NUMERICAL ANALYSIS OF THE CAHN-HILLIARD EQUATION AND APPROXIMATION FOR THE HELE-SHAW PROBLEM, PART I: ERROR ANALYSIS UNDER MINIMUM REGULARITIES *

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Abstract. In this first part of a series, we propose and analyze, under minimum regularity assumptions, a semi-discrete (in time) scheme and a fully discrete mixed finite element scheme for the Cahn-Hilliard equation $u_t + \Delta(\varepsilon\Delta u - \varepsilon^{-1} f(u)) = 0$ arising from phase transition in materials science, where ε is a small parameter known as an "interaction length". The primary goal of this paper is to establish some useful a priori error estimates for the proposed numerical methods, in particular, by focusing on the dependence of the error bounds on ε. Quasi-optimal order error bounds are shown for the semi-discrete and fully discrete schemes under different constraints on the mesh size h and the local time step size k_m of the stretched time grid, and minimum regularity assumptions on the initial function u_0 and domain Ω. In particular, all our error bounds depend on $\frac{1}{\varepsilon}$ only in some lower polynomial order for small ε. The cruxes of the analysis are to establish stability estimates for the discrete solutions, to use a spectrum estimate result of Alikakos and Fusco [3] and Chen [15], and to establish a discrete counterpart of it for a linearized Cahn-Hilliard operator to handle the nonlinear term on the stretched time grid. It is this polynomial dependency of the error bounds that paves the way for us to establish convergence of the numerical solution to the solution of the Hele-Shaw (Mullins-Sekerka) problem (as $\varepsilon \searrow 0$) in Part II [26] of the series.

Key words. Cahn-Hilliard equation, Hele-Shaw (Mullins-Sekerka) problem, phase transition, semi-discrete and fully discrete schemes, mixed finite element method

AMS subject classifications. 65M60, 65M12, 65M15, 35B25, 35K57, 35Q99, 53A10

1. Introduction. This paper is the first part of a series (cf. [26]) which devote to error analysis of a mixed finite element approximation for the Cahn-Hilliard equation and convergence analysis of the numerical solution to the solution of the Hele-Shaw (Mullins-Sekerka) problem. While Part II [26] of this series focuses on the approximation of the Hele-Shaw problem under some stronger regularity assumptions, this paper mainly concerns the error analysis of the mixed finite element method for the Cahn-Hilliard equation under *minimum regularity assumptions* on the domain and the initial data. Specifically, we shall propose and analyze a semi-discrete (in time) method and a fully discrete mixed finite element time-stepping method for the Cahn-Hilliard equation (the super-index ε on u^ε is suppressed for notational brevity)

$$u_t + \Delta(\varepsilon\Delta u - \frac{1}{\varepsilon} f(u)) = 0 \quad \text{in } \Omega_T := \Omega \times (0,T)\,, \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}(\varepsilon\Delta u - \frac{1}{\varepsilon} f(u)) = 0 \quad \text{in } \partial\Omega_T := \partial\Omega \times (0,T)\,, \tag{1.2}$$

$$u = u_0 \quad \text{in } \Omega \times \{0\}\,, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^N$ $(N = 2, 3)$ is a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$. $T > 0$ is a fixed constant, and f is the derivative of a smooth double equal well potential taking its global minimum value 0 at $u = \pm 1$. A typical example of f is

$$f(u) := F'(u) \quad \text{and} \quad F(u) = \frac{1}{4}(u^2 - 1)^2\,.$$

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The existence of bistable states suggests that a nonconvex energy functional is associated with the equation (see the discussion below). In order to achieve broader applicability, in this paper we shall consider more general potentials which satisfy some structural assumptions (see Section 2), and our analysis will be carried out based on these assumptions. We like to remark that nonsmooth potentials have also been considered in the literature for the Cahn-Hilliard equation, for that we refer to [21, 18, 6, 7] and the references therein.

The equation (1.1) was originally introduced by Cahn and Hilliard [12] to describe the complicated phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably. Note that the equation (1.1) differs from the original Cahn-Hilliard equation (see [12]) in the scaling of the time so that t here, called the *fast time*, represents $\frac{t}{\varepsilon}$ in the original formulation.

In the equation, u represents the concentration of one of the two metallic components of the alloy mixture. The parameter ε is an "interaction length", which is small compared to the characteristic dimensions on the laboratory scale. The two boundary conditions in (1.2), the outward normal derivatives of u and $\varepsilon\Delta u - \varepsilon^{-1} f(u)$ vanish on $\partial\Omega$, imply that none of the mixture can pass through the walls of the container Ω; the first condition is the most natural way to ensure that the total "free energy" of the mixture decreases in time, which is required by thermodynamics, when there is no interaction between the alloy and the containing walls. The evolution of the concentration consists of two stages: the first stage (rapid in time) is known as phase separation and the second (slow in time) is known as phase coarsening. At the end of the first stage, fine-scaled phase regions are formed, which are separated by a thin region, usually considered as a hypersurface called the *interface*. At the end of the second stage, the solution will generically tend to a stable state, which minimizes the energy functional associated with (1.1). For more physical background, derivation, and discussion of the Cahn-Hilliard equation and related equations, we refer to [12, 8, 2, 11, 30, 31, 4] and the references therein.

It is well-known that the Cahn-Hilliard equation (1.1) is a gradient flow with the Liapunov energy functional

$$\mathcal{J}_\varepsilon(u) := \int_\Omega \phi_\varepsilon(u)\,\mathrm{d}x \quad \text{and} \quad \phi_\varepsilon(u) = \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\,. \tag{1.4}$$

Here, the energy density $\phi_\varepsilon(u)$ is a nonconvex function. It is also known [1] that the elliptic operator $L_{CH}(u) := \Delta(\varepsilon\Delta u - \varepsilon^{-1} f(u))$ associated with the Cahn-Hilliard equation (1.1) is the representation of the Fréchet derivative $\mathcal{J}_\varepsilon'(u)$ of $\mathcal{J}_\varepsilon(u)$ in the space $H^{-1}(\Omega)$.

Another gradient flow for the same Liapunov energy functional in (1.4) is the Allen-Cahn equation

$$u_t - \varepsilon\Delta u + \frac{1}{\varepsilon} f(u) = 0\,, \tag{1.5}$$

which was originally introduced by Allen and Cahn [4] to describe the motion of antiphase boundaries in crystalline solids (see [12, 31, 25] and references therein). It is also known [1] that the elliptic operator $L_{AC}(u) := -\varepsilon\Delta u + \varepsilon^{-1} f(u)$ associated with the Allen-Cahn equation (1.5) is the representation of the Fréchet derivative $\mathcal{J}_\varepsilon'(u)$ in the space $L^2(\Omega)$. On the other hand, the Cahn-Hilliard equation is known to conserve the total mass because its solution satisfies $\frac{d}{dt}\int_\Omega u(x,t)\mathrm{d}x = 0$, but the Allen-Cahn equation does not conserve the total mass.

In addition to the reason that the Cahn-Hilliard equation is widely accepted as a good model to describe the phase separation and coarsening phenomena in a melted alloy, it has also been extensively studied in the past decade due to its connection to an interesting and complicated free boundary problem which is known as the Mullins-Sekerka problem arising from studying solidification/liquidation of materials of zero specific heat, which is also known as the (two-phase) Hele-Shaw problem arising from the study of the pressure of immiscible fluids in the air [32, 2, 16, 13, 11, 29, 28]. It was first formally shown by Pego [32] that, as $\varepsilon \searrow 0$, the function $w := -\varepsilon\Delta u + \varepsilon^{-1} f(u)$, known as the chemical potential, tends to a limit, which, together with a free boundary $\Gamma := \cup_{0\le t\le T}(\Gamma_t \times \{t\})$, satisfies the following Hele-Shaw (Mullins-Sekerka) problem:

$$\Delta w = 0 \qquad \text{in } \Omega \setminus \Gamma_t,\, t \in [0,T]\,, \tag{1.6}$$

$$\frac{\partial w}{\partial n} = 0 \qquad \text{on } \partial\Omega,\, t \in [0,T]\,, \tag{1.7}$$

