PROPAGATION OF CRACKS IN ELASTIC MEDIA

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0. Introduction

In this paper we study a quasi-stationary model of crack propagation in $2d$ elastic medium. The model is introduced in §1. The stress function ϕ satisfies the biharmonic equation away from the crack $\Gamma(t)$, the free boundary conditions are

$$\phi = \frac{\partial \phi}{\partial n} = 0$$

from both sides of $\Gamma(t)$, and

$$\dot{X}(t) = \Phi(\mid \dot{X}(t) \mid) \vec{J}(X(t)) \tag{0.1}$$

where $X(t)$ is the moving tip of the crack, i.e., the moving end-point of $\Gamma(t)$; the other end-point of $\Gamma(t)$ is fixed at the boundary. Here $\Phi(s)$ is a rather simple explicit function, and $\vec{J}(X)$ is the so-called J-integral. The J-integral depends on the stress tensor in a global way.

The dynamic condition (0.1) expresses the physical fact that as the tip of the crack propagates, energy flows from the stress field in the body to the crack tip.

In §2 we compute the J-integral explicitly in terms of ϕ. In §3 we establish some results on biharmonic functions in a domain $D \backslash \Gamma_0$ where D is the strip $\{(x,y);\ x \in \mathbb{R},\ -a < y < a\}$ and $\Gamma_0 = \{(x,0);\ -\infty < x < 0\}$; we study the Dirichlet problem as well as the singularity at the tip $(0,0)$. These results are needed in the following sections.

In §4 we construct a travelling wave solution to the crack propagation problem in D. The solution is symmetric in y, and the crack propagates along the x-axis, from $x = -\infty$, at a uniform speed. The rest of the paper is concerned with a small perturbation about the travelling wave solution. In §5 we introduce the linearization of the nonlinear perturbed problem. This linearized problem is studied in sections 6-9. In §8 we prove global existence and uniqueness. In sections 8 and 9 we show that if $(x(t), y(t))$ is the tip of the crack for the linearized problem, then $\dot{y}(0+) > 0$ (we also compute $\dot{y}(0+)$ explicitly). The proofs require some auxiliary estimates and these are proved in sections 6, 7.

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1. The crack model

Let Ω be a domain in $\mathbb{R}^2$, representing a homogeneous elastic body. Let $u = (u_i)$, $\epsilon = (\epsilon_{ij})$ and $\sigma = (\sigma_{ij})$ denote the displacement vector, the strain tensor and the stress tensor, respectively. The linear elasticity equations for homogeneous isotropic material consist of the constitutive law

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{kk}\delta_{ij}\right) \tag{1.1}$$

and the equilibrium conditions

$$\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \tag{1.2}$$

provided no body forces are assumed. Here E is the Young modulus, and ν is the Poisson ratio, and the strain-displacement relation is given by

$$\epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad u_{i,j} = \partial_j u_i. \tag{1.3}$$

Suppose there is initially a crack in Ω, given by a non-intersecting curve Γ_0 with initial point on $\partial\Omega$ and terminal point (the "crack tip") $X_0 = (x_0, y_0)$ inside Ω. Under external forces the crack tip will generally propagate, and we shall denote it by $X(t)$. The crack propagation problem consists of finding the displacement u and path $X(t)$ such that

$$\sigma_{ij,j} = 0 \text{ in } \Omega\backslash\Gamma(t) \tag{1.4}$$

where

$$\Gamma(t) = \Gamma_0 \cup \{X = X(s),\ 0 \le s \le t\}, \tag{1.5}$$

$$\sigma_{ij} n_j = 0 \text{ on } \Gamma^{\pm}(t) \text{ (no traction on } \Gamma^{\pm}(t)) \tag{1.6}$$

where $\Gamma^{\pm}(t)$ means both sides of $\Gamma(t)$, and (n_j) is the normal to the curve,

$$u_i = \beta_i \text{ on } \partial_1\Omega, \quad \sigma_{ij} n_j = g_i \text{ on } \partial_2\Omega \tag{1.7}$$

where $\partial\Omega$ is a disjoint union of $\partial_1\Omega$, $\partial_2\Omega$, and

$$\text{an appropriate dynamical equation holds for } X(t),\ X(0) = X_0. \tag{1.8}$$

The dependence on t appears only through the dynamical equation for $X(t)$; the elasticity equations are taken to be time-independent. Thus the model we consider is "quasi-stationary."

To obtain explicit formula for (1.8) we begin with the energy criterion given by Griffith [7]. Accordingly, when a crack is extended, there is a flow of energy from the stress field in the body to the crack tip. This energy is then stored on both faces of the newly enlarged crack, and

$$G = \gamma \tag{1.9}$$

where G is the energy release rate at the crack tip and $\frac{1}{2}\gamma$ is the stored fracture energy (which in the $3d$ case is also called the surface energy).

There is no satisfactory theory for determining γ from basic principles. It is believed that crack propagation proceeds in two stages. In the first stage the speed v of the crack tip increases until it reaches a limit v_∞, and in the second stage the crack tip continues to move with the uniform speed v_∞; we refer to Freund [4] for very detailed explanations and for references. We also refer to a recent work by Slepyan [12] based on a variational principle which explains what happens at the limit speed.

Here we shall deal only with the first stage (i.e., $v < v_\infty$). For some materials [4, p.417],

$$\gamma = \gamma_0 \left(\frac{v_\infty + v}{v_\infty - v} \right)^m$$

where m is a positive integer, and m, γ_0, v_∞ depend on the material. For simplicity we shall henceforth take $m = 1$, i.e.,

$$\gamma = \gamma_0 \frac{v_\infty + v}{v_\infty - v}; \tag{1.10}$$

all our results, however, extend to the case of any $m > 0$.

We next turn to the energy release rate G. Following Rice [11], G can be described in terms of the J-integral, which we proceed to define. Let P be any point in Ω and let Λ be any smooth non-intersecting curve in Ω surrounding P. The J-integral is defined by

$$\vec{J}_\Lambda(P) = \int_\Lambda (W\vec{n} - \vec{s} \cdot Du) \tag{1.11}$$

where $\vec{n}$ is the outward normal, W is the strain energy density given by

$$W = \frac{1}{2}\sigma_{ij}\epsilon_{ij} \tag{1.12}$$

and

$$\vec{s} = (s_i) = (\sigma_{ij}n_j) \tag{1.13}$$

is the traction vector.

By the divergence theorem one can easily verify (cf. [8]) that

$$\int_\gamma (W\vec{n} - \vec{s} \cdot Du) = 0$$

if γ is any curve in Ω such that the region enclosed by γ does not contain cracks.

Consider now the case where P is the crack tip $X(t)$, and Λ is a closed smooth non-intersecting curve with initial and terminal point on the crack $\Gamma(t)$ such that it does not intersect $\Gamma(t)$, except at its endpoints. Denote the diameter of Λ by $\mid \Lambda \mid$. We take any family of such curves Λ with diameter which converges to zero, and let

$$\vec{J}(P) = \lim_{|\Lambda| \to 0} \vec{J}_\Lambda(P). \tag{1.14}$$

Under some smoothness assumptions on u and $X(t)$ it was shown by Gurtin [8] and Ohtsuka [10] that the limit in (1.14) exists and (motivated by the work of Rice [11]) that the energy release rate G at the crack tip P can be expressed by the J-integral as

$$G = \vec{e} \cdot \vec{J}(P)$$

where $\vec{e}$ is the direction of motion of the crack tip, i.e.,

$$G = \frac{\dot{X}(t)}{|\dot{X}(t)|} \cdot \vec{J}(X(t)). \tag{1.15}$$

Combining this with (1.9) we have

$$\frac{\dot{X}(t)}{|\dot{X}(t)|} \cdot \vec{J}(X(t)) = \gamma. \tag{1.16}$$

A necessary condition for (1.16) to hold is that

$$|\vec{J}(X(t))| \geq \gamma; \tag{1.17}$$

if $|\vec{J}(X(t))| < \gamma$, the crack does not propagate.

If

$$|\vec{J}(X(t))| = \gamma, \tag{1.18}$$

then (1.16) implies that

$$\dot{X}(t) = \frac{\vec{J}(X(t))}{\gamma} |\dot{X}(t)|. \tag{1.19}$$

Stump and Le [13] have recently developed a variational principle and used it to establish that

$$|\vec{J}(X(t))| \leq \gamma$$

always holds; furthermore,

$$\begin{array}{l} \text{if } |\vec{J}(X(t))| < \gamma \text{ then there is no propagation,whereas} \\ \text{if } |\vec{J}(X(t))| = \gamma \text{ then the crack tip propagates in the direction } \frac{1}{\gamma}\vec{J}(X(t)). \end{array} \tag{1.20}$$

Combining this with (1.10) we conclude that

(i) If $|\vec{J}(X(t))| \leq \gamma_0$ then the crack does not propagate;

(ii) If $|\vec{J}(X(t))| > \gamma_0$ then the crack propagates with speed $v(t)$ and direction $\dot{X}(t)/|\dot{X}(t)|$ determined by

$$v(t) = v_\infty \frac{(|\vec{J}(X(t))| - \gamma_0)_+}{|\vec{J}(X(t))| + \gamma_0}, \quad v(t) = |\dot{X}(t)| \tag{1.21}$$

and

$$\dot{X}(t) = \frac{v(t)}{\gamma_0} \frac{v_\infty - v(t)}{v_\infty + v(t)} \vec{J}(X(t)). \tag{1.22}$$

Equation (1.22) is the dynamical equation sought in (1.8), but we still need to compute $\vec{J}(X(t))$ more explicitly. This will be done in the next section.

2. Computation of $\vec{J}$

In plane elasticity, (σ_{ij}) satisfies

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} = 0 \tag{2.1}$$

and

$$\frac{\partial \sigma_{21}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} = 0. \tag{2.2}$$

Equations (2.1) and (2.2) imply that there exist functions ψ_1 and ψ_2 such that

$$\sigma_{11} = \frac{\partial \psi_1}{\partial y}, \quad \sigma_{12} = -\frac{\partial \psi_1}{\partial x}, \tag{2.3}$$

$$\sigma_{21} = \frac{\partial \psi_2}{\partial y}, \quad \sigma_{22} = -\frac{\partial \psi_2}{\partial x}. \tag{2.4}$$

Since $\sigma_{12} = \sigma_{21}$,

$$\frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y} = 0, \tag{2.5}$$

so that there is a function ϕ such that

$$\psi_1 = \frac{\partial \phi}{\partial y}, \quad \psi_2 = -\frac{\partial \phi}{\partial x}; \tag{2.6}$$

ϕ is called the *stress function.* We can express the σ_{ij} in the form

$$\sigma_{11} = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_{12} = -\frac{\partial^2 \phi}{\partial x \partial y}, \quad \sigma_{22} = \frac{\partial^2 \phi}{\partial x^2}. \tag{2.7}$$

The stress tensor $\sigma = (\sigma_{ij})$ is uniquely determined by ϕ up to an arbitrary linear function.

We rewrite (1.1) in component form:

$$\begin{aligned}
\sigma_{11} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{11} + \nu(\epsilon_{22}+\epsilon_{33})],\\
\sigma_{22} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{22} + \nu(\epsilon_{11}+\epsilon_{33})],\\
\sigma_{33} &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_{33} + \nu(\epsilon_{11}+\epsilon_{22})],
\end{aligned} \tag{2.8}$$

$$\sigma_{12} = \frac{E}{1+\nu}\epsilon_{12}, \quad \sigma_{13} = \frac{E}{1+\nu}\epsilon_{13}, \quad \sigma_{23} = \frac{E}{1+\nu}\epsilon_{23}.$$

Conversely, the strain tensor is given in terms of the stress tensor by

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22}+\sigma_{33})],\\
\epsilon_{22} &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11}+\sigma_{33})],\\
\epsilon_{33} &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11}+\sigma_{22})],
\end{aligned} \tag{2.9}$$

$$\epsilon_{12} = \frac{1+\nu}{E}\sigma_{12}, \ \ \epsilon_{13} = \frac{1+\nu}{E}\sigma_{13}, \ \ \epsilon_{23} = \frac{1+\nu}{E}\sigma_{23}.$$

Substituting (1.3) into the equilibrium equation (1.2) and applying the divergence we find that

$$\triangle div u = 0. \tag{2.10}$$

In the case of a plane deformation, the last component u_3 of the displacement u is zero, while the components u_1 and u_2 are functions of x and y only, and so $\epsilon_{i3} = 0, \sigma_{13} = \sigma_{23} = 0$.

By (2.7) and (2.8),

$$\begin{aligned} \triangle\phi = \sigma_{11} + \sigma_{22} &= \frac{E}{(1+\nu)(1-2\nu)}(\epsilon_{11} + \epsilon_{22}) \\ &= \frac{E}{(1+\nu)(1-2\nu)} div u \end{aligned}$$

and therefore (2.10) becomes

$$\triangle^2\phi = 0 \ \text{ in } \ \Omega\backslash\Gamma(t). \tag{2.11}$$

Next we derive boundary conditions for ϕ. We begin with the free traction condition

$$\sigma_{ij}^+ n_j = \sigma_{ij}^- n_j = 0 \ \text{ on } \ \Gamma(t). \tag{2.12}$$

By (2.7),

$$\begin{aligned} \sigma_{1j}n_j &= \frac{\partial^2\phi}{\partial y^2}n_1 - \frac{\partial^2\phi}{\partial x\partial y}n_2 \\ &= -\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)\tau_1 - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)\tau_2 = -\frac{\partial}{\partial\tau}\left(\frac{\partial\phi}{\partial y}\right) \end{aligned} \tag{2.13}$$

where $\vec{\tau} = (n_2, -n_1)$ is the tangent vector to $\Gamma(t)$.

Similarly,

$$\sigma_{2j}n_j = -\frac{\partial^2\phi}{\partial x\partial y}n_1 + \frac{\partial^2\phi}{\partial x^2}n_2 = \frac{\partial}{\partial\tau}\left(\frac{\partial\phi}{\partial x}\right). \tag{2.14}$$

If $\phi \in C^1(\overline{\Omega})$, then (2.12) is equivalent to

$$\nabla\phi \equiv const. \ \text{ along } \ \Gamma(t). \tag{2.15}$$

Since the stress σ only depends on the second derivatives of ϕ, any linear addition to ϕ will not change σ.

Let us prescribe a surface force $\vec{s} = (s_1, s_2)$ on $\partial\Omega$. Then by the same calculations as in (2.13), (2.14),

$$s_1 = \sigma_{1j}n_j = -\frac{\partial}{\partial\tau}\left(\frac{\partial\phi}{\partial y}\right), \ \ s_2 = \sigma_{2j}n_j = \frac{\partial}{\partial\tau}\left(\frac{\partial\phi}{\partial x}\right).$$

Hence ϕ_x and ϕ_y are determined up to a constant.

We assumed that the crack initiated at the boundary $\partial\Omega$. Hence once we fix the constants for ϕ and $\partial\phi/\partial n$ on $\Gamma(t)$, this uniquely determines ϕ and $\partial\phi/\partial n$ on $\partial\Omega$. In view of (2.15) we can take

$$\phi = \frac{\partial\phi}{\partial n} = 0 \ \text{ on } \ \Gamma(t). \tag{2.16}$$

Then the boundary conditions on $\partial\Omega$ becomes

$$\phi = f, \ \frac{\partial \phi}{\partial n} = g \ \text{ on } \ \partial\Omega \tag{2.17}$$

where f, g are prescribed functions.

Next we shall compute the J-integral defined by (1.11) in terms of the stress function ϕ.

By (2.8) and (2.9),

$$\sigma_{33} = \frac{\nu E}{(1+\nu)(1-2\nu)}(\epsilon_{11} + \epsilon_{22}) = \nu(\sigma_{11} + \sigma_{22}) = \nu\Delta\phi. \tag{2.18}$$

It follows by (2.9), (2.7) and (2.18) that

$$\begin{aligned} \epsilon_{11} &= \frac{1}{E}[\phi_{22} - \nu(\phi_{11} + \nu\Delta\phi)] \\ &= \frac{1+\nu}{E}[(1-\nu)\phi_{22} - \nu\phi_{11}] \quad \left(\phi_{11} = \frac{\partial^2\phi}{\partial x^2}, \ \phi_{22} = \frac{\partial^2\phi}{\partial y^2}\right), \end{aligned} \tag{2.19}$$

$$\begin{aligned} \epsilon_{22} &= \frac{1}{E}[\phi_{11} - \nu(\phi_{22} + \nu\Delta\phi)] \\ &= \frac{1+\nu}{E}[(1-\nu)\phi_{11} - \nu\phi_{22}], \end{aligned} \tag{2.20}$$

$$\epsilon_{12} = -\frac{1+\nu}{E}\phi_{12} \quad \left(\phi_{12} = \frac{\partial^2\phi}{\partial x \partial y}\right). \tag{2.21}$$

By (1.1) and (1.12), the strain energy density is

$$\begin{aligned} W &= \frac{E}{2(1+\nu)}\left(\epsilon_{ij}\epsilon_{ij} + \frac{\nu}{1-2\nu}\epsilon_{ii}\epsilon_{jj}\right) \\ &= \frac{E}{2(1+\nu)}\left[\epsilon_{11}^2 + \epsilon_{22}^2 + 2\epsilon_{12}^2 + \frac{\nu}{1-2\nu}(\epsilon_{11}+\epsilon_{22})^2\right] \end{aligned}$$

Using (2.19)-(2.21) we then have

$$W = \frac{1+\nu}{2E}\left\{[(1-\nu)\phi_{22} - \nu\phi_{11}]^2 + [(1-\nu)\phi_{11} - \nu\phi_{22}]^2 + 2\phi_{12}^2 + \frac{\nu}{1-2\nu}[(1-\nu)\Delta\phi - \nu\Delta\phi]^2\right\}$$

from which we easily compute that

$$W = \frac{1+\nu}{2E}\{(1-\nu)(\Delta\phi)^2 - 2\phi_{11}\phi_{22} + 2\phi_{12}^2\}. \tag{2.22}$$

Next we evaluate $\vec{s} \cdot Du$. We begin by expressing the displacement vector u in terms of ϕ.

By (2.19) and (2.20),

$$\frac{\partial u_1}{\partial x} = \epsilon_{11} = \frac{1+\nu}{E}[-\phi_{11} + (1-\nu)\Delta\phi] \tag{2.23}$$

and

$$\frac{\partial u_2}{\partial y} = \epsilon_{22} = \frac{1+\nu}{E}[-\phi_{22} + (1-\nu)\Delta\phi]. \tag{2.24}$$

Denote by $(\Delta\phi)^c$ the harmonic conjugate of $\Delta\phi$, and define the holomorphic function

$$\begin{aligned} \Psi(z) &= \int (\Delta\phi + i(\Delta\phi)^c)(z)dz \\ &= p(z) + iq(z). \end{aligned} \tag{2.25}$$

Then p and q are harmonic functions, and

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \Delta\phi, \tag{2.26}$$

$$\frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x} = -(\Delta\phi)^c. \tag{2.27}$$

Substituting (2.26) into (2.23) and into (2.24), we get

$$\frac{\partial u_1}{\partial x} = \frac{1+\nu}{E}\left(-\phi_{11} + (1-\nu)\frac{\partial p}{\partial x}\right) \tag{2.28}$$

and

$$\frac{\partial u_2}{\partial y} = \frac{1+\nu}{E}\left(-\phi_{22} + (1-\nu)\frac{\partial q}{\partial y}\right). \tag{2.29}$$

By integration we then have

$$u_1 = \frac{1+\nu}{E}(-\phi_1 + (1-\nu)p) + h(y) \tag{2.30}$$

and

$$u_2 = \frac{1+\nu}{E}(-\phi_2 + (1-\nu)q) + k(x) \tag{2.31}$$

where h and k are arbitrary functions.

