin{aligned}
G &= \underline{\nabla}\cdot(\rho_0\underline{\mathcal{X}}) + f_e - \sum_i \frac{1}{|\Omega_i|}(\sigma_0\underline{\mathcal{X}}_i, \underline{\nabla}\underline{\Delta}_i)_i\cdot\underline{\Delta}_i\chi_i \\
&= \underline{\nabla}\cdot(\rho_0{}^2\underline{\Gamma}) + f_e - \sum_i \frac{1}{|\Omega_i|}(\sigma_0{}^2\underline{\Gamma}_i, \underline{\nabla}\underline{\Delta}_i)_i\cdot\underline{\Delta}_i\chi_i,
\end{aligned} \tag{3.10a}$$

$$u_0 = -\rho_0\underline{\mathcal{X}}\cdot\underline{\nu} = -\rho_0{}^2\underline{\Gamma}\cdot\underline{\nu}, \tag{3.10b}$$

$$u^0 = \rho^0, \tag{3.10c}$$

$$\underline{G}_i = \underline{\nabla}\cdot(\sigma_0 \underline{\gamma}_i) = \underline{\nabla}\cdot(\sigma_0{}^2 \underline{\Gamma}_i), \tag{3.10d}$$

and

$$v^0 = \sigma^0 \tag{3.10e}$$

in (3.1)-(3.3) to obtain our model (2.4), (2.6)-(2.8), (2.13)-(2.15), where $u=\rho$ and $v=\sigma$.

<u>Theorem:</u>

If $f_e \epsilon L^2(J;L^2(\Omega))$, $\rho_0{}^2 \epsilon H^1(\Omega)$, $\sigma_0{}^2 \epsilon H^1(\Omega_m)$, $\rho^0 \epsilon H^1(\Omega)$, and $\sigma^0 \epsilon H^1(\Omega_m)$ are such that

$$\frac{1}{|\Omega_i|}[(\rho^0, \chi_i) + (\rho^0, \underline{\Delta}_i \chi_i)\cdot\underline{\Delta}_i] = \sigma^0, \qquad \underline{x} \epsilon \partial\Omega_i, \text{ for each } i, \tag{3.11}$$

then the double porosity model has a unique solution $\rho \epsilon H^1(J;L^2(\Omega)) \cap L^2(J;H^2(\Omega))$ and $\sigma \epsilon H^1(J;L^2(\Omega_m)) \cap L^2(J;H^2(\Omega_m))$ which varies continuously with the data:

$$\begin{aligned} &\|\rho\|_{H^1(J;L^2(\Omega))} + \|\rho\|_{L^2(J;H^2(\Omega))} + \|\sigma\|_{H^1(J;L^2(\Omega_m))} + \|\sigma\|_{L^2(J;H^2(\Omega_m))} \\ &\quad \le C\{\|f_e\|_{L^2(J;L^2(\Omega))} + \|\rho_0{}^2\|_{H^1(\Omega)} + \|\sigma_0{}^2\|_{H^1(\Omega_m)} + \|\rho^0\|_{H^1(\Omega)} + \|\sigma^0\|_{H^1(\Omega_m)}\}. \end{aligned} \tag{3.12}$$

where C depends on the H^1-norms of ρ_0 and σ_0, and, for solutions (ρ_j, σ_j) arising from data $(f_{e,j}, \rho_{0,j}, \sigma_{0,j}, \rho^0{}_j, \sigma^0{}_j)$, $j=1$ and 2,

$$\begin{aligned} &\|\rho_1 - \rho_2\|_{H^1(J;L^2(\Omega))} + \|\rho_1 - \rho_2\|_{L^2(J;H^2(\Omega))} \\ &\qquad + \|\sigma_1 - \sigma_2\|_{H^1(J;L^2(\Omega_m))} + \|\sigma_1 - \sigma_2\|_{L^2(J;H^2(\Omega_m))} \\ &\quad \le C\{\|f_{e,1} - f_{e,2}\|_{L^2(J;L^2(\Omega))} + \|\rho_{0,1}{}^2 - \rho_{0,2}{}^2\|_{H^1(\Omega)} + \|\sigma_{0,1}{}^2 - \sigma_{0,2}{}^2\|_{H^1(\Omega_m)} \\ &\qquad + \|\rho_{0,1} - \rho_{0,2}\|_{H^1(\Omega)} + \|\sigma_{0,1} - \sigma_{0,2}\|_{H^1(\Omega_m)} \\ &\qquad + \|\rho^0{}_1 - \rho^0{}_2\|_{H^1(\Omega)} + \|\sigma^0{}_1 - \sigma^0{}_2\|_{H^1(\Omega_m)}\}, \end{aligned} \tag{3.13}$$

where C depends on the H^1-norms of $\rho_{0,2}$ and $\sigma_{0,2}$ as well as on the $L^2(H^2)$-norms of ρ_1 and σ_1.

While higher order regularity for the solution has not been demonstrated, it is trivial to at least see that the solution is smooth on the interior of its domain when the model's coefficients (including the χ_i) and data are smooth.

4. A Finite Element Method

For parameters h and h_i, all i, in (0,1], let $\mathcal{M}_h \subset H^1(\Omega)$ and, for each i, $\mathcal{N}_{i,h_i} \subset H^1_0(\Omega_i)$ be standard Galerkin finite dimensional H^1-approximation spaces of order $R \geq 2$ in h and $S_i \geq 2$ in h_i, respectively. Specifically, we need to be able to approximate $u \in H^r(\Omega)$, $1 \leq r \leq R$, such that

$$\inf_{\varphi \in \mathcal{M}} \|u - \varphi\|_{H^1(\Omega)} \leq C \|u\|_{H^r(\Omega)} h^{r-1}, \tag{4.1}$$

and, for each i, $v \in H^{s_i}(\Omega_i) \cap H^1_0(\Omega_i)$, $1 \leq s_i \leq S_i$, such that

$$\inf_{\psi \in \mathcal{N}_i} \|v - \psi\|_{H^1(\Omega_i)} \leq C \|v\|_{H^{s_i}(\Omega_i)} h_i^{s_i - 1}. \tag{4.2}$$

(We have and will continue to suppress the h and h_i parameters in the notation.) For convenience, we will assume that $\mathcal{M}$ contains the constant functions. We will also need the space

$$\mathcal{N}_i^* = \mathcal{N}_i + \text{span}\{1, \Lambda_{i,1}, \ldots, \Lambda_{i,d}\}.$$

For any positive integer N, let $\Delta t = T/N$. For any function u, we will use the following notation:

$$t_n = n\Delta t,$$

$$u^n = u(t_n),$$

$$u^{n-1/2} = \frac{u^n + u^{n-1}}{2},$$

and

$$\partial u^n = \frac{u^n - u^{n-1}}{\Delta t},$$

where $n = 0, 1, \ldots, N$ (except that $n \geq 1$ in the last two expressions).