$$w = \sigma\kappa \qquad \text{on } \Gamma_t,\, t \in [0,T]\,, \tag{1.8}$$

$$V = \frac{1}{2}\Big[\frac{\partial w}{\partial n}\Big]_{\Gamma_t} \qquad \text{on } \Gamma_t,\, t \in [0,T]\,, \tag{1.9}$$

$$\Gamma_0 = \Gamma_{00} \qquad \text{when } t = 0\,. \tag{1.10}$$

Here

$$\sigma = \int_{-1}^{1} \sqrt{\frac{F(s)}{2}}\, \mathrm{d}s\,.$$

κ and V are, respectively, the mean curvature and the normal velocity of the interface Γ_t, n is the unit outward normal to either $\partial\Omega$ or Γ_t, $[\frac{\partial w}{\partial n}]_{\Gamma_t} := \frac{\partial w^+}{\partial n} - \frac{\partial w^-}{\partial n}$, and w^+ and w^- are respectively the restriction of w in Ω_t^+ and Ω_t^-, the exterior and interior of Γ_t in Ω. Also $u \to \pm 1$ in $\Omega_t^\pm$ for all $t \in [0,T]$, as $\varepsilon \searrow 0$. The rigorous justification of this limit was successfully carried out by Alikakos, Bates and Chen [2] under the assumption that the above Hele-Shaw (Mullins-Sekerka) problem has a classical solution. Later, Chen [16] formulated a weak solution to the Hele-Shaw (Mullins-Sekerka) problem and showed, using an energy method, that the solution of (1.1)-(1.3) approaches, as $\varepsilon \searrow 0$, to a weak solution of the Hele-Shaw (Mullins-Sekerka) problem. Also, using an energy method, Stoth [35] established a global (in time) convergence result for the case of three-dimensional radial symmetry and Dirichlet boundary conditions.

It is clear that the study of the Cahn-Hilliard equation (1.1) is of great value for understanding phase transition and for investigating the Hele-Shaw (Mullins-Sekerka) (free boundary) problem by taking advantage of the fact that the solution of the Cahn-Hilliard equation is known to exist for all times [24]. In particular, this is attractive from the computational point of view. Due to the nonlinearity in the Cahn-Hilliard equation, its solution only can be sought numerically. The primary numerical challenge for solving the Cahn-Hilliard equation results from the presence of the small parameter ε in front of the nonlinear term in the equation. Recall convergence of the Cahn-Hilliard equation to the Hele-Shaw (Mullins-Sekerka) model only when ε is small. On the other hand, the equation becomes a singularly perturbed fourth order "heat" equation for small ε. To resolve the solution numerically, one has to use small (space) mesh size h and (time) step size k, which must be related to the parameter ε. Numerical approaches are often based on a mixed formulation of (1.1)-(1.3) which involves the chemical potential w,

$$u_t = \Delta w \qquad \text{in } \Omega_T\,, \tag{1.11}$$

$$w = \frac{1}{\varepsilon}\, f(u) - \varepsilon\, \Delta u \quad \text{in } \Omega_T\,, \tag{1.12}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T)\,, \tag{1.13}$$

$$u(x,0) = u_0(x) \qquad \forall\, x \in \Omega\,. \tag{1.14}$$

We refer to [24, 9] and refererences therein for more discussions on well-posedness and regularities of the Cahn-Hilliard and the biharmonic problems.

In the past fifteen years, numerical approximations of the Cahn-Hilliard equation with a *fixed* ε have been developed and analyzed by many authors. Elliott and Zheng [24] analyzed a (continuous in time) semi-discrete conforming finite element discretization in one space dimension. Numerical experiments of the method in one space dimension were reported in [21]. Elliott and French [23] proposed a (continuous in time) semi-discrete nonconforming finite element method based on the Morley nonconforming finite element method [10, 17]. Optimal order error estimates were also established for the nonconforming method under the assumption that the solution is smooth. Elliott, French and Milner [22] proposed and analyzed a (continuous in time) semi-discrete splitting finite element method (mixed finite element method) which approximates simultaneously the concentration u and the chemical potential w. Optimal order error estimates were shown under the assumption that the finite element approximation u_h of the concentration u is bounded in L^∞. Later, Du and Nicolaides [19] analyzed a fully discrete splitting finite element method in one space dimension under weaker regularity assumptions on the solution u of the Cahn-Hilliard equation, and established optimal order error estimates by first proving the boundedness of u_h in L^∞. Copetti and Elliott [18] considered the Cahn-Hilliard equation with a nonsmooth logarithmic potential function. A fully discrete splitting finite element method was proposed and convergence of the method was also demonstrated. In one space dimension, French and Jensen [27] analyzed the long time behavior of the (continuous time) semi-discrete conforming *hp*-finite element approximations. Recently, extensive studies have been carried out by Barrett and Blowey, and others on the finite element approximations of the Cahn-Hilliard system for multi-component alloys with constant or degenerate mobility, we refer to [5, 6, 7] and the references therein for detailed expositions.

We like to point out that the results of all papers cited above were established for the Cahn-Hilliard equation with a *fixed* "interaction length" ε. No special effort and attention were given to address issues such as how the mesh sizes h and k depend on ε and how the error bounds depend on ε. In fact, since all those error estimates were derived using a Gronwall inequality type argument at the end of the derivations, it is not hard to check that all error bounds contain a factor $\exp(\frac{T}{\varepsilon})$, which clearly is not very useful when ε is small.

Unlike the numerical works mentioned above, the focus of this series is on approximating the solution of the Cahn-Hilliard equation (1.1) for small ε, which is the case for both applications we are interested in: simulating the second stage of the concentration evolution process for general regularities (Part I), and approximating the solution (including the free boundary) of the Hele-Shaw (Mullins-Sekerka) problem via the Cahn-Hilliard equation (Part II). The primary goal of this paper is to develop a semi-discrete (in time) and a fully discrete approximation based on a mixed variational formulation for the initial-boundary value problem (1.1)-(1.3), and to establish useful error bounds for general regularities, which show growth only in low polynomial order of $\frac{1}{\varepsilon}$, for the proposed schemes under some reasonable constraints on mesh sizes h and k. To our knowledge, such error estimates for the Cahn-Hilliard equation have not been known in the literature. In addition, such error bounds serve as the basis for computing the solution of the Cahn-Hilliard equation and the solution of the Hele-Shaw (Mullins-Sekerka) problem.

The subsequent analysis applies to a general class of admissible double equal well potentials and initial data $u_0 \in H^{2+\ell}$, $\ell = 0, 1$ that can be bounded in terms of negative powers of ε; see the general assumptions (GA_1)-(GA_3) in Sections 2 and 3.

Our fully discrete scheme, based on a mixed variational formulation for u and the chemical potential w, is defined as

$$\big(d_t U^m, \eta_h\big) + \big(\nabla W^m, \nabla \eta_h\big) = 0 \quad \forall \eta_h \in V_h\,, \tag{1.15}$$

$$\varepsilon\,\big(\nabla U^m, \nabla v_h\big) + \frac{1}{\varepsilon}\,\big(f(U^m) - W^m, v_h\big) = 0 \quad \forall v_h \in V_h\,, \tag{1.16}$$

with some starting value $U^0 \in V_h$. Here $V_h \subset H^1(\Omega)$ denotes the continuous piecewise linear finite element space. We consider this discrete system on the equidistant mesh J_k^1, and also on the stretched mesh $J_k^2 := \{t_m\}_{m=0}^M$ of local mesh sizes

$$k_{m+1} \equiv \begin{cases} (m+1)k_0^2, & \text{for } 0 \le t_{m+1} \le \hat{t}_0\,, \\ \gamma k_0, & \text{for } t_{m+1} \ge \hat{t}_0\,, \end{cases} \tag{1.17}$$

with the basic mesh size k_0, and some positive constants γ and $\hat{t}_0 = O(1)$. Notice that both meshes require asymptotically the same amount of computation cost (cf. Section 3).