By (2.21) and the last two equations,

$$\begin{aligned} -\frac{1+\nu}{E}\phi_{12} = \epsilon_{12} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ &= -\frac{1+\nu}{E}\phi_{12} + \frac{1}{2}(1-\nu)\left(\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}\right) + h'(y) + k'(x). \end{aligned}$$

Recalling (2.27) we find that

$$h'(y) + k'(x) = 0.$$

Consequently $h'(y) = -k'(x) = const.$ and thus

$$h(y) = ay + b, \quad k(x) = -ax + c.$$

The form of h and k indicate that they represent a rigid displacement and can thus be disregarded in the analysis of deformation. We shall therefore take $h \equiv k \equiv 0$ (without affecting the physical problem). Then (2.30) and (2.31) become

$$u_1 = \frac{1+\nu}{E}(-\phi_1 + (1-\nu)p), \tag{2.32}$$

$$u_2 = \frac{1+\nu}{E}(-\phi_2 + (1-\nu)q). \tag{2.33}$$

We now compute

$$\begin{aligned} \vec{s}\cdot D_1 u &= s_1 D_1 u_1 + s_2 D_1 u_2 \\ &= \frac{1+\nu}{E}\{s_1[-\phi_{11} + (1-\nu)\triangle\phi] + s_2[-\phi_{12} + (1-\nu)(\triangle\phi)^c]\}. \end{aligned}$$

Recalling that

$$\begin{aligned} s_1 &= \sigma_{1j}n_j = \phi_{22}n_1 - \phi_{12}n_2, \\ s_2 &= \sigma_{2j}n_j = -\phi_{12}n_1 + \phi_{11}n_2, \end{aligned}$$

we have

$$\begin{aligned} \vec{s}\cdot D_1 u &= \frac{1+\nu}{E}\{(\phi_{22}n_1 - \phi_{12}n_2)(-\phi_{11} + (1-\nu)\triangle\phi) + \\ &\qquad (-\phi_{12}n_1 + \phi_{11}n_2)(-\phi_{12} + (1-\nu)(\triangle\phi)^c)\} \\ &= \frac{1+\nu}{E}\{-\phi_{11}\phi_{22}n_1 + \phi_{11}\phi_{12}n_2 + (1-\nu)s_1\triangle\phi + \\ &\qquad \phi_{12}^2 n_1 - \phi_{11}\phi_{12}n_2 + (1-\nu)s_2(\triangle\phi)^c\} \\ &= \frac{1+\nu}{E}\{(\phi_{12}^2 - \phi_{11}\phi_{22})n_1 + (1-\nu)(s_1\triangle\phi + s_2(\triangle\phi)^c)\}. \qquad (2.34) \end{aligned}$$

We now compute the first component of the J-integral by using (2.22) and (2.34):

$$\begin{aligned} J_\Lambda^1 &= \int_\Lambda (Wn_1 - \vec{s}\cdot D_1 u) \\ &= \frac{1-\nu^2}{2E}\int_\Lambda \{(\triangle\phi)^2 n_1 - 2\vec{s}\cdot\vec{\Phi}\} \qquad (2.35) \end{aligned}$$

where

$$\vec{\Phi} = (\triangle\phi, (\triangle\phi)^c). \tag{2.36}$$

Similarly,

$$\begin{aligned} \vec{s}\cdot D_2 u &= s_1 D_2 u_1 + s_2\epsilon_{22} \\ &= \frac{1+\nu}{E}\{s_1[-\phi_{12} - (1-\nu)(\triangle\phi)^c] + s_2[-\phi_{22} - (1-\nu)\triangle\phi]\} \\ &= \frac{1+\nu}{E}\{(\phi_{12}^2 - \phi_{11}\phi_{22})n_2 + (1-\nu)(-s_1(\triangle\phi)^c + s_2\triangle\phi)\}. \qquad (2.37) \end{aligned}$$

Hence

$$\begin{aligned} J_\Lambda^2 &= \int_\Lambda (Wn_2 - \vec{s}\cdot D_2 u) \\ &= \frac{1-\nu^2}{2E}\int_\Lambda \{(\triangle\phi)^2 n_2 - 2\vec{s}\cdot\vec{\Phi}^c\} \qquad (2.38) \end{aligned}$$

where

$$\vec{\Phi}^c = (-(\triangle\phi)^c, \triangle\phi). \tag{2.39}$$

Having computed $\vec{J}$ we can now state the free boundary problem in terms of ϕ:

Problem (P)

Let Ω be a domain in $\mathbb{R}^2$ with smooth (say $C^{4+\alpha}$) boundary and let Γ_0 be a $C^{4+\alpha}$ curve in Ω with one end-point on $\partial\Omega$ and the other, X_0, in Ω. Given continuous functions f, g on $\partial\Omega$ and positive constants γ_0, v_∞, find functions $\phi(x,y,t)$, $X(t) = (x(t), y(t))$ for $t \geq 0$ such that

$$\Delta^2\phi = 0 \text{ in } \Omega\backslash\overline{\Gamma(t)} \tag{2.40}$$

where

$$\Gamma(t) = \Gamma_0 \cup \{X(s);\ 0 \leq s \leq t\}, \quad X(0) = X_0, \tag{2.41}$$

$$\phi = f, \ \frac{\partial\phi}{\partial n} = g \text{ on } \partial\Omega, \tag{2.42}$$

$$\phi = 0, \ \frac{\partial\phi}{\partial n} = 0 \text{ on } \Gamma(t) \text{ (from both sides)}, \tag{2.43}$$

and

$$\dot{X}(t) = \frac{v(t)}{\gamma_0}\frac{v_\infty - v(t)}{v_\infty + v(t)}\vec{J}(X(t)), \ v(t) = \mid \dot{X}(t) \mid, \tag{2.44}$$

where $\vec{J}(X(t))$ is defined by (1.14) and (2.35), (2.38). We observe that if $\mid \vec{J}(X(t)) \mid \leq \gamma_0$, then (2.44) implies that $\dot{X}(t) = 0$.

3. The biharmonic equation in a strip

Let

$$\begin{aligned} D &= \{(x,y);\ x \in \mathbb{R},\ -a < y < a\} \ (a > 0), \\ \Gamma_0 &= \{(x,0);\ -\infty < x < 0\}, \ D_0 = D\backslash\overline{\Gamma}_0. \end{aligned} \tag{3.1}$$

Consider the elliptic problem,

$$\begin{aligned} \Delta^2\psi &= F \text{ in } D_0, \\ \psi &= \frac{\partial\psi}{\partial n} = 0 \text{ on } \partial D_0. \end{aligned} \tag{3.2}$$

Lemma 3.1. *If F is in $L^\infty(D)$, then there exists a unique bounded solution ψ of (3.2) in $H^2_{loc}(D)$, and*

$$\|\psi\|_{L^\infty(D)} \leq C\|F\|_{L^\infty(D)} \tag{3.3}$$

where C is a constant independent of F.

Proof. Consider first the case when F has compact support, and introduce the functional

$$J(v) = \int_{D_0} \mid \Delta v \mid^2 - 2\int_{D_0} Fv, \quad v \in H^2_0(D_0).$$

One can easily show that $J(v)$ attains a minimum, and that any minimizer ψ is a solution of (3.2); we use here the fact that [1]

$$\|v\|^2_{H^2(D)} \leq C\int_D \mid \Delta v \mid^2 . \tag{3.4}$$

Observe that $J(\psi) = \min J(v) \leq J(0) = 0$, so that

$$\int_D \mid \triangle\psi \mid^2 \leq 2 \int_D F\psi. \tag{3.5}$$

To prove uniqueness suppose that ψ_1 is another bounded solution and set $v = \psi - \psi_1$. Then $\triangle v = 0$ in D_0. By a Phragmén-Lindelöf type theorem of Lax [9] (see also [6]) v decays exponentially as $x \to \pm\infty$. By elliptic estimates [2] the same is true for the first three derivatives of v. Hence, if we integrate by parts in

$$\int_{D_R} \triangle v \cdot \triangle v \quad (D_R = D \cap \{\mid x \mid < R\}),$$

we obtain $\int_{D_R} v\triangle^2 v$ plus boundary integrals (on $|x| = R$) which decay exponentially to zero as $R \to \infty$. It follows that $\int_D \mid \triangle v \mid^2 = 0$, so that $\psi - \psi_1 \equiv v \equiv 0$.

The above results are valid also if we replace $H_0^2(D_0)$ by $H_0^2(D)$.

Consider next the case of general F, and let

$$F_j = F\chi_j, \quad \hat{F}_m = \Sigma_{j=-m}^{m-1} F_j$$

where $\chi_j = \chi_j(x)$ is the characteristic function of $\{j \leq x < j+1\}$. Denote by $\hat{\phi}_m$ the bounded solution in $H^2(D)$ of

$$\begin{aligned} \triangle^2\hat{\phi}_m &= \hat{F}_m \text{ in } D, \\ \hat{\phi}_m &= \frac{\partial\hat{\phi}_m}{\partial n} = 0 \text{ on } \partial D \end{aligned}$$

and by ϕ_j the corresponding solution for F_j. Then

$$\hat{\phi}_m = \Sigma_{j=-m}^{m-1}\phi_j.$$

From (3.4), (3.5),

$$\begin{aligned} \|\phi_j\|^2_{H^2(D)} &\leq C\int_D \mid \triangle\phi_j \mid^2 \leq 2C\int_D \phi_j F_j \\ &\leq \frac{1}{2}\int_D (\phi_j)^2 + CM^2 \end{aligned}$$

where

$$M = \|F\|_{L^\infty(D)}.$$

Hence

$$\|\phi_j\|_{H^2(D)} \leq CM. \tag{3.6}$$

Applying the theorem of Lax [9] in half-strips $D \cap \{x > j+1\}$ and $D \cap \{x < j\}$ and using (3.6), we get

$$\|\phi_j(\xi, \cdot)\|_{H^2(D\cap\{x=\xi\})} \leq CMe^{-\lambda|\xi-j|} \quad (\lambda > 0).$$

The same estimate holds for $j < \xi < j+1$, by (3.6) and elliptic estimates. Summing over j and using also elliptic estimates, we get

$$\mid D^j\hat{\phi}_m(x,y) \mid \leq CM \quad (0 \leq j \leq 2). \tag{3.7}$$

Consider now the solution ψ_m of (3.2) corresponding to $F = F_m$. Since the boundary condition for ψ_m are not invariant in x (throughout D_0), the proof of (3.7) cannot be directly extended to ψ_m. We shall, instead, proceed to estimate the difference

$$\tilde{\psi}_m = \psi_m - \hat{\phi}_m.$$

Clearly

$$\triangle^2 \tilde{\psi}_m = 0 \text{ in } D \cap \{x > 0\},$$
$$\tilde{\psi}_m = \frac{\partial \tilde{\psi}_m}{\partial n} = 0 \text{ on } \partial D \cap \{x > 0\}$$

and, by interior-boundary estimates,

$$\|\tilde{\psi}_m\|_{C^2(\cap\{1<x<2\})} \leq C \|\tilde{\psi}_m\|_{L^\infty(D\cap\{0<x<3\})}.$$

By the theorem of Lax [9] (or [6]) and elliptic estimates, we then have for $x > 3$,

$$| D^j \tilde{\psi}_m(x, y) | \leq C \|\tilde{\psi}_m\|_{L^\infty(D\cap\{0<x<3\})} e^{-\lambda x} \quad (0 \leq j \leq 2, \ \lambda > 0).$$

Recalling (3.7) we conclude that

$$| D^j \psi_m(x, y) | \leq C \left[\|\psi_m\|_{L^\infty(D_3)} + M \right] e^{-\lambda x} + CM \quad (0 \leq j \leq 2) \tag{3.8}$$

if $x > 3$. In the same way we can estimate (3.8) for $x < -3$ (with a different $\lambda > 0$).

We shall next estimate $\|\psi_m\|_{L^\infty(D_3)}$.

Take $R > 3$ and let $\eta \in C^\infty(\overline{\Omega})$ be a cutoff function such that

$$0 \leq \eta \leq 1 \text{ in } D, \ \ \eta = 1 \text{ in } D_{2R}, \ \ \eta = 0 \text{ in } D \backslash D_{3R}$$

and

$$| D\eta | \leq \frac{C}{R}, \ \ | D^2 \eta | \leq \frac{C}{R^2}.$$

Substituting $\xi = \eta^4 \psi_m$ into

$$\int_D \triangle \psi_m \cdot \triangle \xi = \int_D \hat{F}_m \xi$$

we get

$$\int_D \eta^4 (\triangle \psi_m)^2 = - \int_D [2 \triangle \psi_m \nabla \eta^4 \cdot \nabla \psi_m + \triangle \psi_m \triangle \eta^4 \cdot \psi_m] + \int_D \hat{F}_m \eta^4 \psi_m.$$

writing $\nabla \eta^4 = 4\eta^3 \nabla \eta$, $\triangle \eta^4 = 12\eta^2 \mid \nabla \eta \mid^2 + 4\eta^3 \triangle \eta$ and using the Cauchy-Schwarz inequality, we find, after some simple calculations, that

$$\int_D \eta^4 \mid \triangle \psi_m \mid^2 \leq \frac{C}{R^2} \int_{D_{3R}} (\mid \nabla \psi_m \mid^2 + \psi_m^2) + CM \int_D \eta^4 \mid \psi_m \mid .$$

Hence

$$\int_{D_{2R}} \mid \triangle \psi_m \mid^2 \leq \frac{C}{R^2} \int_{D_{3R}} (\mid \nabla \psi_m \mid^2 + \psi_m^2) + CR^2 M^2. \tag{3.9}$$

Let $\tilde{\eta}$ be another cutoff function in $C^\infty(\overline{\Omega})$ such that

$$0 \le \tilde{\eta} \le 1 \text{ in } D, \ \ \tilde{\eta} = 1 \text{ in } D_R, \ \ \tilde{\eta} = 0 \text{ in } D \backslash D_{2R}$$

and

$$\mid D\tilde{\eta} \mid \le \frac{C}{R}, \ \mid D^2\tilde{\eta} \mid \le \frac{C}{R^2}.$$

By (3.4),

$$\|\psi_m\|^2_{H^2(D_R)} \le C \int_D \mid \triangle(\tilde{\eta}\psi_m) \mid^2 .$$

Expanding $\triangle(\tilde{\eta}\psi_m)$ and using the Cauchy-Schwarz inequality, we find that

$$\begin{aligned} \|\psi_m\|_{H^2(D_R)} &\le C \int_{D_{2R}} \mid \triangle\psi_m \mid^2 + \frac{C}{R^2} \int_{D_{2R}} (\mid \nabla\psi_m \mid^2 + \psi_m^2) \\ &= \frac{C}{R^2} \int_{D_{3R}} (\mid \nabla\psi_m \mid^2 + \psi_m^2) + CR^2M^2, \text{ by (3.9).} \end{aligned}$$

Choosing R large we then have

$$\|\psi_m\|^2_{H^2(D_R)} \le \frac{2C}{R^2} \int_{D\cap\{R<|x|<3R\}} (\mid \nabla\psi_m \mid^2 + \psi_m^2) + 2CR^2M^2. \tag{3.10}$$

Using (3.8) to estimate the integral on the right-hand side, we obtain

$$\|\psi_m\|^2_{H^2(D_R)} \le \frac{C}{R^2}\|\psi_m\|^2_{L^\infty(D_3)} + C_R M^2. \tag{3.11}$$

Now, by Sobolev's embedding

$$\|\psi_m\|_{L^\infty(D_3)} \le C\|\psi_m\|_{H^2(D_3)},$$

so that (3.11) yields

$$\|\psi_m\|^2_{H^2(D_R)} \le C_R M^2$$

if R is large enough. It follows, in particular, that

$$\|\psi_m\|_{L^\infty(D_3)} \le CM.$$

Using this in (3.8) we find that

$$\mid \psi_m(x,y) \mid \le C\|F\|_{L^\infty(D)}.$$

By elliptic estimates we can now take a subsequence ψ_m which converges to a solution ψ of (3.2), ψ is in $H^2_{loc}(D)$ and (3.3) holds. The uniqueness of the solution ψ is proved as before. □

Lemma 3.2. *Consider the elliptic problem:*

$$\begin{aligned} &\triangle^2\psi = 0 \ \text{ in } D_0, \\ &\psi = f, \ \frac{\partial\psi}{\partial n} = g \ \text{ on } \partial D, \\ &\psi = \frac{\partial\psi}{\partial n} = 0 \ \text{ on } \Gamma_0 \text{ from both sides.} \end{aligned} \tag{3.12}$$

If the first four derivatives of f and the first three derivatives of g are bounded, then there exists a unique bounded solution of (3.12) in $H^2_{loc}(D)$, and

$$||\psi||_{L^\infty(D)} \leq C\,[||f||_{C^4(\partial D)} + ||g||_{C^3(\partial D)}] \tag{3.13}$$

where C is a constant independent of f, g.

Proof. Choose a function ψ_0 which satisfies

$$\psi_0 = f, \ \frac{\partial \psi_0}{\partial n} = g \ \text{ on } \ \partial D$$

and whose first four derivatives are bounded by the right-hand side of (3.13), and apply Lemma 3.1 to $\psi - \psi_0$. □

Later on we shall also need an extension of Lemma 3.2 to

$$\begin{aligned} &\triangle^2\psi = 0 \ \text{ in } \ D_0, \\ &\psi = f, \ \frac{\partial\psi}{\partial n} = g \ \text{ on } \ \partial D, \\ &\psi = 0, \ \frac{\partial\psi}{\partial n} = h \ \text{ on } \ \Gamma_0 \ \text{ from both sides.} \end{aligned} \tag{3.14}$$

Lemma 3.3. *Assume that h is uniformly bounded and belongs to $C^\alpha(\overline{\Gamma}_0)$, where $\frac{1}{2} < \alpha < 1$. If f, g are as in Lemma 3.2 then there exists a unique bounded solution of (3.14) in $H^2_{loc}(D)$, and*

$$||\psi||_{L^\infty(D)} \leq C\,[||h||_{L^\infty(\Gamma_0)} + ||f||_{C^4(\partial D)} + ||g||_{C^3(\partial D)}] \tag{3.15}$$

where C is a constant independent if h, f, g.

The proof depends upon the construction of a special solution to

$$\begin{aligned} &\triangle^2\chi = 0 \ \text{ in } \ \mathbb{R}^2\backslash\overline{\Gamma}_0, \\ &\chi = 0, \ \frac{\partial\chi}{\partial y} = h \ \text{ on } \ \Gamma_0 \ \text{ (from both side).} \end{aligned} \tag{3.16}$$

Lemma 3.4. *Suppose that h is a bounded function on Γ_0 which belongs to $C^\alpha(\overline{\Gamma}_0)$, $0 < \alpha < 1$. Then there exists a bounded solution χ of (3.16) such that $\chi \in C^1(\mathbb{R}^2)$ and*

$$||\chi||_{L^\infty(\{|y| \leq R\})} \leq C_R||h||_{L^\infty(\Gamma_0)}, \tag{3.17}$$

for any $R > 0$. If further $\alpha \in (\frac{1}{2}, 1)$, then $\chi \in H^2_{loc}(\mathbb{R}^2)$.

Proof. The function

$$w(x_1, y_1) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x_1 h(-\eta^2)}{(y_1 - \eta)^2 + x_1^2} d\eta$$

is harmonic in $\{x_1 > 0\}$, and

$$w(0, y_1) = h(-y_1^2).$$

By the conformal mapping

$$z_1 = z^{\frac{1}{2}} \quad (z = x + iy, \ \ z_1 = x_1 + iy_1) \tag{3.18}$$

we get harmonic function

$$W(x, y) = w(x_1, y_1)$$

in $\mathbb{R}^2 \backslash \overline{\Gamma}_0$, satisfying the boundary condition

$$W(x, 0\pm) = h(x) \ \text{ if } \ x \leq 0.$$

We shall prove that the function

$$\chi(x, y) = yW(x, y) \tag{3.19}$$

satisfies all the assumptions of the lemma.

Clearly χ is biharmonic in $\mathbb{R}^2 \backslash \overline{\Gamma}_0$ and vanishes on Γ_0. Since $\partial_y \chi = W + y\partial_y W$, in order to prove that $\partial_y \chi = h$ on Γ_0 we only need to show that

$$y\frac{\partial W}{\partial y} \to 0 \ \text{ if } \ y \to 0, \ x \leq 0. \tag{3.20}$$

By (3.18)

$$| \nabla W | \leq C \mid \nabla w \mid r^{-\frac{1}{2}} \tag{3.21}$$

where $x = r\cos\theta$, $y = r\sin\theta$. Since $h \in C^\alpha$, $w \in C^\alpha(\mathbb{R}^2)$ and, consequently,

$$| \nabla w | \leq \frac{C}{x_1^{1-\alpha}}. \tag{3.22}$$

It follows that

$$| y\nabla W | \leq C\frac{r^{\frac{1}{2}}\sin\theta}{r^{\frac{1-\alpha}{2}}\cos^{1-\alpha}\frac{\theta}{2}} \leq Cr^{\frac{\alpha}{2}}\cos\frac{\theta}{2} \to 0$$

if $y \to 0$, $x \leq 0$. This establishes (3.20) as well as the fact that $\chi \in C^1(\mathbb{R}^2)$ (since $\partial_x \chi = y\partial_x W$).

