THE POISSON FUNCTION OF FINITE ELASTICITY

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THE POISSON FUNCTION OF FINITE ELASTICITY

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ABSTRACT

The Poisson function is introduced to study in a simple tension test the lateral contractive response of compressible and incompressible, isotropic elastic materials in finite strain. The relation of the Poisson function to the classical Poisson's ratio and its behavior for certain constrained materials are discussed. Some experimental results for several elastomers, including two natural rubber compounds of the same kind studied in earlier basic experiments by Rivlin and Saunders, are compared with the derived relations. A special class of compressible materials also is considered. It is proved that the only class of compressible hyperelastic materials whose response functions depend on only the third principal invariant of the deformation tensor is the class first introduced in experiments by Blatz and Ko. Poisson functions for the Blatz-Ko polyurethane elastomers are derived; and our experimental data are reviewed in relation to a volume constraint equation used in their experiments.

1. INTRODUCTION

Isotropic, linear elasticity theory is characterized by two important physical constants: Young's modulus and Poisson's ratio. It is well-known that their definitions are based upon the simple tension test[1]; and, for a specific homogeneous, isotropic and linearly elastic material, both may be found from this experiment [1, §69]. We recall that Poisson's ratio is determined from kinematical measurements alone; and when the material is known to be incompressible, it has the value 1/2.

In isotropic, nonlinear elasticity theory, the traditional material constants play a less important role, but their use in characterization of the mechanical properties of highly elastic materials certainly is of no lesser importance. However, in this case, the material response generally is not described by constants; rather, it is represented by three scalar-valued functions $\beta_\Gamma = \beta_\Gamma(I_1, I_2, I_3)$ of the three principal invariants I_k of the Cauchy-Green deformation tensor $\underset{\sim}{B}$ so that the principal Cauchy stress components t_k are determined by

$$t_k = \beta_0 + \beta_1\lambda_k^2 + \beta_{-1}\lambda_k^{-2}, \quad k = 1, 2, 3. \tag{1.1}$$

[See 3, §47.] Herein λ_k^2, the squared principal stretches, are the

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[1] This does not preclude the use of other testing methods for the determination of these basic moduli. However, the Poisson ratio is defined in terms of strains in a simple extension produced by simple tensile loading. Although the same thing may be done in a simple compression experiment, a compression test usually is avoided because of eccentric loading and stability problems. Of course, compression data sometimes may be obtained by other means [2, p.270].

principal values of $\underset{\sim}{B}$; and $\Gamma = -1, 0, 1$. For an incompressible material, every deformation must satisfy the constant volume constraint relation

$$I_3^{1/2} = \lambda_1\lambda_2\lambda_3 = 1; \tag{1.2}$$

and the constitutive relation (1.1) is replaced by

$$t_k = -p + \beta_1\lambda_k^2 + \beta_{-1}\lambda_k^{-2}, \quad k = 1, 2, 3, \tag{1.3}$$

in which p is an unknown hydrostatic stress, and the two response functions $\beta_\Gamma = \beta_\Gamma(I_1, I_2)$ depend on the invariants indicated. The determination of the response functions for particular materials is a principal problem in experimental mechanics. Of course, it is reasonable to expect that the response functions, or combinations of them, ought to be related in some limit sense to the classical moduli of the linearized theory; and it is natural to ask how the usual physical parameters may be characterized in the general theory. The connection of the response functions with the Lamé constants is made in [3,§50]. This note concerns the definition of the Poisson function and its relation to the classical modulus known as Poisson's ratio.

The Poisson function is defined in §2; and its connection with the simple tension experiment and its behavior under certain constraints are described there. Some experimental results are then presented for demonstration in §3. Experimental data for a urethane elastomer, a certain blend of natural and synthetic rubbers, and two natural rubber compounds of the same kind used in early experiments by Rivlin and Saunders [2], are compared with the universal Poisson function obtained for incompressible materials. Although every incompressible material has the universal constant, natural state limit value 1/2, the converse is shown generally to be false. A specific application to compressible materials is illustrated.

It is proved in §4 that the only class of compressible hyperelastic materials whose response functions depend on only the third principal invariant of the deformation tensor is the class first studied in experiments by Blatz and Ko [4]. Their constitutive equation for foamed, polyurethane rubber is shown to be related to the micro-structural theory of foamed rubbers due to Gent and Thomas [5] and to the well-known controversial, classical molecular theory of elasticity [6]. Poisson functions are derived for the Blatz-Ko polyurethane materials; and our demonstration data are reviewed in relation to an ad hoc Blatz-Ko constitutive equation of volume control in simple tension. It is shown that the natural state Poisson's ratio for every Blatz-Ko material is simply the ratio of the true lateral contractive strain to the true extensional strain for finite deformations and hence may be readily evaluated from measurements of corresponding stretches over the entire range of elastic extensibility of the material in a simple tension experiment. Discussion of some additional related literature is reserved for the end.

2. THE POISSON FUNCTION

It is easy to show from the constitutive equation for isotropic, linearly elastic solids that a simple tension produces a simple extension provided that the shear modulus $\mu_0 \neq 0$ nor ∞, and Poisson's ratio $\nu_o \neq -1$ nor ∞. In fact, on physical grounds, one usually requires $\infty > \mu_o > 0$ and $1/2 \geq \nu_o > 0$; and, in any case, $\nu_o > -1$ is necessary for material stability [1,§70]. The corresponding result for isotropic, nonlinearly elastic solids is not as transparent. Therefore, to begin, it is necessary to recall Batra's theorem [7] that for every isotropic, compressible or incompressible elastic material, a simple tensile loading

$$t_3 = T, \qquad t_1 = t_2 = 0 \tag{2.1}$$

produces a corresponding extensional deformation

$$\lambda_3 = \lambda, \qquad \lambda_1 = \lambda_2, \tag{2.2}$$

provided that the empirical inequalities

$$\beta_1 > 0, \qquad \beta_{-1} \leq 0 \tag{2.3}$$

hold [3, §51]. Actually, the same result obtains under the weaker condition that the Baker-Ericksen inequalities [3] hold.