We assume that there exist positive constants m_0 and σ_j for $j = 1, 2, 3$ such that

$$m_0 := \frac{1}{|\Omega|}\int_\Omega u_0(x)\,\mathrm{d}x \;\in (-1, 1)\,, \tag{1.18}$$

$$\mathcal{J}_\varepsilon(u_0) := \frac{\varepsilon}{2}\,\|\,\nabla u_0\,\|_{L^2}^2 + \frac{1}{\varepsilon}\,\|\,F(u_0)\,\|_{L^1} \le C\,\varepsilon^{-2\sigma_1}\,, \tag{1.19}$$

$$\|\,w_0^\varepsilon\,\|_{H^\ell} := \|\,-\varepsilon\Delta u_0^\varepsilon + \frac{1}{\varepsilon}\,f(u_0^\varepsilon)\,\|_{H^\ell} \le C\,\varepsilon^{-\sigma_{2+\ell}}\,, \quad \ell = 0, 1\,. \tag{1.20}$$

We now summarize our main results in this paper. Let $0 < \beta < \frac{1}{2}$ be an arbitrary number. On the equidistant time mesh $J_k^1 = \{t_m\}_{m=0}^M$ and for $u_0 \in H^2(\Omega)$, we show a convergence rate $O(k^{\frac{1}{2}-\beta})$ for the implicit Euler semi-discretization (see Theorem 3.4), which can be improved to $O(k_0^{1-\beta})$ on the stretched time mesh $J_k^2 = \{t_m\}_{m=0}^M$ (see Theorem 3.6). Theorem 4.3 contains error estimates for the fully discrete approximation of (1.1)-(1.3) on J_k^2, using the continuous piecewise linear mixed finite element. The results in Theorems 3.4 and 3.6 are obtained under general regularity assumptions for (1.1)-(1.3). Moreover, mesh constraints, which relate ε, k_0 and h and under which the above convergence rates hold, are explicitly formulated. The constraints indicate that small values of β severely restrict the size of k_0.

In the case that $u_0 \in H^3(\Omega)$ and either Ω is a convex polygonal domain for $N = 2$ or the boundary $\partial\Omega$ is of class $C^{2,1}$ for $N = 2, 3$, we show quasi-optimal order in k and optimal order in h convergence on the equidistant time mesh for the fully discrete mixed finite element approximation, see Corollaries 3.5 and 4.4.

The analyses to be given below study the effects of temporal and spatial discretization independently for given initial data $u_0 \in H^\ell(\Omega)$, $\ell = 2, 3$; the complexity of initial data (captured in terms of the parameters σ_i, for $i = 1, ..4$), growth ($p > 2$) and degree of the nonmonotonicity ($\delta > 0$) of f, and the value $\varepsilon > 0$ are all taken into account here to draw conclusions for a robust numerical scheme which necessarily relates the different scales ε and k_0, h, under the premise to derive error bounds that depend only polynomially on $\frac{1}{\varepsilon}$. This scenario not only gives practically relevant error bounds for quantities of interest in materials science (i.e., concentration) but also paves the way to approximate the Hele-Shaw (Mullins-Sekerka) problem via the Cahn-Hilliard equation in the second part [26] of this series.

The main result for (1.15)-(1.16) and general f is given in Theorem 4.3, here we present it in a simplified form.

THEOREM 1.1. *Let $\{(U^m, W^m)\}_{m=0}^M$ solve (1.15)-(1.16) on $J_k^{1+\ell} := \{t_m\}_{m=0}^M$ and for $u_0 \in H^{3-\ell}(\Omega)$, $\ell = 0, 1$. Suppose that $\mathcal{T}_h$ is a quasi-uniform triangulation of Ω, allowing for inverse inequalities and H^1-stability of the L^2-projection in the continuous linear finite element space. For any fixed $0 < \beta < \frac{1}{2}$, if the mesh sizes k_0, h and the starting value U^0 satisfy some appropriate constraints (see Theorem 4.3 and Corollary 4.4 for the precise descriptions), then there hold*

$$\text{(i)} \quad \max_{0 \le m \le M} \| u(t_m) - U^m \|_{H^{-1}} + \Big(\sum_{m=1}^M k_m \| u(t_m) - U^m \|_{L^2}^2 \Big)^{\frac{1}{2}} \le \tilde{C} \Big\{ k_0^{\frac{2-\beta}{2}} \varepsilon^{-\nu_1} + h^2 \varepsilon^{-\nu_2} \Big\} ,$$

$$\text{(ii)} \quad \Big(\sum_{m=1}^M k_m \| u(t_m) - U^m \|_{H^1}^2 \Big)^{\frac{1}{2}} \le \tilde{C} \Big\{ k_0^{\frac{2-\beta}{2}} \varepsilon^{-\nu_1} + h\, \varepsilon^{-\nu_2} \Big\} ,$$

where $\tilde{C} = \tilde{C}(T, \Omega, \sigma_i, p, \delta, \beta; \ln(\frac{1}{k_0}))$ is homogeneous in the last argument, and $\nu_j = \nu_j(\sigma_i, p, \delta, \beta)$ for $j = 1, 2$. Here, it is understood that $k_0 = k$ for J_k^1.

To establish the above error estimates, the following three ingredients play a crucial role in our analysis.

- To establish stability estimates for the discrete solutions of the semi-discrete (in time) and the fully discrete schemes.
- To handle the (nonlinear) potential term in the error equation using a spectrum estimate result due to Alikakos and Fusco [3], and Chen [15] for the linearized Cahn-Hilliard operator

$$\mathcal{L}_{CH} := \Delta(\varepsilon \Delta - \frac{1}{\varepsilon} f'(u) I) , \tag{1.21}$$

  where I denotes the identity operator and u is a solution of the Cahn-Hilliard equation (1.1); see Proposition 2.2 for details.
- To establish a discrete counterpart of above spectrum estimate.

We remark that, using a similar approach a parallel study was also carried out by the authors in [25] for the Allen-Cahn equation and the related curvature driven flows. On the other hand, unlike the Allen-Cahn equation which is a gradient flow in L^2, the Cahn-Hilliard equation is a gradient flow only in H^{-1}, which makes the analysis for the Cahn-Hilliard equation in this paper more delicate and complicated than that for the Allen-Cahn equation given in [25].

The paper is organized as follows: In Section 2, we shall derive some a priori estimates for the solution of (1.1)-(1.3), where special attention is given to the dependence of the solution on ε in various norms. In Section 3 we consider the backward Euler semi-discrete (in time) scheme for the Cahn-Hilliard equation and establish some stability estimates for the semi-discrete solution. We then obtain a sub-optimal error bound, which depends on $\frac{1}{\varepsilon}$ only in a low polynomial order for small ε as is summarized in Theorems 3.4-3.6. The spectrum estimate plays a crucial role in the proof. In Section 4, we propose a fully discrete approximation obtained by discretizing the semi-discrete scheme of Section 3 in space using the lowest order Ciarlet-Raviart mixed finite element method. Optimal order error bounds, depending on $\frac{1}{\varepsilon}$ only in a low polynomial order, are shown for the fully discrete method in Theorem 4.3. The main ideas are to establish some stability estimates for the fully discrete solutions, and more importantly to prove a discrete counterpart of the spectrum estimate of [3, 15].

2. Energy estimates for the differential problem. In this section, we derive some energy estimates in various function spaces in terms of negative powers of ε for the solution u the Cahn-Hilliard equation (1.1) for given $u_0 \in H^{2+\ell}(\Omega)$, $\ell = 0, 1$. Here $J = (0, T)$, and $H^k(\Omega)$ denotes the standard Sobolev space of the functions which and their up to kth order derivatives are L^2-integrable. Throughout this paper, the standard space, norm and inner product notation are adopted. Their definitions can be found in [10, 17]. In particular, $(\cdot, \cdot)$ denotes the standard inner product on $L^2(\Omega)$. Also, $c, \tilde{c}_j, C, \tilde{C}, \tilde{C}_j$ are generic positive constants which are independent of ε and the time and space mesh sizes k, k_0 and h.