The estimate (3.17) follows from the integral representation for w. It finally remains to show that if $\alpha > \frac{1}{2}$ then $\chi \in H^2_{loc}(\mathbb{R}^2)$.

In view of the relations

$$\chi_{xx} = yW_{xx}, \ \ \chi_{xy} = yW_{xy} + W_x, \ \ \chi_{yy} = 2W_y + yW_{yy},$$

all we need to show is that

$$\nabla W \in L^2_{loc}(\mathbb{R}^2), \ \text{ and } \ yW_{ij} \in L^2_{loc}(\mathbb{R}^2). \tag{3.23}$$

To prove the first assertion we use (3.22) in (3.21) to get

$$| \nabla W | \leq C\frac{r^{-\frac{1}{2}}}{x_1^{1-\alpha}} \leq C\frac{r^{-1+\frac{\alpha}{2}}}{\cos^{2(1-\alpha)}\frac{\theta}{2}}.$$

Hence, for any disc $B_R = \{r < R\}$,

$$\int_{B_R} | \nabla W |^2 \le C \int_0^R r^{-1+\alpha} dr \int_{-\pi}^{\pi} \frac{d\theta}{\cos^{2(1-\alpha)}\frac{\theta}{2}} < \infty \text{ if } \alpha > \frac{1}{2}.$$

Next, we have

$$| yW_{ij} | \le C|y| \Big[| D^2 w | r^{-1} + | \nabla w | r^{-\frac{3}{2}} \Big]$$

and, by elliptic estimates,

$$| D^2 w | \le \frac{C}{x_1^{2-\alpha}}.$$

It follows that

$$| yW_{ij} | \le C \frac{r^{\frac{\alpha-2}{2}}}{\cos^{1-\alpha}\frac{\theta}{2}}$$

and

$$\int_{B_R} | yW_{ij} |^2 \le C \int_0^R r^{-1+\alpha} dr \int_{-\pi}^{\pi} \frac{d\theta}{\cos^{2(1-\alpha)}\frac{\theta}{2}} < \infty. \ \square$$

Proof of Lemma 3.3. Apply Lemma 3.2 to $\psi - \chi$. $\square$

We conclude this section with a brief analysis of the behavior of solutions of (3.12) near $(0,0)$.

Lemma 3.5. *Let $\psi(x,y)$ be a solution of (3.12) which belongs to $H^2_{loc}(D)$ and which is an even function in y. Then*

$$\frac{\partial^2 \psi}{\partial y^2}(x,0) \text{ is in } C^\infty(\overline{\Gamma}_0) \tag{3.24}$$

and

$$\frac{\partial^3 \psi}{\partial y^3}(x,0) = -G(x)|x|^{-\frac{3}{2}} \text{ in } \Gamma_0 \tag{3.25}$$

where $G(x)$ is in $C^\infty(\overline{\Gamma}_0)$.

Proof. We use polar coordinates and write $\psi = \psi(r,\theta)$. By [14] ψ has a convergent series expansion about $(0,0)$ in $D_0 \cap B_R$, for any $R > 0$ such that B_R lies in D:

$$\psi(r,\theta) = \Sigma_{k=1}^{\infty} r^{\frac{k}{2}+1} [a_k \cos(\frac{k}{2}+1)\theta + b_k \cos(\frac{k}{2}-1)\theta]$$

where

$$\begin{aligned} & a_k \cos(\frac{k}{2}+1)\pi + b_k \cos(\frac{k}{2}-1)\pi = 0, \\ & (\frac{k}{2}+1)a_k \sin(\frac{k}{2}+1)\pi + (\frac{k}{2}-1)b_k \sin(\frac{k}{2}-1)\pi = 0. \end{aligned} \tag{3.26}$$

(If $\psi(x,y)$ is not even in y then the expansion includes also sine terms) . Note that since $\psi \in H^2_{loc}(D)$, the expansion does not include terms with k smaller than 1.

If we compute

$$\frac{\partial^2}{\partial y^2}\left[r^{\frac{k}{2}\pm 1}\cos(\frac{k}{2}\pm 1)\theta\right] \text{ at } y=0,\ x<0$$

and use (3.26), we find that all terms in $\partial^2\psi/\partial y^2$ with k odd vanish at $\theta = \pm\pi$. This proves the assertion (3.24). The proof of (3.25) is similar: all the terms in $\partial^3\psi/\partial y^3$ with k even vanish at $\theta = \pm\pi$. □

4. A travelling wave solution

In this section we solve a crack propagation problem in the strip D; the crack will be given by

$$\Gamma(t) = \{(x,0);\ -\infty < x < x_0(t)\},\quad x_0(0) = 0, \tag{4.1}$$

and the stress function is a bounded function ϕ_0 satisfying:

$$\Delta^2\phi_0 = 0 \text{ in } D\backslash\overline{\Gamma(t)}, \tag{4.2}$$

$$\phi_0 = M,\ \frac{\partial\phi_0}{\partial n} = 0 \text{ on } \partial D, \tag{4.3}$$

$$\phi_0 = 0,\ \frac{\partial\phi_0}{\partial n} = 0 \text{ on } \Gamma(t); \tag{4.4}$$

here $x_0(t)$ is to be determined by (2.44).

We take

$$\phi_0(x,y,t) = U(x - v_0 t, y) \tag{4.5}$$

where U is the bounded solution in $H^2_{loc}(D)$ to

$$\begin{aligned} &\Delta^2 U = 0 \text{ in } D\backslash\overline{\Gamma}_0,\\ &U = M,\ \frac{\partial U}{\partial n} = 0 \text{ on } \partial D,\\ &U = 0,\ \frac{\partial U}{\partial n} = 0 \text{ on } \Gamma_0 \end{aligned} \tag{4.6}$$

and Γ_0 is as in (3.1); the existence and uniqueness of U was established in Lemma 3.2. Clearly ϕ_0 satisfies (4.2)-(4.4) and therefore it only remains to show that v_0 can be chosen so that (2.44) is satisfied.

Physical interpretation of the boundary conditions: It is easy to verify that

$$\phi_0 \to M \text{ if } x \to \infty \tag{4.7}$$

and

$$\phi_0 \to -\frac{2M}{a^3}|y|^3 + \frac{3M}{a^2}y^2 \text{ if } x \to -\infty. \tag{4.8}$$

The traction vector is given by

$$\vec{s}_0 = \left(\frac{\partial^2\phi_0}{\partial y^2}n_1 - \frac{\partial^2\phi_0}{\partial x\partial y}n_2, \frac{\partial^2\phi_0}{\partial x^2}n_2 - \frac{\partial^2\phi_0}{\partial x\partial y}n_1\right), \tag{4.9}$$

and

$$\vec{s}_0(-\infty) = \left(\frac{12M}{a^3}|y| - \frac{6M}{a^2}, 0\right),\ \vec{s}_0(\infty) = 0. \tag{4.10}$$

On the other hand $\vec{s}_0 = 0$ on ∂D, i.e., there is no traction on $y = \pm a$. Thus the crack propagates due only to force $\vec{s}_0(-\infty)$ which is applied at $x = -\infty$.

To determine v_0 we need first evaluate the J-integral at the tip $X_0(t) = (x_0(t), 0)$.

Denote by $\Lambda = \Lambda^+ \cup \Lambda^- \cup \Lambda_l \cup \Lambda_r$ the path shown in Figure 1; Λ_l lies in $\{y = -b\}$ and Λ_r lies in $\{y = b\}$:

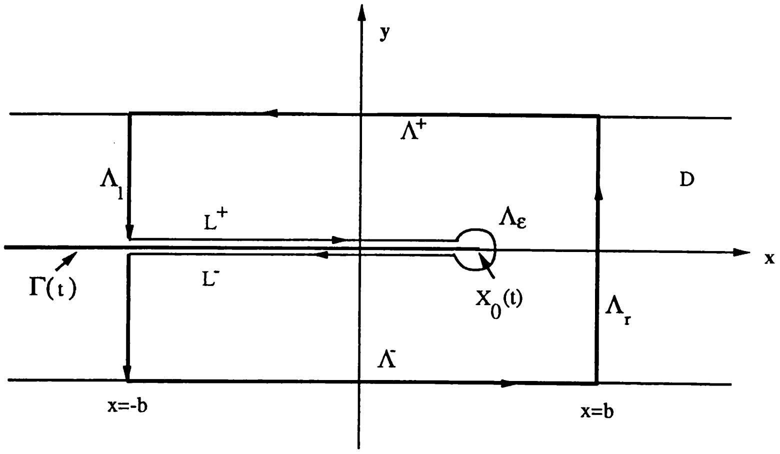

Figure 1

Since J-integral is path-independent, we have

$$J_1(X_0(t)) = \frac{1-\nu^2}{2E}\left\{\int_{\Lambda\cup L^\pm}(\Delta\phi_0)^2 n_1 - 2\int_{\Lambda\cup L^\pm}\vec{s}_0\cdot\vec{\Phi}_0\right\} \tag{4.11}$$

where $L^\pm$ is shown in the figure. Observing that $n_1 = 0$ on $\Lambda^\pm$, $L^\pm$, and

$$\vec{s}_0 = \left(-\frac{\partial}{\partial\tau}\left(\frac{\partial\phi_0}{\partial y}\right), \frac{\partial}{\partial\tau}\left(\frac{\partial\phi_0}{\partial y}\right)\right) = 0 \text{ on } \Lambda^\pm \cup L^\pm, \tag{4.12}$$

we can further write

$$J_1(X_0(t)) = \frac{1-\nu^2}{2E}\left\{\int_{\Lambda_l\cup\Lambda_r}(\Delta\phi_0)^2 n_1 - 2\int_{\Lambda_l\cup\Lambda_r}\vec{s}_0\cdot\vec{\Phi}_0\right\}. \tag{4.13}$$

From (4.9) and

$$\vec{\Phi}_0 = (\Delta\phi_0, (\Delta\phi_0)^c)$$

we have

$$\int_{\Lambda_l}\vec{s}_0\cdot\vec{\Phi}_0 = -\int_{\Lambda_l}\frac{\partial^2\phi_0}{\partial y^2}\Delta\phi_0 + \int_{\Lambda_l}\frac{\partial^2\phi_0}{\partial x\partial y}(\Delta\phi_0)^c \equiv I_1 + I_2. \tag{4.14}$$

Integrating by parts in I_2 and using the Cauchy-Riemann equations, we find that

$$\begin{aligned} I_2 &= -\int_{-a}^{0}\frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial y}(\Delta\phi_0)^c dy - \int_0^a \frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial y}(\Delta\phi_0)^c dy \\ &= -\int_{-a}^{0}\frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial x}(\Delta\phi_0) dy - \int_0^a \frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial x}(\Delta\phi_0) dy \to 0 \qquad \text{as } -b \to -\infty \end{aligned}$$

where the last step follows from (4.8). From (4.8) we also get

$$I_1 \to -2\int_0^a \lim_{-b\to-\infty}\left(\frac{\partial^2\phi_0}{\partial y^2}\right)^2 dy = -2\int_0^a\left(-\frac{12M}{a^3}y + \frac{6M}{a^3}\right)^2 dy = -\frac{24M^2}{a^3},$$

$$\int_{\Lambda_l} (\Delta\phi_0)^2 n_1 \to -\frac{24M^2}{a^3} \qquad \text{if } -b \to -\infty.$$

More simply, from (4.7) we find that

$$\int_{\Lambda_r} (\Delta\phi_0)^2 n_1 - 2\int_{\Lambda_r} \vec{s}_0 \vec{\Phi}_0 \to 0 \qquad \text{as } b \to \infty.$$

Collecting these results, we get

$$J_1(X_0(t)) = \frac{12(1-\nu^2)M^2}{Ea^3}. \tag{4.15}$$

We next evaluate

$$J_2(X_0(t)) = \frac{1-\nu^2}{2E}\left\{\int_{\Lambda\cup L^\pm} (\Delta\phi_0)^2 n_2 - 2\int_{\Lambda\cup L^\pm} \vec{s}_0 \cdot \vec{\Phi}_0^c\right\}; \tag{4.16}$$

here

$$\vec{\Phi}_0^c = (-(\Delta\phi_0)^c, \Delta\phi_0).$$

Since $n_2 = 0$ on $\Lambda_l \cup \Lambda_r$ and $\vec{s}_0 = 0$ on $\Lambda^\pm \cup L^\pm$, we get

$$J_2(X_0(t)) = \frac{1-\nu^2}{2E}\left\{\int_{\Lambda^\pm} (\Delta\phi_0)^2 n_2 + \int_{L^\pm} (\Delta\phi_0)^2 n_2 - 2\int_{\Lambda_l\cup\Lambda_r} \vec{s}_0 \cdot \vec{\Phi}_0^c\right\}.$$

Observing that ϕ_0 is even function in y we deduce that the jump of $\Delta\phi_0$ across L is zero and that

$$\int_{\Lambda^\pm} (\Delta\phi_0)^2 n_2 = 0.$$

Consequently,

$$J_2(X_0(t)) = -\frac{1-\nu^2}{E}\int_{\Lambda_l\cup\Lambda_r} \vec{s}_0 \cdot \vec{\Phi}_0^c.$$

Next, by (4.9),

$$\int_{\Lambda_l} \vec{s}_0 \cdot \vec{\Phi}_0^c = \int_{\Lambda_l} \frac{\partial^2\phi_0}{\partial y^2}(\Delta\phi_0)^c + \int_{\Lambda_l} \frac{\partial^2\phi_0}{\partial x\partial y}\Delta\phi_0 \equiv \tilde{J}_1 + \tilde{J}_2$$

and $\tilde{J}_2 \to 0$ as $-b \to -\infty$, by (4.8). As for $\tilde{J}_1$, we can integrate by parts and proceed similarly to the analysis of I_2 in (4.14):

$$\begin{aligned} \tilde{J}_1 &= -\int_{-a}^{0} \frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial y}(\Delta\phi_0)^c dy - \int_0^a \frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial y}(\Delta\phi_0)^c dy \\ &= -\int_{-a}^{0} \frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial x}(\Delta\phi_0) dy - \int_0^a \frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial x}(\Delta\phi_0) dy \to 0 \text{ as } -b \to -\infty \end{aligned}$$

by (4.8). Similarly one can show that

$$\int_{\Lambda_r} \vec{s}_0 \cdot \vec{\Phi}_0^c \to 0 \text{ if } b \to \infty,$$

so that altogether

$$J_2(X_0(t)) = 0. \tag{4.17}$$

From (2.44), (1.17) and (4.15), (4.17) we see that $\dot{x}_0(t) \geq 0$ and

$$\begin{aligned} v(t) = \dot{x}_0(t) &= v_\infty \frac{(\mid J_1(X_0(t)) \mid -\gamma_0)_+}{\mid J_1(X_0(t)) \mid +\gamma_0} \\ &= v_\infty \frac{((12(1-\nu^2)M^2/Ea^3) - \gamma_0)_+}{(12(1-\nu^2)M^2/Ea^3) + \gamma_0} \equiv v_0 \end{aligned} \tag{4.18}$$

It follows that

$$x_0(t) = v_0 t \text{ and } y_0(t) = 0. \tag{4.19}$$

We summarize:

Theorem 4.1. *There exists a unique bounded travelling wave solution to the special Problem (P) defined by (4.1)-(4.4), where the stress function* ϕ_0 *is given by (4.5) and the crack propagates along the x-axis in accordance with (4.19), where* v_0 *is defined by (4.18).*

5. The linearized problem

We consider a small perturbation of the travelling wave solution constructed in Section 4, and wish to analyze the resulting linearized problem. We take a small $\epsilon > 0$ and define the perturbed boundary conditions:

$$\phi_\epsilon = M + \epsilon, \ \frac{\partial \phi_\epsilon}{\partial n} = 0 \text{ on } \partial D^+ = \{y = a\}, \tag{5.1}$$

$$\phi_\epsilon = M, \ \frac{\partial \phi_\epsilon}{\partial n} = 0 \text{ on } \partial D^- = \{y = -a\}; \tag{5.2}$$

the initial conditions are the same as for the travelling wave solution.

Denote by $\Gamma_\epsilon(t) = \Gamma_0 \cup \{X_\epsilon(s), \ 0 \leq s \leq t\}$ the perturbed crack. Then ϕ_ϵ and $\Gamma_\epsilon(t)$ satisfy:

$$\triangle^2 \phi_\epsilon = 0 \text{ in } D \backslash \overline{\Gamma_\epsilon(t)}, \tag{5.3}$$

$$\phi_\epsilon = 0, \ \frac{\partial \phi_\epsilon}{\partial n} = 0 \text{ on } \Gamma_\epsilon(t), \tag{5.4}$$

and $X_\epsilon(t) = (x_\epsilon(t), y_\epsilon(t))$, the tip of $\Gamma_\epsilon(t)$, travels according to the law (2.44).

We assume that

$$\phi_\epsilon = \phi_0 + \epsilon\phi + o(\epsilon), \tag{5.5}$$

$$X_\epsilon(t) = X_0(t) + \epsilon X(t) + o(\epsilon) \tag{5.6}$$

where $(\phi_0, X_0(t))$ is the solution constructed in Theorem 4.1. Note that

$$\left. \frac{\partial \phi_\epsilon}{\partial \epsilon} \right|_{\epsilon=0} = \phi, \ \left. \frac{\partial X_\epsilon(t)}{\partial \epsilon} \right|_{\epsilon=0} = X(t) \equiv (x(t), y(t)). \tag{5.7}$$

By (5.6) we have

$$\Gamma_\epsilon(t) \to \Gamma(t) \text{ uniformly as } \epsilon \to 0 \tag{5.8}$$

where $\Gamma(t) = \{(x, 0);\ -\infty < x \le v_0 t\}$.

Away from $\Gamma_\epsilon(t)$ and $\Gamma(t)$,

$$0 = \Delta^2 \phi_\epsilon = \Delta^2 \phi_0 + \epsilon \Delta^2 \phi + o(\epsilon) = \epsilon \Delta^2 \phi + o(\epsilon)$$

and, in view of (5.8),

$$\Delta^2 \phi = 0 \text{ in } D \backslash \Gamma(t). \tag{5.9}$$

Clearly, also

$$\phi = 1, \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial D^+, \tag{5.10}$$

$$\phi = 0, \quad \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial D^-. \tag{5.11}$$

Next,

$$0 = \phi_\epsilon(X_\epsilon(s), t) = \phi_\epsilon(x_\epsilon(s), y_\epsilon(s), t) \text{ for } s < t. \tag{5.12}$$

Differentiation of (5.12) with respect to ϵ yields

$$0 = \frac{\partial \phi_\epsilon}{\partial \epsilon} + \frac{\partial \phi_\epsilon}{\partial x}\frac{\partial x_\epsilon}{\partial \epsilon} + \frac{\partial \phi_\epsilon}{\partial y}\frac{\partial y_\epsilon}{\partial \epsilon}.$$

Letting $\epsilon \to 0$ we get

$$0 = \phi + \frac{\partial \phi_0}{\partial x} x(s) + \frac{\partial \phi_0}{\partial y} y(s) \text{ on } \Gamma(t),$$

and, since $\nabla \phi_0 = 0$ on $\Gamma(t)$,

$$\phi = 0 \text{ on } \Gamma(t). \tag{5.13}$$

Also, from the condition

$$\nabla \phi_\epsilon = 0 \text{ on } \Gamma_\epsilon(t),$$

we have

$$0 = \frac{\partial \phi_\epsilon}{\partial y}(x_\epsilon(s), y_\epsilon(s), t) \text{ for } s < t. \tag{5.14}$$

Differentiating with respect to ϵ, we get

$$0 = \frac{\partial}{\partial y}\frac{\partial \phi_\epsilon}{\partial \epsilon} + \frac{\partial^2 \phi_\epsilon}{\partial x \partial y}\frac{\partial x_\epsilon}{\partial \epsilon} + \frac{\partial^2 \phi_\epsilon}{\partial y^2}\frac{\partial y_\epsilon}{\partial \epsilon},$$

and, therefore, as $\epsilon \to 0$,

$$0 = \frac{\partial \phi}{\partial y} + \frac{\partial^2 \phi_0}{\partial x \partial y} x(s) + \frac{\partial^2 \phi_0}{\partial y^2} y(s) \text{ on } \Gamma(t).$$

Since $\partial^2 \phi_0 / \partial x \partial y = 0$ on $\Gamma(t)$, this yields

$$\frac{\partial \phi}{\partial y} = -\left(\frac{\partial^2 \phi_0}{\partial y^2}\right)(v_0 s, 0, t) y(s) \text{ for } s < t. \tag{5.15}$$

Next we derive the motion law for $X(t)$.