It will be useful to define the following elliptic projections. For $u \in H^1(\Omega)$, let $\tilde{u}$ denote the unique function in $\mathcal{M}$ for which

$$(\kappa \underline{\nabla}(u - \tilde{u}), \underline{\nabla}\varphi) = 0, \qquad \text{for all } \varphi \in \mathcal{M}, \tag{4.3a}$$

and

$$\int_\Omega (u - \tilde{u})\, dx = 0. \tag{4.3b}$$

For $v \epsilon H^1_\Lambda(\Omega_i)$, let $\hat{v}$ denote the unique function in $\mathbf{N}_i^*$ for which

$$(\kappa_i \underline{\nabla}(v-\hat{v}), \underline{\nabla}\psi)_i = 0, \qquad \text{for all } \psi \epsilon \mathbf{N}_i, \tag{4.4a}$$

and

$$\hat{v}(\underline{x}) = v(\underline{x}), \qquad \underline{x} \epsilon \partial\Omega_i. \tag{4.4b}$$

If $v \epsilon H^1(\Omega_m)$ is such that $v \epsilon H^1_\Lambda(\Omega_i)$, for all i ($\sigma$ is such a function), then $\hat{v}$ is defined on all of Ω_m, and it is doubly valued on the boundaries of the matrix blocks. Since we will always consider $\hat{v}$ only on an Ω_i, its boundary values can always be considered to be those of its trace from inside Ω_i, and, hence, no confusion should arise.

We shall now describe the finite element method. It will be a straightforward modification of the method in [1]. At each time t_n, we will approximate ρ^n by $U^n \epsilon \mathbf{M}$ and, on Ω_i, σ^n by $V^n \epsilon \mathbf{N}_i^*$. For convenience, start the method with

$$U^0 = \tilde{\rho}^0 \tag{4.5}$$

and

$$V^0 = \hat{\sigma}^0. \tag{4.6}$$

The equations for V^n, $n \geq 1$, will amount to the following:

$$\begin{aligned}(\phi_i \partial V^n, \psi)_i + (\kappa_i \underline{\nabla} V^{n-1/2}, \underline{\nabla}\psi)_i - (2\underline{\gamma}_i V^{n-1/2}, \underline{\nabla}\psi)_i \\ = -(\sigma_0 \underline{\gamma}_i, \underline{\nabla}\psi)_i, \qquad \text{for all } \psi \epsilon \mathbf{N}_i,\end{aligned} \tag{4.7a}$$

$$V^n = \frac{1}{|\Omega_i|}[(U^n, \chi_i) + (U^n, \Delta_i \chi_i)\cdot\Delta_i], \qquad \underline{x} \epsilon \partial\Omega_i. \tag{4.7b}$$

This calculation depends on U^n. In turn, the calculation for U^n will depend on V^n through the matrix source term. We can decouple the two calculations by splitting the boundary condition (4.7b) as

$$V^n = \frac{1}{|\Omega_i|}[(U^{n-1}, \chi_i) + (U^{n-1}, \Delta_i \chi_i)\cdot\Delta_i] + \frac{1}{|\Omega_i|}[(\partial U^n, \chi_i) + (\partial U^n, \Delta_i \chi_i)\cdot\Delta_i]\Delta t, \qquad \underline{x} \epsilon \partial\Omega_i. \tag{4.8}$$

Then, for $n = 1, \ldots, N$,

$$V^n = W^n + \frac{1}{|\Omega_i|}[(\partial U^n, \chi_i)Z_0 + (\partial U^n, \Delta_i \chi_i)\cdot\underline{Z}]\Delta t, \tag{4.9}$$

where $W^n \epsilon \mathbf{N}_i^*$ satisfies

$$\left[\phi_i \frac{W^n - V^{n-1}}{\Delta t}, \psi\right]_i + \left[\kappa_i \underline{\nabla} \frac{W^n + V^{n-1}}{2}, \underline{\nabla}\psi\right]_i - \left[2\underline{\gamma}_i \frac{W^n + V^{n-1}}{2}, \underline{\nabla}\psi\right]_i$$
$$= -(\sigma_0 \underline{\gamma}_i, \underline{\nabla}\psi)_i, \qquad \text{for all } \psi \in \mathbf{N}_i, \tag{4.10a}$$

$$W^n = \frac{1}{|\Omega_i|}[(U^{n-1}, \chi_i) + (U^{n-1}, \Delta_i \chi_i)\cdot\Delta_i], \qquad \underline{x} \in \partial\Omega_i, \tag{4.10b}$$

and where $\underline{Z}=(Z_1,\dots,Z_d)$ and the $Z_j \in \mathbf{N}_i^*$, $j=0,1,\dots,d$, satisfy

$$\left[\phi_i \frac{Z_j}{\Delta t}, \psi\right]_i + \left[\kappa_i \underline{\nabla} \frac{Z_j}{2}, \underline{\nabla}\psi\right]_i - \left[2\underline{\gamma}_i \frac{Z_j}{2}, \underline{\nabla}\psi\right]_i$$
$$= 0, \qquad \text{for all } \psi \in \mathbf{N}_i, \tag{4.11a}$$

$$Z_0 = 1, \qquad \underline{x} \in \partial\Omega_i, \tag{4.11bi}$$

$$Z_j = \Lambda_{i,j}, \qquad \underline{x} \in \partial\Omega_i,\ j=1,\dots,d. \tag{4.11bii}$$

Over the time interval $(t_{n-1}, t_n]$, W^n accounts for the flow in the block arising from the fluid present there at time t_{n-1} (which is V^{n-1}) with no change in the boundary condition, while each Z_j accounts for the flow arising in an empty block experiencing some unit change over its boundary. The combination (4.9) is precisely the solution to (4.7).