With Batra's result in hand, let us assume that a compressible material characterized by (1.1) and (2.3) is subjected to a simple tension (2.1). Then the familiar Young's modulus is defined as the slope of the axial stress/axial stretch function $T = t_3(\lambda)$ evaluated at $\lambda = 1$. However, its determination involves the further assumption that either of the identical transverse stress equations $(2.1)_{2,3}$ may be solved uniquely for the lateral stretch as a function of the axial stretch. In other words, $(2.1)_2$ may be interpreted in a simple tension test as a restriction on the response functions that defines a relation between the longitudinal extension $\lambda \geq 1$ and the lateral contractions $\lambda_1 = \lambda_2 \leq 1$. Hence, their ratio,

$$\alpha(\lambda) \equiv \frac{\lambda_1(\lambda)}{\lambda} , \tag{2.4}$$

defines one kind of <u>lateral contraction function</u> that derives from $(2.1)_2$ and (1.1). Subtraction of the second equation from the first in (1.1) and use of (2.3) shows that in simple tension $0 < \alpha(\lambda) \leq 1$. There exists the possibility that for some response functions the same equations may exhibit several solutions for $\lambda_1(\lambda)$. We consider only those elastic materials for which $\lambda_1(\lambda)$ may be determined uniquely. When this is so, we say that the extension is <u>simple</u>. Thus, if the empirical inequalities are met, it is in this sense that a simple tension produces a simple extension in every compressible, homogeneous and isotropic elastic solid. In linear elasticity theory [1], for example, the null relations $(2.1)_{2,3}$ yield a unique expression for the ratio of the principal transverse contractive and longitudinal engineering strains in terms of the Lamé constants; and this classical squeeze-stretch ratio is commonly known as Poisson's ratio [1, 6].

Recalling that the three principal engineering strains ϵ_k are related to the principal stretches by $\epsilon_k = \lambda_k - 1$, we may define the <u>Poisson function</u> $\nu(\lambda)$ as the ratio of the lateral contractive strain to the extensional strain measured in a simple tension experiment; that is,

$$\nu(\lambda) \equiv -\frac{\epsilon_1}{\epsilon_3} = \frac{1 - \lambda_1}{\lambda_3 - 1} = \frac{1 - \lambda\alpha(\lambda)}{\lambda - 1}\ , \tag{2.5}$$

wherein (2.2) and (2.4) have been used in the last relation. Then, for general homogeneous and isotropic elastic solids, <u>Poisson's ratio</u> ν_o is defined as the value of this function in the undistorted, natural state where $\lambda = 1$:

$$\nu_o \equiv \lim_{\lambda \to 1} \nu(\lambda). \tag{2.6}$$

It follows similarly by Batra's theorem that for the incompressible material (1.3) a simple tension produces an extension (2.2); and the constraint (1.2) determines uniquely the function

$$\lambda_1(\lambda) = \lambda^{-1/2}. \tag{2.7}$$

Thus, a simple tension produces a simple extension in every incompressible, homogeneous and isotropic elastic solid, provided the empirical inequalities (2.3) hold. We have seen that the condition $(2.1)_2$ is essential to the determination of the Poisson function (2.5) for an isotropic and compressible elastic material. On the other hand, in view of (2.7) and the arbitrariness of the stress p in (1.3), the condition $(2.1)_2$ in the case of an incompressible material is irrelative to the determination of the Poisson function, which may be found from the kinematics alone. By use of (2.7) in $(2.5)_2$, we obtain for every incompressible, homogeneous and isotropic material the <u>universal</u> Poisson function[2]

$$\nu(\lambda) = \frac{1}{\lambda + \lambda^{1/2}} . \qquad (2.8)$$

Hence, we may conclude by (2.6) that for every incompressible, isotropic material Poisson's ratio has the unique value $\nu_o = 1/2$. The converse, however, is false, as we shall see in a moment. It may be mentioned that the lateral contraction function (2.4) for the incompressible case becomes $\alpha(\lambda) = \lambda^{-3/2}$; hence, $\alpha(1) = 1$ in the natural state.

The value of Poisson's ratio ν_o is defined by (2.6); hence, it does not necessarily follow that $\nu_o = 1/2$ implies that the isotropic elastic material need be incompressible. We shall illustrate this by a counterexample of a compressible, isotropic material whose Poisson function has the constant value $\nu_o = 1/2$. For this purpose it is useful to recall the results of experiments by Bell [11] for certain homogeneous and isotropic metals in finite (plastic) strain. These data support the following constraint in a variety of deformations:

$$\mathrm{tr}\underset{\sim}{B}^{1/2} = \lambda_1 + \lambda_2 + \lambda_3 = 3. \qquad (2.9)$$

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[2] The function (2.8) and its limit value have been described in different ways by others [8-9]. However, of these, only Pósfalvi [10] derived them in the context of the simple tension test for general incompressible, homogeneous and isotropic hyperelastic materials; but Pósfalvi does nothing with the results. We thank Dr. Joseph D. Walter, Assistant Director of the Firestone Central Research Laboratories, for bringing to our attention the references [9] and [10].

A similar constraint $\operatorname{tr}\underset{\sim}{B} = 3$ has been investigated recently by Ericksen [12] in a study of a constitutive theory for elastic crystals. Details of these applications need not concern us here. Rather, let us consider a homogeneous, isotropic elastic material for which the constraint (2.9) may hold; and let it be subjected to a simple tension to effect, under suitable restrictions on the response functions, an extensional deformation (2.2). Then use of (2.2) in (2.9) yields the unique simple extension relation $2\lambda_1(\lambda) = 3 - \lambda > 0$, which also implies $1 \leq \lambda < 3$. It thus follows by $(2.5)_2$ and (2.6) that the Poisson function for this special class of constrained, compressible, homogeneous and isotropic elastic materials is a constant:

$$\nu(\lambda) = \frac{1}{2} = \nu_o \qquad (2.10)$$

for all $\lambda \in [1, 3)$. If, additionally, the material were assumed incompressible so that (1.2) also must hold for every admissible deformation, it may be seen that only the trivial deformation $\lambda_1 = \lambda_2 = \lambda_3 = 1$ would be possible. For sufficiently small deformations, however, the constraint (2.9) approximates the incompressibility constraint; hence, for small strains, the material behaves initially like an incompressible, isotropic elastic solid.

It is interesting to observe that in every extension (2.2), whatever may be the tractions required for its control in an incompressible material, the Poisson function (2.8) is independent of the elastic response and is valid whether the material be isotropic or not. However, this fact must be viewed with caution.

Control of the deformation (2.2) plainly depends on the nature of the constitutive equation for the stress; and if the homogeneous deformation (2.2) is assigned, this stress distribution may be readily determined. But if the stress is given, conditions needed to assure that the deformation (2.2) is possible, as demonstrated above, must follow from careful examination of the constitutive equation for the prescribed loading situation. In particular, in a simple tension (or compression) test, the kinematic condition $(2.2)_2$ plainly cannot be expected to hold for arbitrary directions in an incompressible, anisotropic material. In such a material, even equal biaxial loading may not produce (2.2). For an isotropic material, we are assured by Batra's theorem that simple tensile loading will effect the deformation (2.2). Therefore, the formula (2.8), though universal for the deformation (2.2), must be viewed indirectly with regard for the nature of the material and of the loading needed to control the deformation. Parallel remarks apply to the Bell constraint (2.9) and the associated value (2.10) for the Poisson function valid in every equi-biaxial deformation (2.2).