Denote by Λ the path shown in the Figure 1. Then

$$J_1(X_\epsilon(t)) = \frac{1-\nu^2}{2E}\left\{\int_\Lambda (\Delta\phi_\epsilon)^2 n_1 + \int_{\Gamma^{\pm}_{\epsilon,b}(t)} (\Delta\phi_\epsilon)^2 n_1 - 2\int_\Lambda \vec{s}_\epsilon \cdot \vec{\Phi}_\epsilon\right\}$$

where $\Gamma_{\epsilon,b}(t) = \Gamma_\epsilon(t) \cap \{x \geq -b\}$.

Since $n_1 = 0$ on ∂D and $\vec{s}_\epsilon = 0$ on ∂D, we have

$$J_1(X_\epsilon(t)) = \frac{1-\nu^2}{2E}\left\{\int_{\Lambda_l \cup \Lambda_r} (\Delta\phi_\epsilon)^2 n_1 + \int_{\Gamma^{\pm}_{\epsilon,b}(t)} (\Delta\phi_\epsilon)^2 n_1 - 2\int_{\Lambda_l \cup \Lambda_r} \vec{s}_\epsilon \cdot \vec{\Phi}_\epsilon\right\}. \tag{5.16}$$

We can easily verify (cf. (4.7), (4.8)) that

$$\phi_\epsilon \to -\frac{\epsilon}{4a^3}y^3 + \frac{3\epsilon}{4a}y + M + \frac{\epsilon}{2} \quad \text{as } x \to \infty \tag{5.17}$$

and

$$\phi_\epsilon \to \begin{cases} -[2(M+\epsilon)/a^3]y^3 + [3(M+\epsilon)/a^2]y^2, & y > 0 \\ [2M/a^3]y^3 + [3M/a^2]y^2, & y < 0 \end{cases} \quad \text{as } x \to -\infty. \tag{5.18}$$

By the method in §4, as $-b \to -\infty$,

$$\begin{aligned} &\int_{\Lambda_l} (\Delta\phi_\epsilon)^2 n_1 - 2\int_{\Lambda_l} \vec{s}_\epsilon \cdot \vec{\Phi}_\epsilon \to \int_{-a}^{a} \lim_{-b\to-\infty}\left(\frac{\partial^2\phi_\epsilon}{\partial y^2}\right)^2 dy \\ &= \frac{12M^2}{a^3} + \frac{12(M+\epsilon)^2}{a^3} = \frac{24}{a^3}M^2 + \frac{24}{a^3}\epsilon + o(\epsilon). \end{aligned}$$

Similarly, as $b \to \infty$,

$$\int_{\Lambda_r} (\Delta\phi_\epsilon)^2 n_1 - 2\int_{\Lambda_r} \vec{s}_\epsilon \cdot \vec{\Phi}_\epsilon \to 3\int_{-a}^{a} \lim_{b\to\infty}\left(\frac{\partial^2\phi_\epsilon}{\partial y^2}\right)^2 = o(\epsilon).$$

Using these relations in (5.16), we get

$$J_1(X_\epsilon(t)) = J_1(X_0(t)) + \frac{12(1-\nu^2)M}{Ea^3}\epsilon + \frac{1-\nu^2}{2E}\int_{\Gamma_{\epsilon,b}(t)^{\pm}} (\Delta\phi_\epsilon)^2 n_1 + o(\epsilon). \tag{5.19}$$

Recall that $\phi_0(x,y) = \phi_0(x,-y)$ and, consequently, $\partial^2\phi_0/\partial y^2$ and $\Delta\phi_0$ are continuous across the crack $\Gamma(t)$. This formally justifies the relation

$$\Delta\phi_\epsilon \to \Delta\phi_0 \text{ from either side of } \Gamma_0(t). \tag{5.20}$$

Along $\Gamma_{\epsilon,b}(t)$ we then have

$$\begin{aligned} (\Delta\phi_\epsilon)^2 n_1 = -(\Delta\phi_\epsilon)^2 \dot{y}_\epsilon(s) &= -\epsilon(\Delta\phi_\epsilon)^2(X_\epsilon(s),t)(\dot{y}(s) + o(1)) \\ &= -\epsilon(\Delta\phi_0)^2(v_0 s, 0, t)(\dot{y}(s) + o(1)), \end{aligned}$$

and we conclude that

$$\int_{\Gamma^{\pm}_{\epsilon,b}(t)} (\Delta\phi_\epsilon)^2 n_1 = o(\epsilon). \tag{5.21}$$

Substituting this into (5.19) we find that

$$J_1(X_\epsilon(t)) = J_1(X_0(t)) + \frac{12(1-\nu^2)M}{Ea^3}\epsilon + o(\epsilon). \tag{5.22}$$

We next evaluate $J_2(X_\epsilon(t))$, which we can write in the form

$$J_2(X_\epsilon(t)) = \frac{1-\nu^2}{2E}\left\{\int_\Lambda (\Delta\phi_\epsilon)^2 n_2 + \int_{\Gamma^\pm_{\epsilon,b}(t)} (\Delta\phi_\epsilon)^2 n_2 - 2\int_\Lambda \vec{s}_\epsilon \cdot \vec{\Phi}^c_\epsilon\right\}. \tag{5.23}$$

We can write

$$\vec{s}_\epsilon = \vec{s}_0 + \epsilon\vec{s} + o(\epsilon),$$
$$\vec{\Phi}^c_\epsilon = \vec{\Phi}^c_0 + \epsilon\vec{\Phi}^c + o(\epsilon)$$

where (cf. (4.9))

$$\vec{s}_0 = \left(\frac{\partial^2\phi_0}{\partial y^2}n_1 - \frac{\partial^2\phi_0}{\partial x\partial y}n_2, \frac{\partial^2\phi_0}{\partial x^2}n_2 - \frac{\partial^2\phi_0}{\partial x\partial y}n_1\right), \tag{5.24}$$

$$\vec{s} = \left(\frac{\partial^2\phi}{\partial y^2}n_1 - \frac{\partial^2\phi}{\partial x\partial y}n_2, \frac{\partial^2\phi}{\partial x^2}n_2 - \frac{\partial^2\phi}{\partial x\partial y}n_1\right), \tag{5.25}$$

and

$$\vec{\Phi}^c_0 = (-(\Delta\phi_0)^c, \Delta\phi_0), \tag{5.26}$$

$$\vec{\Phi}^c = (-(\Delta\phi)^c, \Delta\phi). \tag{5.27}$$

Then we have

$$\begin{aligned} J_2(X_\epsilon(t)) = \frac{1-\nu^2}{2E}\bigg\{&\int_\Lambda (\Delta\phi_0)^2 n_2 + 2\epsilon\int_\Lambda (\Delta\phi_0\cdot\Delta\phi)n_2 - 2\int_\Lambda \vec{s}_0\cdot\vec{\Phi}^c_0 \\ &-2\epsilon\int_\Lambda(\vec{s}_0\cdot\vec{\Phi}^c + \vec{s}\cdot\vec{\Phi}^c_0) + \int_{\Gamma^\pm_{\epsilon,b}(t)}(\Delta\phi_\epsilon)^2 n_2\bigg\} + o(\epsilon). \end{aligned} \tag{5.28}$$

Denote by $[(\Delta\phi_\epsilon)^2]$ the jump of $(\Delta\phi_\epsilon)^2$ across $\Gamma_{\epsilon,b}(t)$ ("plus" above $\Gamma_{\epsilon,b}(t)$ and "minus" below $\Gamma_{\epsilon,b}(t)$). Using this convention we can write

$$\begin{aligned} \int_{\Gamma^\pm_{\epsilon,b}(t)}(\Delta\phi_\epsilon)^2 n_2 &= -\int_{-b/v_0}^t [(\Delta\phi_\epsilon)^2](x_\epsilon(s), y_\epsilon(s), t)\dot{x}_\epsilon(s)ds \\ &= -v_0 f(\epsilon)(1+o(\epsilon)) \end{aligned} \tag{5.29}$$

where

$$f(\epsilon) = \int_{-b/v_0}^t [(\Delta\phi_\epsilon)^2](x_\epsilon(s), y_\epsilon(s), t)ds.$$

Since

$$[\Delta\phi_0](v_0 s, 0, t) = 0 \text{ and } \left[\frac{\partial}{\partial x}\Delta\phi_0(v_0 s, 0, t)\right] = 0,$$

we have

$$f'(\epsilon)\big|_{\epsilon=0} = \int_{-b/v_0}^t 2\left\{\Delta\phi_0[\Delta\phi](v_0 s, 0, t) + \Delta\phi_0\left[\frac{\partial}{\partial y}\Delta\phi_0\right](v_0 s, 0, t)y(s)\right\}ds.$$

Recalling that $\phi_0 = \partial\phi_0/\partial y = 0$ along $\Gamma(t)$, we deduce that

$$f'(0) = 2\int_{-b/v_0}^{t} \frac{\partial^2\phi_0}{\partial y^2}\left\{[\Delta\phi] + \left[\frac{\partial^3\phi_0}{\partial y^3}\right] y(s)\right\} ds.$$

Since $f(0) = 0$, we have $f(\epsilon) = f'(0)\epsilon + o(\epsilon)$. Using this in (5.29) and substituting the result in (5.28), we arrive at the formula

$$J_2(X_\epsilon(t)) = J_2(\phi, y)\epsilon + o(\epsilon) \tag{5.30}$$

where

$$J_2(\phi, y) = \frac{1-\nu^2}{E}\left\{\int_\Lambda (\Delta\phi_0 \cdot \Delta\phi) n_2 - \int_\Lambda \left(\vec{s}_0 \cdot \vec{\Phi}^c + \vec{s} \cdot \vec{\Phi}_0^c\right)\right\} - \frac{v_0(1-\nu^2)}{E}\int_{-b/v_0}^{t} \frac{\partial^2\phi_0}{\partial y^2}\left\{[\Delta\phi] + \left[\frac{\partial^3\phi_0}{\partial y^3}\right] y(s)\right\}(v_0 s, 0, t)ds. \tag{5.31}$$

Having computed J_1, J_2 at the tip $X_\epsilon(t)$, we now express $\dot{X}_\epsilon(t)$ by (2.44):

$$\dot{x}_\epsilon(t) = \frac{v_\epsilon(t)}{\gamma_0}\frac{v_\infty - v_\epsilon(t)}{v_\infty + v_\epsilon(t)} J_1(X_\epsilon(t)), \tag{5.32}$$

$$\dot{y}_\epsilon(t) = \frac{v_\epsilon(t)}{\gamma_0}\frac{v_\infty - v_\epsilon(t)}{v_\infty + v_\epsilon(t)} J_2(X_\epsilon(t)). \tag{5.33}$$

By (5.22), (5.32),

$$v_0 + \epsilon\dot{x}(t) + o(\epsilon) = \frac{v_\epsilon(t)}{\gamma_0}\frac{v_\infty - v_\epsilon(t)}{v_\infty + v_\epsilon(t)} J_1(X_0(t)) + \frac{v_0}{\gamma_0}\frac{v_\infty - v_0}{v_\infty + v_0}\frac{12(1-\nu^2)M}{Ea^3}\epsilon$$

so that

$$\epsilon\dot{x}(t) = J_1(X_0(t))\left[\frac{v_\epsilon}{\gamma_0}\frac{v_\infty - v_\epsilon}{v_\infty + v_\epsilon} - \frac{v_0}{J_1}\right] + \frac{v_0}{\gamma_0}\frac{v_\infty - v_0}{v_\infty + v_0}\frac{J_1(X_0(t))}{M}\epsilon + o(\epsilon)$$

where (4.15) was used. By (2.44)

$$J_1(X_0(t)) = \gamma_0 \frac{v_\infty + v_0}{v_\infty - v_0} \tag{5.34}$$

and, therefore,

$$\epsilon\dot{x}(t) = J_1(X_0(t))\left\{\frac{d}{d\epsilon}\left[\frac{v_\epsilon}{\gamma_0}\frac{v_\infty - v_\epsilon}{v_\infty + v_\epsilon}\right]\bigg|_{\epsilon=0}\right\}\epsilon + \frac{v_0}{M}\epsilon + o(\epsilon).$$

Noting that $(dv_\epsilon/d\epsilon)|_{\epsilon=0} = \dot{x}(t)$, it easily follows that

$$\dot{x}(t) = \frac{J_1(X_0(t))}{\gamma_0}\frac{\dot{x}(t)}{(v_\infty + v_0)^2}(v_\infty^2 - v_0^2 - 2v_\infty v_0) + \frac{v_0}{M}.$$

Using (5.34) once more, we get

$$\dot{x}(t) = \frac{\dot{x}(t)}{v_\infty^2 - v_0^2}(v_\infty^2 - v_0^2 - 2v_\infty v_0) + \frac{v_0}{M},$$

or

$$\dot{x}(t) = \frac{v_\infty^2 - v_0^2}{2v_\infty v_0}.$$

Since $x(0) = 0$ we finally obtain:

$$x(t) = \frac{v_\infty^2 - v_0^2}{2v_\infty v_0} t. \tag{5.35}$$

As for $y(t)$, from (5.33) and (5.30) we get

$$\dot{y}(t) = \frac{v_0}{\gamma_0} \frac{v_\infty - v_0}{v_\infty + v_0} J_2(\phi, y)$$

where $J_2(\phi, y)$ is computed from (5.31) and (5.24)-(5.27).

We summarize the formulation of the linearized problem:

Problem (L).
Given positive constants γ_0, v_∞, M such that

$$\frac{12(1-\nu^2)M^2}{Ea^3} > \gamma_0 \tag{5.36}$$

(i.e., $\dot{x}_0(t) > 0$; see (4.18)), find functions $\phi = \phi(x, y, t)$ and $y(t)$ such that

$$\triangle^2 \phi = 0 \text{ in } D \backslash \overline{\Gamma(t)} \tag{5.37}$$

where $\Gamma(t) = \{(x, 0);\ -\infty < x < v_0 t\}$,

$$\phi = 1,\ \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial D^+, \tag{5.38}$$

$$\phi = 0,\ \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial D^-, \tag{5.39}$$

$$\phi = 0,\ \frac{\partial \phi}{\partial y}(v_0 s, 0, t) = -\left(\frac{\partial^2 \phi_0}{\partial y^2}\right)(v_0 s, 0, t) y(s) \text{ on } \Gamma(t) \tag{5.40}$$

and

$$\dot{y}(t) = \frac{v_0}{\gamma_0} \frac{v_\infty - v_0}{v_\infty + v_0} J_2(\phi, y),\ \ y(0) = 0 \tag{5.41}$$

where $J_2(\phi, y)$ is defined by (5.31), and $(\phi_0, v_0 t)$ is the travelling wave solution constructed in Section 4.

Once the linearized problem is solved, we can compute $x(t)$ from (5.35).

6. Preliminary analysis of the linearized problem

We first simplify the functional $J_2(\phi, y)$ defined in (5.31). By (5.24), (5.25), $\vec{s}_0 = \vec{s} = 0$ on ∂D so that

$$\int_\Lambda (\vec{s}_0 \cdot \vec{\Phi}^c + \vec{s} \cdot \vec{\Phi}_0^c) = \int_{\Lambda_l \cup \Lambda_r} (\vec{s}_0 \cdot \vec{\Phi}^c + \vec{s} \cdot \vec{\Phi}_0^c). \tag{6.1}$$

By (5.24), (5.26),

$$\int_{\Lambda_l} \vec{s}_0 \cdot \vec{\Phi}^c = \int_{-a}^{0} \left[\frac{\partial^2\phi_0}{\partial y^2}(-b,y)(\Delta\phi)^c(-b,y) + \frac{\partial^2\phi_0}{\partial x\partial y}(-b,y)\Delta\phi(-b,y)\right] dy.$$

Integrating by parts,

$$\begin{aligned}\int_{-a}^{a} \frac{\partial^2\phi_0}{\partial y^2}(\Delta\phi)^c dy &= \int_{-a}^{0} \frac{\partial^2\phi_0}{\partial y^2}(\Delta\phi)^c dy + \int_{0}^{a} \frac{\partial^2\phi_0}{\partial y^2}(\Delta\phi)^c dy \\ &= -\int_{-a}^{0} \frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial y}(\Delta\phi)^c dy - \int_{0}^{a} \frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial y}(\Delta\phi)^c dy \\ &= -\int_{-a}^{0} \frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial x}\Delta\phi dy - \int_{0}^{a} \frac{\partial\phi_0}{\partial y}\frac{\partial}{\partial x}\Delta\phi dy\end{aligned}$$

by the Cauchy-Riemann equations.

Also, by integration by parts,

$$\int_{-a}^{a} \left(\frac{\partial^2\phi_0}{\partial x\partial y}\Delta\phi\right)(-b,y)dy = -\int_{-a}^{0} \frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial y}\Delta\phi dy - \int_{0}^{a} \frac{\partial\phi_0}{\partial x}\frac{\partial}{\partial y}\Delta\phi dy.$$

Introducing the notation

$$\nabla^c = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right), \tag{6.2}$$

we get

$$\int_{\Lambda_l} \vec{s}_0 \cdot \vec{\Phi}^c = \int_{\Lambda_l} (\nabla^c\phi_0 \cdot \nabla\Delta\phi)n_1$$

where the right-hand side is defined by

$$\int_{\Lambda_l} (\nabla^c\phi_0 \cdot \nabla\Delta\phi)n_1 = \int_{-a}^{0} (\nabla^c\phi_0 \cdot \nabla\Delta\phi)n_1 + \int_{0}^{a} (\nabla^c\phi_0 \cdot \nabla\Delta\phi)n_1 \tag{6.3}$$

and (n_1, n_2) is the outward normal.

