L^∞-BOUNDS FOR WEAK SOLUTIONS OF AN EVOLUTIONARY EQUATION WITH THE p-LAPLACIAN

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L^∞-Bounds for Weak Solutions of an Evolutionary Equation with the p-Laplacian [1]

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Dedicated to Professor Alois Kufner on the occasion of his 70^{th} *birthday.*

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Abstract

We investigate the smoothing effect of the parabolic part of a quasilinear evolutionary equation on its solution as time evolves. More precisely, the following initial-boundary value problem with Dirichlet boundary conditions is considered:

$$\text{(P)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta_p u + f(x,t,u(x,t)) & \text{for } (x,t) \in \Omega \times (0,T); \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T); \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

Here, Δ_p stands for the negative Dirichlet p-Laplacian defined by $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $2 \leq p < \infty$. The reaction $f : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$\xi \cdot f(x,t,\xi) \leq a|\xi|^q + b|\xi| \quad \text{for a.e. } x \in \Omega \text{ and all } t \in (0,T),\ \xi \in \mathbb{R},$$

where $a > 0$ and $b \geq 0$ are some constants and $q \in [p, p^*)$ is subcritical; $p^* = Np/(N-p)$ if $p < N$ and $p^* \in (p, \infty)$ is arbitrary if $p \geq N$. We assume that Ω is a bounded domain in $\mathbb{R}^N$ and the initial values satisfy only $u_0 \in L^r(\Omega)$ for some $r > (q-p)N/p$, $r \geq 2$. If $u : \Omega \times [0,T) \to \mathbb{R}$ is any weak solution of problem (P), such that the norm $\|u(\,\cdot\,,t)\|_{L^r(\Omega)} \leq c \equiv \text{const}$ is bounded for all $0 \leq t < T$, we show that $u(\,\cdot\,,t) \in L^\infty(\Omega)$ for every $t \in (0,T)$. Moreover, we obtain also an a priori bound on the norm $\|u(\,\cdot\,,t)\|_{L^\infty(\Omega)} \leq C_\varepsilon \equiv \text{const}$ for all times $t \in [\varepsilon, T)$, where $\varepsilon > 0$ is arbitrary.

Keywords: Dirichlet p-Laplacian; degenerate quasilinear parabolic problem; L^∞-smoothing effect; logarithmic Sobolev inequality; ultracontractive C^0-semigroup

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1 Introduction

The smoothing effect of the linear part of a semilinear parabolic equation is a well-known phenomenon that has often been used to obtain the existence and/or uniqueness of solutions to various semilinear parabolic problems. In the most abstract setting of a functional differential equation in a Banach space X, the smoothing effect is expressed by an analytic semigroup of bounded linear operators on X. In the special case when $X = L^2(\Omega)$ ($\Omega \subset \mathbb{R}^N$ open) and e^{-Ht} ($t \geq 0$) is a Markov semigroup on $L^2(\Omega)$ generated by $-H$, where H is a positive definite, selfadjoint operator on $L^2(\Omega)$, the smoothing effect may be expressed through the following property of the semigroup e^{-Ht} called *ultracontractivity*:

(u.c.) for every time $t > 0$, e^{-Ht} is a bounded linear operator from $L^2(\Omega)$ to $L^\infty(\Omega)$.

This property was introduced in the papers by E. B. Davies [10] and E. B. Davies and B. Simon [12] and extensively studied in the monograph by E. B. Davies [11, Chapt. 2, pp. 59–81]. A simple example, when (u.c.) is satisfied, is the positive Dirichlet Laplacian $H = -\Delta$ in $L^2(\Omega)$. Surprisingly enough, we are able to apply similar methods, in particular the logarithmic Sobolev inequalities from [11, Chapt. 2, § 2, pp. 63–71], studied much earlier by L. Gross [20], in order to show that also the nonlinear semigroup of contractions $\mathrm{e}^{\Delta_p t}$ ($t \geq 0$) on $L^2(\Omega)$, generated by the negative Dirichlet p-Laplacian Δ_p, has an analogous smoothing property (ultracontractivity) whenever $2 \leq p < \infty$:

(u.c.′) for every time $t > 0$, $\mathrm{e}^{\Delta_p t}$ maps bounded sets from $L^2(\Omega)$ into bounded sets in $L^\infty(\Omega)$.

As usual, Δ_p stands for the negative Dirichlet p-Laplacian defined by $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$; we consider only $2 \leq p < \infty$. We assume that Ω is a bounded domain in $\mathbb{R}^N$. The same method, based on logarithmic Sobolev inequalities, has been used earlier in Cipriani and Grillo [8] to show the following much weaker result [8, Theorem 1.1, p. 213]: Let $2 \leq p < N$ and $r \geq \max\{2,\, p(p-2)/(N-p)\}$. Then for every time $t > 0$, $\mathrm{e}^{\Delta_p t}$ maps bounded sets from $L^r(\Omega)$ into bounded sets in $L^\infty(\Omega)$. On the other hand, they require $\Omega \subset \mathbb{R}^N$ to be a domain of only finite N-dimensional Lebesgue measure. Recently, Del Pino, Dolbeault, and Gentil [14] applied this method to a related Cauchy problem in $\Omega = \mathbb{R}^N$ for any $1 < p < \infty$.

More generally, we will show an analogous "L^r-to-L^∞" smoothing effect ($r \geq 2$ - a suitable number) for the following quasilinear initial-boundary value problem:

$$\text{(1.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta_p u + f(x,t,u(x,t)) & \text{for } (x,t) \in \Omega \times (0,T); \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times (0,T); \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

We assume that the reaction $f(x,t,\xi)$ satisfies the following measurability and growth conditions:

(**f1**) $f : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., for every $\xi \in \mathbb{R}$, the function $f(\,\cdot\,,\,\cdot\,,\xi) : \Omega \times (0,T) \to \mathbb{R}$ is Lebesgue-measurable, and, for almost all $(x,t) \in \Omega \times (0,T)$, the function $f(x,t,\,\cdot\,) : \mathbb{R} \to \mathbb{R}$ is continuous.

(**f2**) f satisfies $f(\,\cdot\,,\,\cdot\,,\xi) \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$ for every $\xi \in \mathbb{R}$ and the growth condition

$$\xi \cdot f(x,t,\xi) \leq a|\xi|^q + b|\xi| \quad \text{for a.e. } x \in \Omega \text{ and all } t \in (0,T),\ \xi \in \mathbb{R}, \tag{1.2}$$

where $a > 0$ and $b \geq 0$ are some constants and $q \geq p$ is a subcritical exponent to be specified later.

The initial values are assumed to satisfy only $u_0 \in L^r(\Omega)$ for some $r \geq 2$. Assuming suitable restrictions on p, q and r, which are sharp, we are interested in obtaining an a priori $L^\infty(\Omega)$-bound for any weak solution $u : \Omega \times [0,T) \to \mathbb{R}$ of problem (1.1) for all times $t \in (0,T)$. More precisely, if the norm $\|u(\,\cdot\,,t)\|_{L^r(\Omega)} \leq c \equiv \text{const}$ is bounded for all $0 \leq t < T$, then we establish an a priori bound on the norm $\|u(\,\cdot\,,t)\|_{L^\infty(\Omega)} \leq C_\varepsilon \equiv \text{const}$ for all times $t \in [\varepsilon, T)$, where $\varepsilon > 0$ is arbitrary. This result (Theorem 2.1 below) extends earlier results by J. Filo [16], G. M. Lieberman [24, pp. 561–562], and M. M. Porzio [25] (see also E. DiBenedetto [15, Chapt. V, Theorem 3.2, p. 121]). There, a completely different method of proof is used, based on local estimates and Moser's iteration technique. In contrast, we apply the method discovered by L. Gross [20, p. 1066].

2 The main result

Recall that $2 \leq p < \infty$ and Ω is a bounded domain in $\mathbb{R}^N$. Let $p' = p/(p-1)$ be the conjugate exponent; hence $1 < p' \leq 2$. We denote by $W^{-1,p'}(\Omega)$ the dual space of the Sobolev space $W_0^{1,p}(\Omega)$ with respect to the duality induced by the standard inner product in $L^2(\Omega)$. The reader is referred to the monographs Adams and Fournier [1], Kufner [21], and Kufner, John and Fučík [22] for general facts about Sobolev spaces and weak solutions.

Given any initial values $u_0 \in L^2(\Omega)$, a function $u : \Omega \times (0,T) \to \mathbb{R}$ is called a *weak solution* of problem (1.1) if it satisfies the following conditions:

(**u1**) The $L^2(\Omega)$-valued function $t \mapsto u(t) \equiv u(\,\cdot\,,t) : [0,T) \to L^2(\Omega)$ is continuous, i.e., $u \in C([0,T) \to L^2(\Omega))$, and satisfies the initial condition $u(0) = u_0$. In addition, we require also

$$u \in L^p\left((0,T) \to W_0^{1,p}(\Omega)\right) \cap W^{1,p'}\left((0,T) \to W^{-1,p'}(\Omega)\right).$$

(**u2**) The superposition function $(x,t) \mapsto f(x,t,u(x,t)) : \Omega \times (0,T) \to \mathbb{R}$ is locally Lebesgue-integrable, i.e., $f(\,\cdot\,,\,\cdot\,,u) \in L^1_{\mathrm{loc}}(\Omega \times (0,T))$.

(**u3**) The equation

$$
\begin{aligned}
& \int_\Omega u(x,t)\,\phi(x,t)\,\mathrm{d}x - \int_0^t \int_\Omega u(x,s)\,\frac{\partial \phi}{\partial t}\,\mathrm{d}x\,\mathrm{d}s \\
& = -\int_0^t \int_\Omega |\nabla u|^{p-2} (\nabla u \cdot \nabla \phi)\,\mathrm{d}x\,\mathrm{d}s \\
& + \int_0^t \int_\Omega f(x,s,u(x,s))\,\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned} \tag{2.1}
$$

is satisfied for all $t \in (0,T)$ and for all test functions $\phi : \Omega \times [0,T] \to \mathbb{R}$ of class C^1 with a compact support contained in $\Omega \times (0,T)$, i.e., $\phi \in C^1_{\mathrm{c}}(\Omega \times (0,T))$.

Given $1 \le p < N$, we denote by $p^* = Np/(N-p)$ the critical Sobolev exponent for the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. If $N \le p < \infty$, we take $p^* \in (p,\infty)$ arbitrary and as large as needed.