In addition, define for $r \geq 0$

$$H^{-r}(\Omega) := (H^r(\Omega))^* , \qquad H_0^{-r}(\Omega) := \{ \, w \in H^{-r}(\Omega); < w, 1 >_r = 0 \, \} ,$$

where $< \cdot, \cdot >_r$ stands for the dual product between $H^r(\Omega)$ and $H^{-r}(\Omega)$; we denote $L_0^2(\Omega) \equiv H_0^0(\Omega)$. For $v \in L_0^2(\Omega)$, let $v_1 := -\Delta^{-1} v \in H^1(\Omega) \cap L_0^2(\Omega)$ be the solution to

$$\begin{aligned} -\Delta v_1 &= v \quad \text{in } \Omega , \\ \frac{\partial v_1}{\partial n} &= 0 \quad \text{on } \partial\Omega , \end{aligned}$$

and define $\Delta^{-\frac{1}{2}} v$ as

$$\Delta^{-\frac{1}{2}} v := \nabla v_1 = -\nabla \Delta^{-1} v \, .$$

We make the following general assumptions on the derivative f of the potential function F:

General Assumption 1 (GA_1)

1) $f = F'$, for $F \in C^4(\mathbf{R})$, such that $F(\pm 1) = 0$, and $F > 0$ elsewhere.
2) $f'(u)$ satisfies for some finite $2 < p \leq \frac{2(N-1)}{N-2}$ and positive numbers $\tilde{c}_i > 0$, $i = 0, .., 3$,

$$\tilde{c}_1 \, | \, u \, |^{p-2} - \tilde{c}_0 \leq f'(u) \leq \tilde{c}_2 \, | \, u \, |^{p-2} + \tilde{c}_3 \, .$$

3) There exist $0 < \gamma_1 \leq 1$, $\gamma_2 > 0$ and $\delta > 0$ such that for all $|a| \leq 2$

$$\big(f(a) - f(b), a - b \big) \geq \gamma_1 \, \big(f'(a)(a-b), a - b \big) - \gamma_2 \, | \, a - b \, |^{2+\delta} \, .$$

Remark: It is trivial to check that $(\mathrm{GA}_1)_2$ implies

$$-(f'(u)v,v) \le \tilde{c}_0 \| v \|_{L^2}^2 , \quad \forall v \in L^2(\Omega) , \tag{2.1}$$

which will be utilized several times in the paper.

Example: The potential function $F(u) = \frac{1}{4}(u^2-1)^2$, consequently, $f(u) = u^3 - u$, is often used in physical and geometrical applications [4, 12, 8, 2, 16]. For readers' convenience, we verify $(\mathrm{GA}_1)_1$-$(\mathrm{GA}_1)_3$ for the case in the following. First, $(\mathrm{GA}_1)_1$ holds trivially. Since $f'(u) = 3u^2 - 1$, $(\mathrm{GA}_1)_2$ holds with $\tilde{c}_1 = \tilde{c}_2 = 3$ and $\tilde{c}_0 = \tilde{c}_3 = 1$. A direct calculation gives

$$f(a) - f(b) = (a-b)\big[f'(a) + (a-b)^2 - 3(a-b)a\big] . \tag{2.2}$$

Hence, $(\mathrm{GA}_1)_3$ holds with $\gamma_1 = 1, \gamma_2 = 3$, and $\delta = 1$. Also, (2.1) holds with $\tilde{c}_0 = 1$.

In order to trace dependence of the solution on the small parameter $\varepsilon > 0$, we assume that the initial function u_0 satisfies the following conditions:

General Assumption 2 (GA_2)

There exist positive ε-independent constants m_0 and σ_j, $j = 1, 2, 3$ such that (1.18)-(1.20) hold.

Lemma 2.1. *Suppose that f satisfies (GA_1), and $u_0 \in H^2(\Omega)$ satisfies (1.18)-(1.19) in (GA_2). Then, the following estimates hold for the solution (u, w) of (1.11)-(1.14):*

$$\text{(i)} \qquad \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x = m_0 \in (-1,1), \quad \forall\, t \ge 0 ,$$

$$\text{(ii)} \qquad \operatorname*{ess\,sup}_{[0,\infty]} \Big\{ \frac{\varepsilon}{2} \| \nabla u \|_{L^2}^2 + \frac{1}{\varepsilon} \| F(u) \|_{L^1} \Big\} + \left\{ \begin{array}{c} \int_0^\infty \| u_t(s) \|_{H^{-1}}^2 \,\mathrm{d}s \\ \\ \int_0^\infty \| \nabla w(s) \|_{L^2}^2 \,\mathrm{d}s \end{array} \right\} = \mathcal{J}_\varepsilon(u_0) ,$$

$$\text{(iii)} \qquad \int_0^\infty \| \Delta u(s) \|^2 \,\mathrm{d}s \le C\, \varepsilon^{-(2\sigma_1 + 3)} ,$$

$$\text{(iv)} \qquad \operatorname*{ess\,sup}_{[0,\infty]} \| \Delta^{-1} u_t \|_{L^2}^2 + \left\{ \begin{array}{c} \varepsilon \int_0^\infty \| u_t(s) \|_{L^2}^2 \,\mathrm{d}s \\ \\ \varepsilon \int_0^\infty \| \Delta w(s) \|_{L^2}^2 \,\mathrm{d}s \end{array} \right\} \le C\, \varepsilon^{-\max\{2\sigma_1(p-1)+p+1, 2\sigma_2\}} ,$$

$$\text{(v)} \qquad \operatorname*{ess\,sup}_{[0,\infty]} \| \Delta u \|_{L^2}^2 \le C\, \rho_1(\varepsilon) ,$$

$$\text{(vi)} \qquad \operatorname*{ess\,sup}_{[0,\infty]} \tau(t) \| u_t \|_{H^{-1}}^2 + \varepsilon \int_0^\infty \tau(s) \| \nabla u_t \|_{L^2}^2 \,\mathrm{d}s \le C\, \varepsilon^{-(2\sigma_1+3)} ,$$

$$\text{(vii)} \qquad \operatorname*{ess\,sup}_{[0,\infty]} \tau(t) \| u_t \|_{L^2}^2 + \varepsilon \int_0^\infty \tau(s) \| \Delta u_t \|_{L^2}^2 \,\mathrm{d}s \le C\, \varepsilon^{-\max\{2\sigma_1(p-1)+p+4, 2\sigma_2+1\}} ,$$

$$\text{(viii)} \qquad \int_0^\infty \| \Delta^{-2} u_{tt}(s) \|_{L^2}^2 \,\mathrm{d}s \le C\, \rho_2(\varepsilon) ,$$

$$\text{(ix)} \qquad \int_0^\infty \tau(s) \| \Delta^{-1} u_{tt}(s) \|_{H^{-1}}^2 \,\mathrm{d}s \le C\, \rho_2(\varepsilon) ,$$

where

$$\rho_1(\varepsilon) := \varepsilon^{-\max\{2\sigma_1(p-1)+p+3, 2\sigma_1+3, 2(\sigma_2+1)\}} ,$$
$$\rho_2(\varepsilon) := \varepsilon^{-\max\{2(\sigma_1(2p-3)+p+1), 2\sigma_2+1+2\sigma_1(p-2)+p\}} ,$$

and $\tau \equiv \tau(t) = \min\{t, \widehat{t}_0\}$, *for any fixed small number* $0 < \widehat{t}_0 \ll 1$.

Proof. (i) The assertion follows immediately from integrating (1.1) over Ω and using the boundary condition (1.2).

(ii) This assertion is the immediate consequence of the basic energy law associated with the Cahn-Hilliard equation

$$\frac{d}{dt}\mathcal{J}_\varepsilon(u(t)) = \begin{cases} -\| u_t(t) \|_{H^{-1}}^2, \\ \\ -\| \nabla w(t) \|_{L^2}^2, \end{cases} \tag{2.3}$$

where

$$\mathcal{J}_\varepsilon(u) := \int_\Omega \Big[\varepsilon|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\Big]\, \mathrm{d}x \qquad \forall\, t \geq 0\,. \tag{2.4}$$

(iii) Multiply (1.11) by u, (1.12) by $-\Delta u$, and add these equations. Integration by parts on the nonlinear term and (2.1) lead to

$$\frac{1}{2}\frac{d}{dt}\| u \|_{L^2}^2 + \varepsilon \| \Delta u \|_{L^2}^2 \leq -\frac{1}{\varepsilon}\,(f'(u), |\,\nabla u\,|^2) \leq \frac{\tilde{c}_0}{\varepsilon}\,\| \nabla u \|_{L^2}^2\,.$$

The assertion then follows from (ii).