Similarly,

$$\int_{\Lambda_l} \vec{s} \cdot \vec{\Phi}_0^c = \int_{\Lambda_l} (\nabla^c\phi \cdot \nabla\Delta\phi_0)n_1$$

and

$$\int_{\Lambda_r} (\vec{s}_0 \cdot \vec{\Phi}^c + \vec{s} \cdot \vec{\Phi}_0^c) = \int_{\Lambda_r} (\nabla^c\phi_0 \cdot \nabla\Delta\phi + \nabla^c\phi \cdot \nabla\Delta\phi_0)n_1.$$

Using all this in (6.1) and substituting the result into (5.31), we obtain

$$\begin{aligned} J_2(\phi,y) = \frac{1-\nu^2}{E}&\left\{\int_{\Lambda}(\Delta\phi_0 \cdot \Delta\phi)n_2 - \int_{\Lambda_l \cup \Lambda_r} (\nabla^c\phi_0 \cdot \nabla\Delta\phi + \nabla^c\phi \cdot \nabla\Delta\phi_0)n_1\right\} \\ &-\frac{v_0(1-\nu^2)}{E}\int_{-b/v_0}^{t} \frac{\partial^2\phi_0}{\partial y^2}\left\{[\Delta\phi] + \left[\frac{\partial^3\phi_0}{\partial y^3}\right] y(s)\right\}(v_0 s, 0, t)ds. \end{aligned} \tag{6.4}$$

Since $\nabla^c\phi_0 = \nabla\phi = 0$ on ∂D, we can also write

$$\begin{aligned} J_2(\phi,y) = \frac{1-\nu^2}{E}&\left\{\int_{\Lambda}(\Delta\phi_0 \cdot \Delta\phi)n_2 - \int_{\Lambda} (\nabla^c\phi_0 \cdot \nabla\Delta\phi + \nabla^c\phi \cdot \nabla\Delta\phi_0)n_1\right\} \\ &-\frac{v_0(1-\nu^2)}{E}\int_{-b/v_0}^{t} \frac{\partial^2\phi_0}{\partial y^2}\left\{[\Delta\phi] + \left[\frac{\partial^3\phi_0}{\partial y^3}\right] y(s)\right\}(v_0 s, 0, t)ds. \end{aligned} \tag{6.5}$$

By Lemma 3.5

$$\frac{\partial^2\phi_0}{\partial y^2}(v_0 s,0,t)\equiv K(s,t) \text{ is } C^\infty \text{ function in } (s,t), \tag{6.6}$$

and

$$\left[\frac{\partial^3\phi_0}{\partial y^3}\right](v_0 s,0,t)\equiv -G(s,t)(t-s)^{-\frac{3}{2}} \tag{6.7}$$

where $G(s,t)$ is a C^∞ function in (s,t). The singularity of $[\partial^3\phi_0/\partial y^3](v_0 s,0,t)$ at $s=t$ will cause difficulties when we try to analyze the last integral in (6.5). To overcome these difficulties we shall work with a new function ψ, defined by

$$\psi=\phi+\frac{\partial\phi_0}{\partial y}y(t). \tag{6.8}$$

Then

$$\triangle^2\psi=0 \text{ in } D\backslash\Gamma(t), \tag{6.9}$$

$$\psi=1,\ \frac{\partial\psi}{\partial y}=\frac{\partial^2\phi_0}{\partial y^2}y(t) \text{ on } \partial D^+, \tag{6.10}$$

$$\psi=0,\ \frac{\partial\psi}{\partial y}=\frac{\partial^2\phi_0}{\partial y^2}y(t) \text{ on } \partial D^-, \tag{6.11}$$

$$\psi=0,\ \frac{\partial\psi}{\partial y}=\frac{\partial^2\phi_0}{\partial y^2}(y(t)-y(s)) \text{ on } \Gamma(t), \tag{6.12}$$

and

$$\dot y(t)=\frac{v_0}{\gamma_0}\frac{v_\infty-v_0}{v_\infty+v_0}\tilde J(\psi,y),\ y(0)=0 \tag{6.13}$$

where

$$\begin{aligned}\tilde J(\psi,y)=&\frac{1-\nu^2}{E}\left\{\int_{\Lambda(t)}(\triangle\phi_0\cdot\triangle\psi)n_2-\int_{\Lambda(t)}(\nabla^c\phi_0\cdot\nabla\triangle\psi+\nabla^c\psi\cdot\nabla\triangle\phi_0)n_1\right\}\\ &-\frac{1-\nu^2}{E}y(t)\left\{\int_{\Lambda(t)}\left(\triangle\phi_0\cdot\triangle\frac{\partial\phi_0}{\partial y}\right)n_2-\int_{\Lambda(t)}(\nabla^c\phi_0\cdot\nabla\triangle\frac{\partial\phi_0}{\partial y}+\nabla^c\frac{\partial\phi_0}{\partial y}\cdot\nabla\triangle\phi_0)n_1\right\}\\ &-\frac{v_0(1-\nu^2)}{E}\int_{-b/v_0}^{t}K(s,t)\left\{[\triangle\psi]+G(s,t)\frac{y(t)-y(s)}{(t-s)^{\frac{3}{2}}}\right\}ds;\end{aligned} \tag{6.14}$$

here $\Lambda(t)$ consists of the four segments

$$\partial D^\pm\cap\{-b\le x\le b+v_0t\},\ \{x=-b,\ |y|\le a\},\ \{x=b+v_0t,\ |y|\le a\}.$$

For any $T>0$ and $0<\alpha<1$, introduce the space

$$C_0^{1+\alpha}[0,T]=\{y(s)\in C^{1+\alpha}[0,T],\ y(0)=0\}.$$

with the usual $C^{1+\alpha}$ norm.

Take any y in $C_0^{1+\alpha}[0,T]$, extend it by zero to $-\infty < s < 0$, and define

$$h(s,t) = \left(\frac{\partial^2 \phi_0}{\partial y^2}\right)(v_0 s, 0, t)(y(t) - y(s)). \tag{6.15}$$

Then h is Lipschitz continuous in $s \in (-\infty, t]$ and in $C^{1+\alpha}$ for $s \neq 0$. By Lemma 3.3 there exists a unique bounded solution ψ to (6.9)-(6.12). We define $\tilde{J}(\psi, y)$ by (6.14), and a function $\tilde{y}(t)$ by

$$\frac{d}{dt}\tilde{y}(t) = \frac{v_0}{\gamma_0}\frac{v_\infty - v_0}{v_\infty + v_0}\tilde{J}(\psi, y), \;\; \tilde{y}(0) = 0. \tag{6.16}$$

Defining a mapping $\mathcal{M}$ by

$$\tilde{y} = \mathcal{M}y \tag{6.17}$$

we shall prove (in the next two sections) that $\mathcal{M}$ has a unique fixed point. This will establish existence and uniqueness of solution to the linearized problem.

7. Auxiliary estimates

Take $0 < T < 1$. For any $\delta \in (0, a)$ we surround the interval $\{0 \leq x \leq v_0 t, \; y = 0\}$ by a rectangle

$$D_\delta = \{-\delta < x < v_0 t + \delta, \; |y| < \delta\}$$

and set

$$D_R^\delta = \left(\overline{D} \cap \{-R \leq x \leq v_0 t + R\}\right) \setminus (D_\delta \cup \Gamma(t)).$$

By Lemma 3.3

$$||\psi||_{L^\infty(D)} \leq C(1 + ||y||_{C^0[0,T]}), \tag{7.1}$$

and by elliptic estimates we then also have

$$||\psi||_{C^4(D_R^\delta)} \leq C_{R,\delta}(1 + ||y||_{C^0[0,T]}) \tag{7.2}$$

where $C_{R,\delta}$ is a constant independent of T. Since $y(0) = 0$,

$$|y(t)| \leq T||y||_{C^1[0,T]}, \tag{7.3}$$

so that

$$||\psi||_{C^4(\Lambda(t))} \leq C_{R,\delta}(1 + T||y||_{C^1[0,T]}). \tag{7.4}$$

We next proceed to estimate $[\Delta\psi](v_0 s, 0, t)$ for $-b/v_0 \leq s \leq t$. For any fixed $t \in [0,T]$, let

$$\tilde{\psi} = \psi - \chi$$

where χ is defined by Lemma 3.4 with h given by (6.15) (as a function of s). By Lemma 3.4,

$$||\chi||_{L^\infty(D)} \leq C||y||_{C^0[0,T]} \tag{7.5}$$

so that, by (7.1), also

$$||\tilde{\psi}||_{L^\infty(D)} \leq C(1 + ||y||_{C^0[0,T]}). \tag{7.6}$$

Since $\tilde{\psi} \in H^2_{loc}(D)$ we have [14]:

$$\tilde{\psi}(r,\theta) = \Sigma_{k=1}^{\infty} r^{\frac{k}{2}+1}[a_k \cos(\frac{k}{2}+1)\theta + b_k \cos(\frac{k}{2}-1)\theta + c_k \sin(\frac{k}{2}-1)\theta + d_k \sin(\frac{k}{2}+1)\theta] \quad (7.7)$$

for (r,θ) in the disc $B_\delta(t)$ with radius δ and center at the tip $(v_0 t, 0)$, where the coefficients $a_k, \dots$ are defined by

$$a_k = \frac{2}{\pi \delta^{\frac{k}{2}+1}} \int_{-\pi}^{\pi} \tilde{\psi}(\delta,\theta) \cos(\frac{k}{2}+1)\theta d\theta, \dots \ ;$$

δ is any positive number in the interval $(0,a)$ (so that $B_\delta(t) \subset D$). It follows that

$$|a_k| \le C\delta^{-(\frac{k}{2}+1)} \|\tilde{\psi}\|_{L^\infty(D)};$$

the same bound holds for b_k, c_k, d_k.

Noting that

$$| \triangle\tilde{\psi}(r,\pi) - \triangle\tilde{\psi}(r,-\pi) | \le C(|c_1| + |d_1|)r^{-\frac{1}{2}} + C\Sigma_{k=2}^{\infty} r^{\frac{k}{2}-1}(|a_k| + |b_k| + |c_k| + |d_k|),$$

for $r \le \frac{\delta}{2}$ we get

$$| [\triangle\tilde{\psi}](v_0 s, 0, t) | \le C\|\tilde{\psi}\|_{L^\infty(D)} \left(1 + \frac{1}{\sqrt{t-s}}\right) \text{ if } t - \frac{\delta}{2v_0} < s < t. \quad (7.8)$$

Next, by elliptic estimates we get

$$| [\triangle\tilde{\psi}](v_0 s, 0, t) | \le C(1 + \|\chi\|_{C^2} + \|y\|_{C^0[0,T]} + \|\tilde{\psi}\|_{L^\infty(D)}) \text{ if } -\frac{b}{v_0} \le s \le t - \frac{\delta}{2v_0} \quad (7.9)$$

where the C^2 norm of χ is taken over the intervals $\partial D^{\pm}$, $D \cap \{x = -b\}$ and $D \cap \{x = v_0 t - \delta/8\}$. From the integral representation formula for χ we easily obtain

$$\|\chi\|_{C^2} \le C\|y\|_{C^0[0,T]}.$$

Using this in (7.9), combining the result with (7.8), and recalling (7.6), (7.3), we obtain:

Lemma 7.1. *The following inequality holds:*

$$| [\triangle\psi](v_0 s, 0, t) | \le C(1 + \sqrt{T}\|y\|_{C^1[0,T]}) \left(1 + \frac{1}{\sqrt{t-s}}\right) + | [\triangle\chi](v_0 s, 0, t) | . \quad (7.10)$$

To estimate $[\triangle\chi](v_0 s, 0, t)$ we shall use the notation of Lemma 3.4, replacing however Γ_0 by $\Gamma(t)$. Thus the origin in the z-plane is replaced by $z_0 = v_0 t + i \cdot 0$,

$$z_1 = (z - z_0)^{\frac{1}{2}}, \ \ W(x,y) = w(x_1, y_1), \quad (7.11)$$

and

$$w(0, y_1) = W(z_1^2 + z_0)|_{x_1=0} = h(-\frac{1}{v_0}y_1^2 + t, t) \equiv \tilde{h}(y_1) \quad (7.12)$$

where

$$h(s,t) = \frac{\partial^2 \phi_0}{\partial y^2}(v_0 s, 0, t)(y(t) - y(s)), \quad -\infty < s < t. \tag{7.13}$$

Clearly $h \in C^{1+\alpha}$ except at $s = 0$ (since $\dot{y}(0+)$ might not be equal to zero). This means that

$$w(0, y_1) \in C^{1+\alpha} \text{ except at } y_1 = \pm\sqrt{v_0 t}.$$

Lemma 7.2. *Let $\hat{w}$ be bounded solution of*

$$\begin{aligned} &\triangle \hat{w} = 0 \text{ in } \mathbb{R}^2 \cap \{x > 0\}, \\ &\hat{w}(0, y) = \hat{h}(y), \quad y \in \mathbb{R} \end{aligned}$$

where $\hat{h}$ is a bounded $C^{1+\alpha}$ function in $\mathbb{R}\backslash\{0\}$,

$$\begin{aligned} &\hat{h}(y) = h_0|y| \text{ if } |y| \leq 2R \ (R \geq 1), \\ &||\hat{h}||_{C^0(\mathbb{R})} \leq C|h_0|R. \end{aligned}$$

Then

$$|\ \hat{w}_x(0, y)\ | \leq C|h_0| \log\frac{2R}{|y|} \ \textit{for} \ |y| \leq R$$

where C is a positive constant independent of h_0, R.

Proof. By scaling we may assume that $R = 1$. The function $\zeta = \hat{w} - h_0|y|$ satisfies:

$$\begin{aligned} &\triangle \zeta = 0 \text{ in } (\mathbb{R}^2 \cap \{x > 0\})\backslash\{y = 0\}, \\ &\zeta(0, y) = 0 \text{ in } |y| < 2. \end{aligned}$$

Extend ζ as an odd function into $\{x < 0\}$. Then the extended function is harmonic in the strip $\{0 < |y| < 2\}$ and

$$\zeta_y(x, 0+) - \zeta_y(x, 0-) = -2h_0 \text{ if } x \neq 0.$$

By Lemma 4.2 of [5] we deduce that

$$|\ \zeta_x(0, y)\ | \leq C|h_0|(1 + |\log|y|\ |)$$

and the lemma follows. □

We now break $\tilde{h}$ into two functions: $\tilde{h}_0 + \tilde{h}_1$ where $\tilde{h}_0$ contains the jumps on the derivative of $\tilde{h}$ at $y_1 = \pm\sqrt{v_0 t}$:

$$\begin{aligned} \tilde{h}_0(y_1) &= \tilde{h}'(\sqrt{v_0 t}+)(y_1 - \sqrt{v_0 t})_+ - \tilde{h}'(\sqrt{v_0 t}-)(y_1 - \sqrt{v_0 t})_- \\ &- \tilde{h}'(-\sqrt{v_0 t}+)(y_1 + \sqrt{v_0 t})_+ - \tilde{h}'(-\sqrt{v_0 t}-)(y_1 + \sqrt{v_0 t})_- \end{aligned} \tag{7.14}$$

if $|y_1| \leq 4\sqrt{v_0 T} + \sqrt{b}$ and $\tilde{h}_0$ remains bounded at infinity. We can write $w = w_0 + w_1$ where w_j is the bounded harmonic function in $\mathbb{R}\backslash\{x \geq 0\}$ with $w_j(0, y_1) = \tilde{h}_j(y_1)$. Clearly

$$||\tilde{h}_0||_{C^{1+\alpha}\{\sqrt{v_0 t} \leq y_1 \leq R\}} \leq C_R \max\{|\ \tilde{h}'(\pm\sqrt{v_0 t}\pm)\ |\}$$

and $\tilde{h}_1 \in C^{1+\alpha}(\mathbb{R})$.

From the relation

$$\tilde{h}'(y_1) = -\frac{2}{v_0} y_1 h'(-\frac{1}{v_0} y_1^2 + t, t)$$

we deduce that

$$||\tilde{h}'||_{C^0[-\sqrt{v_0 t}, \sqrt{v_0 t}]} \leq C\sqrt{T} ||y||_{C^1[0,T]}, \tag{7.15}$$

since $|y_1| \leq \sqrt{v_0 T}$. Similarly we can estimate the Hölder coefficient of $\tilde{h}'$ and thus get:

$$\begin{aligned} ||\tilde{h}||_{C^{1+\alpha}[-\sqrt{v_0 t}, \sqrt{v_0 t}]} &= ||\tilde{h}||_{C^0[-\sqrt{v_0 t}, \sqrt{v_0 t}]} + ||\tilde{h}'||_{C^0[-\sqrt{v_0 t}, \sqrt{v_0 t}]} + ||\tilde{h}'||_{C^\alpha[-\sqrt{v_0 t}, \sqrt{v_0 t}]} \\ &\leq C\sqrt{T} ||y||_{C^{1+\alpha}[0,T]} \end{aligned} \tag{7.16}$$

where we used the fact that $\tilde{h}(0) = h(t,t) = 0$ and, therefore,

$$||\tilde{h}||_{C^0[-\sqrt{v_0 t}, \sqrt{v_0 t}]} \leq T ||\tilde{h}'||_{C^0[-\sqrt{v_0 t}, \sqrt{v_0 t}]}.$$

For $|y_1| > \sqrt{v_0 t}$, $y(s) = 0$ so that

$$\tilde{h}(y_1) = \frac{\partial^2 \phi_0}{\partial y^2}(-y_1^2 + v_0 t, 0, t) y(t).$$

Consequently,

$$||\tilde{h}||_{C^{1+\alpha}\{\sqrt{v_0 t} \leq |y_1| \leq R\}} \leq C_R |y(t)| \leq C_R T ||y||_{C^1[0,T]}. \tag{7.17}$$

From the form of $\tilde{h}_0$ in (7.14) and from (7.16) it follows that $|\tilde{h}_0'|$ is bounded by the right hand side of (7.16). Since further $\tilde{h}_0$ is piecewise linear, the estimates (7.16), (7.17) hold for the function $\tilde{h}_1 = \tilde{h} - \tilde{h}_0$ and (since $\tilde{h}_1$ is continuously differentiable)

$$||\tilde{h}_1||_{C^{1+\alpha}[-R,R]} \leq C_R \sqrt{T} ||y||_{C^{1+\alpha}[0,T]}.$$

Then, by $C^{1+\alpha}$ estimates for harmonic functions,

$$||\nabla w_1||_{L^\infty(B_R)} \leq CR \left[||\tilde{h}_1||_{C^{1+\alpha}[-2R,2R]} + ||w_1||_{L^\infty(\mathbb{R})} \right] \leq C_R \sqrt{T} ||y||_{C^{1+\alpha}[0,T]}.$$

As for ∇w_0, it can be estimated by Lemma 7.2. Using also (7.14), (7.16) and (7.17), we find that

$$| \nabla w_0(0, y_1) | \leq C\sqrt{T} ||y||_{C^{1+\alpha}[0,T]} \left(1 + \left| \log \frac{C}{|y_1^2 - v_0 t|} \right| \right).$$

Recalling that $\triangle \chi = 2W_y$ and estimating W_y from relations (7.11), (7.12) (cf. the proof of Lemma 3.4), we find that

$$| \triangle \chi(v_0 s, 0, t) | \leq C\sqrt{T} ||y||_{C^{1+\alpha}[0,T]} \left(1 + \left| \log \frac{1}{|s|} \right| \right)$$

if $-b/v_0 \leq s \leq t$. Combining this with Lemma 7.1 we conclude:

Lemma 7.3. *The following inequality holds:*

$$|[\triangle \psi](v_0 s, 0, t)| \leq C \left(1 + \sqrt{T} ||y||_{C^{1+\alpha}[0,T]} \right) \left(1 + \left| \log \frac{1}{|s|} \right| \right) \left(1 + \frac{1}{\sqrt{t-s}} \right) \tag{7.18}$$

for $-b/v_0 \le s \le t$, where C is a constant independent of y.

We next establish:

Lemma 7.4 *The function $\psi(x, y, \cdot)$ is continuously differentiable in t, and*

$$\|\psi_t\|_{L^\infty(D\times[0,T])} \le C\|y\|_{C^1[0,T]} \tag{7.19}$$

where C is independent of T.

Proof. Introduce the function ψ_1 by

$$\psi_1(x, y, t + \Delta t) \equiv \psi(x + v_0\Delta t, y, t + \Delta t).$$

From (6.9)-(6.12) we deduce that the function

$$\psi_{\Delta t} = \frac{1}{\Delta t}(\psi_1 - \psi)$$

satisfies

$$\begin{aligned}
&\Delta^2\psi_{\Delta t} = 0 \;\text{ in }\; D\backslash\overline{\Gamma(t)},\\
&\psi_{\Delta t} = 0,\; \frac{\partial}{\partial y}\psi_{\Delta t} = \frac{1}{\Delta t}\left\{\frac{\partial^2\phi_0}{\partial y^2}(x, \pm a, t)[y(t + \Delta t) - y(t)]\right\} \;\text{ on }\; \partial D^{\pm},\\
&\psi_{\Delta t} = 0 \;\text{ on }\; \Gamma(t),\\
&\frac{\partial}{\partial y}\psi_{\Delta t} = \frac{1}{\Delta t}\left\{\frac{\partial^2\phi_0}{\partial y^2}(v_0 s, 0, t)[y(t + \Delta t) - y(s + \Delta t) - y(t) + y(s)]\right\} \;\text{ on }\; \Gamma(t).
\end{aligned} \tag{7.20}$$

In the boundary conditions we have used the fact that

$$\phi_0(v_0 s - v_0\Delta t, 0, t + \Delta t) = \phi_0(v_0 s, 0, t).$$

Denote by $\chi_{\Delta t}$ the function χ introduced in Lemma 3.4 for the boundary data

$$h(v_0 s) = \frac{\partial}{\partial y}\psi_{\Delta t}(v_0 s, 0, t),$$

and let

$$\tilde{\psi}_{\Delta t} = \psi_{\Delta t} - \chi_{\Delta t}.$$

By Lemma 3.2

$$\|\tilde{\psi}_{\Delta t}\|_{L^\infty(D)} \le C\|y\|_{C^1[0,T]}. \tag{7.21}$$

By elliptic estimates we also have

$$\|\tilde{\psi}_{\Delta t}\|_{C^{1+\alpha}(\Omega)} \le C(\Omega)\|y\|_{C^{1+\alpha}[0,T]} \quad (0 < \alpha < 1)$$

for any closed subdomain Ω of $\overline{D}$ which does not contain the tip $(v_0 t, 0)$. Near the tip we have an expansion similar to (7.7) and, by (7.21), the coefficients are bounded by

$$C\delta^{-(\frac{k}{2}+1)}\|y\|_{C^1[0,T]}\,.$$

It follows that

$$\|\tilde{\psi}_{\Delta t}\|_{C^{1+\alpha}(B_{\frac{\delta}{2}}(t))} \le C\|y\|_{C^1[0,T]} \text{ if } 0 < \alpha < \frac{1}{2}. \tag{7.22}$$

From (7.21), (7.22) we see that any sequence $\Delta t \to 0$ has a subsequence such that

$$\tilde{\psi}_{\Delta t}(\cdot, t) \to \tilde{\psi}^*(\cdot, t) \text{ in } C^{1+\alpha}_{loc}(\overline{D}) \tag{7.23}$$

for any $0 < \alpha < \frac{1}{2}$, for some function $\tilde{\psi}^*$.