Throughout the rest of this article we impose the following restrictions on the numbers p, q and r: We assume $p,q,r \in [2,\infty)$ and

$$
p \le q < \min\left\{ p^*,\, p\left(1 + \frac{r}{N}\right) \right\}. \tag{2.2}
$$

We set

$$
r_{p,q} \stackrel{\mathrm{def}}{=} (q-p)\,\frac{N}{p} \tag{2.3}
$$

and observe that, owing to $p \le q < p^*$, we have $0 \le r_{p,q} < q$. Therefore, without loss of generality, in all our results below we may restrict ourselves to the case $r_{p,q} < r \le q$. Consequently, (2.2) may be replaced by

$$
\max\{p,r\} \le q < p\left(1 + \frac{r}{N}\right). \tag{2.4}
$$

It is easy to see that (2.4) implies (2.2). All our results below remain valid if r is replaced by any number $\overline{r} \ge r$ while p and q are held fixed. In case $r_{p,q} \ge 2$, we may choose r arbitrarily close to $r_{p,q}$.

The following theorem is our main result.

Theorem 2.1 *Let $p,q,r \in [2,\infty)$ verify (2.4). Assume that f satisfies hypotheses* (**f1**) *and* (**f2**). *Let u be a weak solution of problem* (1.1) *such that*

$$
\|u(\,\cdot\,,t)\|_{L^r(\Omega)} \le c \equiv \mathrm{const} < \infty \quad \text{for all } 0 \le t < T. \tag{2.5}
$$

Then we have $u(\cdot\,,t) \in L^\infty(\Omega)$ *for every* $t \in (0,T)$. *Moreover, if* $\varepsilon \in (0,T)$ *is arbitrary, there exists a constant* $C_\varepsilon \geq 0$ *such that*

$$\|u(\cdot\,,t)\|_{L^\infty(\Omega)} \leq C_\varepsilon \quad \textit{for all times } t \in [\varepsilon, T). \tag{2.6}$$

Proof. The proof is given in Sections 3 through 7 below. The article concludes with Appendix A which contains the antiderivatives of a few important elementary functions.

The conclusion of this theorem remains valid under somewhat weaker hypotheses on the function u described in Remark 2.4 below.

In Porzio [25], Theorem 2.1 was proved only for $r = 2$ and by a completely different method; see also the monograph by DiBenedetto [15, Chapt. V, Theorem 3.2, p. 121]. Under this restriction, condition (2.4) reads

$$p \leq q < p\left(1 + \frac{2}{N}\right). \tag{2.7}$$

This restriction on the exponent q in the growth condition (1.2) is much too strong in case the a priori bound (2.5) is available for some $r \geq 2$. In Filo [16] and Lieberman [24, pp. 561–562], Theorem 2.1 was proved under the same hypotheses as ours (2.4) for $p \leq N$, whereas for $p > N$ both authors needed a condition stronger than (2.4), namely,

$$p \leq q < p + r \quad \text{with} \quad p + r < p\left(1 + \frac{r}{N}\right). \tag{2.8}$$

We postpone a more detailed discussion on these differences, optimality of condition (2.4), and possible generalizations to other degenerate parabolic problems ([16, 24, 26]) until Section 9.

In the stationary case, i.e., when the functions u and f in Theorem 2.1 are independent from time t, our theorem renders the following slight improvement of a result due to Anane [3, Théorème A.1, p. 96].

Corollary 2.2 *Let* $p, q, r \in [2, \infty)$ *verify* (2.4). *Assume that* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function such that* $f(\cdot\,,\xi) \in L^1_{\mathrm{loc}}(\Omega)$ *for every* $\xi \in \mathbb{R}$, *and the following inequality holds in analogy with* (1.2)*:*

$$\xi \cdot f(x,\xi) \leq a|\xi|^q + b|\xi| \quad \textit{for a.e. } x \in \Omega \textit{ and all } \xi \in \mathbb{R}.$$

Assume that $u \in W_0^{1,p}(\Omega)$ *satisfies* $f(\cdot\,,u) \in L^1_{\mathrm{loc}}(\Omega)$ *and*

$$\int_\Omega |\nabla u|^{p-2}(\nabla u \cdot \nabla \phi)\,\mathrm{d}x = \int_\Omega f(x,u(x))\,\phi\,\mathrm{d}x \quad \textit{for all } \phi \in C^1_{\mathrm{c}}(\Omega).$$

Then $u \in L^\infty(\Omega)$ *and there exists a constant* $C_0 > 0$ *such that* $\|u\|_{L^\infty(\Omega)} \leq C_0$, *where* C_0 *depends solely upon* a, b, N, p, r, *and the norm* $\|u\|_{L^r(\Omega)}$.

The corresponding analogue of Remark 2.4 applies to this corollary as well.

Some remarks are in order.

Remark 2.3 As far as the definition of a weak solution of problem (1.1) is concerned, while still requiring conditions (**u1**) and (**u2**), we may replace (**u3**) by the following equivalent condition:

(**u3'**) The equation

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(x,t)\,\phi(x,t)\,\mathrm{d}x - \int_\Omega u(x,t)\,\frac{\partial \phi}{\partial t}\,\mathrm{d}x \\ &= -\int_\Omega |\nabla u|^{p-2} (\nabla u \cdot \nabla \phi)\,\mathrm{d}x + \int_\Omega f(x,t,u(x,t))\,\phi(x,t)\,\mathrm{d}x \end{aligned} \tag{2.9}$$

is satisfied for almost every $t \in (0,T)$ and for all test functions $\phi : \Omega \times [0,T] \to \mathbb{R}$ of class C^1 with a compact support contained in $\Omega \times [0,T]$, i.e., $\phi \in C^1_{\mathrm{c}}(\Omega \times [0,T])$.

By the density of the space of test functions $C^1_{\mathrm{c}}(\Omega \times [0,T])$ in the Fréchet space $C([0,T) \to L^2(\Omega))$ and in the Banach spaces $L^p\left((0,T) \to W_0^{1,p}(\Omega)\right)$ and $W^{1,p'}\left((0,T) \to W^{-1,p'}(\Omega)\right)$, we conclude that equation (2.9) entails

$$\begin{aligned} \int_\Omega \frac{\partial u}{\partial t}\,\phi(x)\,\mathrm{d}x = &-\int_\Omega |\nabla u|^{p-2} (\nabla u \cdot \nabla \phi)\,\mathrm{d}x \\ &+ \int_\Omega f(x,t,u(x,t))\,\phi(x)\,\mathrm{d}x \end{aligned} \tag{2.10}$$

for almost every $t \in (0,T)$ and for all $\phi \in C^1_{\mathrm{c}}(\Omega)$.

Remark 2.4 It will become clear from our proof of Theorem 2.1 below that the conclusion of this theorem remains valid even if condition (**u2**) is replaced by the growth condition (1.2) and condition (**u3**) is replaced by the following weaker one:

(**u3"**) The inequality

$$\begin{aligned} \int_\Omega \frac{\partial u}{\partial t}\,u(x,t)\,\phi(x)\,\mathrm{d}x + &\int_\Omega |\nabla u|^{p-2} (\nabla u \cdot \nabla (u\phi))\,\mathrm{d}x \\ &\le \int_\Omega f(x,t,u(x,t))\,u(x,t)\,\phi(x)\,\mathrm{d}x \end{aligned} \tag{2.11}$$

holds true for almost every $t \in (0,T)$ and for all $\phi \in C^1(\overline{\Omega})$ satisfying $\phi \ge 0$ in Ω.

The corresponding statement applies also to Corollary 2.2.

Remark 2.5 The growth condition (1.2) is too weak to guarantee the existence of a weak solution to problem (1.1) in Theorem 2.1. The following stronger condition however does guarantee the existence, locally in time, for any initial data $u_0 \in L^r(\Omega)$:

$$|f(x,t,\xi)| \le a|\xi|^{q-1} + b \quad \text{for a.e. } x \in \Omega \text{ and all } t \in (0,T'),\ \xi \in \mathbb{R}, \tag{2.12}$$

where $a > 0$ and $b \ge 0$ are some constants and $T' \in (0,T]$ is small enough. This claim follows from a combination of Theorem III.2.6, p. 140, in Barbu [4] and the proof of Theorem 3.10.1, p. 189, in Vrabie [27]. An alternative approach from Boccardo, Murat, and Puel [5, 6] or Cirmi and Porzio [9] based on a priori L^∞-bounds for any solution can be adapted also to the present setting to obtain local (in time) existence. In our present article we establish such a priori bounds. We leave the details to the reader.

3 Ultracontractive C^0-semigroups ($p = 2$)

The idea of using the logarithmic Sobolev inequalities in order to establish an "L^p-to-L^q" smoothing effect ($1 < p < q < \infty$) of an analytic semigroup of bounded linear operators on $L^2(\Omega)$, modelling a diffusion process in an arbitrary domain $\Omega \subset \mathbb{R}^N$, is due to L. Gross [20, p. 1066], the proof of Theorem 1. In the paper of Davies and Simon [12], this idea was developed further: An equivalence relation was established between the "L^2-to-L^∞" smoothing effect of the semigroup e^{-Ht} ($t \ge 0$) on $L^2(\Omega)$, termed ultracontractivity (see (u.c.)), and the corresponding logarithmic Sobolev inequality for the (infinitesimal) generator $-H$ of this semigroup. To describe this phenomenon in details below, we use the monograph by Davies [11, Chapt. 2, pp. 59–81].

Let e^{-Ht} ($t \ge 0$) be a symmetric Markov C^0-semigroup on $L^2(\Omega)$ with the generator $-H$. The reader is referred to [11, Chapt. 1, § 4, pp. 21–25] for the definition and basic properties of a symmetric Markov semigroup. In particular, H is a positive definite, selfadjoint operator on $L^2(\Omega)$. Assume that the semigroup e^{-Ht} ($t \ge 0$) is ultracontractive with

$$\|\mathrm{e}^{-Ht}\|_{L^2\to L^\infty} \le \mathrm{e}^{M(t)} \quad \text{for all } t > 0, \tag{3.1}$$

where $M : (0,\infty) \to \mathbb{R}$ is a monotonically decreasing continuous function. Here, $\|\mathrm{e}^{-Ht}\|_{L^2\to L^\infty}$ denotes the norm of the bounded linear operator e^{-Ht} from $L^2(\Omega)$ to $L^\infty(\Omega)$. Let Q denote the quadratic form associated with the operator H,

$$Q(f) \stackrel{\text{def}}{=} \int_\Omega (Hf)\,\overline{f}\,\mathrm{d}x, \quad f \in \mathrm{dom}(H),$$

where $\mathrm{dom}(H)$ stands for the domain of H. The domain of Q is the Friedrichs energy space $\mathrm{Quad}(H)$. Then $0 \le f \in \mathrm{Quad}(H) \cap L^1(\Omega) \cap L^\infty(\Omega)$ implies $f^2 \log f \in L^1(\Omega)$ and the logarithmic Sobolev inequality

$$\int_\Omega f^2 \log f\,\mathrm{d}x \le \varepsilon\,Q(f) + M(\varepsilon)\,\|f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \log \|f\|_{L^2(\Omega)} \tag{3.2}$$

is valid for all $\varepsilon > 0$. This result is taken from [11, Theorem 2.2.3, p. 64].