Let

$$Q_{i,j} = \frac{1}{|\Omega_i|}\left\{\left[\phi_i \frac{Z_j}{\Delta t}, 1\right]_i + \left[\left[\phi_i \frac{Z_j}{\Delta t}, \Delta_i\right]_i + \left[\kappa_i \underline{\nabla} \frac{Z_j}{2} - Z_j \underline{\gamma}_i, \underline{\nabla}\Delta_i\right]_i\right]\cdot\Delta_i\right\}\chi_i,$$
$$j = 0,1,\dots,d, \tag{4.12}$$

and $\underline{Q}_i=(Q_{i,1},\dots,Q_{i,d})$. The matrix source term should be approximated as follows:

$$f_i^{n-1/2} \approx -\frac{1}{|\Omega_i|}\Big\{ (\phi_i \partial V^n, 1)_i + [(\phi_i \partial V^n, \Delta_i)_i$$
$$+ (\kappa_i \underline{\nabla} V^{n-1/2} - (2V^{n-1/2} - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\Delta_i)_i]\cdot\Delta_i\Big\}\chi_i$$

$$= -\frac{1}{|\Omega_i|}\Big\{\left[\phi_i \frac{W^n - V^{n-1}}{\Delta t}, 1\right]_i + \Big[\left[\phi_i \frac{W^n - V^{n-1}}{\Delta t}, \underline{\Delta}_i\right]_i$$
$$+ \left[\kappa_i \underline{\nabla} \frac{W^n + V^{n-1}}{2} - (W^n + V^{n-1} - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\Delta_i\right]_i\Big]\cdot\Delta_i\Big\}\chi_i$$

$$-\frac{1}{|\Omega_i|}\Big\{(\partial U^n, \chi_i)Q_{i,0} + (\partial U^n, \Delta_i\chi_i)\cdot\underline{Q}_i\Big\}\Delta t. \tag{4.13}$$

The equations for U^n, $n \geq 1$, can now be expressed as

$$(\phi\partial U^n, \varphi) + \sum_i \frac{1}{|\Omega_i|}\Big\{(\partial U^n, \chi_i)(Q_{i,0}, \varphi) + (\partial U^n, \underline{\Delta}_i\chi_i)\cdot(\underline{Q}_i, \varphi)\Big\}\Delta t$$
$$+ (\kappa\underline{\nabla}U^{n-1/2}, \underline{\nabla}\varphi) - (2\underline{\gamma}U^{n-1/2}, \underline{\nabla}\varphi)$$
$$= -(\rho_0\underline{\gamma}, \underline{\nabla}\varphi) + (f_e^{n-1/2}, \varphi)$$
$$- \sum_i \frac{1}{|\Omega_i|}\Bigg\{\left[\left[\phi_i\frac{W^n - V^{n-1}}{\Delta t}, 1\right]_i(\chi_i, \varphi) + \left[\left[\phi_i\frac{W^n - V^{n-1}}{\Delta t}, \underline{\Delta}_i\right]_i\right.\right.$$
$$\left.\left. + \left[\kappa_i\underline{\nabla}\frac{W^n + V^{n-1}}{2} - (W^n + V^{n-1} - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\underline{\Delta}_i\right]_i\right]\cdot(\underline{\Delta}_i\chi_i, \varphi)\Bigg\},$$
$$\text{for all } \varphi \in \mathcal{M}. \qquad (4.14)$$

This has completed the method. In summary, (4.5) and (4.6) give the values of U^0 and V^0. A single factorization (for each i) can be used to obtain the Z_j from (4.11), and then the $Q_{i,0}$ and $\underline{Q}_i$ can be calculated from (4.12). Now, successively for $n=1,\ldots,N$, solve (4.10) for W^n, (4.14) for U^n, and then define V^n by (4.9).

Note that only the block problems (4.10), (4.11) that sit over the quadrature points of the fracture calculation (4.14) need be computed. The block problems are independent of each other, so they can be solved in parallel. Hence, as in [1], this is a field-scale method.

Before going on to an analysis of the convergence of the method, we should remark that the linear systems that arise in (4.3), (4.4), (4.10), (4.11), and (4.14) are not singular. This is known for the first four systems, provided only that Δt is not too large. Since uniqueness implies existence, for (4.14) it is sufficient to verify that

$$\left[\phi\frac{U}{\Delta t}, \varphi\right] + \sum_i \frac{1}{|\Omega_i|}\Big\{(U, \chi_i)(Q_{i,0}, \varphi) + (U, \underline{\Delta}_i\chi_i)\cdot(\underline{Q}_i, \varphi)\Big\}$$
$$+ \left[\kappa\underline{\nabla}\frac{U}{2}, \underline{\nabla}\varphi\right] - (\underline{\gamma}U, \underline{\nabla}\varphi)$$
$$= 0, \qquad \text{for all } \varphi \in \mathcal{M}, \qquad (4.15)$$

has $U=0$ as its only solution in $\mathcal{M}$. If we set $\varphi=U$ in (4.15), then for Δt not too large, the first and last two terms on the left side above taken together are positive-definite. For similar reasons, the other term is at least positive-semidefinite. (This reflects the stabilizing effect of the matrix which is more fully explored in [1] for its simpler model.) We can easily see this fact by writing the term out via (4.12). With $\Lambda_{i,0}=1$,

$$\frac{1}{|\Omega_i|}\Big\{(U,\chi_i)(Q_{i,0},U)+(U,\Delta_i\chi_i)\cdot(\underline{Q}_i,U)\Big\}$$

$$=\frac{1}{|\Omega_i|^2}\sum_{j=0}^{d}(U,\Lambda_{i,j}\chi_i)\Big\{\sum_{k=0}^{d}\Big[\Big[\phi_i\frac{Z_j}{\Delta t},\Lambda_{i,k}\Big]_i+\Big[\kappa_i\underline{\nabla}\frac{Z_j}{2}-Z_j\underline{\gamma}_i,\underline{\nabla}\Lambda_{i,k}\Big]_i\Big]$$

$$\times(\Lambda_{i,k}\chi_i,U)\Big\}$$

$$=\frac{1}{|\Omega_i|^2}\sum_{j=0}^{d}\sum_{k=0}^{d}(U,\Lambda_{i,j}\chi_i)\Big[\Big[\phi_i\frac{Z_j}{\Delta t},Z_k\Big]_i+\Big[\kappa_i\underline{\nabla}\frac{Z_j}{2}-Z_j\underline{\gamma}_i,\underline{\nabla}Z_k\Big]_i\Big](U,\Lambda_{i,k}\chi_i), \tag{4.16}$$

by (4.11). Now, the expression in square brackets on the far right side of (4.16) is a $(d+1)\times(d+1)$-tensor which is the sum of $1+2d$ terms. $1+d$ of these terms are outer products of $(d+1)$-vectors (and so are positive-semidefinite), while the other d terms are dominated by these, so long as Δt is not too large. Hence, (4.16) is nonnegative, and $U=0$ is the only solution to (4.15).