On the other hand from the proof of Lemma 3.4 we can deduce that

$$\chi_{\Delta t} \to \chi^* \text{ in } C^0_{loc}(\overline{D})$$

where χ^* is the bounded solution to

$$\begin{aligned}
&\Delta^2 \chi^* = 0 \text{ in } \mathbb{R}^2 \backslash \overline{\Gamma(t)}, \\
&\chi^* = 0, \ \frac{\partial \chi^*}{\partial y} = \frac{\partial^2 \phi_0}{\partial y^2}[y'(t) - y'(s)] \text{ on } \Gamma(t).
\end{aligned}$$

Combining this fact with (7.23) we find that, for a subsequence $\Delta t \to 0$,

$$\psi_{\Delta t}(\cdot, t) \to \psi^*(\cdot, t) \text{ uniformly in compact subsets of } \overline{D} \tag{7.24}$$

where ψ' is a bounded solution to

$$\begin{aligned}
&\Delta^2 \psi^* = 0 \text{ in } D \backslash \overline{\Gamma(t)}, \\
&\psi^* = 0, \ \frac{\partial \psi^*}{\partial y} = \left(\frac{\partial^2 \phi_0}{\partial y^2}\right)(x, \pm a, t) y'(t) \text{ on } \partial D^{\pm}, \\
&\psi^* = 0, \ \frac{\partial \psi^*}{\partial y} = \left(\frac{\partial^2 \phi_0}{\partial y^2}\right)(v_0 s, 0, t)[y'(t) - y'(s)] \text{ on } \Gamma(t).
\end{aligned} \tag{7.25}$$

By Lemma 3.3 ψ^* is the unique bounded solution of (7.25), and

$$\|\psi^*(\cdot, t)\|_{L^\infty(D)} \le C\|y\|_{C^1[0,T]}. \tag{7.26}$$

Therefore (7.24) holds for all $\Delta t \to 0$.

By uniqueness for (7.25) we also deduce that $\psi^*(x, y, t)$ is continuous in t. Since

$$\psi^* = v_0 \psi_x + \psi_t \text{ in } D,$$

we further conclude that

$$\|v_0 \psi_x + \psi_t\|_{L^\infty(D)} \le C\|y\|_{C^1[0,T]}.$$

Similarly one can show that ψ_x satisfies

$$\begin{aligned}
&\Delta^2 \psi_x = 0 \text{ in } D \backslash \overline{\Gamma(t)}, \\
&\psi_x = 0, \ \frac{\partial \psi_x}{\partial y} = \left(\frac{\partial^3 \phi_0}{\partial x \partial y^2}\right) y(t) \text{ on } \partial D^{\pm}, \\
&\psi_x = 0 \text{ on } \Gamma(t), \\
&\frac{\partial \psi_x}{\partial y} = \left(\frac{\partial^3 \phi_0}{\partial x \partial y^2}\right)[y(t) - y(s)] - \frac{1}{v_0}\left(\frac{\partial^2 \phi_0}{\partial y^2}\right) y'(s) \text{ on } \Gamma(t)
\end{aligned}$$

and ψ_x is continuous in t; furthermore

$$\|\psi_x(\cdot, t)\|_{L^\infty(D)} \leq C\|y\|_{C^1[0,T]}.$$

Combining this with the results for ψ^*, the assertion of the lemma follows. □

From Lemma 7.4 and elliptic estimates it follows that ψ is smooth in (x, y, t) for $(x, y) \in D$, $0 < t < T$, and

$$\frac{\partial}{\partial t}\Delta\psi = \Delta\psi_t. \tag{7.27}$$

We conclude this section with one more lemma:

Lemma 7.5. *The following estimate holds:*

$$|[\Delta\psi_t](v_0 s, 0, t)| \leq C\left(1 + \|y\|_{C^{1+\alpha}[0,T]}\right)(1 + |\log|s||)\left(1 + \frac{1}{\sqrt{t-s}}\right) \tag{7.28}$$

for $-b/v_0 \leq s \leq t$, where C is a constant independent of T (Recall that $T \leq 1$.).

Proof. By (6.9)-(6.12) and Lemma 7.4,

$$\begin{aligned}
&\Delta^2\psi_t = 0 \ \text{ in } \ D\backslash\overline{\Gamma(t)},\\
&\psi_t = 0, \ \frac{\partial\psi_t}{\partial y} = \left(\frac{\partial^3\phi_0}{\partial t\partial y^2}\right)y(t) + \left(\frac{\partial^2\phi_0}{\partial y^2}\right)y'(t) \ \text{ on } \ \partial D^{\pm},\\
&\psi_t = 0 \ \text{ on } \ \Gamma(t),\\
&\frac{\partial\psi_t}{\partial y} = \left(\frac{\partial^3\phi_0}{\partial t\partial y^2}\right)[y(t) - y(s)] + \left(\frac{\partial^2\phi_0}{\partial y^2}\right)y'(t) \ \text{ on } \ \Gamma(t).
\end{aligned}$$

Considering t as a parameter, we can apply the proof of (7.18) to ψ_t since the boundary conditions are Lipschitz continuous. Since however the present function h is

$$h(s,t) = \left(\frac{\partial^3\phi_0}{\partial t\partial y^2}\right)[y(t) - y(s)] + \left(\frac{\partial^2\phi_0}{\partial y^2}\right)y'(t),$$

and it does not generally vanish at the tip (i.e., $h(t,t) \neq 0$ in general), we do not get as in (7.18) a factor $\sqrt{T}$ in front of

$$\|y\|_{C^{1+\alpha}[0,T]}. \quad \square$$

8. Existence and uniqueness

For any $y \in C^{1+\alpha}[0,T]$ we have defined $(\mathcal{M}y)(t)$ by (6.17). Using Lemma 7.1 we easily find that

$$\left|\dot{\mathcal{M}}(y)(t)\right| \leq C(1 + \sqrt{T}\|y\|_{C^{1+\alpha}[0,T]}). \tag{8.1}$$

We next proceed to estimate the C^α norm of $\dot{\mathcal{M}}(y)$. Let $0 < t_1 < t_2 < T$. By (6.16),

$$\left|\dot{\mathcal{M}}(y)(t_1) - \dot{\mathcal{M}}(y)(t_2)\right| \leq C\left|\tilde{J}(\psi, y)(t_1) - \tilde{J}(\psi, y)(t_2)\right|.$$

Using (7.1), (7.2) we easily find that

$$\left|\dot{\mathcal{M}}(y)(t_1) - \dot{\mathcal{M}}(y)(t_2)\right| \leq C\left(1 + \|y\|_{C^1[0,T]}\right)|t_1 - t_2| + I \tag{8.2}$$

where

$$I = \left| \int_{-b/v_0}^{t_1} K(t_1) \left[[\Delta\psi](v_0 s, t_1) - G(t_1) \frac{y(t_1) - y(s)}{(t_1 - s)^{\frac{3}{2}}} \right] ds \right.$$
$$\left. - \int_{-b/v_0}^{t_2} K(t_2) \left[[\Delta\psi](v_0 s, t_2) - G(t_2) \frac{y(t_2) - y(s)}{(t_2 - s)^{\frac{3}{2}}} \right] ds \right| \tag{8.3}$$

and

$$K(t_j) = K(t_j, s), \;\; G(t_j) = G(t_j, s), \;\; [\Delta\psi](v_0 s, t_j) = [\Delta\psi](v_0 s, 0, t_j).$$

We can write

$$I \leq I_1 + I_2 + I_3 \tag{8.4}$$

where

$$I_1 = \int_{-b/v_0}^{t_1} |K(t_1)[\Delta\psi](v_0 s, t_1) - K(t_2)[\Delta\psi](v_0 s, t_2)| ds,$$

$$I_2 = \int_{-b/v_0}^{t_1} \left| K(t_1)G(t_1) \frac{y(t_1) - y(s)}{(t_1 - s)^{\frac{3}{2}}} - K(t_2)G(t_2) \frac{y(t_2) - y(s)}{(t_2 - s)^{\frac{3}{2}}} \right| ds,$$

$$I_3 = \int_{t_1}^{t_2} \left| [\Delta\psi](v_0 s, t_2) + G(t_2) \frac{y(t_2) - y(s)}{(t_2 - s)^{\frac{3}{2}}} \right| ds.$$

Using Lemmas 7.3, 7.5, we easily get

$$|I_1| \leq C \left(1 + \|y\|_{C^{1+\alpha}[0,T]}\right) \sqrt{t_2 - t_1}.$$

Next,

$$I_2 = \int_{-b/v_0}^{t_1} \int_0^1 \left| \frac{d}{d\tau} \left[K(t_2 + \tau(t_1 - t_2)) G(t_2 + \tau(t_1 - t_2)) \frac{y(t_2 + \tau(t_1 - t_2)) - y(s)}{(t_2 + \tau(t_1 - t_2) - s)^{\frac{3}{2}}} \right] \right| d\tau ds.$$

Applying the derivative $d/d\tau$ first to KG and then to the quotient, we find that

$$I_2 \leq C\|y\|_{C^1[0,T]}(t_2 - t_1)^{\frac{3}{2}} + C\|y\|_{C^1[0,T]} \int_{-b/v_0}^{t_1} \int_0^1 \frac{|t_1 - t_2|}{(t_2 + \tau(t_1 - t_2) - s)^{\frac{3}{2}}} d\tau ds$$

so that

$$I_2 \leq C\|y\|_{C^1[0,T]} \sqrt{t_2 - t_1}.$$

Finally, by Lemma 7.3

$$I_3 \leq C \left(1 + \|y\|_{C^{1+\alpha}[0,T]}\right) \int_{t_1}^{t_2} |\log s| \frac{1}{\sqrt{t_2 - t_1}} ds$$
$$C_\epsilon \left(1 + \|y\|_{C^{1+\alpha}[0,T]}\right) (t_2 - t_1)^{\frac{1}{2} - \epsilon}$$

for any $\epsilon > 0$.

Combining the estimates on the I_j and recalling (8.3), (8.4), we get the Hölder estimate

$$\left| \dot{\mathcal{M}}(y)(t_1) - \dot{\mathcal{M}}(y)(t_2) \right| \leq C_\epsilon \left(1 + \|y\|_{C^{1+\alpha}[0,T]}\right) |t_1 - t_2|^{\frac{1}{2} - \epsilon} \tag{8.5}$$

for any $0 < \epsilon < \frac{1}{2}$.

Choosing $\alpha \in (0, \frac{1}{2})$ we deduce from (8.1) and (8.5) that

$$\mathcal{M} \text{ maps } C_0^{1+\alpha}[0,T] \text{ into itself.} \tag{8.6}$$

We next prove:

Lemma 8.1. *If T is small enough then $\mathcal{M}$ is a contraction.*

Proof. Let y_1, y_2 be two functions in $C_0^{1+\alpha}[0,T]$ and denote by ψ_1 and ψ_2 corresponding bounded solutions of (6.9)-(6.12). Let $y = y_1 - y_2$, $\psi = \psi_1 - \psi_2$. By the proof of (8.1) we get

$$\|\dot{\mathcal{M}}(y)\|_{C^0[0,T]} \le C\sqrt{T}\|y\|_{C^{1+\alpha}[0,T]}.$$

From (8.5) with $2\epsilon = \frac{1}{2} - \alpha$ we also have

$$\|\dot{\mathcal{M}}(y)\|_{C^\alpha[0,T]} \le CT^\epsilon\|y\|_{C^{1+\alpha}[0,T]}$$

and, since $\mathcal{M}(y)(0) = 0$,

$$\|\mathcal{M}(y)\|_{C^0[0,T]} \le CT\|\dot{\mathcal{M}}(y)\|_{C^0[0,T]}.$$

Combining these estimates we see that

$$\|\mathcal{M}(y_1) - \mathcal{M}(y_2)\|_{C^{1+\alpha}[0,T]} = \|\mathcal{M}(y)\|_{C^{1+\alpha}[0,T]} \le CT^\epsilon\|y_1 - y_2\|_{C^{1+\alpha}[0,T]}$$

and the lemma follows. □

Theorem 8.2. *There exists a unique bounded solution of (6.9)-(6.14) for all $t > 0$, with $y \in C^{1+\alpha}[0,T]$ for all $T > 0$ $(0 < \alpha < \frac{1}{2})$.*

Proof. From (8.6) and Lemma 8.1 it follows that $\mathcal{M}$ has a unique fixed point in $C^{1+\alpha}[0,T]$ if T is small enough. A step-by-step argument establishes uniqueness for $0 < t < t_0$ of solutions with $y \in C^{1+\alpha}[0,t_0]$; here t_0 is any positive number.

A step-by-step argument can also be used to prove global existence. Indeed, a quick look at the proofs of Lemmas 7.1 and 7.3-7.5 shows that the lemmas remain valid for all $0 \le t \le T$ with constants C which depend on T. Thus if a solution $y(t)$ is already known to exist for $0 \le t \le T_0$ and to belong to $C^{1+\alpha}[0,T_0]$, we can extend it to a larger interval, say $0 \le t \le T_0 + \delta$, by repeating the previous existence proof; we only need to slightly change the definition of $\tilde{y}$ in (6.16), by taking

$$\tilde{y}(T_0) = y(T_0);$$

the constant δ depends on T_0, but it remains uniformly positive for T_0 is any bounded interval. □

Theorem 8.3. *The initial velocity of the crack tip is upward; more precisely,*

$$\dot{y}(0+) = \frac{6\sqrt{3}M}{a^3}\rho_0 \quad \textit{where } \rho_0 = \frac{v_0}{\gamma_0}\frac{v_\infty - v_0}{v_\infty + v_0}\frac{1-\nu^2}{E}. \tag{8.7}$$

In the remaining part of this section we shall prove:

Lemma 8.4.

$$|\dot{y}(0+)| = \frac{6\sqrt{3}M}{a^3}\rho_0. \tag{8.8}$$

In §9 we shall prove that $\dot{y}(0+) > 0$, and this together with (8.8) complete the proof of Theorem 8.3.

Proof of Lemma 8.4. By (6.5) and (5.41) we have

$$\dot{y}(0+) = \rho_0 \left\{ \int_{\Lambda \cup \Gamma_b^{\pm}(0)} (\triangle\phi_0 \cdot \triangle\phi) n_2 - \int_{\Lambda} (\nabla^c \phi_0 \cdot \nabla\triangle\phi + \nabla^c\phi \cdot \nabla\triangle\phi_0)\, n_1 \right\} \tag{8.9}$$

where $\Gamma_b(0) = \{(x,0);\ -b < x < 0\}$. Define a function ϕ^τ by

$$\phi^\tau(x,y) = \phi(x,-y).$$

Then $\phi + \phi^\tau = \phi_0/M$. By elementary calculation, if we replace ϕ_0 by ϕ^τ in the right-hand side of (8.8), then each integral is equal to zero. Hence,

$$\dot{y}(0+) = M\rho_0 \left\{ \int_{\Lambda} (\triangle\phi)^2 n_2 + \int_{\Gamma_b(0)} [(\triangle\phi)^2] n_2 - 2\int_{\Lambda} \vec{s} \cdot \vec{\Phi}^c \right\}.$$

Since the integral

$$\int_{\gamma} \left\{ (\triangle\phi)^2 n_2 - 2\vec{s} \cdot \vec{\Phi}^c \right\}$$

along any closed curve γ is independent of γ provided γ does not intersect $\Gamma(0)$, we then obtain

$$\dot{y}(0+) = M\rho_0 \lim_{\epsilon \to 0} \int_{\partial B_\epsilon} \left\{ (\triangle\phi)^2 n_2 - 2\vec{s} \cdot \vec{\Phi}^c \right\} \tag{8.10}$$

where B_ϵ is the disc $\{x^2 + y^2 < \epsilon^2\}$ and ∂B_ϵ is its boundary.

To compute the right-hand side of (8.9) we use the representation (7.7) in B_ϵ and the conditions $\phi = \partial_y \phi = 0$ if $y = 0, x < 0$. Then we get

$$\phi = r^{3/2} \left[\alpha(\cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta) + \beta(\sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta) \right] + O(r^2). \tag{8.11}$$

It will be convenient to use complex variables:

$$z = x + iy = re^{i\theta},$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

so that

$$\triangle = 4\frac{\partial^2}{\partial z \partial \bar{z}}.$$

We can then rewrite (8.10) in the form

$$\phi = Az^{3/2} + \bar{A}\bar{z}^{3/2} + Bz\bar{z}^{1/2} + \bar{B}z^{1/2}\bar{z} + O(|z|^2)$$

where

$$A = \frac{1}{2}(\alpha - i\beta), \;\; B = \frac{1}{2}(3\alpha - i\beta).$$

Also

$$\frac{\partial \phi}{\partial z} = \frac{3}{2} A z^{1/2} + B\bar{z}^{1/2} + \frac{1}{2}\bar{B} z^{-1/2}\bar{z} + O(|z|), \;\; \frac{\partial \phi}{\partial \bar{z}} = \overline{\left(\frac{\partial \phi}{\partial z}\right)} \tag{8.12}$$

and

$$\Delta \phi = 2B\bar{z}^{-1/2} + 2\bar{B} z^{-1/2} + O(1). \tag{8.13}$$

Hence

$$\int_{\partial B_\epsilon} (\Delta\phi)^2 n_2 = 4 \int_{\partial B_\epsilon} \left(B^2 \bar{z}^{-1} + \bar{B}^2 z^{-1} + 2|B|^2|z|^{-1}\right) n_2 |dz| + O(\epsilon).$$

Using the relations

$$z = \epsilon e^{i\theta}, \;\; n_2 = \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

we easily compute

$$\int_{\partial B_\epsilon} (\Delta\phi)^2 n_2 = 2i(B^2 - \bar{B}^2)2\pi + O(\epsilon) = 12\alpha\beta\pi + O(\epsilon). \tag{8.14}$$

To evaluate the integral with $\vec{s}\cdot\Phi^c$ in (8.9) we write

$$\begin{aligned} \vec{s} &= \left(\frac{\partial^2\phi}{\partial y^2}\cos\theta - \frac{\partial^2\phi}{\partial x\partial y}\sin\theta, \; \frac{\partial^2\phi}{\partial x^2}\sin\theta - \frac{\partial^2\phi}{\partial x\partial y}\cos\theta\right) \\ &= \left(\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial y}\right), -\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial x}\right)\right). \end{aligned}$$

It follows that

$$\int_{\partial B_\epsilon} \vec{s}\cdot\vec{\Phi}^c = -\int_{-\pi}^{\pi} \left\{\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial y}\right)(\Delta\phi)^c + \frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial x}\right)\Delta\phi\right\} d\theta.$$

By integration by parts,

$$\begin{aligned} \int_{\partial B_\epsilon} \vec{s}\cdot\vec{\Phi}^c &= \int_{-\pi}^{\pi} \left\{\frac{\partial\phi}{\partial y}\frac{\partial}{\partial\theta}(\Delta\phi)^c + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\theta}\Delta\phi\right\} d\theta \\ &= \int_{\partial B_\epsilon} \frac{\partial\phi}{\partial y}\frac{\partial}{\partial r}\Delta\phi|dz| + \frac{1}{\epsilon}\int_{\partial B_\epsilon} \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\theta}\Delta\phi|dz| \equiv I_1 + I_2 \end{aligned} \tag{8.15}$$

where we have used the Cauchy-Riemann equations

$$\frac{\partial}{\partial r}\Delta\phi = \frac{1}{r}\frac{\partial}{\partial\theta}(\Delta\phi)^c, \;\; \frac{1}{r}\frac{\partial}{\partial\theta}\Delta\phi = -\frac{\partial}{\partial r}(\Delta\phi)^c.$$