For each $t > 0$, e^{-Ht} is an integral operator on $L^2(\Omega)$,

$$(\mathrm{e}^{-Ht} f)(x) = \int_\Omega K(x,y;t) f(y)\,\mathrm{d}x, \quad x \in \Omega,\ f \in L^2(\Omega), \tag{3.3}$$

with a nonnegative kernel $K : \Omega \times \Omega \times (0,\infty) \to \mathbb{R}$, by [11, Lemma 2.1.2, p. 59]. Moreover, if $\Omega \subset \mathbb{R}^N$ has finite N-dimensional Lebesgue measure $|\Omega|_N$, then the kernel has the representation

$$0 \le K(x,y;t) = \sum_{n=1}^{\infty} \exp(-E_n t)\, \phi_n(x)\, \phi_n(y) \tag{3.4}$$

where the infinite series converges uniformly on $\Omega \times \Omega \times [\alpha, \infty)$ for any $\alpha > 0$; see [11, Theorem 2.1.4, p. 60]. Here, $E_1, E_2, \dots, E_n, \dots$ are the eigenvalues of H, repeated according to their multiplicity, with the associated eigenfunctions ϕ_n ($n = 1, 2, \dots$).

For the Dirichlet Laplacian $H = -\Delta$ in $L^2(\Omega)$ we have

$$0 \le K(x,y;t) \le (4\pi t)^{-N/2}\, \mathrm{e}^{-(x-y)^2/4t} \quad \text{for } x, y \in \Omega \text{ and } t > 0, \tag{3.5}$$

see Davies [11, Example 2.1.9, p. 63]. Consequently,

$$\|\mathrm{e}^{-Ht}\|_{L^1 \to L^\infty} \le (4\pi t)^{-N/2} \tag{3.6}$$

which yields, by [11, Lemma 2.1.2, p. 59],

$$\|\mathrm{e}^{-Ht}\|_{L^2 \to L^\infty} \le (8\pi t)^{-N/4} \quad \text{for all } t > 0. \tag{3.7}$$

Hence, inequality (3.1) holds with $M(t) = -(N/4)\log(8\pi t)$ and, whenever $0 \le f \in W_0^{1,2}(\Omega) \cap L^1(\Omega) \cap L^\infty(\Omega)$, then also $f^2 \log f \in L^1(\Omega)$ and the logarithmic Sobolev inequality (3.2) becomes

$$\begin{aligned} \int_\Omega f^2 \log f\,\mathrm{d}x \le \varepsilon \int_\Omega |\nabla f|^2\,\mathrm{d}x - \frac{N}{4}\log(8\pi\varepsilon)\,\|f\|^2_{L^2(\Omega)} \\ + \|f\|^2_{L^2(\Omega)} \log \|f\|_{L^2(\Omega)} \end{aligned} \tag{3.8}$$

for all $\varepsilon > 0$. It is an easy exercise to show that if inequality (3.8) holds for $\varepsilon = 1$ then it holds for all $\varepsilon > 0$ as well. Indeed, one may use the substitution $x = \varepsilon^{1/2} y$ in $\mathbb{R}^N$ as the constants in this inequality are independent from the domain $\Omega \subset \mathbb{R}^N$.

Finally, given any $2 \le r < \infty$, from inequality (3.2) one can derive

$$\begin{aligned} \int_\Omega g^r \log g\,\mathrm{d}x \le \varepsilon \int_\Omega (Hg)\, g^{r-1}\,\mathrm{d}x + \frac{2}{r} M\!\left(\frac{2\varepsilon}{r'}\right) \|g\|^r_{L^r(\Omega)} \\ + \|g\|^r_{L^r(\Omega)} \log \|g\|_{L^r(\Omega)} \end{aligned} \tag{3.9}$$

for every $g \in \mathcal{D}_+ \stackrel{\text{def}}{=} \bigcup_{t>0} \mathrm{e}^{-Ht}(L^1(\Omega) \cap L^\infty(\Omega))_+$; see the proof of Lemma 2.2.6 in [11, p. 67]. As usual, $r' = r/(r-1)$ and $X_+ = \{f \in X : f \ge 0\}$ denotes the positive cone in an ordered Banach space X.

For the Dirichlet Laplacian $H = -\Delta$ in $L^2(\Omega)$ inequality (3.9) becomes

$$\int_\Omega g^r \log g \,\mathrm{d}x \le \varepsilon \int_\Omega |\nabla g|^2 g^{r-2} \,\mathrm{d}x - \frac{N}{2r} \log\left(\frac{16\pi\varepsilon}{r}\right) \|g\|^r_{L^r(\Omega)} + \|g\|^r_{L^r(\Omega)} \log \|g\|_{L^r(\Omega)} \tag{3.10}$$

for all $2 \le r < \infty$ and for every $g \in \mathcal{D}_+ = \bigcup_{t>0} \mathrm{e}^{\Delta t}(L^1(\Omega) \cap L^\infty(\Omega))_+$. This inequality follows also directly from (3.8) by setting $f = g^{r/2}$.

4 A time-dependent norm of the solution

We wish to apply the logarithmic Sobolev inequality (3.10) to inequality (2.11) with the test function ϕ replaced by the new, time-dependent function $|u(\,\cdot\,,t)|^{\varrho(t)}$, where $\varrho : [0,T) \to \mathbb{R}_+$ is a continuously differentiable function with the derivative $\varrho'(t) > 0$ for all $0 \le t < T$. As usual, we set $\mathbb{R}_+ = [0,\infty)$. The function ϱ will be specified later, in Section 7.

We need the following technical identity which is essentially due to Gross [20, Lemma 1.1, p. 1065]; see also Cipriani and Grillo [8, Lemma 3.2, p. 220] or Davies [11, Lemma 2.2.2, p. 64].

Lemma 4.1 *Let $r : [0,T) \to [2,\infty)$ be a continuously differentiable function, where $0 < T \le \infty$. Assume that $g : [0,T) \to L^2(\Omega)$ verifies condition* (**u1**) *in place of u and, in addition, $g(t) \in L^\infty(\Omega)$ for every $t \in [0,T)$ and the norm $\|g(t)\|_{L^\infty(\Omega)}$ is bounded in $[0,T)$. Then the function $t \mapsto \|g(t)\|^{r(t)}_{L^{r(t)}(\Omega)}$ is locally absolutely continuous in $[0,T)$ and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|g(t)\|^{r(t)}_{L^{r(t)}(\Omega)} = r(t) \int_\Omega |g(t)|^{r(t)-2} g(t)\, g'(t) \,\mathrm{d}x + r'(t) \int_\Omega |g(t)|^{r(t)} \log |g(t)| \,\mathrm{d}x \tag{4.1}$$

for a.e. $t \in (0,T)$.

Here, we have used the notation $g'(t) = \frac{\partial g}{\partial t}(\,\cdot\,,t)$. As usual, we identify the spaces $L^1([0,T) \to L^1(\Omega)) \equiv L^1(\Omega \times [0,T))$ and similar ones.

Let u be any weak solution of problem (1.1). By Remark 2.4, u satisfies inequality (2.11) with the test function $\phi \in C^1(\overline{\Omega})$, $\phi \ge 0$ in Ω, chosen as follows:

First, let $n \in \mathbb{N} = \{1, 2, \dots\}$ be arbitrary, but fixed. For $(x,t) \in \Omega \times [0,T)$ define

$$u^{(n)}(x,t) \stackrel{\text{def}}{=} \begin{cases} -n & \text{if } u(x,t) < -n; \\ u(x,t) & \text{if } |u(x,t)| \le n; \\ n & \text{if } u(x,t) > n. \end{cases} \tag{4.2}$$

With a help from [19, Theorem 7.8, p. 153], it is easy to see that also $u^{(n)}$ verifies condition (**u1**). Hence, there exists a sequence of functions $\{\psi_m^{(n)}\}_{m=1}^{\infty} \subset C^1\left(\overline{\Omega} \times [0,T]\right)$ such that

- $0 \le \psi_m^{(n)}(x,t) \le n^{\varrho(t)}$ for all $m = 1, 2, \dots$ and $(x,t) \in \Omega \times [0,T)$;
- $u\,\psi_m^{(n)} \to u^{(n)}\,|u^{(n)}|^{\varrho}$ as $m \to \infty$, in the Banach space
$$C([0,T'] \to L^2(\Omega)) \cap L^p\left((0,T') \to W_0^{1,p}(\Omega)\right) \qquad \text{for each } T' \in (0,T);$$
- $\psi_m^{(n)}(x,t) \to |u^{(n)}(x,t)|^{\varrho(t)}$ as $m \to \infty$, pointwise for almost all $(x,t) \in \Omega \times [0,T)$.

We replace ϕ in inequality (2.11) by $\psi_m^{(n)}(\,\cdot\,,t)$, for a.e. $t \in (0,T)$, and then let $m \to \infty$, thus arriving at
$$\begin{aligned}&\int_\Omega \frac{\partial u}{\partial t}\, u^{(n)}(x,t)\,|u^{(n)}(x,t)|^{\varrho(t)}\,\mathrm{d}x\\ &+ \int_\Omega |\nabla u|^{p-2}\left[\nabla u \cdot \nabla(u^{(n)}\,|u^{(n)}|^{\varrho})\right]\mathrm{d}x\\ &\le \int_\Omega f(x,t,u(x,t))\,u^{(n)}(x,t)\,|u^{(n)}(x,t)|^{\varrho(t)}\,\mathrm{d}x\end{aligned}$$
for almost every $t \in (0,T)$ and for all $n = 1, 2, \dots$. By our definition of $u^{(n)}$, combined with [19, Theorem 7.8, p. 153], the last inequality finally reads
$$\begin{aligned}&\int_\Omega \frac{\partial u^{(n)}}{\partial t}\, u^{(n)}(x,t)\,|u^{(n)}(x,t)|^{\varrho(t)}\,\mathrm{d}x\\ &+ \int_\Omega |\nabla u^{(n)}|^{p-2}\left[\nabla u^{(n)} \cdot \nabla(|u^{(n)}|^{\varrho}\,u^{(n)})\right]\mathrm{d}x\\ &\le \int_\Omega f(x,t,u^{(n)}(x,t))\,u^{(n)}(x,t)\,|u^{(n)}(x,t)|^{\varrho(t)}\,\mathrm{d}x\end{aligned} \tag{4.3}$$
for almost every $t \in (0,T)$ and for all $n = 1, 2, \dots$.