5. Convergence of the Method

Following [10], for $n=0,1,\ldots,N$, consider each approximation error as the sum of two pieces: $\rho^n-U^n=(\rho^n-\tilde{\rho}^n)+(\tilde{\rho}^n-U^n)$ and $\sigma^n-V^n=(\sigma^n-\hat{\sigma}^n)+(\hat{\sigma}^n-V^n)$. It is known that

$$\left|\frac{\partial^k(\rho-\tilde{\rho})}{\partial t^k}\right|_{H^\ell(\Omega)}\le\left|\frac{\partial^k\rho}{\partial t^k}\right|_{H^r(\Omega)}h^{r-\ell},\qquad k=0,1\ ;\ \ell=0 \text{ and } \ell=-1 \text{ if } R\ge 3\ ;\ 1\le r\le R, \tag{5.1a}$$

$$\|\underline{\nabla}(\rho-\tilde{\rho})\|_{L^2(\Omega)}\le\|\rho\|_{H^r(\Omega)}h^{r-1},\qquad 1\le r\le R, \tag{5.1b}$$

$$\left|\frac{\partial^k(\sigma-\hat{\sigma})}{\partial t^k}\right|_{L^2(\Omega_m)}\le\sum_i\left|\frac{\partial^k\sigma}{\partial t^k}\right|_{H^{s_i}(\Omega_i)}h_i^{s_i},\qquad k=0,1\ ;\ 1\le s_i\le S_i \text{ (all } i), \tag{5.2a}$$

and

$$\|\underline{\nabla}(\sigma-\hat{\sigma})\|_{L^2(\Omega_m)}\le\sum_i\|\sigma\|_{H^{s_i}(\Omega_i)}h_i^{s_i-1},\qquad 1\le s_i\le S_i \text{ (all } i). \tag{5.2b}$$

At first glance, it may appear that the linear functions $\overline{\sigma}_i$ where $\sigma-\overline{\sigma}_i\in H^1{}_0(\Omega_i)$ must be included in the bounding norms of (5.2). However, $\|\underline{\nabla}\overline{\sigma}_i\|^2{}_{L^2(\Omega_i)}\le\|\underline{\nabla}\sigma\|^2{}_{L^2(\Omega_i)}$, so Poincaré's inequality shows that this is not necessary.

It remains to estimate the errors

$$\zeta^n=\tilde{\rho}^n-U^n\in\mathcal{M}$$

and

$$\xi^n=\hat{\sigma}^n-V^n\in\mathcal{N}_i^*\ \text{(all } i),$$

which satisfy an equation that is similar to (3.5). We will derive it in stages below.

For $n=1,\ldots,N$, let us first combine the weak form of the average of the equations for ρ at times t_n and t_{n-1} ((2.8), (2.13), and (2.14)) with the defining relation for $\tilde{\rho}$ (4.3a). After some manipulation, this equation is

$$
\begin{aligned}
&(\phi\partial\tilde{\rho}^n, \varphi) + (\kappa\underline{\nabla}\tilde{\rho}^{n-1/2}, \underline{\nabla}\varphi) \\
&\quad = ((2\tilde{\rho}^{n-1/2} - \rho_0)\underline{\gamma}, \underline{\nabla}\varphi) + (f_e^{n-1/2}, \varphi) \\
&\qquad - \sum_i \frac{1}{|\Omega_i|}\Big[\Big\{(\phi_i\partial\hat{\sigma}^n, 1)_i + [(\phi_i\partial\hat{\sigma}^n, \underline{\Delta}_i)_i \\
&\qquad\qquad + (\kappa_i\underline{\nabla}\hat{\sigma}^{n-1/2} - (2\hat{\sigma}^{n-1/2} - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\underline{\Delta}_i)_i]\cdot\underline{\Delta}_i\Big\}\chi_i, \varphi\Big] \\
&\qquad - (\phi(\rho_t^{n-1/2} - \partial\tilde{\rho}^n), \varphi) + (2(\rho^{n-1/2} - \tilde{\rho}^{n-1/2})\underline{\gamma}, \underline{\nabla}\varphi) \\
&\qquad - \sum_i \frac{1}{|\Omega_i|}\Big[\Big\{(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), 1)_i + [(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), \underline{\Delta}_i)_i \\
&\qquad\qquad - (\sigma^{n-1/2} - \hat{\sigma}^{n-1/2}, \underline{\nabla}\cdot\kappa_i\underline{\nabla}\underline{\Delta}_i)_i \\
&\qquad\qquad - (2(\sigma^{n-1/2} - \hat{\sigma}^{n-1/2})\underline{\gamma}_i, \underline{\nabla}\underline{\Delta}_i)_i]\cdot\underline{\Delta}_i\Big\}\chi_i, \varphi\Big], \\
&\qquad\qquad\qquad \text{for all } \varphi\in\mathcal{M}, \qquad (5.3)
\end{aligned}
$$

where (4.4b) has been used in integrating $(\kappa_i\underline{\nabla}(\sigma^{n-1/2}-\hat{\sigma}^{n-1/2}), \underline{\nabla}\underline{\Delta}_i)_i$ by parts.

Next, we can find a weak form for the average of the equation for σ (2.4) at times t_n and t_{n-1}. We will take a test function $\psi\in\mathcal{N}_i^*$ for which $\overline{\psi}_i$ is the linear function associated to it (i.e., integrate against $\psi-\overline{\psi}_i\in\mathcal{N}_i\subset H^1{}_0(\Omega_i)$). Combined with the defining relation (4.4a), after some manipulation, we see that

$$
\begin{aligned}
&(\phi_i\partial\hat{\sigma}^n, \psi)_i + (\kappa_i\underline{\nabla}\hat{\sigma}^{n-1/2}, \underline{\nabla}\psi)_i \\
&\quad = (2(\hat{\sigma}^{n-1/2} - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\psi)_i \\
&\qquad + \frac{1}{|\Omega_i|}\Big[\Big\{(\phi_i\partial\hat{\sigma}^n, 1)_i + [(\phi_i\partial\hat{\sigma}^n, \underline{\Delta}_i)_i \\
&\qquad\qquad + (\kappa_i\underline{\nabla}\hat{\sigma}^{n-1/2} - (2\hat{\sigma}^{n-1/2} - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\underline{\Delta}_i)_i]\cdot\underline{\Delta}_i\Big\}\chi_i, \overline{\psi}_i\Big] \\
&\qquad - (\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), \psi)_i + (2(\sigma^{n-1/2} - \hat{\sigma}^{n-1/2})\underline{\gamma}_i, \underline{\nabla}\psi)_i \\
&\qquad + \frac{1}{|\Omega_i|}\Big[\Big\{(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), 1)_i + [(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), \underline{\Delta}_i)_i \\
&\qquad\qquad - (\sigma^{n-1/2} - \hat{\sigma}^{n-1/2}, \underline{\nabla}\cdot\kappa_i\underline{\nabla}\underline{\Delta}_i)_i \\
&\qquad\qquad - (2(\sigma^{n-1/2} - \hat{\sigma}^{n-1/2})\underline{\gamma}_i, \underline{\nabla}\underline{\Delta}_i)_i]\cdot\underline{\Delta}_i\Big\}\chi_i, \overline{\psi}_i\Big], \\
&\qquad\qquad\qquad \text{for all } \psi\in\mathcal{N}_i^*, \qquad (5.4)
\end{aligned}
$$