(iv) We formally differentiate (1.11)-(1.12) in time,

$$u_{tt} - \Delta w_t = 0\,, \tag{2.5}$$

$$w_t = \frac{1}{\varepsilon}\, f'(u)u_t - \varepsilon\, \Delta u_t\,. \tag{2.6}$$

Testing (2.5) with $\Delta^{-2}u_t$, (2.6) with $-\Delta u_t$, and using Young's inequality we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\| \Delta^{-1}u_t \|_{L^2}^2 + \varepsilon \| u_t \|_{L^2}^2 &= -\frac{1}{\varepsilon}\,(f'(u)u_t, \Delta^{-1}u_t) \\ &\leq \frac{1}{\varepsilon}\,\| f'(u) \|_{L^3}\, \| u_t \|_{L^2}\, \| \Delta^{-1}u_t \|_{L^6} \\ &\leq \frac{\varepsilon}{2}\,\| u_t \|_{L^2}^2 + \frac{C}{\varepsilon^3}\,\| f'(u) \|_{L^3}^2\, \| \Delta^{-1}u_t \|_{L^6}^2\,. \end{aligned} \tag{2.7}$$

The last term in (2.7) can be bounded by

$$\frac{1}{\varepsilon^3}\,\| f'(u) \|_{L^3}^2\, \| \Delta^{-\frac{1}{2}}u_t \|_{L^2}^2 \leq \frac{1}{\varepsilon^3}\Big(\tilde{c}_2\, \| u \|_{L^{3(p-2)}}^{2(p-2)} + \tilde{c}_3\Big)\| \Delta^{-\frac{1}{2}}u_t \|_{L^2}^2\,. \tag{2.8}$$

Coming back to (2.7), and integrating over $[0, \infty)$ then gives the result.

(v) We multiply (1.11) by u, (1.12) by $-\Delta u$, and use (2.1) to get

$$\begin{aligned} \varepsilon \| \Delta u \|_{L^2}^2 &\leq \frac{1}{\varepsilon}\,\| \Delta^{-1}u_t \|_{L^2}^2 - \frac{8}{\varepsilon}\,\big(f'(u)\nabla u, \nabla u\big) \\ &\leq \frac{1}{\varepsilon}\,\| \Delta^{-1}u_t \|_{L^2}^2 + \frac{8\tilde{c}_0}{\varepsilon}\,\| \nabla u \|_{L^2}^2\,. \end{aligned} \tag{2.9}$$

The assertion follows from (ii) and (iv).

(vi) Testing (2.5) with $-\tau\Delta^{-1}u_t$, (2.6) with τu_t, and using Young's inequality give

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\tau\| u_t \|_{H^{-1}}^2\Big) + \varepsilon\tau\, \| \nabla u_t \|_{L^2}^2 &\leq \frac{1}{2}\,\| \Delta^{-\frac{1}{2}}u_t \|_{L^2}^2 - \frac{\tau}{\varepsilon}\,(f'(u), |\,u_t\,|^2) \\ &\leq \frac{1}{2}\,\| \Delta^{-\frac{1}{2}}u_t \|_{L^2}^2 + \frac{\tilde{c}_0\tau}{\varepsilon}\,\| u_t \|_{L^2}^2 \\ &\leq \frac{\varepsilon\tau}{2}\,\| \nabla u_t \|_{L^2}^2 + \Big(\frac{1}{2} + \frac{\tilde{c}_0^2\tau}{2\varepsilon^3}\Big)\| u_t \|_{H^{-1}}^2\,. \end{aligned}$$

We then obtain (vi) from (ii).

(vii) Multiplying (2.5) by τu_t, (2.6) by $-\tau\Delta u_t$ leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\tau\| u_t \|_{L^2}^2\Big) + \varepsilon\tau\,\| \Delta u_t \|_{L^2}^2 &= \frac{1}{2}\| u_t \|_{L^2}^2 + \frac{\tau}{\varepsilon}\,(f'(u)u_t, \Delta u_t) \qquad (2.10)\\ &\leq \frac{1}{2}\| u_t \|_{L^2}^2 + \frac{C\tau}{\varepsilon}\,\| f'(u) \|_{L^3}\,\| \nabla u_t \|_{L^2}\,\| \Delta u_t \|_{L^2}\\ &\leq \frac{1}{2}\| u_t \|_{L^2}^2 + \frac{C\tau}{\varepsilon^3}\,\| f'(u) \|_{L^3}^2\,\| \nabla u_t \|_{L^2}^2 + \frac{\varepsilon\tau}{2}\,\| \Delta u_t \|_{L^2}^2 .\end{aligned}$$

Using an argument similar to (2.8), the fact that $H^1(\Omega) \hookrightarrow L^{3(p-2)}(\Omega)$ and (vi) we get the assertion (vii).

(viii) Testing (2.5) with $\Delta^{-4}u_{tt}$ and (2.6) with $-\Delta^{-3}u_{tt}$ leads to

$$\begin{aligned}\| \Delta^{-2}u_{tt} \|_{L^2}^2 &\leq \varepsilon^2\,\| u_t \|_{L^2}^2 + \frac{2}{\varepsilon}\,(f'(u)u_t, \Delta^{-3}u_{tt})\\ &\leq \varepsilon^2\,\| u_t \|_{L^2}^2 + \frac{2}{\varepsilon}\,\| f'(u) \|_{L^2}\,\| u_t \|_{L^2}\,\| \Delta^{-3}u_{tt} \|_{L^\infty}\\ &\leq \frac{C}{\varepsilon^2}\,\Big(\tilde{c}_2\,\| u \|_{L^{2(p-2)}}^{2(p-2)} + (\tilde{c}_3)^2\Big)\,\| u_t \|_{L^2}^2 + \frac{1}{2}\,\| \Delta^{-2}u_{tt} \|_{L^2}^2 .\end{aligned}$$

The assertion (viii) then follows from $H^1(\Omega) \hookrightarrow L^{2(p-2)}(\Omega)$ and (ii), (iv).

(ix) Multiplying (2.5) by $-\tau\Delta^{-3}u_{tt}$, (2.6) by $-\tau\Delta^{-2}u_{tt}$ gives

$$\begin{aligned}\tau\| \Delta^{-\frac{3}{2}}u_{tt} \|_{L^2}^2 + \frac{\varepsilon}{2}\frac{d}{dt}\Big(\tau\| \Delta^{-\frac{1}{2}}u_t \|_{L^2}^2\Big) &= -\frac{\tau}{\varepsilon}\,(f'(u)u_t, \Delta^{-2}u_{tt}) + \frac{\varepsilon}{2}\| \Delta^{-\frac{1}{2}}u_t \|_{L^2}^2\\ &\leq \frac{C}{\varepsilon^2}\,\| f'(u) \|_{L^3}^2\,\| u_t \|_{L^2}^2 + \frac{\tau}{2}\,\| \Delta^{-\frac{3}{2}}u_{tt} \|_{L^2}^2 + \frac{\varepsilon}{2}\| \Delta^{-\frac{1}{2}}u_t \|_{L^2}^2 . \qquad (2.11)\end{aligned}$$

Then, the above inequality, (iv) and (ii) imply the assertion. □

The above estimates are derived under the minimum regularity assumption $u_0 \in H^2(\Omega)$. They show the strong dependency of the solution on negative powers of ε in high norms. On the other hand, we show in the following that the estimates will improve drastically if the initial data $u_0 \in H^3(\Omega)$ and the boundary $\partial\Omega \in C^{2,1}$ are considered. Alternatively, the subsequent results also hold for convex polygonal domains in the case $N = 2$.