Now, by (8.11)

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial\bar{z}} = \frac{3}{2}\left(Az^{1/2} + \bar{A}\bar{z}^{1/2}\right) + B\bar{z}^{1/2} + \bar{B}z^{1/2} + \frac{1}{2}\left(\bar{B}z^{-1/2}\bar{z} + B\bar{z}^{-1/2}z\right) + O(|z|),$$

$$\frac{\partial\phi}{\partial y} = i\left(\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial\bar{z}}\right) = \frac{3}{2}i\left(Az^{1/2} - \bar{A}\bar{z}^{1/2}\right) + i\left(B\bar{z}^{1/2} - \bar{B}z^{1/2}\right) + \frac{i}{2}\left(\bar{B}z^{-1/2}\bar{z} - B\bar{z}^{-1/2}z\right) + O(|z|),$$

and by (8.12)

$$\frac{\partial}{\partial r}\Delta\phi = \frac{\partial}{\partial z}\Delta\phi \cdot \frac{\partial z}{\partial r} + \frac{\partial}{\partial \bar{z}}\Delta\phi \cdot \frac{\partial \bar{z}}{\partial r} = -\bar{B}z^{-1}\bar{z}^{-1/2} - B\bar{z}^{-1}z^{-1/2} + O(|z|^{-1}),$$

$$\frac{\partial}{\partial \theta}\Delta\phi = \frac{\partial}{\partial z}\Delta\phi \cdot \frac{\partial z}{\partial \theta} + \frac{\partial}{\partial \bar{z}}\Delta\phi \cdot \frac{\partial \bar{z}}{\partial \theta} = -i\bar{B}z^{-1/2} + iB\bar{z}^{-1/2} + O(1).$$

Substituting these expressions into the right-hand side of (8.14) we obtain, after simple calculation,

$$I_1 = -9\alpha\beta\pi + O(\epsilon), \;\; I_2 = 3\alpha\beta\pi + O(\epsilon).$$

Hence

$$\int_{\partial B_\epsilon} \vec{s} \cdot \vec{\Phi}^c = -6\alpha\beta\pi + O(\epsilon)$$

and together with (8.13) we get, from (8.9),

$$\dot{y}(0+) = 24M\rho_0\alpha\beta\pi. \tag{8.16}$$

To compute α we consider the function $\tilde{\phi} = \phi + \phi^r$. From (8.10) we deduce the expansion

$$\begin{aligned} \tilde{\phi} &= 2r^{3/2}\alpha(\cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta) + O(r^2), \\ &= \alpha(z^{3/2} + \bar{z}^{3/2}) + 3\alpha(z\bar{z}^{1/2} + z^{1/2}\bar{z}) + O(|z|^2). \end{aligned}$$

Since $\tilde{\phi} = \phi_0/M$, the computation of the J-integral in (4.15) gives

$$J_1 = \lim_{\epsilon\to 0} \frac{1-\nu^2}{2E} \int_{\partial B_\epsilon} \left\{ (\Delta\tilde{\phi})^2 n_1 - 2\vec{\tilde{s}} \cdot \vec{\tilde{\Phi}} \right\} = \frac{12(1-\nu^2)}{Ea^3}. \tag{8.17}$$

On the other hand, proceeding as before to evaluate the limit of the integral in (8.16) we find that

$$\int_{\partial B_\epsilon} (\Delta\tilde{\phi})^2 n_1 \to 72\alpha^2\pi, \;\; \int_{\partial B_\epsilon} \vec{\tilde{s}} \cdot \vec{\tilde{\Phi}} \to -36\alpha^2\pi \;\text{ as }\; \epsilon \to 0,$$

so that, by (8.16),

$$\alpha^2 = \frac{1}{6\pi a^3}. \tag{8.18}$$

Next we consider the function $\phi^* = \phi - \phi^r$. We compute the J-integral in two ways. First we take the integral along the curve Λ shown in Figure 1 and let $b \to \infty$. We then obtain, analogously to (4.15),

$$J_1 = \frac{9(1-\nu^2)}{Ea^3}. \tag{8.19}$$

Next we compute J_1 from the formula

$$J_1 = \lim_{\epsilon\to 0} \frac{1-\nu^2}{2E} \int_{\partial B_\epsilon} \left\{ (\Delta\phi^*)^2 n_1 - 2\vec{s^*} \cdot \vec{\Phi^*} \right\}.$$

Noting that in B_ϵ

$$\begin{aligned} \phi^* &= 2\beta r^{3/2}(\sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta) + O(r^{5/2}), \\ &= -i\beta(z^{3/2} - \bar{z}^{3/2}) - i\beta(z\bar{z}^{1/2} - z^{1/2}\bar{z}) + O(|z|^{5/2}), \end{aligned}$$

we obtain

$$\int_{\partial B_\epsilon} (\Delta\phi^*)^2 n_1 \to -8\beta^2\pi, \quad \int_{\partial B_\epsilon} \vec{s^*} \cdot \vec{\Phi^*} \to -12\beta^2\pi \text{ as } \epsilon \to 0,$$

so that

$$J_1 = 8\beta^2\pi\frac{1-\nu^2}{E}.$$

Comparing with (8.18) we find that

$$\beta^2 = \frac{9}{8\pi a^3}. \tag{8.20}$$

Finally, substituting (8.17) and (8.19) into (8.15), the assertion (8.8) follows. □

9. $\dot{y}(0+) > 0$

For any $-\infty < \lambda < b$, let

$$D_b = D \cap \{x < b\}, \quad \Gamma_\lambda = \{(x,0);\ x < \lambda\}.$$

Consider the following problem:

$$\begin{aligned}
&\Delta^2\phi = 0 \ \text{ in } \ D_b\backslash\Gamma_\lambda, \\
&\phi = 1, \ \ \phi_y = 0 \ \text{ on } \ \partial D_b\backslash\{x=b\}, \\
&\phi = 1, \ \ \phi_{xx} = 0 \ \text{ on } \ \{x = b\}, \\
&\phi = 0, \ \ \phi_y = 0 \ \text{ on } \ \Gamma_\lambda \text{ (from both sides).}
\end{aligned} \tag{9.1}$$

By the analysis of §3 one can establish the existence of a unique bounded solution $\phi = \phi_{\lambda,b}$ in $H^2_{loc}(D_b)$ of the system (9.1). Furthermore $\phi_{\lambda,b} \in C^0(\overline{D}_b)$ and then also, by a standard argument, $\phi_{\lambda,b}$ is continuous in (x,y,λ,b) for $(x,y) \in \overline{D}_b$, $-\infty < \lambda < b < \infty$. Since, near the tip $(\lambda, 0)$,

$$\phi_{\lambda,b} = 2\alpha_{\lambda,b}r^{3/2}\left(\cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta\right) + O(r^2) \tag{9.2}$$

where (r,θ) are polar coordinates about $(\lambda,0)$, we conclude that

$$\alpha_{\lambda,b} \text{ is continuous with respect to } (\lambda, b). \tag{9.3}$$

Lemma 9.1. $\alpha_{\lambda,b} \neq 0$ *for all* $-\infty < \lambda < b < \infty$.

Proof. The proof is based on arguments used in proving Lemma 8.4. Set, for simplicity, $\phi = \phi_{\lambda,b}$ and let $\tilde{\phi}$ be the bounded solution in $H^2_{loc}(D_b)$ of the system which is identical to (9.1) except for the boundary condition on $\{x = b\}$:

$$\tilde{\phi}_x = 0; \ \ \tilde{\phi}_{xxx} = 0 \ \text{ on } \ \{x = b\}.$$

As in the case of the J-integral one can show that the integral

$$\int_\gamma \left[(\Delta\phi \cdot \Delta\tilde{\phi})n_1 - (\vec{s}\cdot\vec{\tilde{\Phi}} + \vec{\tilde{s}}\cdot\vec{\Phi})\right]$$

is equal to zero if γ encloses a region in D which does not contain cracks.

Let $\Lambda_{b,c}$ be the curve $\Lambda^l_{b,c} \cup \Lambda^r_{b,c} \cup \Lambda^{\pm}_{b,c} \cup L^{\pm}_c$ shown in Figure 2 when $L^{\pm}_c$ converge to the interval $\{(x,0);\ -c < x < \lambda\}$.

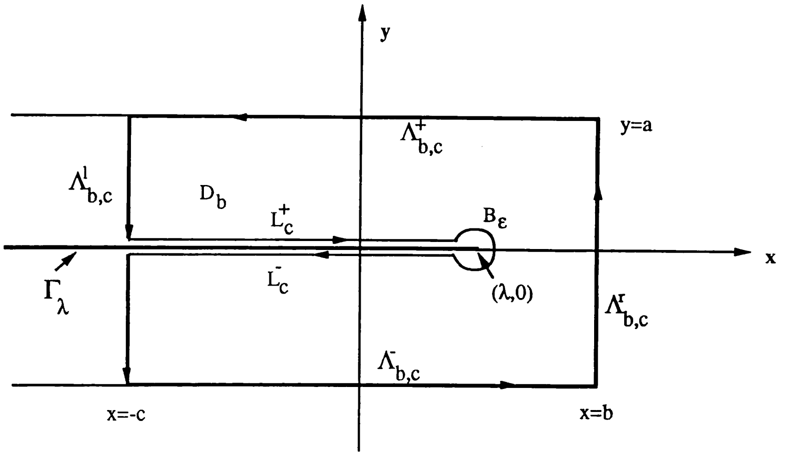

Figure 2

Since $\tilde{\phi}$ is even in y, we have

$$\tilde{\phi} = 2\tilde{\alpha} r^{3/2} \left(\cos \frac{3}{2}\theta + 3 \cos \frac{1}{2}\theta \right) + O(r^2)$$

near $(\lambda, 0)$. Proceeding as in the derivation of (8.17), we find that

$$J \equiv \lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\lambda,0)} \left\{ (\Delta\phi \cdot \Delta\tilde{\phi}) n_1 - (\vec{s} \cdot \vec{\tilde{\Phi}} + \vec{\tilde{s}} \cdot \vec{\Phi}) \right\} = 72\pi \alpha_{\lambda,b} \tilde{\alpha}. \tag{9.4}$$

On the other hand, by the path independence property of the integral mentioned above, we also have

$$J = \int_{\Lambda_{b,c}} \left\{ (\Delta\phi \cdot \Delta\tilde{\phi}) n_1 - (\vec{s} \cdot \vec{\tilde{\Phi}} + \vec{\tilde{s}} \cdot \vec{\Phi}) \right\}.$$

Since $n_1 = 0$, $\vec{s} = \vec{\tilde{s}} = 0$ on $\Lambda^{\pm}_{b,c} \cup \Gamma_\lambda$, we have

$$\begin{aligned} J &= \int_{\Lambda^l_{b,c} \cup \Lambda^r_{b,c}} \left\{ (\Delta\phi \cdot \Delta\tilde{\phi}) n_1 - (\vec{s} \cdot \vec{\tilde{\Phi}} + \vec{\tilde{s}} \cdot \vec{\Phi}) \right\} \\ &= -2 \int_0^a (\Delta\phi \cdot \Delta\tilde{\phi})(-c, y) dy + 2 \int_0^a \left(\frac{\partial^2 \phi}{\partial y^2} \Delta\tilde{\phi} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} \Delta\phi \right) (-c, y) dy \\ &\quad -2 \int_0^a \left(\frac{\partial^2 \phi}{\partial x \partial y} (\Delta\tilde{\phi})^c + \frac{\partial^2 \tilde{\phi}}{\partial x \partial y} (\Delta\phi)^c \right) (-c, y) dy \\ &\quad + \int_{-a}^a (\Delta\phi \cdot \Delta\tilde{\phi})(b, y) dy - \int_{-a}^a \left(\frac{\partial^2 \phi}{\partial y^2} \Delta\tilde{\phi} + \frac{\partial^2 \tilde{\phi}}{\partial y^2} \Delta\phi \right) (b, y) dy \\ &\quad + \int_{-a}^a \left(\frac{\partial^2 \phi}{\partial x \partial y} (\Delta\tilde{\phi})^c + \frac{\partial^2 \tilde{\phi}}{\partial x \partial y} (\Delta\phi)^c \right) (b, y) dy \end{aligned}$$

where we have used the fact that ϕ, $\tilde{\phi}$ are even and $(\Delta\phi)^c$, $(\Delta\tilde{\phi})^c$ are odd with respect to y.

Observe that $\partial^2\phi/\partial y^2 = \Delta\phi = 0$ at $x = b$. Integrating by parts and using the boundary conditions on ϕ and $\tilde{\phi}$, and finally taking $-c \to -\infty$, we get

$$\begin{aligned} J &= 2\int_0^a \left(\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\tilde{\phi}}{\partial y^2}\right)(-\infty, y)dy - \int_{-a}^a \left(\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\Delta\tilde{\phi}\right)(b, y)dy - \int_{-a}^a \left(\frac{\partial\tilde{\phi}}{\partial x}\frac{\partial}{\partial x}\Delta\phi\right)(b, y)dy \\ &= 2\int_0^a \left(-\frac{12}{a^3}y + \frac{6}{a^2}\right)^2 dy = \frac{24}{a^3}. \end{aligned}$$

Combining this with (9.4), the assertion of the lemma follows. □

Set

$$\delta = b - \lambda, \quad \phi_\delta(x, y) = \phi_{\lambda,b}(x + b, y). \tag{9.5}$$

Lemma 9.2. *There holds:*

$$\|\phi_\delta\|_{L^\infty} \le C$$

where C is a constant independent of λ, δ.

Proof. If the assertion is not true then there is a sequence $\delta_j \to 0$ such that

$$M_j \equiv \|\phi_{\delta_j}\| \to \infty$$

and, since ϕ_{δ_j} is bounded as $x \to -\infty$, there is a point (x_j, y_j) in $D \cap \{x < 0\} \cap \{y \ge 0\}$ such that

$$M_j = \phi_{\delta_j}(x_j, y_j)$$

(we assume for definiteness that $\sup|\phi_{\delta_j}| = \sup\phi_{\delta_j}$). Since $\phi_\delta(x, y)$ is even in y, we can take $y_j \ge 0$.

Consider first the case where

$$x_j \to 0, \quad y_j \not\to a, \quad \text{and} \quad \frac{y_j}{|x_j|} \to \infty \tag{9.6}$$

and introduce the functions

$$\phi_j(x, y) = \frac{1}{M_j}\phi_{\delta_j}(|x_j|x, y_j + |x_j|y).$$

Then $\phi_j(-1, 0) = 1$ and

$$|\phi_j| \le 1 \;\text{ in }\; D_j = \{(x, y);\; x < 0, -y_j/|x_j| < y < (a - y_j)/|x_j|\}.$$

By elliptic estimates, $\phi_j \to \bar{\phi}$ uniformly in compact subsets of $\overline{\mathbb{R}^2_-}$ where $\mathbb{R}^2_- = \{(x, y) \in \mathbb{R}^2,\ x < 0\}$, and $\bar{\phi}$ is the solution to

$$\begin{aligned} &\Delta\bar{\phi} = 0 \;\text{ in }\; \mathbb{R}^2_- \\ &\bar{\phi} = 0, \;\; \bar{\phi}_{xx} = 0 \;\text{ on }\; \{x = 0\}. \end{aligned} \tag{9.7}$$

Also

$$\bar{\phi}(-1, 0) = 1. \tag{9.8}$$

Since $|\bar{\phi}| \le 1$, by scaling we deduce that

$$D^j\bar{\phi}(x,y) = O(r^{-j}) \text{ if } r \to \infty.$$

Using these estimates we find that

$$0 = \int_{\mathbb{R}^2_- \cap \{r<1/\epsilon\}} \bar{\phi}\Delta^2\bar{\phi} = \int_{\mathbb{R}^2_- \cap \{r<1/\epsilon\}} |\Delta\bar{\phi}|^2 + I_\epsilon$$

where $I_\epsilon \to 0$ if $\epsilon \to 0$. Consequently $\bar{\phi} \equiv 0$, which is a contradiction to (9.8).

In the case where

$$y_j/|x_j| \text{ remain bounded as } x_j \to 0,$$

we work with the sequence

$$\phi_j(x,y) = \frac{1}{M_j}\phi_{\delta_j}(|x_j|x, |x_j|y).$$

Let $\mu = \lim -\delta_j/|x_j| \in [-\infty, 0]$. Then $\bar{\phi}$ satisfies (9.7) in $\mathbb{R}^2_-\backslash\bar{\Gamma}_\mu$, and on Γ_μ we have $\bar{\phi} = \bar{\phi}_y = 0$.

If $\mu = 0$ or $-\infty$, one can verify, as in the case of (9.6), that $\bar{\phi} \equiv 0$. If $-\infty < \mu < 0$, then $\bar{\phi} = O(\bar{r}^{3/2})$ near $(\mu, 0)$ and, by scaling,

$$D^j\bar{\phi}(x,y) = O(\bar{r}^{3/2-j}) \text{ if } \bar{r} \to 0$$

where $\bar{r}$ is the distance to $(\mu, 0)$. We then deduce that $\bar{\phi} \equiv 0$ by integrating $\bar{\phi}\Delta^2\bar{\phi}$ over $(\mathbb{R}^2_-\backslash\bar{\Gamma}_\mu) \cap \{\epsilon < \bar{r} < 1/\epsilon\}$ and taking $\epsilon \to 0$. On the other hand, $\phi_j(-1, y_j/|x_j|) = 1$. By assumption, we may set $\bar{y} = \lim y_j/|x_j| \in [0, \infty)$. Since $\phi_j \to \bar{\phi}$ in $C^0_{loc}(\mathbb{R}^2_-)$, we get $\bar{\phi}(-1, \bar{y}) = 1$, which is a contradiction.

Similarly we can treat the case where $x_j \to 0$, $y_j \to a$.

Finally, if $|x_j|$ remains uniformly positive, then we use the scaling

$$\phi_j(x,y) = \frac{1}{M_j}\phi_{\delta_j}(x - x_j, y),$$

and derive a contradiction in a simpler way. □

Lemma 9.3. *For λ close to b, $\alpha_{\lambda,b} > 0$.*

Proof. Let

$$\Phi_\delta(x,y) = \phi_\delta(\delta x, \delta y).$$

By Lemma 9.2 and elliptic estimates,

$$\Phi_{\delta_j} \to \Phi_0 \text{ uniformly in compact subsets of } \overline{\mathbb{R}^2_-}$$

where Φ_0 is a bounded solution of

$$\begin{aligned} &\Delta^2\Phi_0 = 0 \text{ in } \mathbb{R}^2_-\backslash\bar{\Gamma}_{-1}, \\ &\Phi_0 = 1, \ \frac{\partial^2\Phi_0}{\partial x^2} = 0 \text{ on } \{x = 0\}, \\ &\Phi_0 = 0, \ \frac{\partial\Phi_0}{\partial y} = 0 \text{ on } \Gamma_{-1}. \end{aligned} \tag{9.9}$$

As in the proof of Lemma 9.2, one can verify that this bounded solution to (9.9) is unique and, consequently

$$\Phi_\delta \to \Phi_0 \text{ in } C^0_{loc}(\overline{\mathbb{R}^2_-}). \tag{9.10}$$

Observe that near $(-1,0)$,

$$\Phi_\delta = 2\alpha_\delta r^{3/2}\left(\cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta\right) + O(r^2)$$

where

$$\alpha_\delta = \alpha_{\lambda,b}\delta^{3/2}.$$

Hence, in order to complete the proof of Lemma 9.3 it suffices to show that, near the tip $(-1,0)$,

$$\Phi_0(x,y) = \alpha_0 r^{3/2}\left(\cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta\right) + O(r^2) \text{ with } \alpha_0 > 0. \tag{9.11}$$

Let w be the bounded solution to

$$\begin{aligned} &\triangle w = 0 \text{ in } \mathbb{R}^2_-\backslash\overline{\Gamma}_{-1}, \\ &w = 1 \text{ on } \{x=0\}, \\ &w = 0 \text{ on } \Gamma_{-1} \text{ (from both sides).} \end{aligned}$$

By the maximum principle, $0 < w < 1$. Further, near $(-1,0)$

$$w = a_1 r^{1/2}\cos\frac{\theta}{2} + O(r^{3/2}) \text{ where } a_1 > 0. \tag{9.12}$$

Indeed, this follows from the fact the function $\tilde{w}(z) = w(z^2-1)$ is harmonic in $Re(z) > 0$ and vanishes on $Re(z) = 0$.