where we recall that

$$\overline{\psi}_i = \frac{1}{|\Omega_i|}[(\overline{\psi}_i, \chi_i) + (\overline{\psi}_i, \Delta_i\chi_i)\cdot\Delta_i]. \tag{5.5}$$

We can write the defining equation for V^n (4.7a) in a similar manner:

$$\begin{aligned}
&(\phi_i\partial V^n, \psi)_i + (\kappa_i\underline{\nabla}V^{n-1/2}, \underline{\nabla}\psi)_i \\
&\quad = ((2V^{n-1/2} - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\psi)_i \\
&\qquad + \frac{1}{|\Omega_i|}\Big[\Big\{(\phi_i\partial V^n, 1)_i + [(\phi_i\partial V^n, \underline{\Delta}_i)_i \\
&\qquad\qquad + (\kappa_i\underline{\nabla}V^{n-1/2} - (2V^{n-1/2} - \sigma_0)\underline{\gamma}_i, \underline{\nabla}\Delta_i)_i]\cdot\Delta_i\Big\}\chi_i, \overline{\psi}_i\Big], \\
&\qquad\qquad \text{for all } \psi\in\mathcal{N}_i^*.
\end{aligned} \tag{5.6}$$

We are now ready to derive the equation for ζ^n and ξ^n, $n=1,\ldots,N$. Add the difference of (5.3) and (4.14) (with (4.13)) to the sum on i of the difference of (5.4) and (5.6). The result is

$$\begin{aligned}
&(\phi\partial\zeta^n, \varphi) + \sum_i(\phi_i\partial\xi^n, \psi)_i + (\kappa\underline{\nabla}\zeta^{n-1/2}, \underline{\nabla}\varphi) + \sum_i(\kappa_i\underline{\nabla}\xi^{n-1/2}, \underline{\nabla}\psi)_i \\
&\quad = (2\zeta^{n-1/2}\underline{\gamma}, \underline{\nabla}\varphi) + \sum_i(2\xi^{n-1/2}\underline{\gamma}_i, \underline{\nabla}\psi)_i \\
&\qquad - \sum_i\frac{1}{|\Omega_i|}\Big[\Big\{(\phi_i\partial\xi^n, 1)_i + [(\phi_i\partial\xi^n, \underline{\Delta}_i)_i \\
&\qquad\qquad + (\kappa_i\underline{\nabla}\xi^{n-1/2} - 2\xi^{n-1/2}\underline{\gamma}_i, \underline{\nabla}\Delta_i)_i]\cdot\Delta_i\Big\}\chi_i, \varphi - \overline{\psi}_i\Big] \\
&\qquad - (\phi(\rho_t^{n-1/2} - \partial\tilde{\rho}^n), \varphi) - \sum_i(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), \psi)_i \\
&\qquad + (2(\rho^{n-1/2} - \tilde{\rho}^{n-1/2})\underline{\gamma}, \underline{\nabla}\varphi) + \sum_i(2(\sigma^{n-1/2} - \hat{\sigma}^{n-1/2})\underline{\gamma}_i, \underline{\nabla}\psi)_i \\
&\qquad - \sum_i\frac{1}{|\Omega_i|}\Big[\Big\{(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), 1)_i + [(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), \underline{\Delta}_i)_i \\
&\qquad\qquad - (\sigma^{n-1/2} - \hat{\sigma}^{n-1/2}, \underline{\nabla}\cdot\kappa_i\underline{\nabla}\Delta_i)_i \\
&\qquad\qquad - (2(\sigma^{n-1/2} - \hat{\sigma}^{n-1/2})\underline{\gamma}_i, \underline{\nabla}\Delta_i)_i]\cdot\Delta_i\Big\}\chi_i, \varphi - \overline{\psi}_i\Big], \\
&\qquad\qquad \text{for all } \varphi\in\mathcal{M} \text{ and } \psi\in\mathcal{N}_i^* \text{ (all i)}, n=1,\ldots,N.
\end{aligned} \tag{5.7}$$

(Remember that $\overline{\psi}_i$ is the linear function associated to $\psi|_{\Omega_i}$.)

It will be convenient to write down the boundary and initial conditions. In light of (4.4b), the difference of (2.6) and (4.7b) yields the boundary condition

$$\xi^n = \sigma^n - V^n$$
$$= \frac{1}{|\Omega_i|}[(\zeta^n, \chi_i) + (\zeta^n, \Delta_i\chi_i)\cdot\Delta_i] + \frac{1}{|\Omega_i|}[(\rho^n - \tilde{\rho}^n, \chi_i) + (\rho^n - \tilde{\rho}^n, \Delta_i\chi_i)\cdot\Delta_i],$$
$$\underline{x}\in\partial\Omega_i,\ n=1,\dots,N. \quad (5.8)$$

The initial conditions are

$$\zeta^0 = 0, \qquad \underline{x}\in\Omega, \quad (5.9)$$

and

$$\xi^0 = 0, \qquad \underline{x}\in\Omega_m, \quad (5.10)$$

by (4.5) and (4.6).