On the other hand, contraction functions certainly may be defined in terms of other experiments; and, for distinction, these may be named <u>apparent</u> Poisson functions ν_a. For an incompressible material, the apparent Poisson function will be the same as (2.8) in any experiment for which (2.2) holds; but the loading needed to control the deformation will be determined by the particular constitutive equation for the material. In equal triaxial extension of a cube of any incompressible material, the only solution is the trivial solution $\lambda_1 = \lambda_2 = \lambda_3 = 1$; hence, for this case (2.8) yields the apparent value $\nu_a(\lambda) = \nu_o = 1/2$. We are reminded, however, that nonuniqueness of a pure homogeneous deformation is possible in all around tension of an incompressible material. Rivlin [13] has shown, for example, that for a uniform tension $T > 0$ on all faces of a cube of neo-Hookean material for which $\beta_1 = \mu_o$ is constant and $\beta_{-1} = 0$, seven possible states exist. The trivial state $\lambda_1 = \lambda_2 = \lambda_3 = 1$ is always a solution for which $\nu_a = 1/2$. This state is stable provided that $T/\mu_o < 2$. The state $\lambda_1 = \lambda_2$, $0 < \lambda_3 = \lambda < T/\mu_o$, and two others obtained by cyclic permutation of the λ_k's, are stable equilibrium solutions; and the apparent Poisson function is the same as (2.8). The remaining three solutions are unstable. Although each solution has the same apparent Poisson function (2.8), it can not be measured in these unstable states. Other examples may be easily constructed.

3. SOME EXPERIMENTAL RESULTS

Experimental data obtained from at least two specimens of each of three considerably different elastomers are presented here. One is a polyether, polyurethane elastomer; another is a carbon-black reinforced, sulfur-cured blend of natural and synthetic rubbers; and a third variety is a natural gum rubber. The procedure for obtaining the axial and transverse stretch data is straightforward. Specimens having straight sides of length 42 mm and width of 6 mm were die stamped from thin sheets of uniform thickness of 1 to 3 mm. Each sample was quasistatically elongated in a tensile loading frame. The test was stopped periodically to allow measurement of the specimen width to the nearest

0.01 mm with a Gaertner traversing microscope equipped with a digital readout; and, at that time, the elongation was measured to the nearest 0.1 mm with a linear variable differential transformer fastened to the loading actuator. These techniques permitted reasonably accurate determination of the transverse and axial stretches suitable for demonstration purposes here. For the sake of clarity in diagrams presented below, not all the data values collected will be shown.

It is seen that the Poisson function (2.8) for an incompressible material is a monotonically decreasing function for which $0 < \nu(\lambda) \leq 1/2$ in simple tension. The graph of (2.8) is shown in Figs. 1 and 2 together with tensile test stretch data for the three kinds of elastomers described above. It is seen that the urethane follows the universal function very nicely, particularly for axial stretches $\lambda > 1.5$, roughly. Although the data for the carbon-black reinforced blend of natural and synthetic rubbers, as shown in Fig. 1, follows the trend of the universal graph, its deviation at the larger deformations is evident. Two compounds of natural gum rubber of the same[3] kind used in the basic experiments by Rivlin and Saunders [2, p.285] were fabricated from their recipes provided for compounds described as A and B. Fig. 2 shows that our compound A is exceptional in its comparison with the kinematical relation (2.8), while our compound B, though well-behaved, falls below and virtually parallel to the master curve. The scatter in the data

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[3] There was a minor difference; the antioxidant *nonox* used in [2] was replaced by another hindered phenol type antioxidant, tradename American Cyanamid AO2246.

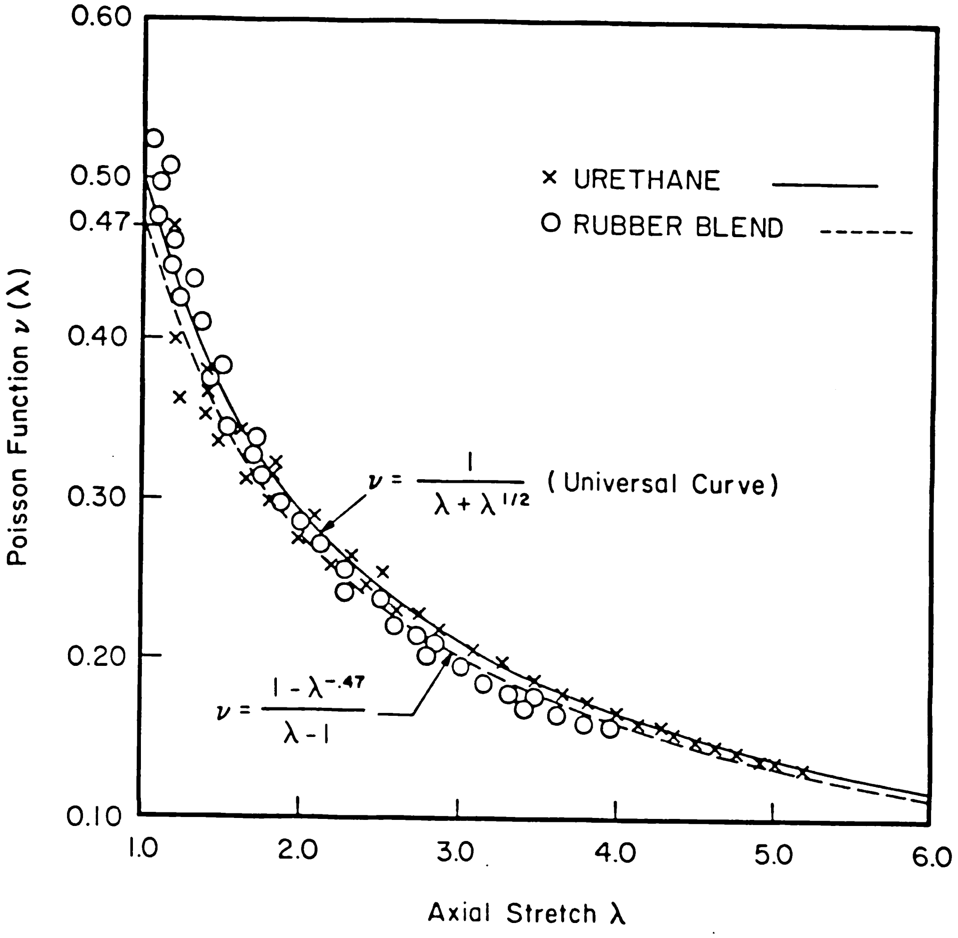

Fig. 1. Comparison of extension data for two elastomers with the Poisson function (2.8) for an incompressible material in a simple tension test.