LEMMA 2.2. *Suppose that f satisfies (GA$_1$), and $u_0 \in H^3(\Omega)$ satisfies (GA$_2$), and $\partial\Omega$ is of class $C^{2,1}$. Then the solution of (1.11)-(1.14) satisfies the following estimates:*

(i) $\displaystyle \frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x = m_0 \in (-1,1), \quad \forall\, t \geq 0\,,$

(ii) $\displaystyle \operatorname*{ess\,sup}_{[0,\infty]}\,\| u_t \|_{H^{-1}}^2 + \varepsilon\int_0^\infty \| \nabla u_t \|_{L^2}^2\,\mathrm{d}s \leq C\,\varepsilon^{-\max\{2\sigma_1+3, 2\sigma_3\}}\,,$

(iii) $\displaystyle \operatorname*{ess\,sup}_{[0,\infty]}\,\| \nabla u \|_{L^2}^2 + \varepsilon\int_0^\infty \| u(s) \|_{H^3}^2\,\mathrm{d}s \leq C\,\varepsilon^{-\{2\sigma_1(p-1)+p+4\}}\,,$

(iv) $\displaystyle \int_0^\infty \| \Delta^{-1}u_{tt} \|_{H^{-1}}^2\,\mathrm{d}s \leq C\,\tilde{\rho}_2(\varepsilon)\,,$

where

$$\tilde{\rho}_2(\varepsilon) := \varepsilon^{-\max\{2\sigma_1(2p-3)+2p+2,\, 2\sigma_1(p-2)+2\sigma_2+p+1,\, 2\sigma_3-1\}}\,.$$

Proof. (i) The proof of assertion (i) is trivial.

(ii) It is same as step (vi) in the proof of Lemma 2.1 except for omitting the time weight.

(iii) We multiply (1.11) by $-\Delta u$, (1.12) by $\Delta^2 u$, and integrate by parts.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\| \nabla u \|_{L^2}^2 + \varepsilon \| \nabla\Delta u \|_{L^2}^2 &= \frac{1}{\varepsilon}\left(f'(u)\nabla u, \nabla\Delta u\right)\\ &\le \frac{1}{\varepsilon^3}\| f'(u) \|_{L^3}^2 \| \nabla u \|_{L^6}^2 + \frac{\varepsilon}{2}\| \nabla\Delta u \|_{L^2}^2 .\end{aligned}$$

The assertion follows from integration over times $0 \le s < \infty$ and applying (iii) of Lemma 2.1.

(iv) This estimate follows directly from (2.11). ▯

We conclude this section by citing the following result of [3, 15] on a low bound estimate of the spectrum of the linearized Cahn-Hilliard operator $\mathcal{L}_{CH}$ in (1.14). The estimate plays an important role in our error analysis.

PROPOSITION 2.3. *Suppose that (GA_1) holds. Then there exists a positive constant C_0 such that the principle eigenvalue of the linearized Cahn-Hilliard operator $\mathcal{L}_{CH}$ in (1.14) satisfies for small $\varepsilon > 0$*

$$\lambda_{CH} \equiv \inf_{0\not\equiv\psi\in H^1(\Omega)} \frac{\varepsilon \| \nabla\psi \|_{L^2}^2 + \frac{1}{\varepsilon}\left(f'(u)\psi, \psi\right)}{\| \Delta^{-\frac{1}{2}}\psi \|_{L^2}^2} \ge -C_0 ,$$

or equivalently

$$\lambda_{CH} \equiv \inf_{\substack{0\not\equiv\psi\in H^1(\Omega)\\ \Delta w=\psi}} \frac{\varepsilon \| \nabla\psi \|_{L^2}^2 + \frac{1}{\varepsilon}\left(f'(u)\psi, \psi\right)}{\| \nabla w \|_{L^2}^2} \ge -C_0 .$$

3. Error analysis for a semi-discrete (in time) approximation. We start this section with a weak formulation of (1.11)-(1.14): Find $(u(t), w(t)) \in [H^1(\Omega)]^2$ such that for almost every $t \in (0, T)$

$$(u_t, \eta) + (\nabla w, \nabla\eta) = 0 \qquad \forall\, \eta \in H^1(\Omega)\,, \tag{3.1}$$

$$\varepsilon\,(\nabla u, \nabla v) + \frac{1}{\varepsilon}\left(f(u), v\right) = (w, v) \quad \forall\, v \in H^1(\Omega)\,, \tag{3.2}$$

$$u(x, 0) = u_0(x) \quad \forall\, x \in \Omega\,. \tag{3.3}$$

Note that $(u_t, 1) = 0$, that is, the mass $\big(u(t), 1\big) = \big(u_0^\varepsilon, 1\big)$ is conserved for all $t \ge 0$.

A semi-discrete mixed formulation via implicit Euler method on the time mesh $J_k^1 := \{t_m\}_{m=0}^M$ reads: Find $\big\{(u^m, w^m)\big\}_{m=1}^M \in [H^1(\Omega)]^2$ such that for every $0 \le m \le M$

$$(d_t u^{m+1}, \eta) + (\nabla w^{m+1}, \nabla\eta) = 0 \qquad \forall\, \eta \in H^1(\Omega)\,, \tag{3.4}$$

$$\varepsilon\,(\nabla u^{m+1}, \nabla v) + \frac{1}{\varepsilon}\left(f(u^{m+1}), v\right) = (w^{m+1}, v) \quad \forall\, v \in H^1(\Omega)\,, \tag{3.5}$$

with $u^0 = u_0$. Here $J_k^1 := \{t_m\}_{m=0}^M$ is a quasi-uniform partition of $[0, T]$ of mesh size $k := \frac{T}{M}$. Also, $d_t u^{m+1} := (u^{m+1} - u^m)/k$.

It turns out from the subsequent analysis that this scheme on the time mesh J_k^1 only performs sub-optimal (see Theorem 3.4) for general regularities (see Lemma 2.1) and quasi-optimal (see Corollary 3.5) under additional assumptions on regularity of the problem (see Lemma 2.2). The reason for the sub-optimal convergence in the case of general regularities is the lack of an estimate for $\Delta^{-1}u_{tt}$ in $L^2(J,H^{-1}(\Omega))$.

In order to construct an optimally convergent time discretization scheme for (1.11)-(1.12) in the case $u_0 \in H^2(\Omega)$, we suggest to compute iterates u^{m+1} of (3.4)-(3.5) on a stretched mesh $J_k^2 := \{t_m\}_{m=0}^M$, of local mesh sizes

$$k_{m+1} \equiv \begin{cases} (m+1)k_0^2, & \text{for } 0 \leq t_{m+1} \leq \hat{t}_0\,, \\ \gamma k_0, & \text{for } t_{m+1} \geq \hat{t}_0\,, \end{cases} \tag{3.6}$$

with the basic mesh size k_0 and some positive constants γ, and $\hat{t}_0 = O(1)$; see (Chapter 10 of [33]). Obviously, this grid structure is very fine near the origin, with increasing mesh size at increasing times, and requires $O(k_0^{-1})$ iteration steps to overcome the critical time interval $[0,\hat{t}_0]$. It will be proved in Theorem 3.6 that the benefit of using the stretched mesh J_k^2 is that it results in quasi-optimal error bounds.

For equidistant meshes, the scheme (3.4)-(3.5) has been used mostly in the literature (see [25, 19] and the reference therein). However, a verification of an estimate that corresponds to (ii) of Lemma 2.1 for the semi-discrete solution is not immediate, since $(\mathrm{GA}_1)_1$ has no evident discrete analogy. The necessity of this result will be clear in the subsequent error analysis, for that we make the final general assumption on f which applies to both meshes J_k^1 and J_k^2:

General Assumption 3 (GA_3)

Suppose that there exists $\alpha_0 \geq 0$, $0 < \gamma_3 < 1$, and $\tilde{c}_4 > 0$ such that f satisfies for any $0 < k_m \leq \varepsilon^{\alpha_0}$ and any set of discrete (in time) functions $\{\phi^m\}_{m=0}^M \in H^1(\Omega)$

$$\begin{aligned} \gamma_3 \sum_{m=1}^{\ell} k_m \Big(\| \, d_t\phi^m \, \|_{H^{-1}}^2 + k_m\varepsilon \| \, \nabla d_t \phi^m \, \|_{L^2}^2 \Big) \qquad\qquad\qquad\qquad \qquad (3.7)\\ + \frac{1}{\varepsilon} \sum_{m=1}^{\ell} k_m \big(f(\phi^m), d_t\phi^m \big) + \tilde{c}_4 \mathcal{J}_\varepsilon(\phi^0) \geq \frac{\tilde{c}_4}{\varepsilon} \| \, F(\phi^\ell) \, \|_{L^1} \quad \forall\, \ell \leq M. \end{aligned}$$

A direct consequence of (3.7) are the following stability estimates for the scheme (3.4)-(3.5) to be valid on both meshes J_k^1 and J_k^2. Moreover, additional estimates in strong norms are shown for the stretched mesh, which indicates its stabilizing effect.