We are now ready to derive our error estimates. First, take $\varphi=\zeta^{n-1/2}$ and $\psi=\xi^{n-1/2}$ in (5.7). In this case,

$$\overline{\psi}_i = \frac{1}{|\Omega_i|}[(\zeta^{n-1/2}, \chi_i) + (\zeta^{n-1/2}, \Delta_i\chi_i)\cdot\Delta_i]$$
$$+ \frac{1}{|\Omega_i|}[(\rho^{n-1/2} - \tilde{\rho}^{n-1/2}, \chi_i) + (\rho^{n-1/2} - \tilde{\rho}^{n-1/2}, \Delta_i\chi_i)\cdot\Delta_i]$$
$$- \frac{\delta_{n,1}}{2|\Omega_i|}[(\rho^0 - \tilde{\rho}^0, \chi_i) + (\rho^0 - \tilde{\rho}^0, \Delta_i\chi_i)\cdot\Delta_i], \quad (5.11)$$

where $\delta_{n,\ell}$ is the Kronecker delta symbol. The terms in (5.7) containing the expression $\varphi-\overline{\psi}_i$ do not vanish; however, the effect of $\varphi=\zeta^{n-1/2}$ cancels with the effect of the first term on the right side of (5.11). Hence, (5.7) becomes

$$\frac{1}{2\Delta t}[(\phi\zeta^n, \zeta^n) - (\phi\zeta^{n-1}, \zeta^{n-1})] + \frac{1}{2\Delta t}\sum_i[(\phi_i\xi^n, \xi^n)_i - (\phi_i\xi^{n-1}, \xi^{n-1})_i]$$
$$+ (\kappa\underline{\nabla}\zeta^{n-1/2}, \underline{\nabla}\zeta^{n-1/2}) + \sum_i(\kappa_i\underline{\nabla}\xi^{n-1/2}, \underline{\nabla}\xi^{n-1/2})_i$$
$$= (2\zeta^{n-1/2}\underline{\gamma}, \underline{\nabla}\zeta^{n-1/2}) + \sum_i(2\xi^{n-1/2}\underline{\gamma}_i, \underline{\nabla}\xi^{n-1/2})_i$$
$$+ \sum_i\frac{1}{|\Omega_i|}\Big[\Big\{(\phi_i\partial\xi^n, 1)_i + [(\phi_i\partial\xi^n, \Delta_i)_i$$
$$+ (\kappa_i\underline{\nabla}\xi^{n-1/2} - 2\xi^{n-1/2}\underline{\gamma}_i, \underline{\nabla}\Delta_i)_i]\cdot\Delta_i\Big\}\chi_i,$$
$$\frac{1}{|\Omega_i|}[(\rho^{n-1/2} - \tilde{\rho}^{n-1/2}, \chi_i) + (\rho^{n-1/2} - \tilde{\rho}^{n-1/2}, \Delta_i\chi_i)\cdot\Delta_i]$$
$$- \frac{\delta_{n,1}}{2|\Omega_i|}[(\rho^0 - \tilde{\rho}^0, \chi_i) + (\rho^0 - \tilde{\rho}^0, \Delta_i\chi_i)\cdot\Delta_i]\Big]$$
$$- (\phi(\rho_t^{n-1/2} - \partial\tilde{\rho}^n), \zeta^{n-1/2}) - \sum_i(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), \xi^{n-1/2})_i$$

$$+ \left(2(\rho^{n-1/2} - \tilde{\rho}^{n-1/2})\underline{\gamma}, \underline{\nabla}\zeta^{n-1/2}\right) + \sum_i \left(2(\sigma^{n-1/2} - \hat{\sigma}^{n-1/2})\underline{\gamma}_i, \underline{\nabla}\xi^{n-1/2}\right)_i$$

$$+ \sum_i \frac{1}{|\Omega_i|}\Bigg[\Big\{\left(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), 1\right)_i + \left[\left(\phi_i(\sigma_t^{n-1/2} - \partial\hat{\sigma}^n), \underline{\Delta}_i\right)_i$$

$$- \left(\sigma^{n-1/2} - \hat{\sigma}^{n-1/2}, \underline{\nabla}\cdot\kappa_i\underline{\nabla}\underline{\Delta}_i\right)_i$$

$$- \left(2(\sigma^{n-1/2} - \hat{\sigma}^{n-1/2})\underline{\gamma}_i, \underline{\nabla}\underline{\Delta}_i\right)_i\right]\cdot\underline{\Delta}_i\Big\}\chi_i,$$

$$\frac{1}{|\Omega_i|}\left[\left(\rho^{n-1/2} - \tilde{\rho}^{n-1/2}, \chi_i\right) + \left(\rho^{n-1/2} - \tilde{\rho}^{n-1/2}, \underline{\Delta}_i\chi_i\right)\cdot\underline{\Delta}_i\right]$$

$$- \frac{\delta_{n,1}}{2|\Omega_i|}\left[\left(\rho^0 - \tilde{\rho}^0, \chi_i\right) + \left(\rho^0 - \tilde{\rho}^0, \underline{\Delta}_i\chi_i\right)\cdot\underline{\Delta}_i\right]\Bigg]$$

$$\le C\Big\{\|\rho_t^{n-1/2} - \partial\tilde{\rho}^n\|^2_{H^{-1}(\Omega)} + \|\sigma_t^{n-1/2} - \partial\hat{\sigma}^n\|^2_{H^{-1}(\Omega_m)}$$

$$+ \delta_{n,1}[1 + (\Delta t)^{-1}]\|\rho^0 - \tilde{\rho}^0\|^2_{L^2(\Omega)}$$

$$+ \|\rho^{n-1/2} - \tilde{\rho}^{n-1/2}\|^2_{L^2(\Omega)} + \|\sigma^{n-1/2} - \hat{\sigma}^{n-1/2}\|^2_{L^2(\Omega_m)}$$

$$+ \left(\phi\zeta^{n-1/2}, \zeta^{n-1/2}\right) + \sum_i \left(\phi_i\xi^{n-1/2}, \xi^{n-1/2}\right)_i\Big\}$$

$$+ {}^1/_2\Big\{\left(\kappa\underline{\nabla}\zeta^{n-1/2}, \underline{\nabla}\zeta^{n-1/2}\right) + \sum_i \left(\kappa_i\underline{\nabla}\xi^{n-1/2}, \underline{\nabla}\xi^{n-1/2}\right)_i\Big\}$$

$$+ \frac{\delta_{n,1}}{8\Delta t}\sum_i \left(\phi_i\xi^1, \xi^1\right)_i$$

$$+ \sum_i \frac{1}{|\Omega_i|^2}\Bigg[\Big\{\left(\phi_i\partial\xi^n, 1\right)_i + \left(\phi_i\partial\xi^n, \underline{\Delta}_i\right)_i\cdot\underline{\Delta}_i\Big\}\chi_i,$$

$$\left(\rho^{n-1/2} - \tilde{\rho}^{n-1/2}, \chi_i\right) + \left(\rho^{n-1/2} - \tilde{\rho}^{n-1/2}, \underline{\Delta}_i\chi_i\right)\cdot\underline{\Delta}_i\Bigg]. \quad (5.12)$$