Finally, in Lemma 4.1 let us take $q(t) = 2 + \varrho(t)$ for $0 \le t < T$. We apply Lemma 4.1 to the first term on the left-hand side in (4.3), the identity
$$\nabla(|u^{(n)}|^{\varrho}\,u^{(n)}) = (1+\varrho)|u^{(n)}|^{\varrho}\,\nabla u^{(n)}$$
to the second one, and the growth condition (1.2) to the term on the right-hand side, thus arriving at the following estimate:
$$\begin{aligned}&\frac{1}{2+\varrho}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u^{(n)}(x,t)|^{2+\varrho(t)}\,\mathrm{d}x\\ &= \int_\Omega \frac{\partial u^{(n)}}{\partial t}\,u^{(n)}\,|u^{(n)}|^{\varrho}\,\mathrm{d}x + \frac{\varrho'(t)}{2+\varrho(t)}\int_\Omega |u^{(n)}|^{2+\varrho}\,\log|u^{(n)}|\,\mathrm{d}x\\ &\le -(1+\varrho)\int_\Omega |\nabla u^{(n)}|^p\,|u^{(n)}|^{\varrho}\,\mathrm{d}x + \int_\Omega |u^{(n)}|^{\varrho}\left(a\,|u^{(n)}|^q + b\,|u^{(n)}|\right)\mathrm{d}x\\ &+ \frac{\varrho'(t)}{2+\varrho(t)}\int_\Omega |u^{(n)}|^{2+\varrho}\,\log|u^{(n)}|\,\mathrm{d}x\end{aligned}$$

for a.e. $t \in (0,T)$.

In order to simplify our notation, we drop the upper index "(n)" in $u^{(n)}(x,t)$ and write only $u(x,t)$ instead, assuming only that the function u verifies condition (**u1**), $|u| \leq n$ in $\Omega \times [0,T)$ for some $n \in \mathbb{N}$, and inequality (4.4) holds for a.e. $t \in (0,T)$. Hence, the last inequality reads as follows (cf. CIPRIANI and GRILLO [8, Lemma 3.5, p. 223]).

Lemma 4.2 *For each $n = 1, 2, \dots,$ the function $u = u^{(n)}$ verifies condition* (**u1**) *and the inequality*

$$\begin{aligned} &\frac{1}{2+\varrho} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u(x,t)|^{2+\varrho(t)}\, \mathrm{d}x + (1+\varrho) \int_\Omega |\nabla u|^p\, |u|^\varrho\, \mathrm{d}x \\ &\leq \int_\Omega |u|^\varrho \left(a\, |u|^q + b\, |u|\right)\, \mathrm{d}x + \frac{\varrho'(t)}{2+\varrho(t)} \int_\Omega |u|^{2+\varrho} \log |u|\, \mathrm{d}x \end{aligned} \tag{4.4}$$

holds for a.e. $t \in (0,T)$.

The two summands on the right-hand side of (4.4) will be estimated next, in Sections 5 and 6. Recall that $\varrho : [0,T) \to \mathbb{R}_+$ is a continuously differentiable function with $\varrho'(t) > 0$ for all $0 \leq t < T$.

5 L^r-estimates for $r \geq p \geq 2$

Recalling our hypothesis (2.4), let us denote $\varrho_0 \stackrel{\text{def}}{=} r - 2\ (\geq 0)$ and consider any $\varrho \in [\varrho_0, \infty)$. We introduce the abbreviations

$$E \equiv E(\varrho) \stackrel{\text{def}}{=} \frac{q-2}{p\left(1 + \frac{2+\varrho}{N}\right) - q} \quad \text{and} \quad E_0 \stackrel{\text{def}}{=} E(\varrho_0) = \frac{q-2}{p\left(1 + \frac{r}{N}\right) - q}\,. \tag{5.1}$$

Clearly, we have $0 \leq E \leq E_0 < \infty$ and $E(\varrho) \searrow 0$ as $\varrho \nearrow \infty$.

We begin by estimating the integral $\int_\Omega |u|^{q+\varrho}\, \mathrm{d}x$ on the right-hand side of inequality (4.4).

Lemma 5.1 *Let $a \in (0,\infty)$ and let $p, q, r \in [2,\infty)$ verify* (2.4)*. Then there exists a constant $C_1 > 0$ such that the estimate*

$$a \int_\Omega |u|^{q+\varrho}\, \mathrm{d}x \leq \frac{1}{4}(1+\varrho) \int_\Omega |\nabla u|^p\, |u|^\varrho\, \mathrm{d}x + C_1\, \|u\|_{L^{2+\varrho}(\Omega)}^{q+\varrho+(q-p)E(\varrho)} \tag{5.2}$$

holds for every $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for all $\varrho \in [\varrho_0, \infty)$.

Proof. We use inequalities of Gagliardo-Nirenberg-type; see, e.g., ADAMS and FOURNIER [1], Chapt. 5, or A. FRIEDMAN [17], Part 1, Theorem 9.3, p. 24. We treat only the case $q > 2$, leaving a number of obvious amendments for $q = 2$ to the reader. Recall that $2 \le p \le q < p^*$. There exists a constant $c_{p,q} > 0$ such that the following inequality holds, whenever $2 \le R \le p \le Q \le q$:

$$\|v\|_{L^Q(\Omega)} \le c_{p,q} \, \|\nabla v\|^{\theta}_{L^p(\Omega)} \cdot \|v\|^{1-\theta}_{L^R(\Omega)} \quad \text{for all } v \in W^{1,p}_0(\Omega) \tag{5.3}$$

where

$$\theta \stackrel{\text{def}}{=} \left(\frac{1}{R} - \frac{1}{Q} \right) \left(\frac{1}{R} - \frac{1}{p} + \frac{1}{N} \right)^{-1} . \tag{5.4}$$

Notice that $0 \le \theta < 1$ (with $\theta > 0$ if $R < Q$) and

$$\frac{1}{Q} = \theta \left(\frac{1}{p} - \frac{1}{N} \right) + \frac{1-\theta}{R} .$$

Now we rewrite inequality (5.3) as

$$\int_\Omega |v|^Q \, \mathrm{d}x \le c_{p,q}^Q \left(\int_\Omega |\nabla v|^p \, \mathrm{d}x \right)^{\theta Q/p} \left(\int_\Omega |v|^R \, \mathrm{d}x \right)^{(1-\theta)Q/R} \tag{5.5}$$

for all $v \in W^{1,p}_0(\Omega)$.

Next, let us take any $\varrho \in \mathbb{R}_+$ and $u \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. We substitute

$$Q = \frac{q+\varrho}{1+\frac{\varrho}{p}} = p \cdot \frac{q+\varrho}{p+\varrho} \quad \text{and} \quad R = \frac{2+\varrho}{1+\frac{\varrho}{p}} = p \cdot \frac{2+\varrho}{p+\varrho}$$

together with $v = |u|^{\varrho/p} u$ in inequality (5.5), thus arriving at

$$\begin{aligned} \int_\Omega |u|^{q+\varrho} \, \mathrm{d}x \le{}& c_{p,q}^{p(q+\varrho)/(p+\varrho)} \left(1 + \frac{\varrho}{p} \right)^{\theta p(q+\varrho)/(p+\varrho)} \\ & \times \left(\int_\Omega |\nabla u|^p \, |u|^\varrho \, \mathrm{d}x \right)^{\theta(q+\varrho)/(p+\varrho)} \|u\|^{(1-\theta)(q+\varrho)}_{L^{2+\varrho}(\Omega)} \end{aligned} \tag{5.6}$$

where, by equation (5.4),

$$\theta = \frac{1}{q+\varrho} \cdot \frac{q-2}{1 - \frac{2+\varrho}{p+\varrho}\left(1 - \frac{p}{N}\right)} = \frac{1}{q+\varrho} \cdot \frac{q-2}{p\left(1 + \frac{2+\varrho}{N}\right) - 2} . \tag{5.7}$$

Again, notice that $0 < \theta < 1$ ($\theta > 0$ because $q > 2$) and

$$\frac{1}{q+\varrho} = \theta \cdot \frac{1 - \frac{p}{N}}{p+\varrho} + \frac{1-\theta}{2+\varrho} .$$

We have $E > 0$ owing to $q > 2$. Observe that the reciprocal exponent of the integral on the right-hand side of (5.6),

$$P \stackrel{\text{def}}{=} \frac{p+\varrho}{\theta(q+\varrho)} = \frac{p\left(1 + \frac{2+\varrho}{N}\right) - 2}{q-2} = 1 + \frac{p\left(1 + \frac{2+\varrho}{N}\right) - q}{q-2} = 1 + \frac{1}{E} , \tag{5.8}$$

satisfies

$$P \geq P_0 \stackrel{\text{def}}{=} 1 + \frac{p\left(1+\frac{r}{N}\right)-q}{q-2} = 1 + \frac{1}{E_0} > 1.$$

Its conjugate exponent is qiven by

$$P' = \frac{P}{P-1} = 1 + \frac{q-2}{p\left(1+\frac{2+\varrho}{N}\right)-q} = 1 + E \tag{5.9}$$

and satisfies

$$P' \leq P_0' = \frac{P_0}{P_0-1} = 1 + \frac{q-2}{p\left(1+\frac{r}{N}\right)-q} = 1 + E_0 < \infty.$$

Below we will need also the exponent

$$(1-\theta)(q+\varrho)P' = q+\varrho+\frac{(q-p)(q-2)}{p\left(1+\frac{2+\varrho}{N}\right)-q} = q+\varrho+(q-p)E. \tag{5.10}$$

We estimate the right-hand side of (5.6) by Young's inequality using the pair of conjugate exponents (P,P'), for any $0<\varepsilon<\infty$,

$$\begin{aligned}&\int_\Omega |u|^{q+\varrho}\,\mathrm{d}x \leq \\ &c_{p,q}^{p(q+\varrho)/(p+\varrho)}\left[\frac{\varepsilon}{P}\left(1+\frac{\varrho}{p}\right)^p\int_\Omega |\nabla u|^p\,|u|^\varrho\,\mathrm{d}x + \frac{\varepsilon^{-P'/P}}{P'}\,\|u\|_{L^{2+\varrho}(\Omega)}^{q+\varrho+(q-p)E}\right].\end{aligned} \tag{5.11}$$