Lemma 3.1. *For $k_m \leq \varepsilon^{\alpha_0}$ and $u_0 \in H^2(\Omega)$, the solution of the scheme (3.4)-(3.5) satisfies the following estimates on both meshes J_k^1 and J_k^2.*

$$\text{(i)} \quad \frac{1}{|\Omega|} \int_\Omega u^m \,\mathrm{d}x = m_0 \in (-1,1), \quad \forall\, m \geq 0\,,$$

$$\begin{aligned} \text{(ii)} \quad & \max_{0 \leq m \leq M} \Big\{ \varepsilon \, \| \, \nabla u^m \, \|_{L^2}^2 + \frac{1}{\varepsilon} \, \| \, F(u^m) \, \|_{L^1} \Big\} \\ & + \sum_{m=1}^{M} k_m \Big\{ \| \, \nabla w^m \, \|_{L^2}^2 + \| \, d_t u^m \, \|_{H^{-1}}^2 + \varepsilon k_m \, \| \, \nabla d_t u^m \, \|_{L^2}^2 \Big\} \leq 2\tilde{c}_4 \, \mathcal{J}_\varepsilon(u^0)\,, \end{aligned}$$

$$\text{(iii)} \quad \sum_{m=0}^{M} k_m \| \, \Delta u^m \, \|_{L^2}^2 \leq C\varepsilon^{-(2\sigma_1+3)}\,.$$

Moreover, there hold for J_k^1

$$\text{(iv)} \quad \max_{1\le m\le M} \|\, \Delta^{-1} d_t u^m \,\|_{L^2}^2 + \varepsilon k \sum_{m=1}^{M} \|\, d_t u^m \,\|_{L^2}^2 \le C\, \varepsilon^{-\max\{2\sigma_1(p-1)+p+1, 2\sigma_2\}}\,,$$

$$\text{(v)} \quad \max_{0\le m\le M} \|\, \Delta u^m \,\|_{L^2}^2 \le C\, \rho_1(\varepsilon)\,,$$

and for J_k^2

$$\begin{aligned}\text{(vi)} \quad & \max_{1\le m\le M} \Big\{ \|\, \tilde d_t u^m \,\|_{H^{-1}}^2 + \varepsilon \,\|\, \nabla u^m \,\|_{L^2}^2 \Big\} + \frac{k_0^2}{2} \sum_{m=1}^{M} \Big\{ \|\, \tilde d_t^2 u^m \,\|_{H^{-1}}^2 \\ & \qquad + \frac{\varepsilon k_0^2}{2} \|\, \nabla \tilde d_t u^m \,\|_{L^2}^2 \Big\} + \frac{\varepsilon}{2} \sum_{m=1}^{M} k_m \,\|\, \nabla \tilde d_t u^m \,\|_{L^2}^2 \\ & \qquad\qquad \le C \Big\{ \varepsilon^{-(2\sigma_1+3)} + \ln\Big(\frac{1}{k_0}\Big) \varepsilon^{-\{2\sigma_1(p-1)+p+\frac{3}{2}\}} \Big\}\,,\end{aligned}$$

$$\begin{aligned}\text{(vii)} \quad & \max_{1\le m\le M} \Big\{ \|\, \tilde d_t u^m \,\|_{L^2}^2 + \varepsilon \,\|\, \Delta u^m \,\|_{L^2}^2 \Big\} + \frac{k_0^2}{2} \sum_{m=1}^{M} \Big\{ \|\, \tilde d_t^2 u^m \,\|_{L^2}^2 + \varepsilon \,\|\, \Delta \tilde d_t u^m \,\|_{L^2}^2 \Big\} \\ & \qquad + \frac{\varepsilon}{2} \sum_{m=1}^{M} k_m \,\|\, \Delta \tilde d_t u^m \,\|_{L^2}^2 \le C\varepsilon\, \tilde\rho_1(\varepsilon)\,,\end{aligned}$$

where $\tilde d_t \varphi^{m+1} := \frac{1}{k_0} \left\{ \varphi^{m+1} - \varphi^m \right\}$*, and*

$$\begin{aligned}\tilde\rho_1(\varepsilon) = \max\Big\{ & \ln\Big(\frac{1}{k_0}\Big)\, \varepsilon^{-((2\sigma_1+1)(p-2)+4)} \Big\{ \varepsilon^{-(2\sigma_1+3)} + \ln\Big(\frac{1}{k_0}\Big)\, \varepsilon^{-(2\sigma_1(p-1)+p+\frac{3}{2})} \Big\} \\ & + C\Big\{ \varepsilon^{-2((\sigma_1+\sigma_2)+1)} + \varepsilon^{-(2\sigma_1(p-2)+2\sigma_1+p+1)} \Big\}, \rho_1(\varepsilon) \Big\}\,.\end{aligned}$$

In addition, under the assumptions of Lemma 2.2, there also holds for the mesh J_k^1

$$\text{(viii)} \quad \max_{1\le m\le M} \|\, d_t u^m \,\|_{H^{-1}}^2 + \varepsilon\, k \sum_{m=1}^{M} \|\, \nabla d_t u^m \,\|_{L^2}^2 \le C\, \varepsilon^{-\max\{2\sigma_1+3, 2\sigma_3\}}\,, \tag{3.8}$$

$$\text{(ix)} \quad \max_{0\le m\le M} \|\, \nabla u^m \,\|_{L^2}^2 + \varepsilon\, k \sum_{m=0}^{M} \|\, u^m \,\|_{H^3}^2 \le C\, \varepsilon^{-\{2\sigma_1(p-1)+p+4\}}\,. \tag{3.9}$$

Proof. The proof of (i) is trivial, setting $\eta = 1$ in (3.4). To see (ii), we choose $(\eta, v) = (w^{m+1}, d_t u^{m+1})$ in (3.4)-(3.5). The assertion then follows from (GA_3) and the inequality $\|\, d_t u^{m+1} \,\|_{H^{-1}} \le \|\, \nabla w^{m+1} \,\|_{L^2}$.

The proof of (iii) is similar to that of (iii) in Lemma 2.1. We choose $(\eta, v) = (u^{m+1}, -\Delta u^{m+1})$ in (3.4)-(3.5) and arrive at

$$\begin{aligned}\frac{1}{2}\, d_t \|\, u^{m+1} \,\|_{L^2}^2 + \frac{k_{m+1}}{2} \|\, d_t u^{m+1} \,\|_{L^2}^2 + \frac{\varepsilon}{2} \|\, \Delta u^{m+1} \,\|_{L^2}^2 &= -\frac{1}{\varepsilon} \left(f'(u^{m+1}), |\, \nabla u^{m+1} \,|^2 \right) \\ &\le \frac{\tilde c_0}{\varepsilon} \|\, \nabla u^{m+1} \,\|_{L^2}^2\,.\end{aligned}$$

The assertion then follows from (ii).