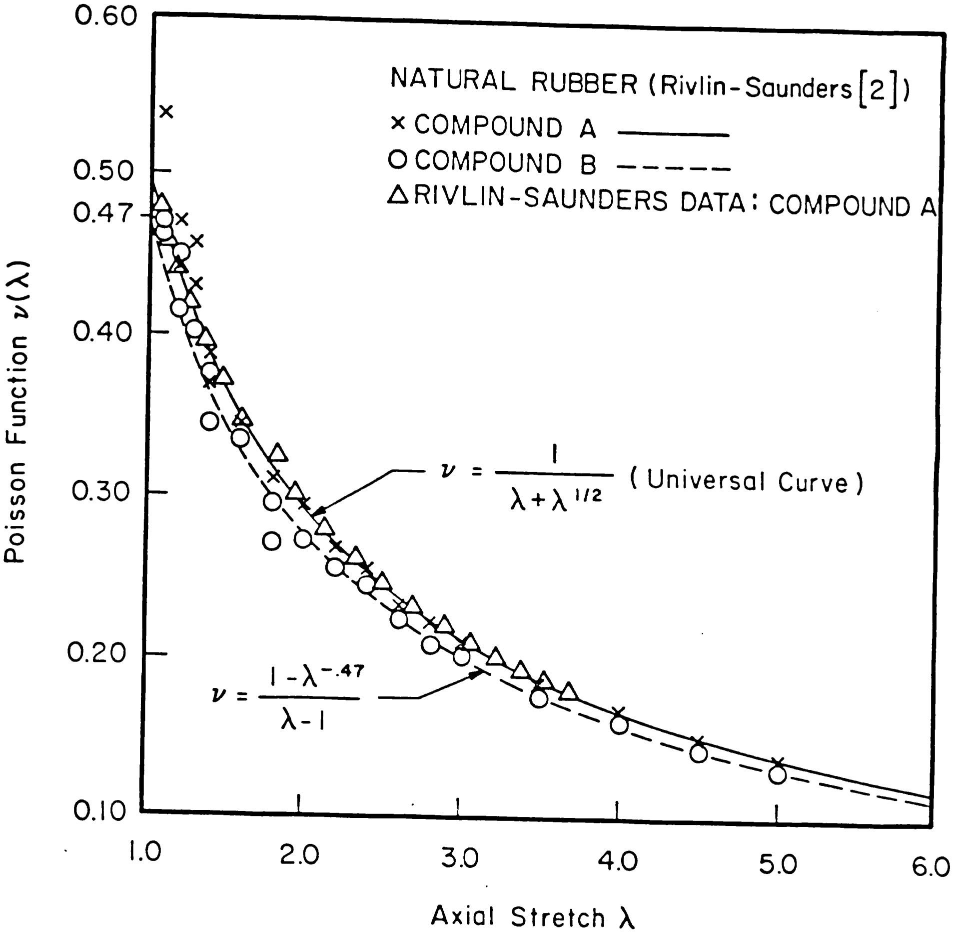

Fig. 2. Comparison of extension data for two rubber compounds with the Poisson function (2.8) for an incompressible material in a simple tension test. Data by Rivlin and Saunders[2] for compound A also is shown.

for small deformations was typical for all the samples; and we feel no need to provide explanation for it. The dotted curves shown in Figs. 1 and 2 have an analytical basis which will be explained in the next section; it suffices to mention here that these curves approximate the best fit for the data. The extension data obtained by Rivlin and Saunders for their compound A also is shown in Fig. 2. It is found that these data, for the same reason noted later, fit the universal relation (2.8). Our data for the same material is essentially coincident with theirs, except at small deformations, as noted before.

The same data may be viewed differently in Figs. 3 and 4, which emphasize the incompressibility relation (2.7) in simple tension. The data are to be compared with the line whose slope is one. The response appears to be about the same as described for Fig. 1 and 2, except that the small amount of scatter evident for the smaller stretches appears diminished in Figs. 3 and 4. It is quite clear from both graphs that the data for the urethane and the natural gum compound A fall reasonably close to the kinematical function described; therefore, these materials are virtually incompressible. The special rubber blend and the natural rubber compound B exhibit almost incompressible response that we shall examine again further on. The data for the Rivlin-Saunders compound A also is shown in Fig. 4. However, it must be mentioned that Rivlin and Saunders did not confirm by any tests in [2] that the incompressibility constraint actually was obeyed by either compound they studied. Since they used the incompressibility condition to compute from measured values of λ alone the values for I_2 provided in Table 6 in [2]; it is not surprising

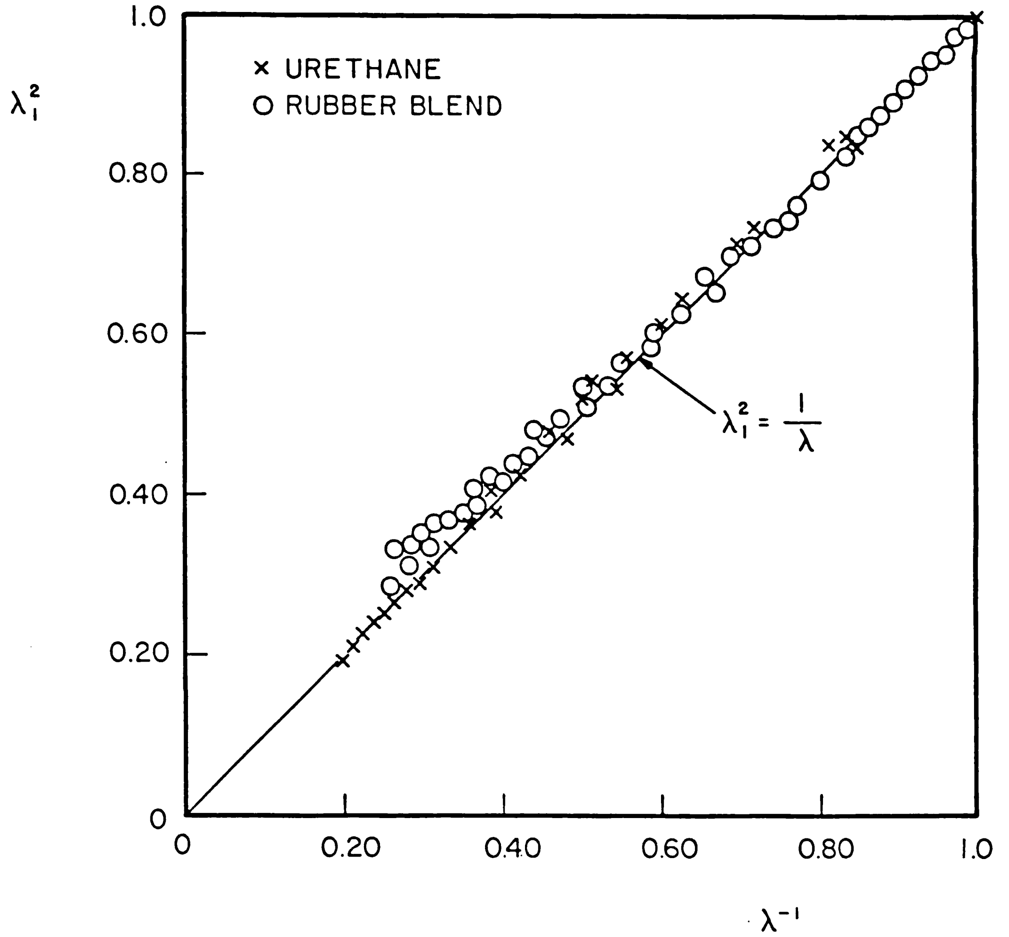

Fig. 3. Comparison of stretch data for two elastomers with the incompressibility condition $\lambda_1^2 = \lambda^{-1}$ in a simple tension test. See (2.7).