Now, sum on n from 1 to m. After some manipulation, we see that

$$\left(\phi\zeta^m, \zeta^m\right) + \sum_i \left(\phi_i\xi^m, \xi^m\right)_i$$

$$+ \sum_{n=1}^m \left(\kappa\underline{\nabla}\zeta^{n-1/2}, \underline{\nabla}\zeta^{n-1/2}\right)\Delta t + \sum_{n=1}^m \sum_i \left(\kappa_i\underline{\nabla}\xi^{n-1/2}, \underline{\nabla}\xi^{n-1/2}\right)_i \Delta t$$

$$\le C\Big\{\sum_{n=1}^m \|\rho_t^{n-1/2} - \partial\tilde{\rho}^n\|^2_{H^{-1}(\Omega)}\Delta t + \sum_{n=1}^m \|\sigma_t^{n-1/2} - \partial\hat{\sigma}^n\|^2_{H^{-1}(\Omega_m)}\Delta t$$

$$+ \|\rho - \tilde{\rho}\|^2_{L^\infty(J;L^2(\Omega))} + \|\sigma - \hat{\sigma}\|^2_{L^\infty(J;L^2(\Omega_m))}$$

$$+ \sum_{n=1}^m \left(\phi\zeta^{n-1/2}, \zeta^{n-1/2}\right)\Delta t + \sum_{n=1}^m \sum_i \left(\phi_i\xi^{n-1/2}, \xi^{n-1/2}\right)_i \Delta t\Big\}$$

$$+ {}^1/_4\sum_i \left(\phi_i\xi^1, \xi^1\right)_i$$

$$+ 2\sum_{n=1}^{m}\sum_{i}\frac{1}{|\Omega_i|^2}\Big[\{(\phi_i\partial\xi^n,1)_i + (\phi_i\partial\xi^n,\Delta_i)_i\cdot\Delta_i\}\chi_i,$$

$$(\rho^{n-1/2}-\tilde{\rho}^{n-1/2},\chi_i) + (\rho^{n-1/2}-\tilde{\rho}^{n-1/2},\Delta_i\chi_i)\cdot\Delta_i\Big]\Delta t. \tag{5.13}$$

The last term above can be summed by parts. The result is

$$2\sum_{i}\frac{1}{|\Omega_i|^2}\big[\{(\phi_i\xi^m,1)_i+(\phi_i\xi^m,\Delta_i)_i\cdot\Delta_i\}\chi_i,\ (\rho^m-\tilde{\rho}^m,\chi_i)+(\rho^m-\tilde{\rho}^m,\Delta_i\chi_i)\cdot\Delta_i\big]$$

$$- 2\sum_{n=1}^{m}\sum_{i}\frac{1}{|\Omega_i|^2}\Big[\{(\phi_i\xi^{n-1/2},1)_i + (\phi_i\xi^{n-1/2},\Delta_i)_i\cdot\Delta_i\}\chi_i,$$

$$(\partial\rho^n-\partial\tilde{\rho}^n,\chi_i) + (\partial\rho^n-\partial\tilde{\rho}^n,\Delta_i\chi_i)\cdot\Delta_i\Big]\Delta t$$

$$\le C\Big\{\|\rho-\tilde{\rho}\|^2_{L^\infty(J;L^2(\Omega))} + \sum_{n=1}^{m}\|\partial\rho^n-\partial\tilde{\rho}^n\|^2_{H^{-1}(\Omega)}\Delta t$$

$$+ \sum_{n=1}^{m}\sum_{i}(\phi_i\xi^{n-1/2},\xi^{n-1/2})_i\Delta t\Big\}$$

$$+ {}^1\!/_4\sum_{i}(\phi_i\xi^m,\xi^m)_i, \tag{5.14}$$

where $\chi_i\epsilon H^1(\Omega)$, for all i, has been used to obtain the optimal bound on $\partial\rho^n-\partial\tilde{\rho}^n$.

Finally, if Δt is not too large, the discrete Gronwall inequality can be applied to (5.13), (5.14) to yield the estimates

$$\max_{0\le n\le N}\|\zeta^n\|^2_{L^2(\Omega)} + \max_{0\le n\le N}\|\xi^n\|^2_{L^2(\Omega_m)}$$

$$+ \sum_{n=1}^{N}\|\underline{\nabla}\zeta^{n-1/2}\|^2_{L^2(\Omega)}\Delta t + \sum_{n=1}^{N}\|\underline{\nabla}\xi^{n-1/2}\|^2_{L^2(\Omega_m)}\Delta t$$

$$\le C\Big\{\sum_{n=1}^{N}[\|\rho_t^{n-1/2}-\partial\rho^n\|^2_{H^{-1}(\Omega)} + \|\partial\rho^n-\partial\tilde{\rho}^n\|^2_{H^{-1}(\Omega)}]\Delta t$$

$$+ \sum_{n=1}^{N}\|\sigma_t^{n-1/2}-\partial\hat{\sigma}^n\|^2_{H^{-1}(\Omega_m)}\Delta t$$

$$+ \|\rho-\tilde{\rho}\|^2_{L^\infty(J;L^2(\Omega))} + \|\sigma-\hat{\sigma}\|^2_{L^\infty(J;L^2(\Omega_m))}\Big\}$$

$$\le C\Big\{\|\rho-\tilde{\rho}\|^2_{L^\infty(J;L^2(\Omega))} + \|\rho_t-\tilde{\rho}_t\|^2_{L^2(J;H^{-1}(\Omega))}$$

$$+ \|\sigma-\hat{\sigma}\|^2_{L^\infty(J;L^2(\Omega_m))} + \|\sigma_t-\hat{\sigma}_t\|^2_{L^2(J;H^{-1}(\Omega_m))}$$

$$+ \Big[\Big\|\frac{\partial^3\rho}{\partial t^3}\Big\|^2_{L^2(J;H^{-1}(\Omega))} + \Big\|\frac{\partial^3\sigma}{\partial t^3}\Big\|^2_{L^2(J;H^{-1}(\Omega_m))}\Big](\Delta t)^4\Big\}. \tag{5.15}$$

It is also valuable to take the test functions $\varphi=\partial\zeta^n$ and $\psi=\partial\xi^n$ in (5.7). Then

$$\overline{\psi}_i = \frac{1}{|\Omega_i|}[(\partial\zeta^n, \chi_i) + (\partial\zeta^n, \Delta_i\chi_i)\cdot\Delta_i]$$

$$+ \frac{1}{|\Omega_i|}[(\partial\rho^n - \partial\tilde{\rho}^n, \chi_i) + (\partial\rho^n - \partial\tilde{\rho}^n, \Delta_i\chi_i)\cdot\Delta_i]$$

$$+ \frac{\delta_{n,1}}{\Delta t|\Omega_i|}[(\rho^0 - \tilde{\rho}^0, \chi_i) + (\rho^0 - \tilde{\rho}^0, \Delta_i\chi_i)\cdot\Delta_i]. \tag{5.16}$$