Next let v be the bounded solution to

$$\begin{aligned} \triangle v &= 0 \text{ in } \mathbb{R}^2_-\backslash\overline{\Gamma}_{-1}, \\ v &= 0 \text{ on } \{x=0\}, \\ v^\pm &= (1+x)w^\pm_y \text{ on } \Gamma^\pm_{-1} \end{aligned}$$

where $v^\pm$, $w^\pm$ are defined in $\{y \gtrless 0\}$. Note that $(1+x)w^\pm_y$ is in $C^\alpha(\Gamma^\pm_{-1})$ for any $0 < \alpha < 1/2$.

We next introduce the function

$$\phi_1 = (1+x)w + yv.$$

It satisfies:

$$\begin{aligned} \triangle^2\phi_1 &= 0 \text{ in } \mathbb{R}^2_-\backslash\overline{\Gamma}_{-1}, \\ \phi_1 &= 1 \text{ on } \{x=0\}, \\ \frac{\partial^2\phi_1}{\partial x^2} &= 2w_x \text{ on } \{x=0\}, \\ \phi_1 &= 0, \ \frac{\partial\phi_1}{\partial y} = 0 \text{ on } \Gamma_{-1}. \end{aligned} \tag{9.13}$$

Observe that $\phi_1 \in C^1(\mathbb{R}^2_-)$ and ϕ_1 grows linearly at ∞ (since w and v are bounded). Further,

$$\phi_1 = a_1(1+x)^{3/2} + O((1+x)^{5/2}) \ \text{ if } \ y = 0, -1 < x < 0. \tag{9.14}$$

We note by the maximum principle, that $w_x > 0$.

To get rid of the $2w_x$ in (9.13) we shall construct two auxiliary functions: w_2, the bounded solution to

$$\begin{aligned} &\triangle w_2 = 0 \ \text{ in } \ \mathbb{R}^2_- \backslash \overline{\Gamma}_{-1}, \\ &w_2 + \frac{\partial w_2}{\partial x} + y\frac{\partial w_2}{\partial y} = -w_x \ \text{ on } \ \{x = 0\}, \\ &w_2 = 0 \ \text{ on } \ \Gamma_{-1} \ \text{(from both sides)} \end{aligned} \tag{9.15}$$

and v_2, a linearly bounded solution to

$$\begin{aligned} &\triangle v_2 = 0 \ \text{ in } \ \mathbb{R}^2_- \backslash \overline{\Gamma}_{-1}, \\ &v_2 = -yw_2 \ \text{ on } \ \{x = 0\}, \\ &v_2 = x(1+x)\left(\frac{\partial w_2}{\partial y}\right)^{\pm} \ \text{ on } \ \Gamma^{\pm}_{-1}. \end{aligned} \tag{9.16}$$

We construct w_2 by solving truncated problems in $\mathbb{R}^2_- \cap \{x^2 + y^2 < \rho^2\}$ with

$$w_2 = 0 \ \text{ on } \ x^2 + y^2 = \rho^2,$$

and observe, by the maximum principle, that

$$-A < w_2 < 0 \quad (A = \sup|w_x|).$$

We also construct v_2 by truncations, and use the Phragmén-Lindelöf theorem to deduce that

$$|v_2| = O(r) \ \text{ as } \ r \to \infty.$$

Using conformal mapping and the maximum principle we prove, as in (9.12), that near the tip $(-1, 0)$

$$w_2 = a_2 r^{1/2} \cos\frac{\theta}{2} + O(r^{3/2}) \ \text{ where } \ a_2 < 0. \tag{9.17}$$

Since w_2 is bounded, (9.15) gives

$$w_2 + y\frac{\partial w_2}{\partial y}\bigg|_{x=0} = O\left(\frac{1}{|y|}\right) \ \text{ if } \ |y| \to \infty$$

and, by integration,

$$|w_2(0,y)| \le C\frac{\log|y|}{|y|} \ \text{ if } \ |y| \to \infty.$$

Therefore, by the Phragmén-Lindelöf theorem, for any $\epsilon > 0$,

$$|w_2| \le C\frac{1+r^{\epsilon}}{1+r} \ \text{ in } \ \mathbb{R}^2_-$$

from which we deduce that

$$\left| x(1+x)\left(\frac{\partial w_2}{\partial y}\right)^{\pm} \right| \le C(1+|x|^{\epsilon}) \text{ on } \Gamma_{-1}^{\pm}.$$

Therefore (again by the Phragmén-Lindelöf theorem), v_2 is bounded sublinearly:

$$|v_2| \le C(1+r^{\epsilon}) \text{ in } \mathbb{R}^2_-.$$

Now let

$$\phi_2 = x(1+x)w_2 + y^2 w_2 + y v_2.$$

Then

$$|\phi_2| \le C(1+r^{1+\epsilon}) \text{ in } \mathbb{R}^2_-$$

and, as easily seen, $\phi_2 = O(r^{3/2})$ near the tip $(-1,0)$ so that

$$\phi_2 \in H^2_{loc}(\mathbb{R}^2_-). \tag{9.18}$$

The function ϕ_2 satisfies

$$\begin{aligned} \triangle^2 \phi_2 &= 0 \text{ in } \mathbb{R}^2_- \backslash \overline{\Gamma}_{-1}, \\ \phi_2 &= 0 \text{ on } \{x=0\}, \\ \frac{\partial^2 \phi_2}{\partial x^2} &= -2w_x \text{ on } \{x=0\}, \\ \phi_2 = 0, \ \frac{\partial \phi_2}{\partial y} &= 0 \text{ on } \Gamma_{-1}. \end{aligned}$$

It follows that the function

$$\hat{\Phi} \equiv \phi_1 + \phi_2$$

satisfies the system (9.9). Furthermore, using (9.14), (9.17) we find, by restricting the expansion near $(-1,0)$,

$$\hat{\Phi} = a r^{3/2}\left(\cos\frac{3}{2}\theta + 3\cos\frac{1}{2}\theta\right) + O(r^2)$$

to $\{(x,0),\ -1 < x < 0\}$, that

$$a = a_1 - a_2 > 0.$$

Thus if we can prove that $\hat{\Phi} \equiv \Phi_0$ then assertion (9.11) follows and the proof of Lemma 9.3 is complete.

Set $\tilde{\Phi} = \hat{\Phi} - \Phi_0$. Then

$$\begin{aligned} &\triangle^2 \tilde{\Phi} = 0 \text{ in } \mathbb{R}^2_- \backslash \overline{\Gamma}_{-1}, \\ &\tilde{\Phi} = 0, \ \frac{\partial^2 \tilde{\Phi}}{\partial x^2} = 0 \text{ on } \{x=0\}, \\ &\tilde{\Phi} = 0, \ \frac{\partial \tilde{\Phi}}{\partial y} = 0 \text{ on } \Gamma_{-1}. \end{aligned}$$

Take odd extension of $\tilde{\Phi}$ across $x = 0$. This extension is biharmonic in

$$\mathbb{R}_0^2 \equiv \mathbb{R}_-^2 \backslash \{(x,0);\ x < -1 \text{ or } x > 1\},$$

and

$$\tilde{\Phi} = O(r^{1+\epsilon}) \text{ if } r \to \infty, \quad \tilde{\Phi} \in H^2_{loc}(\mathbb{R}^2).$$

Let

$$V_0(x,y) = \frac{1}{2}\int_0^y \triangle\tilde{\Phi}(x,s)ds.$$

Since $\triangle\tilde{\Phi} = O(r^{-1/2})$ near $(\pm 1, 0)$, the integral is well defined in $\mathbb{R}_0^2$. It is clear that

$$\frac{\partial}{\partial y}\triangle V_0 \equiv 0$$

and, consequently,

$$\triangle V_0 \equiv b_0(x) \text{ in } \mathbb{R}_0^2.$$

Setting

$$V_1(x) = \int_0^x \int_0^\xi b_0(s)dsd\xi,$$

the function $V \equiv V_0 - V_1$ is harmonic in $\mathbb{R}_0^2$. One can easily verify that the function

$$W = \tilde{\Phi} - yV$$

is also harmonic in $\mathbb{R}_0^2$, and

$$\tilde{\Phi} = yV + W \text{ in } \mathbb{R}_0^2;$$

the function V is bounded by $O(r^\epsilon)$ as $r \to \infty$, and $V(0,y) = 0$. By the Phragmén-Lindelöf theorem we deduce that $V \to 0$ at ∞. It follows (by the maximum principle) that $V \equiv 0$. Since W is harmonic and $W = W_y = 0$ on $\{(x,0);\ x < -1\}$ (since the same is true for $\tilde{\Phi}$), it follows that $W \equiv 0$. Hence $\hat{\Phi} - \Phi_0 \equiv \tilde{\Phi} \equiv 0$. □

We shall now prove the same result for $\beta_{\lambda,b}$, where $\beta_{\lambda,b}$ is defined by the expansion near $(\lambda, 0)$,

$$\psi_{\lambda,b} = 2\beta_{\lambda,b}r^{3/2}\left(\sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta\right) + O(r^{5/2}), \tag{9.19}$$

and $\psi_{\lambda,b}$ is the bounded solution in $H^2_{loc}(D_b)$ to

$$\begin{aligned} &\triangle^2\psi = 0 \text{ in } D_b\backslash\Gamma_\lambda, \\ &\psi = 1,\ \psi_y = 0 \text{ on } \partial D_b^+\backslash\{x = b\}, \\ &\psi = -1,\ \psi_y = 0 \text{ on } \partial D_b^-\backslash\{x = b\}, \\ &\psi = -\frac{1}{2a^3}y^3 + \frac{3}{2a}y,\ \psi_{xx} = 0 \text{ on } \{x = b\}, \\ &\psi = 0,\ \psi_y = 0 \text{ on } \Gamma_\lambda. \end{aligned} \tag{9.20}$$

Notice that $\psi_{\lambda,b}$ is odd in y. As in (9.3),

$$\beta_{\lambda,b} \text{ is continuous with respect to } (\lambda, b). \tag{9.21}$$

Lemma 9.4. *$\beta_{\lambda,b} \neq 0$ for $-\infty < \lambda < b < \infty$.*

Proof. Let $\tilde{\psi}$ be the bounded solution in $H^2_{loc}(D_b)$ satisfying (9.20) except for the boundary conditions on $\{x = b\}$ which are replaced by

$$\tilde{\psi}_x = 0, \ \ \tilde{\psi}_{xxx} = 0.$$

Near $(\lambda, 0)$,

$$\tilde{\psi} = 2\tilde{\beta} r^{3/2} \left(\sin \frac{3}{2}\theta + \sin \frac{1}{2}\theta \right) + O(r^{5/2}).$$

As before,

$$J \equiv \lim_{\epsilon \to} \int_{\partial B_\epsilon(\lambda,0)} \left\{ (\Delta\psi \cdot \Delta\tilde{\psi}) n_1 - (\vec{s} \cdot \vec{\tilde{\Phi}} + \vec{\tilde{s}} \cdot \vec{\Phi}) \right\} = 16\pi \beta_{\lambda,b} \tilde{\beta}, \tag{9.22}$$

and (by the path independence of J-integrals)

$$J = \int_{\Lambda_{b,c}} \left\{ (\Delta\psi \cdot \Delta\tilde{\psi}) n_1 - (\vec{s} \cdot \vec{\tilde{\Phi}} + \vec{\tilde{s}} \cdot \vec{\Phi}) \right\}.$$

If we perform integration by parts in the last integral and let $-c \to -\infty$, we obtain

$$\begin{aligned} J = {} & 2\int_0^a \left(\frac{\partial^2 \psi}{\partial y^2} \frac{\partial^2 \tilde{\psi}}{\partial y^2} \right) (-\infty, y) dy + \int_{-a}^a (\Delta\psi \cdot \Delta\tilde{\psi})(b, y) dy \\ & - \int_{-a}^a \left(\frac{\partial^2 \psi}{\partial y^2} \Delta\tilde{\psi} + \frac{\partial^2 \tilde{\psi}}{\partial y^2} \Delta\psi \right) (b, y) dy \\ & - \int_{-a}^a \left(\frac{\partial \psi}{\partial x} \frac{\partial}{\partial x} \Delta\tilde{\psi} + \frac{\partial \tilde{\psi}}{\partial x} \frac{\partial}{\partial x} \Delta\psi \right) (b, y) dy. \end{aligned}$$

Using the boundary conditions at $x = b$, we find that

$$J = \frac{18}{a^3},$$

and upon comparing with (9.22) we conclude that $\beta_{\lambda,b} \neq 0$. □

The proof of Lemma 9.2 shows that

$$||\psi_{\lambda,b}||_{L^\infty} \leq C$$

where C is independent of λ, b. We shall need, however, also a gradient estimate:

Lemma 9.5. *There holds:*

$$||\nabla \psi_{\lambda,b}||_{L^\infty(D_b)} \leq C \tag{9.23}$$

where C is a constant independent of λ, b.

Proof. Let

$$\tilde{\psi}(x, y) = \psi_{\lambda,b}(x, y) - \left(-\frac{1}{2a^3} y^3 + \frac{3}{2a} y \right) \tag{9.24}$$

and extend $\tilde{\psi}$ into $\{x > b\}$ by odd reflection. Then

$$\begin{aligned}
&\Delta^2\tilde{\psi} = 0 \text{ in } D\backslash\{\overline{\Gamma}_\lambda \cup \overline{\Gamma}^*_{-\lambda}\}, \\
&\tilde{\psi} = 0, \ \tilde{\psi}_y = 0 \text{ on } \partial D, \\
&\tilde{\psi} = 0, \ \tilde{\psi}_y = \begin{cases} -3/2a & \text{on } \Gamma_\lambda \\ 3/2a & \text{on } \Gamma^*_{-\lambda} \end{cases}
\end{aligned} \tag{9.25}$$

where $\Gamma^*_{-\lambda}$ is the reflection of Γ_λ about $x = b$.

As easily verified (see [3]) the function

$$q = |\nabla\tilde{\psi}|^2 - (\Delta\tilde{\psi})\tilde{\psi}$$

is subharmonic in $D\backslash(\overline{\Gamma}_\lambda \cup \overline{\Gamma}^*_{-\lambda})$ and, therefore,

$$q(x,y) \le \sup_{\partial\{D\backslash(\overline{\Gamma}_\lambda\cup\overline{\Gamma}^*_{-\lambda})\}} q + \sup_{|y|\le a} q(\pm\infty, y). \tag{9.26}$$

By the boundary conditions in (9.25) we see that the right-hand side of (9.26) is bounded by a constant C independent of λ, b. Since $\tilde{\psi}(x,0) = 0$ ($\tilde{\psi}$ is odd in y), (9.26) implies that

$$|\nabla\tilde{\psi}(x,0)| \le C, \ \ \lambda < x < 2b - \lambda. \tag{9.27}$$

We write $\tilde{\psi}$ in $\{0 < y < a\}$ in the form $yW + \tilde{\psi}_1$ where W is a bounded harmonic function satisfying the boundary conditions:

$$W(x,0) = \tilde{\psi}_y(x,0), \ \ W(x,a) = 0;$$

then $\tilde{\psi}_1$ is a bounded biharmonic function satisfying:

$$\begin{aligned}
&\tilde{\psi}_1 = \frac{\partial}{\partial y}\tilde{\psi}_1 = 0 \text{ at } y = 0, \\
&\tilde{\psi}_1 \text{ and } \frac{\partial}{\partial y}\tilde{\psi}_1 \text{ are smooth at } y = a.
\end{aligned}$$

Using (9.27) to estimate $\nabla(yW)$ and using elliptic estimates to estimate $\nabla\tilde{\psi}_1$, we find that

$$\|\nabla\tilde{\psi}\|_{L^\infty(D\cap\{y>0\})} \le C,$$

from which the assertion (9.23) readily follows. □

Lemma 9.6. *For λ close to b, $\beta_{\lambda,b} > 0$.*

Proof. Let $\delta = b - \lambda$ and choose a coordinate system with $(b,0)$ at the origin. Consider the blow-up sequence

$$\Psi_\delta(x,y) = \frac{1}{\delta}\tilde{\psi}(\delta x, \delta y). \tag{9.28}$$

Using Lemma 9.5 and elliptic estimates we deduce that

$$\Psi_\delta \to \Psi_0 \text{ in } C^0_{loc}(\overline{\mathbb{R}^2_-}) \tag{9.29}$$

where Ψ_0 is a solution to

$$\begin{aligned}
&\Delta^2\Psi_0 = 0 \text{ in } \mathbb{R}^2_-\backslash\overline{\Gamma}_{-1},\\
&\Psi_0 = 0, \ \frac{\partial^2\Psi_0}{\partial x^2} = 0 \text{ on } x = 0, \qquad\qquad (9.30)\\
&\Psi_0 = 0, \ \frac{\partial\Psi_0}{\partial y} = -\frac{3}{2a} \text{ on } \Gamma_{-1} \text{ (from both sides)},
\end{aligned}$$

and

$$\sup_{\mathbb{R}^2_-} |\nabla\Psi_0| \leq C < \infty.$$

Since $\Psi_0 = O(r)$ as $r \to \infty$, it follows, from the proof that $\hat{\Phi}_0 \equiv \Phi_0$, that such a solution is unique.

Let w is the unique bounded solution to

$$\begin{aligned}
&\Delta w = 0 \text{ in } \mathbb{R}^2_-\backslash\overline{\Gamma}_{-1},\\
&w = 0 \text{ on } x = 0,\\
&w = -\frac{3}{2a} \text{ on } \Gamma_{-1} \text{ (from both sides,)};
\end{aligned}$$

then, by uniqueness,

$$\Psi_0 = yw. \qquad (9.31)$$

By the maximum principle $w > -3/2a$, and therefore

$$\hat{\Psi}_0(x,y) \equiv yw(x,y) + \frac{3}{2a}y > 0 \text{ in } \{y > 0\}. \qquad (9.32)$$

Since $\hat{\Psi}_0$ is biharmonic in $\mathbb{R}^2_-\backslash\overline{\Gamma}_{-1}$, odd in y, and

$$\hat{\Psi}_0 = \frac{\partial\hat{\Psi}_0}{\partial y} = 0 \text{ on } \Gamma_{-1},$$

we have

$$\hat{\Psi}_0 = 2\beta_0 r^{3/2}\left(\sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta\right) + O(r^{5/2})$$

near $(-1,0)$ and, by (9.32),

$$\beta_0 > 0.$$

Introducing the function

$$\hat{\Psi}_\delta = \Psi_\delta + \frac{3}{2a}y$$

and its expansion near $(-1,0)$:

$$\hat{\Psi}_\delta = 2\beta_\delta r^{3/2}\left(\sin\frac{3}{2}\theta + \sin\frac{1}{2}\theta\right) + O(r^{5/2})$$

we then have

$$\beta_\delta \to \beta_0 > 0 \text{ if } \delta \to 0.$$

On the other hand, from (9.19), (9.24), (9.28), (9.29) and (9.31), (9.32) we deduce that

$$\beta_\delta = \beta_{\lambda,b}\delta^{1/2},$$

and therefore $\beta_{\lambda,b} > 0$ if λ is close to b. □

We have proved so far that

$$\alpha_{\lambda,b} \text{ and } \beta_{\lambda,b} \text{ are continuous in } (\lambda, b) \text{ for } -\infty < \lambda < b < \infty, \text{ and never vanish,}$$

and $\alpha_{\lambda,b} > 0$, $\beta_{\lambda,b} > 0$ if $b - \lambda$ is small. It follows that

$$\alpha_{\lambda,b} > 0 \;\; \beta_{\lambda,b} > 0 \;\text{ for all } -\infty < \lambda < b < \infty.$$

Taking $\lambda = 0$ and $b \to \infty$ we deduce that

$$\alpha = \lim \alpha_{0,b} \geq 0, \;\; \beta = \lim \beta_{0,b} \geq 0.$$

Recalling (8.17), (8.19) we see that $\alpha > 0$, $\beta > 0$ and thus, by (8.15), $\dot{y}(0+) > 0$. □

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