Here, formulas (5.8), (5.9) and (5.10) have been employed. We choose

$$\varepsilon = \frac{(1+\varrho)P}{4\,a\,c_{p,q}^{p(q+\varrho)/(p+\varrho)}}\left(1+\frac{\varrho}{p}\right)^{-p} \tag{5.12}$$

and observe that

$$\begin{aligned}&a\,c_{p,q}^{p(q+\varrho)/(p+\varrho)}\,\frac{\varepsilon^{-P'/P}}{P'}\\ &= \frac{1}{1+E}\left(a\,c_{p,q}^{p(q+\varrho)/(p+\varrho)}\right)^{1+E}\left(\frac{4}{(1+\varrho)(1+E^{-1})}\right)^E\left(1+\frac{\varrho}{p}\right)^{pE}\\ &\leq \frac{1}{4}\left(4\,a\,c_{p,q}^{p(q+\varrho)/(p+\varrho)}\right)^{1+E}\left(1+\frac{\varrho}{p}\right)^{pE} \leq C_1 < \infty\end{aligned} \tag{5.13}$$

with a constant C_1 independent from $\varrho \geq \varrho_0$. The last claim follows from

$$pE\left(1+\frac{\varrho}{p}\right) = \frac{(q-2)(p+\varrho)}{p\left(1+\frac{2+\varrho}{N}\right)-q},$$

where

$$\begin{aligned}\frac{p+\varrho}{p\left(1+\frac{2+\varrho}{N}\right)-q} &\in \left[\frac{N}{p},\,\frac{p+r-2}{p\left(1+\frac{r}{N}\right)-q}\right]\\ &= \left[\frac{N}{p},\,\frac{N}{p}\left(1+\frac{q-p+\frac{p}{N}(p-2)}{p\left(1+\frac{r}{N}\right)-q}\right)\right],\end{aligned}$$

combined with

$$\left(1+\frac{\varrho}{p}\right)^{1/[1+(\varrho/p)]} \longrightarrow 1 \quad \text{as } \varrho \searrow 0 \text{ or } \varrho \nearrow \infty.$$

Finally, applying (5.12) and (5.13) to the right-hand side of (5.11) we arrive at (5.2) as desired. ■

Using Hölder's inequality we find

$$\begin{aligned}\int_\Omega |u|^{1+\varrho}\,\mathrm{d}x &\le \left(\int_\Omega |u|^{2+\varrho}\,\mathrm{d}x\right)^{(1+\varrho)/(2+\varrho)} \left(\int_\Omega \mathrm{d}x\right)^{1/(2+\varrho)} \\ &= |\Omega|_N^{1/(2+\varrho)}\, \|u\|_{L^{2+\varrho}(\Omega)}^{1+\varrho}\end{aligned} \tag{5.14}$$

where $|\Omega|_N = \int_\Omega \mathrm{d}x$ denotes the N-dimensional Lebesgue measure of $\Omega \subset \mathbb{R}^N$. Finally, we combine inequalities (5.2) and (5.14) to get

$$\begin{aligned}&\int_\Omega |u|^\varrho (a|u|^q + b|u|)\,\mathrm{d}x \\ &\le \frac{1}{4}(1+\varrho)\int_\Omega |\nabla u|^p\,|u|^\varrho\,\mathrm{d}x + C_1\,\|u\|_{L^{2+\varrho}(\Omega)}^{q+\varrho+(q-p)E} + b\,|\Omega|_N^{1/(2+\varrho)}\,\|u\|_{L^{2+\varrho}(\Omega)}^{1+\varrho} \\ &\le \frac{1}{4}(1+\varrho)\int_\Omega |\nabla u|^p\,|u|^\varrho\,\mathrm{d}x + C_1\,\|u\|_{L^{2+\varrho}(\Omega)}^{q+\varrho+(q-p)E} + C_2\,\|u\|_{L^{2+\varrho}(\Omega)}^{1+\varrho}\end{aligned} \tag{5.15}$$

where

$$C_2 = b\,(\max\{|\Omega|_N, 1\})^{1/2} \ge 0.$$

6 Logarithmic estimates for $p \ge 2$

Now we estimate the second summand on the right-hand side of (4.4). Recall that $\varrho : [0,T) \to \mathbb{R}_+$ is a continuously differentiable function with $\varrho'(t) > 0$ for all $0 \le t < T$. We treat the semilinear case $p = 2$ first.

Lemma 6.1 *We have*

$$\begin{aligned}&\frac{\varrho'}{2+\varrho}\int_\Omega |u|^{2+\varrho}\,\log|u|\,\mathrm{d}x \\ &\le \frac{1}{4}(1+\varrho)\int_\Omega |\nabla u|^2\,|u|^\varrho\,\mathrm{d}x + \frac{N\varrho'}{2(2+\varrho)^2}\,\log\left(\frac{\varrho'}{4\pi(1+\varrho)}\right)\|u\|_{L^{2+\varrho}(\Omega)}^{2+\varrho} \\ &+ \frac{\varrho'}{2+\varrho}\,\|u\|_{L^{2+\varrho}(\Omega)}^{2+\varrho}\,\log\|u\|_{L^{2+\varrho}(\Omega)}\end{aligned} \tag{6.1}$$

for every $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for all $0 \le t < T$.

Proof. We use the logarithmic Sobolev inequality (3.10) to estimate

$$
\begin{aligned}
&\int_\Omega |u|^{2+\varrho} \log|u| \,\mathrm{d}x \le \varepsilon \int_\Omega |\nabla u|^2 \, |u|^\varrho \,\mathrm{d}x \\
&- \frac{N}{2(2+\varrho)} \log\left(\frac{16\pi\varepsilon}{2+\varrho}\right) \|u\|^{2+\varrho}_{L^{2+\varrho}(\Omega)} + \|u\|^{2+\varrho}_{L^{2+\varrho}(\Omega)} \log \|u\|_{L^{2+\varrho}(\Omega)}
\end{aligned}
\tag{6.2}
$$

for any $0 < \varepsilon < \infty$. Now we take

$$\varepsilon = \frac{(2+\varrho)(1+\varrho)}{4\varrho'(t)} > 0$$

and use the expression

$$\frac{16\pi\varepsilon}{2+\varrho} = \frac{4\pi(1+\varrho)}{\varrho'(t)} > 0$$

to get inequality (6.1). ■

The lemma below for $p > 2$ will be used later with $\varepsilon : [0,T) \to (0,\infty)$ being a continuous function rather than a constant.

Lemma 6.2 *Let $2 < p < \infty$. Then there exists a constant $C_3 > 0$ such that the estimate*

$$
\begin{aligned}
&\frac{\varrho'}{2+\varrho} \int_\Omega |u|^{2+\varrho} \log|u| \,\mathrm{d}x \\
&\le \frac{1}{4}(1+\varrho) \int_\Omega |\nabla u|^p \, |u|^\varrho \,\mathrm{d}x \\
&+ C_3 \left(\frac{\varepsilon\, \varrho'}{(2+\varrho)(1+\varrho)^{2/p}} \right)^{p/(p-2)} \|u\|^{\varrho}_{L^{2+\varrho}(\Omega)} \\
&+ \varphi_\varepsilon(\varrho)\, \varrho' \, \|u\|^{2+\varrho}_{L^{2+\varrho}(\Omega)} + \frac{\varrho'}{2+\varrho} \|u\|^{2+\varrho}_{L^{2+\varrho}(\Omega)} \log \|u\|_{L^{2+\varrho}(\Omega)}
\end{aligned}
\tag{6.3}
$$

holds for every $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and for all $0 \le t < T$ and $0 < \varepsilon < \infty$. Here, we have denoted

$$\varphi_\varepsilon(\varrho) \stackrel{\text{def}}{=} - \frac{N}{2(2+\varrho)^2} \log\left(\frac{16\pi\varepsilon}{2+\varrho}\right) \quad \textit{for } \varrho \in \mathbb{R}_+. \tag{6.4}$$

Proof. Again, we begin with inequality (6.2). The first term on the right-hand side is estimated by Young's inequality,

$$|\nabla u|^2 \le \frac{2}{p}\eta |\nabla u|^p + \frac{p-2}{p}\eta^{-2/(p-2)}$$

for any $0 < \eta < \infty$, which entails

$$\int_\Omega |\nabla u|^2 \, |u|^\varrho \,\mathrm{d}x \le \frac{2}{p}\eta \int_\Omega |\nabla u|^p \, |u|^\varrho \,\mathrm{d}x + \frac{p-2}{p}\eta^{-2/(p-2)} \int_\Omega |u|^\varrho \,\mathrm{d}x.$$

Setting

$$\eta = \frac{p}{8\varepsilon}\left(\frac{\varrho'}{(2+\varrho)(1+\varrho)}\right)^{-1}$$

and recalling $\varrho'(t) > 0$ we arrive at

$$\begin{aligned}
&\frac{\varrho'}{2+\varrho}\int_\Omega |\nabla u|^2\, |u|^\varrho \,\mathrm{d}x \\
&\le \frac{1}{4\varepsilon}(1+\varrho)\int_\Omega |\nabla u|^p\, |u|^\varrho \,\mathrm{d}x \\
&+ \frac{p-2}{2}\cdot\frac{\varrho'}{2+\varrho}\left(\frac{p}{8\varepsilon}\right)^{-2/(p-2)}\left(\frac{\varrho'}{(2+\varrho)(1+\varrho)}\right)^{2/(p-2)}\int_\Omega |u|^\varrho \,\mathrm{d}x \\
&\le \frac{1}{4\varepsilon}(1+\varrho)\int_\Omega |\nabla u|^p\, |u|^\varrho \,\mathrm{d}x \\
&+ \frac{p-2}{2}\left(\frac{8\varepsilon}{p}\right)^{2/(p-2)}\left(\frac{\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)}\int_\Omega |u|^\varrho \,\mathrm{d}x.
\end{aligned} \tag{6.5}$$

The last integral is estimated similarly as in (5.14),

$$\begin{aligned}
\int_\Omega |u|^\varrho \,\mathrm{d}x &\le \left(\int_\Omega |u|^{2+\varrho}\,\mathrm{d}x\right)^{\varrho/(2+\varrho)}\left(\int_\Omega \mathrm{d}x\right)^{2/(2+\varrho)} \\
&= |\Omega|_N^{2/(2+\varrho)}\, \|u\|^\varrho_{L^{2+\varrho}(\Omega)}.
\end{aligned} \tag{6.6}$$

Next, from inequalities (6.2), (6.5) and (6.6) we deduce

$$\begin{aligned}
&\frac{\varrho'}{2+\varrho}\int_\Omega |u|^{2+\varrho}\,\log|u| \,\mathrm{d}x \\
&\le \frac{1}{4}(1+\varrho)\int_\Omega |\nabla u|^p\, |u|^\varrho \,\mathrm{d}x \\
&+ \frac{p-2}{2}\left(\frac{8}{p}\right)^{2/(p-2)}\left(\frac{\varepsilon\,\varrho'}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)} |\Omega|_N^{2/(2+\varrho)}\, \|u\|^\varrho_{L^{2+\varrho}(\Omega)} \\
&- \frac{N}{2}\cdot\frac{\varrho'}{(2+\varrho)^2}\,\log\left(\frac{16\pi\varepsilon}{2+\varrho}\right)\|u\|^{2+\varrho}_{L^{2+\varrho}(\Omega)} \\
&+ \frac{\varrho'}{2+\varrho}\,\|u\|^{2+\varrho}_{L^{2+\varrho}(\Omega)}\,\log\|u\|_{L^{2+\varrho}(\Omega)}
\end{aligned}$$

for any $0 < \varepsilon < \infty$. Finally, we arrive at inequality (6.3) by taking the constant

$$C_3 = \frac{p-2}{2}\left(\frac{8}{p}\right)^{2/(p-2)}\cdot\max\{|\Omega|_N, 1\} > 0$$

and the function $\varphi_\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ as defined in (6.4). ■

7 Estimates for the time-dependent norm

Now we estimate the entire right-hand side of (4.4). We take $\varrho : [0,T) \to \mathbb{R}_+$ continuously differentiable with $\varrho'(t) > 0$ for all $0 \le t < T$ and $\varepsilon : [0,T) \to (0, 1/8\pi]$ continuous, but otherwise both ϱ and ε arbitrary, to be specified later. We treat only the case $p > 2$, leaving a few necessary modifications for $p = 2$ to the reader: The estimate (6.3) needs to be replaced by (6.1) and the constant δ below $(0 < \delta \leq 1/8\pi)$ by 4δ.