The terms in (5.7) containing $\varphi-\overline{\psi}_i$ estimate directly , since we now have (5.15). It is known how to treat all the other terms; in particular, the four gravitational terms containing the expressions $\underline{\nabla}\partial\zeta^n$ and $\underline{\nabla}\partial\xi^n$ can be treated by a summation by parts argument analogous to the one given above. The final result is

$$\sum_{n=1}^{N}\|\partial\zeta^n\|^2_{L^2(\Omega)}\Delta t + \sum_{n=1}^{N}\|\partial\xi^n\|^2_{L^2(\Omega_m)}\Delta t$$

$$+ \max_{0\le n\le N}\|\underline{\nabla}\zeta^n\|^2_{L^2(\Omega)} + \max_{0\le n\le N}\|\underline{\nabla}\xi^n\|^2_{L^2(\Omega_m)}$$

$$\le C\Big\{\sum_{n=1}^{N}[\|\rho_t^{n-1/2} - \partial\rho^n\|^2_{L^2(\Omega)} + \|\partial\rho^n - \partial\tilde{\rho}^n\|^2_{L^2(\Omega)}]\Delta t$$

$$+ \sum_{n=1}^{N}[\|\sigma_t^{n-1/2} - \partial\sigma^n\|^2_{L^2(\Omega_m)} + \|\partial\sigma^n - \partial\hat{\sigma}^n\|^2_{L^2(\Omega_m)}]\Delta t$$

$$+ \|\rho - \tilde{\rho}\|^2_{L^\infty(J;L^2(\Omega))} + \|\sigma - \hat{\sigma}\|^2_{L^\infty(J;L^2(\Omega_m))}\Big\}$$

$$\le C\Big\{\|\rho - \tilde{\rho}\|^2_{L^\infty(J;L^2(\Omega))} + \|\rho_t - \tilde{\rho}_t\|^2_{L^2(J;L^2(\Omega))}$$

$$+ \|\sigma - \hat{\sigma}\|^2_{L^\infty(J;L^2(\Omega_m))} + \|\sigma_t - \hat{\sigma}_t\|^2_{L^2(J;L^2(\Omega_m))}$$

$$+ \Big[\Big\|\frac{\partial^3\rho}{\partial t^3}\Big\|^2_{L^2(J;L^2(\Omega))} + \Big\|\frac{\partial^3\sigma}{\partial t^3}\Big\|^2_{L^2(J;L^2(\Omega_m))}\Big](\Delta t)^4\Big\}. \tag{5.17}$$

The following theorem is a combination of the estimates (5.1), (5.2), (5.15), and (5.17).

<u>Theorem:</u>

If the data and solution of the double porosity model are sufficiently smooth, and if Δt is not too large, then the solution of the finite element method approximates the solution of the model as follows:

$$\max_{0\le n\le N} \|\rho^n - U^n\|_{L^2(\Omega)} + \max_{0\le n\le N} \|\sigma^n - V^n\|_{L^2(\Omega_m)}$$

$$\le C\Big\{\big[\|\rho\|_{L^\infty(J;H^r(\Omega))} + \|\rho_t\|_{L^2(J;\overline{H}^{r-k}(\Omega))}\big]h^r$$

$$+ \sum_i \big[\|\sigma\|_{L^\infty(J;H^{s_i}(\Omega_i))} + \|\sigma_t\|_{L^2(J;H^{s_i}(\Omega_i))}\big]h_i^{s_i}$$

$$+ \Big[\Big\|\frac{\partial^3\rho}{\partial t^3}\Big\|_{L^2(J;H^{-1}(\Omega))} + \Big\|\frac{\partial^3\sigma}{\partial t^3}\Big\|_{L^2(J;H^{-1}(\Omega_m))}\Big](\Delta t)^2\Big\}, \tag{5.18}$$

$$\max_{0\le n\le N} \|\rho^n - U^n\|_{H^1(\Omega)} + \max_{0\le n\le N} \|\sigma^n - V^n\|_{H^1(\Omega_m)}$$

$$\le C\Big\{\big[\|\rho\|_{L^\infty(J;H^r(\Omega))} + \|\rho_t\|_{L^2(J;\overline{H}^{r-1}(\Omega))}\big]h^{r-1}$$

$$+ \sum_i \big[\|\sigma\|_{L^\infty(J;H^{s_i}(\Omega_i))} + \|\sigma_t\|_{L^2(J;\overline{H}^{s_i-1}(\Omega_i))}\big]h_i^{s_i-1}$$

$$+ \Big[\Big\|\frac{\partial^3\rho}{\partial t^3}\Big\|_{L^2(J;L^2(\Omega))} + \Big\|\frac{\partial^3\sigma}{\partial t^3}\Big\|_{L^2(J;L^2(\Omega_m))}\Big](\Delta t)^2\Big\}, \tag{5.19}$$

$$\Big\{\sum_{n=1}^N \|\rho_t^{n-1/2} - \partial U^n\|^2_{L^2(\Omega)}\Delta t\Big\}^{1/2} + \Big\{\sum_{n=1}^N \|\sigma_t^{n-1/2} - \partial V^n\|^2_{L^2(\Omega_m)}\Delta t\Big\}^{1/2}$$

$$\le C\Big\{\big[\|\rho\|_{L^\infty(J;H^r(\Omega))} + \|\rho_t\|_{L^2(J;H^r(\Omega))}\big]h^r$$

$$+ \sum_i \big[\|\sigma\|_{L^\infty(J;H^{s_i}(\Omega_i))} + \|\sigma_t\|_{L^2(J;H^{s_i}(\Omega_i))}\big]h_i^{s_i}$$

$$+ \Big[\Big\|\frac{\partial^3\rho}{\partial t^3}\Big\|_{L^2(J;L^2(\Omega))} + \Big\|\frac{\partial^3\sigma}{\partial t^3}\Big\|_{L^2(J;L^2(\Omega_m))}\Big](\Delta t)^2\Big\}, \tag{5.20}$$

where $1\le r\le R$, $1\le s_i\le S_i$ (all i), $k=1$ (except $k=0$ if $R=2$), and $\overline{H}^{\ell} = \overline{H}^{\max(1,\ell)}$.

Note that the error estimates are optimal with respect to the discretization parameters. The regularity required of ρ and σ is also optimal, except in (5.18), where σ (and ρ if $R=2$) must be slightly smoother.

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