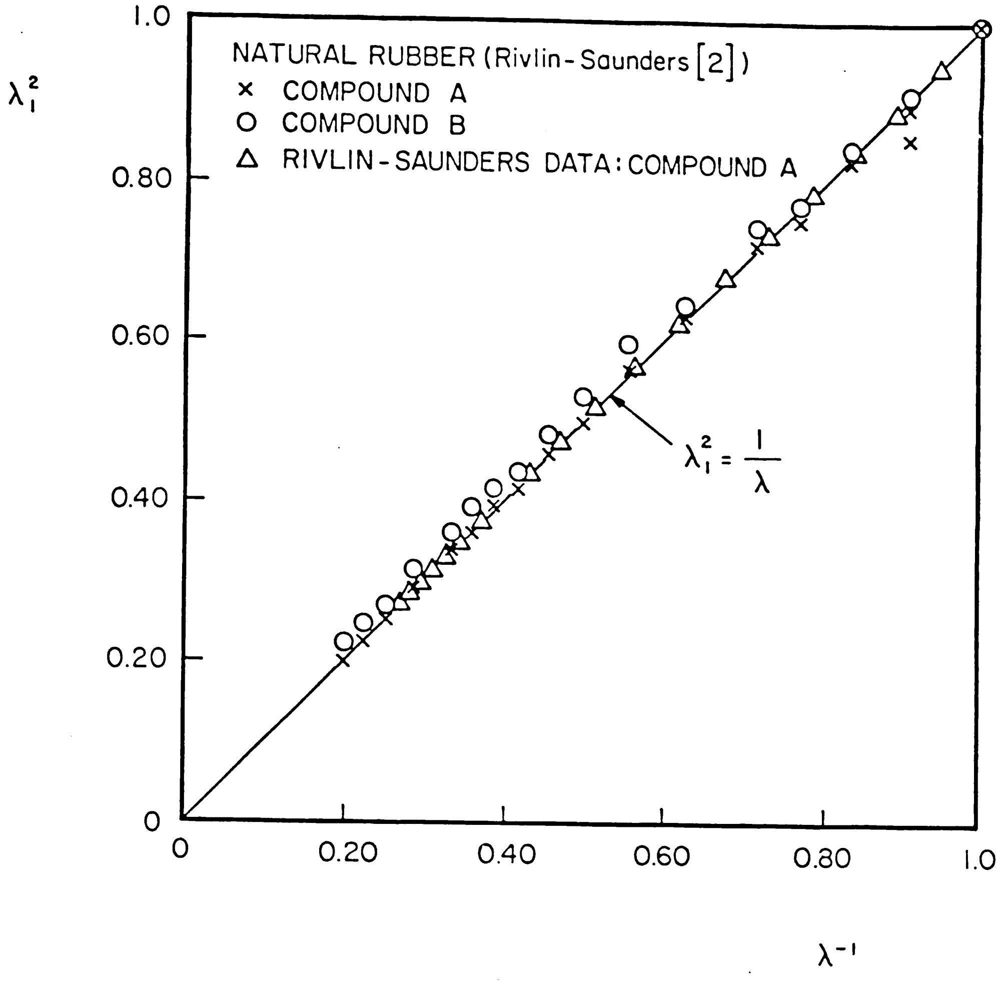

Fig. 4. Comparison of stretch data for two rubber compounds with the incompressibility condition $\lambda_1^2 = \lambda^{-1}$ in a simple tension test. See (2.7). Data by Rivlin and Saunders [2] for compound A also is shown.

that our calculation of values of λ_1 corresponding to their tabulated values for λ and I_2 results in their data falling smack on the line in Fig. 4. The same applies to Fig. 2. The representation of our stretch data in the universal plot in Fig. 2, however, is a genuine experimental result that demonstrates the incompressibility of the natural rubber compound A. The response of compound B is another matter that will be discussed later.

4. EXAMPLE FOR COMPRESSIBLE RUBBERS

Let us consider a class of compressible, isotropic hyperelastic materials with strain energy function $W = W(J_1,J_2,J_3)$ per unit undeformed volume, and whose response functions in (1.1) depend on J_3 alone: $\beta_\Gamma = \beta_\Gamma(J_3)$. Herein we have introduced the invariants

$$J_1 \equiv I_1 = \mathrm{tr}\underset{\sim}{B}, \quad J_2 \equiv I_2/I_3 = \mathrm{tr}\underset{\sim}{B}^{-1}, \quad J_3 \equiv I_3^{1/2} = \det\underset{\sim}{F}. \qquad (4.1)$$

Then the following relations will be obtained for this hyperelastic material [3,§86]:

$$\beta_0(J_3) = \frac{\partial W}{\partial J_3}, \quad \beta_1(J_3) = \frac{2}{J_3}\frac{\partial W}{\partial J_1}, \quad \beta_{-1}(J_3) = -\frac{2}{J_3}\frac{\partial W}{\partial J_2}. \qquad (4.2)$$

Bearing in mind the assumed functional dependence, it may be seen that these relations will hold if and only if $2\partial W/\partial J_1 = \alpha$ and $2\partial W/\partial J_2 = \beta$ are constants. Thus, introducing these in (4.2) and writing $\partial W/\partial J_3 \equiv W_3(J_3)$, we obtain the response functions

$$\beta_0 = W_3(J_3), \quad \beta_1 = \frac{\alpha}{J_3}, \quad \beta_{-1} = -\frac{\beta}{J_3}. \qquad (4.3)$$

It is known that $\beta_1(1) - \beta_{-1}(1) = \alpha + \beta = \mu_o$, the usual constant shear modulus in the undistorted, natural state of the material [3, §50]. Thus, upon introducing $\alpha \equiv \mu_o f$ and $\beta \equiv \mu_o(1 - f)$, where f is another constant, and substituting (4.3) into (1.1), we reach the general form of the constitutive equation for our compressible, hyperelastic material:

$$\underset{\sim}{T} = W_3(J_3)\,\underset{\sim}{1} + \frac{\mu_o f}{J_3}\,\underset{\sim}{B} - \frac{\mu_o(1 - f)}{J_3}\,\underset{\sim}{B}^{-1}. \qquad (4.4)$$

This equation was first introduced in an altogether different way by Blatz and Ko [4]. It may be seen from (4.3) that the empirical inequalities (2.3) are satisfied for the Blatz-Ko material if and only if $\mu_o > 0$ and $0 < f \leq 1$. These conditions were not noted in [4]; however, they are essential in the biaxial deformation problems described there.