We add the estimates in (5.15) and (6.3) and apply the result to the right-hand side of (4.4), thus arriving at

$$
\begin{aligned}
&\frac{1}{2+\varrho} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u|^{2+\varrho}\,\mathrm{d}x + \frac{1}{2}(1+\varrho) \int_\Omega |\nabla u|^p\,|u|^\varrho\,\mathrm{d}x \\
&\leq C_1 \|u\|_{L^{2+\varrho}(\Omega)}^{q+\varrho+(q-p)E(\varrho)} + C_2 \|u\|_{L^{2+\varrho}(\Omega)}^{1+\varrho} \\
&+ C_3 \left(\frac{\varepsilon(t)\,\varrho'(t)}{(2+\varrho)(1+\varrho)^{2/p}} \right)^{p/(p-2)} \|u\|_{L^{2+\varrho}(\Omega)}^{\varrho} \\
&+ \varphi_{\varepsilon(t)}(\varrho(t))\,\varrho'(t)\,\|u\|_{L^{2+\varrho}(\Omega)}^{2+\varrho} + \frac{\varrho'(t)}{2+\varrho} \|u\|_{L^{2+\varrho}(\Omega)}^{2+\varrho} \log \|u\|_{L^{2+\varrho}(\Omega)}
\end{aligned}
\tag{7.1}
$$

for a.e. $t \in (0,T)$. Finally, using the identity

$$
\begin{aligned}
&\frac{1}{2+\varrho} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u|^{2+\varrho}\,\mathrm{d}x = \\
&\|u\|_{L^{2+\varrho}(\Omega)}^{1+\varrho} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^{2+\varrho}(\Omega)} + \frac{\varrho'}{2+\varrho} \|u\|_{L^{2+\varrho}(\Omega)}^{2+\varrho} \log \|u\|_{L^{2+\varrho}(\Omega)}
\end{aligned}
$$

for a.e. $t \in (0,T)$, from inequality (7.1) we derive

$$
\begin{aligned}
&\|u\|_{L^{2+\varrho}(\Omega)}^{1+\varrho} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^{2+\varrho}(\Omega)} + \frac{1}{2}(1+\varrho) \int_\Omega |\nabla u|^p\,|u|^\varrho\,\mathrm{d}x \\
&\leq C_1 \|u\|_{L^{2+\varrho}(\Omega)}^{Z(\varrho)+\varrho} + C_2 \|u\|_{L^{2+\varrho}(\Omega)}^{1+\varrho} \\
&+ C_3 \left(\frac{\varepsilon(t)\,\varrho'(t)}{(2+\varrho)(1+\varrho)^{2/p}} \right)^{p/(p-2)} \|u\|_{L^{2+\varrho}(\Omega)}^{\varrho} + \varphi_{\varepsilon(t)}(\varrho(t))\,\varrho'(t)\,\|u\|_{L^{2+\varrho}(\Omega)}^{2+\varrho}
\end{aligned}
\tag{7.2}
$$

for a.e. $t \in (0,T)$. Here, we have denoted

$$
Z \equiv Z(\varrho) \stackrel{\text{def}}{=} q + (q-p)E(\varrho) = q + \frac{(q-p)(q-2)}{p\left(1 + \frac{2+\varrho}{N}\right) - q}
\tag{7.3}
$$

$$
\text{and} \quad Z_0 \stackrel{\text{def}}{=} Z(\varrho_0) = q + \frac{(q-p)(q-2)}{p\left(1 + \frac{r}{N}\right) - q}\,.
\tag{7.4}
$$

Notice that, by (5.1), for all $\varrho \geq \varrho_0$ one has $0 \leq E(\varrho) \leq E_0 < \infty$ which implies $q \leq Z(\varrho) \leq Z_0 < \infty$ as well. It is easy to see that inequality (7.2) entails

$$
\begin{aligned}
&\frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left(\|u\|_{L^{2+\varrho}(\Omega)}^2 \right) + \frac{1}{2}(1+\varrho) \int_\Omega |\nabla u|^p\,|u|^\varrho\,\mathrm{d}x \\
&\leq C_1 \|u\|_{L^{2+\varrho}(\Omega)}^{Z(\varrho)} + C_2 \|u\|_{L^{2+\varrho}(\Omega)} \\
&+ C_3 \left(\frac{\varepsilon(t)\,\varrho'(t)}{(2+\varrho)(1+\varrho)^{2/p}} \right)^{p/(p-2)} + \varphi_{\varepsilon(t)}(\varrho(t))\,\varrho'(t)\,\|u\|_{L^{2+\varrho}(\Omega)}^{2}
\end{aligned}
\tag{7.5}
$$

for a.e. $t \in (0,T)$.

It remains to show that the norm $\nu(t) \stackrel{\text{def}}{=} \|u(\,\cdot\,,t)\|_{L^{2+\varrho(t)}(\Omega)}$ $(0 \le t < T)$ stays bounded for all $0 \le t < S$, where $S \in (0,T)$ is a suitable number and the upper bound on $\nu(t)$ $(0 \le t < S)$ depends solely on an upper bound on

$$U_0 \stackrel{\text{def}}{=} \max\left\{\|u(\,\cdot\,,0)\|_{L^r(\Omega)},\, 1\right\}.$$

Then also

$$\|u(\,\cdot\,,S)\|_{L^\infty(\Omega)} \le \liminf_{t\to S-} \|u(\,\cdot\,,t)\|_{L^{2+\varrho(t)}(\Omega)} < \infty \tag{7.6}$$

as desired. In fact, we will be able to take S arbitrarily small, $0 < S < T$. Subsequently, applying condition (2.5), we may replace U_0 by

$$\max\left\{\sup_{0\le t<T} \|u(\,\cdot\,,t)\|_{L^r(\Omega)},\, 1\right\}$$

and the initial condition for $u(\,\cdot\,,t)$ at $t = 0$ by that at $t = t_0$ for any $t_0 \in [0, T - S)$, thus obaining $u(\,\cdot\,,s) \in L^\infty(\Omega)$ with $\|u(\,\cdot\,,s)\|_{L^\infty(\Omega)}$ uniformly bounded for $S \le s < T$.

Our strategy in estimating $\nu(t)$ $(0 \le t < S)$ is based on a suitable choice of the functions $\varepsilon(t)$ and $\varrho(t)$ in inequality (7.5). But first we weaken and simplify this inequality as follows. By an argument with sub- and supersolutions applied to (7.5) for the square norm $\|u(\,\cdot\,,t)\|^2_{L^{2+\varrho(t)}(\Omega)}$, we have

$$\|u(\,\cdot\,,t)\|^2_{L^{2+\varrho(t)}(\Omega)} \le \tilde U(t) \quad \text{for every } 0 \le t < \min\{T, T_{\max}\}, \tag{7.7}$$

where $\tilde U : [0, T_{\max}) \to \mathbb{R}_+$ is the solution of the initial value problem

$$\begin{cases} \dfrac{1}{2}\cdot\dfrac{\mathrm{d}}{\mathrm{d}t}\tilde U(t) = (C_1 + C_2)\,\tilde U(t)^{Z(\varrho(t))/2} \\ \qquad\qquad + C_3\left(\dfrac{\varepsilon(t)\,\varrho'(t)}{(2+\varrho)(1+\varrho)^{2/p}}\right)^{p/(p-2)} \\ \qquad\qquad + \varphi_{\varepsilon(t)}(\varrho(t))\,\varrho'(t)\,\tilde U(t) \quad \text{for } 0 \le t < T_{\max}; \\ \tilde U(0) = \tilde U_0 \stackrel{\text{def}}{=} \max\left\{\|u(\,\cdot\,,0)\|^2_{L^r(\Omega)},\, 1\right\} \ge 1, \end{cases} \tag{7.8}$$

with $[0, T_{\max})$ being the maximal time interval of existence $(0 < T_{\max} \le \infty)$. This means that if $T_{\max} < \infty$ then

$$\tilde U(t) \nearrow \infty \quad \text{as} \quad t \nearrow T_{\max}.$$

Since we are interested in S small enough only, we may replace T by $\min\{T, T_{\max}\}$ without loss of generality, i.e., we may assume $T \le T_{\max}$. Notice that, by our hypotheses $\varrho'(t) > 0$ and $0 < 8\pi\varepsilon(t) \le 1$, combined with $C_1 > 0$ (Lemma 5.1), we have $\tilde U'(t) > 0$ for all $0 \le t < T$, and therefore also $\tilde U(t) > 1$ for all $0 < t < T$. Thus, in inequality (7.7), we are allowed to use the function $U(t) = \tilde U(t)^{1/2}$ which yields

$$\|u(\,\cdot\,,t)\|_{L^{2+\varrho(t)}(\Omega)} \le U(t) \quad \text{for every } 0 \le t < T. \tag{7.9}$$

Furthermore, by (7.2) or (7.8), the function $U(t)$ satisfies

$$(7.10) \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} U(t) = (C_1 + C_2)\, U(t)^{Z(\varrho(t))-1} \\ \qquad\qquad + C_3 \left(\dfrac{\varepsilon(t)\, \varrho'(t)}{(2+\varrho)(1+\varrho)^{2/p}} \right)^{p/(p-2)} U(t)^{-1} \\ \qquad\qquad + \varphi_{\varepsilon(t)}(\varrho(t))\, \varrho'(t)\, U(t) \quad \text{for } 0 \le t < T; \\ U(0) = U_0 \stackrel{\text{def}}{=} \max\left\{ \|u(\,\cdot\,,0)\|_{L^r(\Omega)},\, 1 \right\} \ge 1. \end{cases}$$

As above, our hypotheses $\varrho'(t) > 0$ and $0 < 8\pi\varepsilon(t) \le 1$ guarantee $U'(t) \ge 0$ and $U(t) \ge 1$ for all $0 \le t < T$.