Experiments by Blatz and Ko [4] on a certain compressible, foamed, polyurethane rubber revealed the specific response functions

$$\beta_0 = \mu_o, \quad 0 < \beta_1 \ll 1, \quad \beta_{-1} = -\mu_o/J_3, \qquad (4.5)$$

where β_1 was considered negligible so that $f = 0$, very nearly, and $W_3 = \mu_o = 32$ psi. Thus, in general terms, (4.4) reduces to the following constitutive equation for the Blatz-Ko foamed, polyurethane rubber:

$$\underset{\sim}{T} = \mu_o[\underset{\sim}{1} - J_3^{-1}\,\underset{\sim}{B}^{-1}]. \tag{4.6}$$

For the simple tension (2.1), (2.2) holds and $J_3 = \lambda_1^2\lambda$. It follows that (4.6) yields

$$T = \mu_o(1 - \lambda_1^{-2}\lambda^{-3}), \qquad \lambda_1(\lambda) = \lambda^{-1/4}. \tag{4.7}$$

The extension, therefore, is simple. Application of (2.5) delivers the Poisson function

$$\nu(\lambda) = \frac{1 - \lambda^{-1/4}}{\lambda - 1}\,. \tag{4.8}$$

We thus find by (2.6) that the foamed, polyurethane rubber (4.6) has a Poisson ratio $\nu_o = 1/4$, which is, in fact, the experimental value found by Blatz and Ko. However, they made no connection of their data with (4.8); rather, they used a clever ad hoc rule described below to determine ν_o.

The linearized form of (4.6) will be considered next. First we note that $E_o = 5\mu_o/2$ is the usual Young's modulus for this model; and for a sufficiently small engineering strain $\underset{\sim}{\epsilon}$, it can be easily shown that

$$\underset{\sim}{B} = \underset{\sim}{1} + 2\underset{\sim}{\epsilon}, \qquad J = 1 + \theta, \tag{4.9}$$

in which $\theta = \mathrm{tr}\underset{\sim}{\epsilon}$ describes the small change in volume per unit initial volume. Then, to the first order in $\underset{\sim}{\epsilon}$, (4.6) becomes

$$\underset{\sim}{T} = \frac{2E_o}{5}\,[\theta\underset{\sim}{1} + 2\underset{\sim}{\epsilon}]. \tag{4.10}$$

We thus recover the linearized, uni-constant equation for general isotropic, foamed elastic materials derived by Gent and Thomas [5] from a simple micro-structural model consisting of a network of thin extensible rubber cords connected by rigid joints. It may be noted that (4.10) is the same equation obtained from linear, isotropic elasticity theory with $\nu_o = 1/4$, i.e. with equal Lamé constants $\lambda_o = \mu_o$; it is the constitutive equation of the controversial 19th century rari-constant elasticity model that evolved from molecular theories of elasticity due by Poisson and Cauchy [8, 16]. On the contrary side, we are reminded of Wertheim's many experiments on metals for which he claimed a universal average value $\nu_o = 1/3$. The rari-constant theorists and experimentists were careful always to exclude from the uni-constant theory unusual materials that they believed ought not to be treated as elastic; caoutchouc was an example often cited [8, 16]. It is strangely coincidental that Gent and Thomas [5] found in their experiments on foamed natural rubber the average value $\nu_o = 1/3$, as compared with their predicted universal value of 1/4.

There is a third empirical condition, $\beta_0 \leq 0$, that also should be respected [3,§51]. We see from $(4.5)_1$ that the Blatz-Ko foamed, polyurethane rubber model fails to satisfy it [14]. It is possible, of course, that this results from the fact that a foamed rubber is not a homogeneous, materially uniform and isotropic continuum. Nonetheless, the test data share good agreement with this model. It should be emphasized also that the data for the Blatz-Ko compressible, solid polyurethane rubber material described below support all of the empirical inequalities.

It is also interesting, though apparently not well-known, that in the construction of their more general constitutive equation (4.4), which essentially is designed to reduce to the Mooney-Rivlin model when $\nu_o = 1/2$, Blatz and Ko [4] invoked the following additional ad hoc constitutive assumption of volume control in a simple tension:

$$J_3 = \lambda^{1 - 2\nu_o}. \tag{4.11}$$

It follows by Batra's theorem that $\lambda_1 = \lambda_2$ in the simple tension; hence, (4.11) yields the unique relation

$$\lambda_1(\lambda) = \lambda^{-\nu_o}. \tag{4.12}$$

Therefore, the extension is indeed simple. This must hold in a simple tension of every Blatz-Ko material (4.4) for which (2.3) holds. Thus, the Poisson function for every such material is given by

$$\nu(\lambda) = \frac{1 - \lambda^{-\nu_o}}{\lambda - 1}. \tag{4.13}$$

It is readily seen that for small strains (4.12) may be linearized to $\epsilon_1 = -\nu_o\epsilon_3$; hence, the constant exponent ν_o in (4.11) to (4.13) is the classical Poisson ratio for the material. Of course, the same thing derives from (2.6). Thus, the occurrence of Poisson's ratio $\nu_o = 1/4$ in $(4.7)_2$ and (4.8) is not coincidental. Notice also that for this case the value $\nu_o = 1/2$ reduces (4.11) to the incompressibility condition (1.2) in simple tension.

A material whose response in a simple tension test fails to obey the rule (4.12) can not be a candidate for the Blatz-Ko model. Therefore, when attempting to model the constitutive behavior of an elastomer, the experimenter may find it helpful to first confirm the volume control relation (4.12) by plotting a graph of $\log(1/\lambda_1)$ against $\log \lambda$, which is a straight line of slope ν_o. For illustration, the tensile test data for the elastomers considered earlier are plotted accordingly in Figs. 5 and 6. A least squares fit of the data with straight lines through the origin shows in Fig. 5 that the urethane has a Poisson ratio $\nu_o = 0.493$, while the rubber blend satisfies $\nu_o = 0.468$. Similar tests on a second urethane and a second rubber blend, which differed from the others only slightly in their formulation and processing, yielded the same basic response with the respective values $\nu_0 = 0.463$ and $\nu_o = 0.459$.