Next, we compute the solution of problem (7.10) by the change of variable $\varrho = \varrho(t) \ge \varrho_0$ in $U(t) = \hat{U}(\varrho(t))$ for all t small enough, say, $0 \le t < T'_{\max}$, provided $\varepsilon \equiv \varepsilon_\delta : [0,T) \to (0,\infty)$ and $\varrho \equiv \varrho_\delta : [0,T) \to \mathbb{R}_+$ are chosen as follows, where $0 < \delta \le 1/8\pi$ is a constant: The function ε is the superposition $\varepsilon(t) \stackrel{\text{def}}{=} \hat{\varepsilon}(\varrho(t), U(t))$ defined for every $0 \le t < T$, where

$$(7.11) \qquad \hat{\varepsilon}(\varrho, U) \stackrel{\text{def}}{=} \begin{cases} \delta\, U^{-2[Z(\varrho)-p]/p} & \text{if } \varrho_0 \le \varrho < \infty \text{ and } U > 0; \\ \delta\, U^{-2(q-p)/p} & \text{if } \varrho = \infty \text{ and } U > 0. \end{cases}$$

Recall that $Z(\varrho) \searrow q$ as $\varrho \nearrow \infty$, by (7.3). Hence, $\hat{\varepsilon} : [\varrho_0, \infty] \times (0,\infty) \to (0,\infty)$ is continuous. The function ϱ is constructed in a more complicated manner depending on U. We take the pair of functions (ϱ, U) to be the unique solution of the system of coupled equations (7.10) and the following one,

$$(7.12) \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \varrho(t) = \dfrac{1}{\delta}\, U(t)^{Z(\varrho)-2}\, (2+\varrho)(1+\varrho)^{2/p} \quad \text{for } 0 \le t < T'_{\max}; \\ \varrho(0) = \varrho_0 \;(= r - 2 \ge 0). \end{cases}$$

Similarly as above, $[0, T'_{\max})$ is the maximal time interval of existence ($0 < T'_{\max} \le T$), i.e., if $T'_{\max} < T$ then

$$\varrho(t) \nearrow \infty \quad \text{as} \quad t \nearrow T'_{\max}.$$

We set formally $\varrho(t) = \infty$ for $T'_{\max} \le t < T$. Hence, the function $\varepsilon : [0,T) \to (0,\infty)$ is continuous and satisfies $0 < 8\pi\varepsilon(t) \le 1$ for all $0 \le t < T$.

Naturally, we wish to show that there exists a number $0 < \hat{\delta} \le 1/8\pi$ such that indeed $T'_{\max} < T$ whenever $0 < \delta < \hat{\delta}$. This will guarantee (7.6), i.e.,

$$(7.13) \qquad \|u(\,\cdot\,,S)\|_{L^\infty(\Omega)} \le \liminf_{t \to S-} \|u(\,\cdot\,,t)\|_{L^{2+\varrho(t)}(\Omega)} \le \liminf_{t \to S-} U(t) = U(S) < \infty$$

with $S = T'_{\max} < T$ whenever $0 < \delta < \hat{\delta}$.

Lemma 7.1 *Let $0 < \delta \le 1/8\pi$ be arbitrary and set*

$$C = C_1 + C_2 + C_3 > 0.$$

Given the choice of $\varepsilon(t) = \hat{\varepsilon}(\varrho(t), U(t))$ *for every* $0 \le t < T$*, the pair of functions* $(\varrho, U) : [0, T) \to \mathbb{R}_+ \times (0, \infty)$ *satisfies equations* (7.10) *and* (7.12) *for all* $0 \le t < T'_{\max}$ $(\le T)$ *if and only if* $U(t) = \hat{U}(\varrho(t))$ *holds for all* $0 \le t < T'_{\max}$*, where* $\hat{U} : [\varrho_0, \infty) \to (0, \infty)$ *is the (unique) solution of the initial value problem*

$$(7.14) \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\varrho} \hat{U}(\varrho) = \left[\dfrac{C\,\delta}{(2+\varrho)(1+\varrho)^{2/p}} + \varphi_\delta(\varrho) \right] \hat{U}(\varrho) \\ \qquad + \dfrac{N[Z(\varrho) - p]}{p(2+\varrho)^2} \hat{U}(\varrho) \cdot \log \hat{U}(\varrho) \quad \textit{for } \varrho_0 \le \varrho < \infty; \\ \hat{U}(\varrho_0) = U_0 = \max\left\{ \|u(\,\cdot\,, 0)\|_{L^r(\Omega)},\, 1 \right\} \ge 1, \end{cases}$$

and ϱ *is the (unique) solution of*

$$(7.15) \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \varrho(t) = \dfrac{1}{\delta} \hat{U}(\varrho)^{Z(\varrho)-2} (2+\varrho)(1+\varrho)^{2/p} \quad \textit{for } 0 \le t < T'_{\max}; \\ \varrho(0) = \varrho_0 = r - 2 \ge 0. \end{cases}$$

It is easy to see that problem (7.14) is equivalent to the following inhomogeneous linear problem for the unknown function $V(\varrho) \stackrel{\text{def}}{=} \log \hat{U}(\varrho)$:

$$(7.16) \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\varrho} V(\varrho) = \dfrac{C\,\delta}{(2+\varrho)(1+\varrho)^{2/p}} + \varphi_\delta(\varrho) \\ \qquad + \dfrac{N[Z(\varrho) - p]}{p(2+\varrho)^2} V(\varrho) \quad \text{for } \varrho_0 \le \varrho < \infty; \\ V(\varrho_0) = V_0 \stackrel{\text{def}}{=} \max\left\{ \log \|u(\,\cdot\,, 0)\|_{L^r(\Omega)},\, 0 \right\} \ge 0. \end{cases}$$

Proof of Lemma 7.1. In order to apply our choices, equations (7.11) and (7.12), to problem (7.10), we first calculate the expression

$$\frac{\varepsilon(t)\,\varrho'(t)}{(2+\varrho)(1+\varrho)^{2/p}} = U(t)^{(p-2)Z(\varrho(t))/p} \qquad \text{for } 0 \le t < T'_{\max}$$

and then insert it into (7.10), thus arriving at

$$(7.17) \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} U(t) = \left[C + \dfrac{1}{\delta} (2+\varrho)(1+\varrho)^{2/p} \varphi_{\varepsilon(t)}(\varrho(t)) \right] U(t)^{Z(\varrho(t))-1} \\ \qquad \text{for } 0 \le t < T'_{\max}; \\ U(0) = U_0 = \max\left\{ \|u(\,\cdot\,, 0)\|_{L^r(\Omega)},\, 1 \right\} \ge 1. \end{cases}$$

By the standard theory for systems of ordinary differential equations, the system of equations (7.12) and (7.17) for the pair of unknown functions (ϱ, U) of time t, with $\varepsilon(t) = \hat{\varepsilon}(\varrho(t), U(t))$ for every $0 \le t < T$, possesses a unique solution on a maximal interval $0 \le t < T'_{\max}$ for some $T'_{\max} \in (0, T]$.

We apply (7.11) and (7.12) again, to get

$$\varphi_{\varepsilon(t)}(\varrho(t)) = \varphi_\delta(\varrho(t)) + \frac{N[Z(\varrho(t)) - p]}{p(2+\varrho(t))^2}\, \log U(t),$$
$$U(t)^{Z(\varrho(t))-2} = \frac{\delta\, \varrho'(t)}{(2+\varrho)(1+\varrho)^{2/p}}\,,$$

respectively, and consequently, problem (7.17) becomes

$$\begin{cases} \dfrac{1}{\varrho'(t)} \cdot \dfrac{\mathrm{d}}{\mathrm{d}t} U(t) = \left[\dfrac{C\,\delta}{(2+\varrho)(1+\varrho)^{2/p}} + \varphi_\delta(\varrho(t)) \right] U(t) \\ \qquad\qquad + \dfrac{N[Z(\varrho(t)) - p]}{p(2+\varrho(t))^2}\, U(t) \cdot \log U(t) \quad \text{for } 0 \le t < T'_{\max}; \\ U(0) = U_0 = \max\left\{ \|u(\,\cdot\,,0)\|_{L^r(\Omega)},\, 1 \right\} \ge 1. \end{cases} \tag{7.18}$$

As the function $\varrho(t)$ satisfies $\varrho'(t) > 0$ for all $t \in [0, T'_{\max})$, we are allowed to replace the time variable t in problem (7.18) above by ϱ, thus turning it into the new independent variable for an equivalent initial value problem, namely, problem (7.14). Finally, problem (7.15) is obtained from (7.12). ■

8 Proof of Theorem 2.1 (completed)

Our next goal is to show that

$$(\varrho(t), U(t)) \nearrow (\infty, U_\infty) \quad \text{as } t \nearrow T'_{\max}, \tag{8.1}$$

where $1 \le U_\infty < \infty$ and $0 < T'_{\max} < T$, whenever $\delta > 0$ is small enough, say, $0 < \delta < \hat{\delta}$ for some $0 < \hat{\delta} \le 1/8\pi$. This claim can be derived from equations (7.14) and (7.15) as follows.

Problem (7.14) is equivalent to the inhomogeneous linear problem (7.16) with the coefficients

$$A(\varrho) \stackrel{\text{def}}{=} \frac{N[Z(\varrho) - p]}{p(2+\varrho)^2} \quad \text{and} \quad B(\varrho) \stackrel{\text{def}}{=} \frac{C\,\delta}{(2+\varrho)(1+\varrho)^{2/p}} + \varphi_\delta(\varrho)$$

defined for $\varrho_0 \le \varrho < \infty$. Recall that the expressions $\varphi_\delta(\varrho)$ and $Z(\varrho)$ have been defined in (6.4) and (7.3), respectively. The antiderivatives of $\varphi_\delta(\varrho)$ and $A(\varrho)$ are calculated in Appendix A. As we take only $0 < \delta \le 1/8\pi$, we have $A(\varrho) > 0$ and $B(\varrho) > 0$ together with

$$\int_{\varrho_0}^{\infty} A(\varrho)\, \mathrm{d}\varrho < \infty \quad \text{and} \quad \int_{\varrho_0}^{\infty} B(\varrho)\, \mathrm{d}\varrho < \infty.$$

It follows that

$$\begin{aligned} 0 \le V(\varrho) &\le V_0 \exp\left(\int_{\varrho_0}^{\varrho} A(\sigma)\, \mathrm{d}\sigma \right) + \int_{\varrho_0}^{\varrho} \exp\left(\int_{\tau}^{\varrho} A(\sigma)\, \mathrm{d}\sigma \right) B(\tau)\, \mathrm{d}\tau \\ &\le \left(V_0 + \int_{\varrho_0}^{\infty} B(\tau)\, \mathrm{d}\tau \right) \exp\left(\int_{\varrho_0}^{\infty} A(\sigma)\, \mathrm{d}\sigma \right) \equiv \hat{C}_\delta(V_0) < \infty \end{aligned} \tag{8.2}$$

for all $\varrho_0 \le \varrho < \infty$. The constant $\hat{C}_\delta(V_0) > 0$ depends solely upon N, $|\Omega|_N$, p, q, r, δ, and the upper bound $U_0 = \exp(V_0) \ge 1$ on the initial norm

$$\|u(\,\cdot\,,0)\|_{L^r(\Omega)} \le U_0.$$

So we have verified $U_\infty \le \exp\left(\hat{C}_\delta(V_0)\right) < \infty$ as desired in (8.1).

It remains to show that $\varrho(t) \nearrow \infty$ as $t \nearrow T'_{\max}$. From problem (7.12) with $U(t) \ge U_\infty \ge 1$ for $0 \le t < T'_{\max}$, we deduce

$$\text{(8.3)} \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\,\varrho(t) \ge \dfrac{1}{\delta}\,(2+\varrho)(1+\varrho)^{2/p} & \text{for } 0 \le t < T'_{\max}; \\ \varrho(0) = 0. \end{cases}$$

Now let $\tilde{\varrho} \equiv \tilde{\varrho}_\delta :\ [0, \tilde{T}_{\max}) \to \mathbb{R}_+$ be the solution of the corresponding initial value problem

$$\text{(8.4)} \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\,\tilde{\varrho}(t) = \dfrac{1}{\delta}\,(2+\tilde{\varrho})(1+\tilde{\varrho})^{2/p} & \text{for } 0 \le t < \tilde{T}_{\max}; \\ \tilde{\varrho}(0) = 0, \end{cases}$$

where $[0, \tilde{T}_{\max})$ is the maximal time interval of existence ($0 < \tilde{T}_{\max} \equiv \tilde{T}_{\max,\delta} \le \infty$). It is easy to see that $\tilde{T}_{\max,\delta} = \delta \cdot \tilde{T}_{\max,1} < \infty$ whenever $0 < \delta \le 1/8\pi$. The number $\tilde{T}_{\max,1}$ depends solely upon p. We compare equations (8.3) and (8.4) to conclude that $T'_{\max} \le \tilde{T}_{\max,\delta} = \delta \cdot \tilde{T}_{\max,1}$ whenever $0 < \delta \le 1/8\pi$. Thus, fixing

$$\hat{\delta} = \min\left\{ T/\tilde{T}_{\max,1},\ 1/8\pi \right\}$$

we can achieve $T'_{\max} < T$ whenever $0 < \delta < \hat{\delta}$. Moreover, we get $T'_{\max} \to 0+$ as $\delta \to 0+$. This proves (7.13) with $S = T'_{\max} < T$ arbitrarily small, $0 < S < T$.

Returning to the original notation in Lemma 4.2, $u = u^{(n)}$ with $n \in \mathbb{N}$ fixed, (7.13) reads

$$\text{(8.5)} \qquad \|u^{(n)}(\,\cdot\,,S)\|_{L^\infty(\Omega)} \le \liminf_{t \to S-} \|u^{(n)}(\,\cdot\,,t)\|_{L^{2+\varrho(t)}(\Omega)} \le \liminf_{t \to S-} U(t) = U(S) < \infty$$

with $S = T'_{\max} < T$ whenever $0 < \delta < \hat{\delta}$. Notice that the initial condition U_0 for $U(t)$ at $t = 0$, starting from problem (7.10), can be taken independent from $n \in \mathbb{N}$,

$$U_0 \stackrel{\text{def}}{=} \max\left\{ \|u(\,\cdot\,,0)\|_{L^r(\Omega)},\ 1 \right\} \ge 1.$$

This choice makes the function $U(t)$ of $t \in [0,T)$ independent from n, as well. The conclusion of Theorem 2.1 follows immediately from (8.5) by taking any $n \ge U(S)$; namely, then one has $u(\,\cdot\,,S) = u^{(n)}(\,\cdot\,,S)$ in $L^\infty(\Omega)$.

Notice that, applying condition (2.5), we may replace U_0 by

$$U_0 \stackrel{\text{def}}{=} \max\left\{ \sup_{0 \le t < T} \|u(\,\cdot\,,t)\|_{L^r(\Omega)},\ 1 \right\}, \quad 1 \le U_0 \le \max\{c, 1\},$$

and the initial condition for $u(\,\cdot\,,t)$ at $t=0$ by that at $t=t_0$ for any $t_0 \in [0,T-S)$, thus obaining $u(\,\cdot\,,s) \in L^\infty(\Omega)$ with

$$\|u(\,\cdot\,,s)\|_{L^\infty(\Omega)} \leq U(S) < \infty \quad \text{whenever } S \leq s < T.$$

Recall that $S>0$ may be chosen arbitrarily small, but the function $U : [0,T) \to (0,\infty)$ will depend on this choice at every time $t \in [0,T)$. ■

9 Discussion

The condition

$$q < p\left(1 + \frac{r}{N}\right)$$

in (2.4) seems to be sharp for Theorem 2.1 to be valid. This claim can be inferred from the parabolic interpolation inequality (3.1) in DiBenedetto [15, Chapt. I, Prop. 3.1, p. 7] which corresponds to our inequality (5.6) derived from the Gagliardo-Nirenberg inequality (5.3). For $p=r=2$ the optimality of this condition is discussed in [15, Chapt. V, Remark 3.1, p. 122].

Condition (2.8) imposed in Filo [16] and Lieberman [24, pp. 561–562] for $p>N$ is stronger than our (2.4) for the following reason: In [24, Lemma 3.3, p. 557], on the right-hand side of inequality (3.6), the power N/p in the factor $\left(\int_\Omega |\nabla h|^p \,\mathrm{d}x\right)^{N/p}$ for $p>N$ is simply too high to yield an optimal result as compared to our choice of θ in the Gagliardo-Nirenberg inequality (5.6).

The method presented in our proof of Theorem 2.1 provides an analogous result also for a class of so-called doubly nonlinear parabolic equations with sources studied in C. Chen [7], Filo [16], Lieberman [24], and Porzio [26]. This class contains degenerate parabolic problems treated in our present work. We will publish the details elsewhere.

Last but not least, our logarithmic estimates in Section 6 would have been somewhat simpler if we had employed a sharper version of a logarithmic Sobolev inequality, namely,

$$\int_\Omega |f|^p \log(|f|^p)\,\mathrm{d}x \leq \frac{N}{p} \log\left(\mathcal{L}_p \int_\Omega |\nabla f|^p \,\mathrm{d}x\right) \quad \text{provided} \quad \int_\Omega |f|^p \,\mathrm{d}x = 1,$$

obtained recently in Del Pino and Dolbeault [13] for $1<p<N$, and in Agueh, Ghoussoub, and Kang [2] or Gentil [18] for any $1<p<\infty$. Here, $\mathcal{L}_p$ is a positive constant that can be computed explicitly,

$$\mathcal{L}_p = \frac{p}{N}\left(\frac{p-1}{\mathrm{e}}\right)^{p-1} \pi^{-p/2} \left(\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(N\frac{p-1}{p}+1\right)}\right)^{p/N}.$$

A. Some antiderivatives

We have used the following formulas for the function $\varphi_\varepsilon(\varrho)$ of $\varrho \in \mathbb{R}_+$, defined by (6.4), and for its antiderivative $\Phi_\varepsilon(\varrho)$ as well, where

$$\Phi_\varepsilon(\varrho) \stackrel{\text{def}}{=} \frac{N}{2(2+\varrho)} \left[\log\left(\frac{16\pi\varepsilon}{2+\varrho}\right) - 1\right] \qquad \text{for } \varrho \in \mathbb{R}_+. \tag{A.1}$$

We compute

$$\begin{aligned} \frac{2}{N} \cdot \frac{\mathrm{d}}{\mathrm{d}\varrho} \Phi_\varepsilon(\varrho) &= \log(16\pi\varepsilon) \frac{\mathrm{d}}{\mathrm{d}\varrho}[(2+\varrho)^{-1}] \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}\varrho}[(2+\varrho)^{-1} \log(2+\varrho)] - \frac{\mathrm{d}}{\mathrm{d}\varrho}[(2+\varrho)^{-1}] \\ &= -\log(16\pi\varepsilon) \frac{1}{(2+\varrho)^2} + \frac{1}{(2+\varrho)^2} \log(2+\varrho) \\ &= \frac{2}{N} \cdot \varphi_\varepsilon(\varrho). \end{aligned} \tag{A.2}$$

Notice that

$$\Phi_\varepsilon(\varrho_0) = \frac{N}{2r} \left[\log\left(\frac{16\pi\varepsilon}{r}\right) - 1\right] \qquad \text{with } \varrho_0 = r - 2$$

and

$$\Phi_\varepsilon(\infty) \stackrel{\text{def}}{=} \lim_{\varrho\to\infty} \Phi_\varepsilon(\varrho) = 0. \tag{A.3}$$

We have employed also the expression

$$\exp\left(\Phi_\varepsilon(\varrho)\right) = \left(\frac{16\pi\varepsilon/\mathrm{e}}{2+\varrho}\right)^{N/2(2+\varrho)} \qquad \text{for } \varrho \in \mathbb{R}_+. \tag{A.4}$$

Finally, we have used also the antiderivative of the function

$$\begin{aligned} \psi(\varrho) \stackrel{\text{def}}{=} \frac{Z(\varrho) - p}{(2+\varrho)^2} &= (q-p) \frac{1 + E(\varrho)}{(2+\varrho)^2} \\ &= \frac{q-p}{(2+\varrho)^2} + \frac{(q-2)(q-p)N/p}{(2+\varrho)^2[2+\varrho-(q-p)N/p]} \end{aligned}$$

of $\varrho \in \mathbb{R}_+$. Recalling eq. (7.3) we compute its antiderivative $\Psi(\varrho)$ as follows. We have $\psi(\varrho) = \frac{\mathrm{d}}{\mathrm{d}\varrho} \Psi(\varrho)$ where

$$\begin{aligned} \Psi(\varrho) &\stackrel{\text{def}}{=} \frac{p-2}{2+\varrho} + \frac{p(q-2)}{N(q-p)} \log\left(1 - \frac{(q-p)N}{p(2+\varrho)}\right) \\ &= \frac{p-2}{2+\varrho} + \frac{p(q-2)}{N(q-p)} \log\left[\frac{N}{p(2+\varrho)}\left(p\left(1 + \frac{2+\varrho}{N}\right) - q\right)\right]. \end{aligned}$$

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