The data for the Rivlin-Saunders natural rubber compound A are shown in Fig.6. However, as noted earlier, because they measured only λ and, in effect, used the incompressibility condition to compute λ_1, one should expect, as seen in Fig. 6, that their converted data should follow perfectly the ideal line for which $\nu_o = 0.5$. It is seen, however, that our corresponding data for the natural rubber compound A also enjoys excellent correlation with the volume control relation for incompressible materials. The compound A yielded, among all the elastomers we studied, the best fit correlation with $\nu_o = 0.499$. The natural rubber compound B, on the other hand, produced in our tests the best fit $\nu_o = 0.466$. The values of ν_o found in this manner were then used

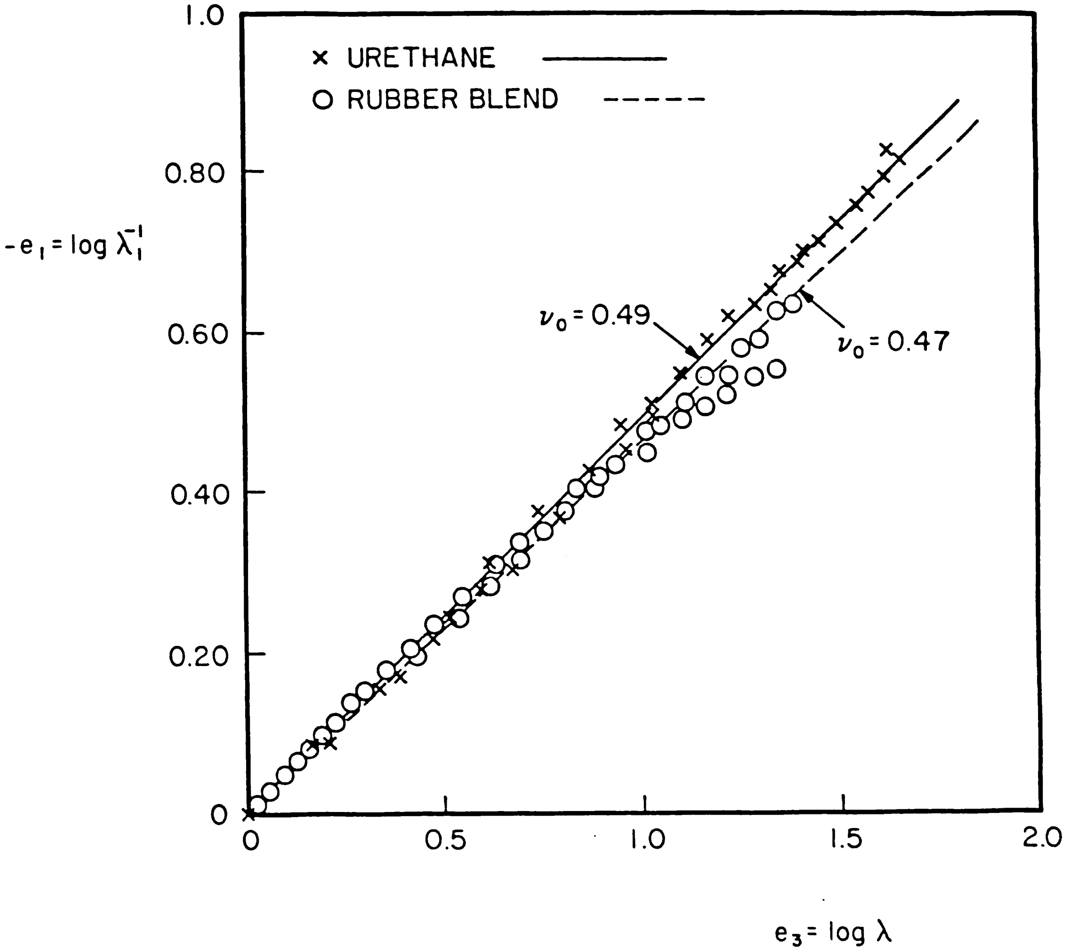

Fig. 5. Best fit comparison of tensile test data for two elastomers with the Blatz-ko volume control relation (4.12), and evaluation of Poisson's ratio for the materials.

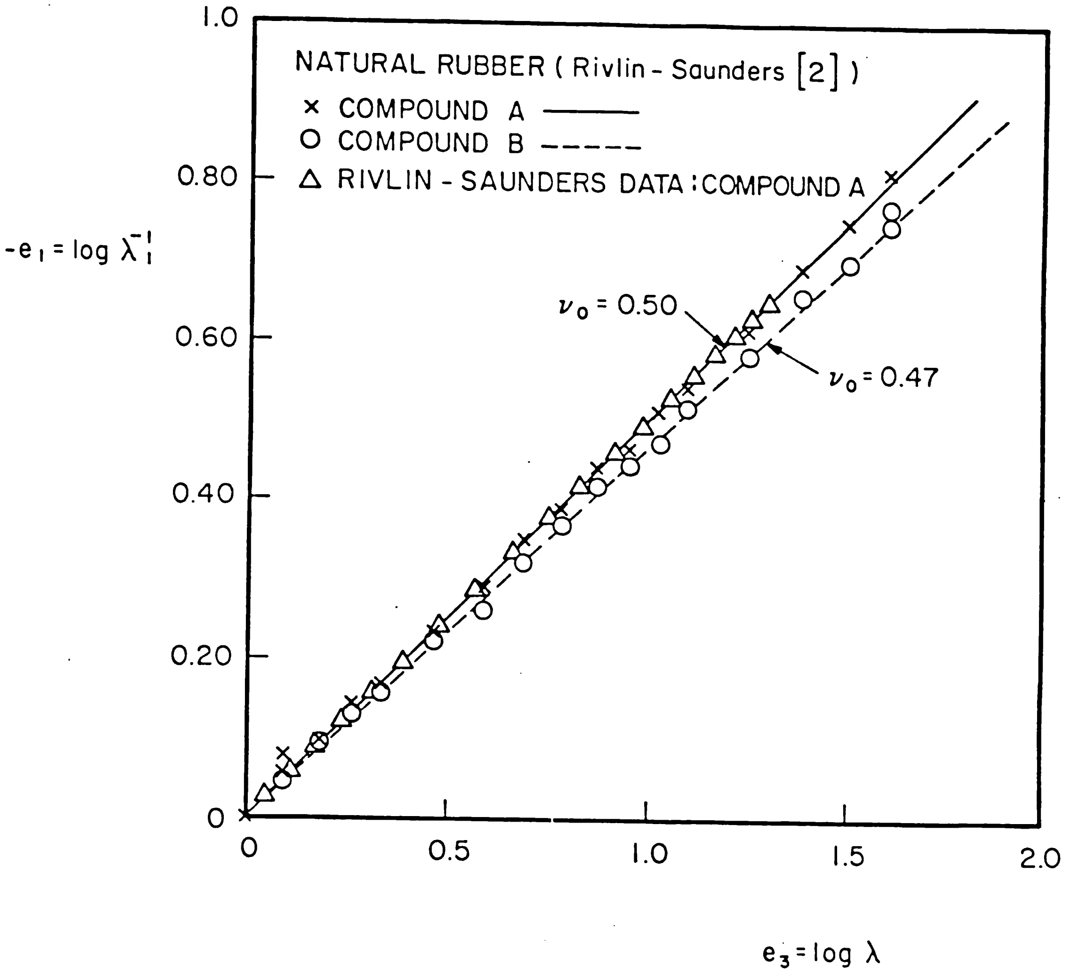

Fig. 6. Best fit comparison of tensile test data for two rubber compounds with the Blatz - Ko volume control relation (4.12), and evaluation of Poisson's ratio for the materials. Data by Rivlin and Saunders [2] for compound A also is shown.

in (4.13), and the corresponding best fit graphs of their Poisson functions $\nu(\lambda)$ were plotted in Figs. 1 and 2. Of course, the curves for the urethane and the rubber compound A lay so close to the master curve that we let this curve represent their behavior, as shown therein. Although the elastomers for which $\nu_o \neq 0.5$, approximately, may thus be viewed as candidates for a Blatz-Ko constitutive model, considerable further evaluation would be necessary to establish this.

Based upon their volume control relation (4.11), Blatz and Ko graphed the straight line of $\log J_3$ against $\log\lambda$ and from its slope $1 - 2\nu_o$ determined for their foamed, polyurethane rubber the value $\nu_o = 0.25$; but they apparently were unable to apply the same method to their solid, polyurethane rubber. By an altogether different and unrelated argument, they arrived at the value $\nu_o = 0.463$. We encountered no serious difficulties in our graphical evaluations of ratios of similar value for other varieties of rubber based upon (4.6). Evaluation by Blatz and Ko [4] of the tension data for their solid, polyurethane showed that $f = 1$ and $\mu_o = 34$ psi. Thus, in general terms, the reduced form of the Blatz-Ko constitutive relation (4.4) for their solid polyurethane rubber may be written as

$$\underset{\sim}{T} = W_3(J_3)\,\underset{\sim}{1} + \frac{\mu_o}{J_3}\,\underset{\sim}{B}, \tag{4.14}$$

subject to the further empirical inequality $\beta_o = W_3(J_3) \leq 0$.

Finally, it may be observed that the true strain e_k in any direction k is defined by $e_k = \log \lambda_k$. Consequently, the volume control relation (4.12) may be rewritten as

$$\nu_o = - \frac{e_1}{e_3} , \qquad (4.15)$$

which reveals that in finite strain Poisson's ratio for every Blatz-Ko model is the ratio of the true lateral contractive strain to the true extensional strain. Therefore, Figs. 5 and 6 actually are plots of the true transverse strain e_1 versus the true axial strain e_3 in a simple tension test. This simple fact has apparently gone unnoticed by others.

5. CONCLUSION

The tensile test possibly is the most important among all simple experiments used to characterize the phenomenological behavior of solid materials; and its application to rubbery materials provides an excellent opportunity for instruction in some interesting aspects of nonlinear elasticity. With this objective in mind, some data for the so-called Poisson's ratio (sic) as a function of engineering strain for an unspecified rubber material was illustrated by Coakham, Eastwood and Evans [15]; however, they provided no explanation or discussion of the phenomenon. This almost casual indication of a substantial variation in the lateral contraction ratio for rubber in finite strain prompted, we feel, an inaccurate critique by Lindley [8] a year later.

Lindley observed correctly that Poisson's ratio is a material constant, hence independent of the strain from the natural state. But his subsequent remark that its definition is valid only for small strains, so that its use is inappropriate at large strains, is imprecise. He naturally assumes that regardless of the nonlinear constitutive description of the material, the transverse strains in a simple tension test must be equal; and based upon this assertion and the incompressibility of rubber, which he justifies in terms of the bulk modulus rather than the volume constraint for infinitesimal strains, he provides an equation in terms of (possibly) finite engineering strains that characterizes reasonably the data shown in [15]. We agree with Lindley that the experiment in [15] and in our own Figs. 1 and 2 above, is an inappropriate method for evaluation of Poisson's ratio for natural rubber, though it may be an excellent demonstration in support of its incompressibility. However, since Poisson's ratio $\nu_o = 1/2$ for every incompressible, isotropic elastic material, these data obviously are not intended for this evaluation in the first place.

The fact that one can indeed define a Poisson function, or some other lateral contraction function, that accurately demonstrates the variation in the lateral contractive response of elastomers over a wide range of deformation in a simple tension test, as shown in Figs. 1 and 2, apparently is unappreciated by Lindley [8] and ignored by Coakham, Eastwood and Evans [15]. Of course, not every elastomer need be incompressible; and based upon the Blatz-Ko volume control relation (4.11), it is seen in Figs. 5 and 6 that in special circumstances the kinematical data for finite deformations may be plotted in a manner that does allow for easy evaluation of their Poisson's ratio in the natural state. In fact, our equation (4.15), demonstrated by rough experiments, refutes Lindley's remark that use of Poisson's ratio is appropriate only for small strains. Moreover, Anand [17] has found that (4.15) arises naturally in a linear theory of isotropic elasticity that uses the true strain $\underset{\sim}{e} \equiv \log \underset{\sim}{B}^{1/2}$ as a deformation measure for moderately large strains. He showed that Hencky's constitutive equation for the Kirchhoff stress shares good agreement with a variety of experimental data for moderately large deformations defined by stretches of roughly 1.3 to 1.4. Although it may be tempting to adopt (4.15) as the definition of Poisson's ratio for large deformations, we caution that this rule applies only to the class of materials for which (4.12) holds in a simple tension test. The Blatz-Ko material (4.6), in view of $(4.7)_2$, the linear Hencky model [17], by definition, and all others for which (4.12) may be valid, belong to this class. The definition (2.5), on the other hand, extends to all isotropic elastic materials that respect the empirical inequalities.

Finally, we are reminded that in numerical work involving elastomeric materials which often are <u>assumed</u> ideally incompressible, a value of ν_0 <u>close</u> to 0.5 commonly is used to avoid computational difficulties. But it may be useful to first evaluate the actual lateral contractive response for the special elastomeric material of interest. Indeed, it may happen that a plot of the kind used in Figs. 5 and 6 may provide useful data for a more appropriate and realistic estimate of Poisson's ratio for elastomers studied in numerical work.

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