To show assertion (iv), we first apply the difference operator d_t to (3.4)-(3.5),

$$(d_t^2 u^{m+1},\eta) + \varepsilon\,(\nabla d_t w^{m+1},\nabla\eta) = 0 \quad \forall\,\eta\in H^1(\Omega)\,, \tag{3.10}$$

$$\varepsilon(\nabla d_t u^{m+1},\nabla v) + \frac{1}{\varepsilon}\,(d_t f(u^{m+1}),v) = (d_t w^{m+1},v) \quad \forall\,v\in H^1(\Omega)\,. \tag{3.11}$$

For $(\eta,v) = (\Delta^{-2}d_t u^{m+1}, -\Delta^{-1}d_t w^{m+1})$, using the Mean Value Theorem on $d_t f(u^{m+1})$ leads to

$$\begin{aligned}
&\frac{1}{2}\,d_t\|\,\Delta^{-1}d_t u^{m+1}\,\|_{L^2}^2 + \frac{k}{2}\,\|\,\Delta^{-1}d_t^2 u^{m+1}\,\|_{L^2}^2 + \varepsilon\,\|\,d_t u^{m+1}\,\|_{L^2}^2 \\
&\quad = -\frac{2}{\varepsilon}\,(f'(\xi)d_t u^{m+1},\Delta^{-1}d_t u^{m+1}) \\
&\quad \le \frac{\varepsilon}{2}\,\|\,d_t u^{m+1}\,\|_{L^2}^2 + \frac{C}{\varepsilon^3}\,\|\,f'(\xi)\,\|_{L^3}^2\,\|\,\Delta^{-1}d_t u^{m+1}\,\|_{L^6}^2\,.
\end{aligned} \tag{3.12}$$

Here, ξ is a value between u^m and u^{m+1}. For the following step, we introduce $u^{-1}\in H^1(\Omega)$ such that $\frac{1}{|\,\Omega\,|}\int_\Omega u^{-1}\,\mathrm{d}x = m_0$, as the solution of

$$(\Delta^{-1}d_t u^0,\varphi) = \big(-\varepsilon\,\Delta u^0 + \frac{1}{\varepsilon}\,f(u^0),\varphi\big)\,, \tag{3.13}$$

for all $\varphi\in\big\{\chi\in H^1(\Omega);\ (\varphi,1)=0\big\}$.

Then, summation over $0\le m\le M$, and Cauchy's inequality, together with (ii) imply the result.

(v) We test (3.4)-(3.5) by $(-\Delta^{-1}u^{m+1},u^{m+1})$,

$$\varepsilon\,\|\,\Delta u^{m+1}\,\|_{L^2}^2 \le \frac{1}{\varepsilon}\,\|\,\Delta^{-1}d_t u^{m+1}\,\|_{L^2}^2 - \frac{2}{\varepsilon}\,\big(f'(u^{m+1}),|\,\nabla u^{m+1}\,|^2\big)\,,$$

the assertion follows from (2.1), (ii) and (iv).

Finally, the estimates (viii)-(ix) can be shown using similar arguments to those in the proof of (ii)-(iii) of Lemma 2.2 with some straightforward modifications.

We now analyze solutions $\big\{u^m\big\}_{m=0}^M$ which are obtained from the mesh J_k^2. The estimates (i)-(iii) remain valid for the mesh J_k^2. Instead of (iv) and (v), we find the stronger results (vi) and (vii). Rewrite (3.4) as

$$\tilde d_t u^{m+1} - (m+1)k_0\Delta w^{m+1} = 0\,. \tag{3.14}$$

(vi) We apply $\tilde d_t$ to this equation and find

$$\tilde d_t^2 u^{m+1} - mk_0\Delta\tilde d_t w^{m+1} - \Delta w^{m+1} = 0\,.$$

We test the above equation with $-\Delta^{-1}\tilde d_t u^{m+1}$ and find the counterpart of (3.12)

$$\begin{aligned}
&\frac{1}{2}\,\tilde d_t\|\,\Delta^{-\frac12}\tilde d_t u^{m+1}\,\|_{L^2}^2 + \frac{k_0}{2}\,\|\,\Delta^{-\frac12}\tilde d_t^2 u^{m+1}\,\|_{L^2}^2 + \varepsilon mk_0\,\|\,\nabla\tilde d_t u^{m+1}\,\|_{L^2}^2 \\
&\quad + \frac{\varepsilon}{2}\,\tilde d_t\|\,\nabla u^{m+1}\,\|_{L^2}^2 + \frac{\varepsilon k_0}{2}\,\|\,\nabla\tilde d_t u^{m+1}\,\|_{L^2}^2 \\
&\quad = -\frac{mk_0}{\varepsilon}\,\big(f'(\xi),|\,\tilde d_t u^{m+1}\,|^2\big) + \frac{1}{\varepsilon}\,\big(f'(u^{m+1})\nabla u^{m+1},\nabla\Delta^{-1}\tilde d_t u^{m+1}\big)\,.
\end{aligned} \tag{3.15}$$

Here, ξ is a value between u^m and u^{m+1}. Using (2.1), the last line can be bounded by

$$\begin{aligned}\frac{\tilde{c}_0 m k_0}{\varepsilon} \| \tilde{d}_t u^{m+1} \|_{L^2}^2 + \frac{C}{\varepsilon} \| f'(u^{m+1}) \|_{L^3} \| \nabla u^{m+1} \|_{L^2} \| \nabla \Delta^{-1} \tilde{d}_t u^{m+1} \|_{L^6} \\ \leq \frac{\varepsilon m k_0}{4} \| \nabla \tilde{d}_t u^{m+1} \|_{L^2}^2 + \frac{C m k_0}{\varepsilon^3} \| \nabla \Delta^{-1} \tilde{d}_t u^{m+1} \|_{L^2}^2 \\ + \frac{1}{\varepsilon} \| f'(u^{m+1}) \|_{L^3} \| \nabla u^{m+1} \|_{L^2} \| \nabla \Delta^{-1} \tilde{d}_t u^{m+1} \|_{L^6} .\end{aligned}$$

Let $\delta > 0$. Using (2.8) the last contribution is bounded by

$$\begin{aligned}\frac{C\delta}{\varepsilon^{5/2}} \Big(\tilde{c}_2 \| u^{m+1} \|_{L^{3(p-2)}}^{2(p-2)} + \tilde{c}_3 \Big) \| \nabla u^{m+1} \|_{L^2}^2 + \frac{\sqrt{\varepsilon}}{\delta} \| \tilde{d}_t u^{m+1} \|_{L^2}^2 \\ \leq C\delta\, \varepsilon^{-\{(2\sigma_1+1)(p-2)+2\sigma_1+\frac{7}{2}\}} + \frac{\varepsilon m k_0}{4} \| \nabla \tilde{d}_t u^{m+1} \|_{L^2}^2 + \frac{C}{k_0 m \delta^2} \| \nabla \Delta^{-1} \tilde{d}_t u^{m+1} \|_{L^2}^2 .\end{aligned}$$

We insert this into (3.15) and multiply by k_0, finally sum over m from 1 to ℓ ($\leq M$). Note that $| \sum_{m=1}^M \frac{1}{m} - \ln \frac{1}{M} | < 1$, and

$$\begin{aligned}\sum_{m=1}^M k_m \| \Delta^{-\frac{1}{2}} \tilde{d}_t u^m \|_{L^2}^2 &= \sum_{m=1}^M k_m \big(k_0 m \big)^2 \| \Delta^{-\frac{1}{2}} d_t u^m \|_{L^2}^2 \\ &\leq \sum_{m=1}^M k_m \| \Delta^{-\frac{1}{2}} d_t u^m \|_{L^2}^2 .\end{aligned}$$

From (i) and discrete Gronwall's inequality, we find for the choice $\delta^2 = \ln\big(\frac{1}{k_0}\big)$,

$$\begin{aligned}\frac{1}{2} \| \Delta^{-\frac{1}{2}} \tilde{d}_t u^\ell \|_{L^2}^2 + \frac{k_0^2}{2} \sum_{m=1}^M \| \Delta^{-\frac{1}{2}} \tilde{d}_t^2 u^m \|_{L^2}^2 + \frac{\varepsilon}{4} \sum_{m=1}^M k_m \| \nabla \tilde{d}_t u^m \|_{L^2}^2 \\ + \frac{\varepsilon}{2} \| \nabla u^\ell \|_{L^2}^2 + \frac{\varepsilon k_0^2}{2} \sum_{m=1}^M \| \nabla \tilde{d}_t u^{m+1} \|_{L^2}^2 \qquad (3.16) \\ \leq C \Big\{ \varepsilon^{-(2\sigma_1+3)} + \ln\Big(\frac{1}{k_0}\Big) \varepsilon^{-\{2\sigma_1(p-1)+p+\frac{3}{2}\}} \Big\} + \frac{1}{2} \Big\{ \| \Delta^{-\frac{1}{2}} \tilde{d}_t u^1 \|_{L^2}^2 + \varepsilon \| \nabla u^1 \|_{L^2}^2 \Big\} .\end{aligned}$$

By (ii), the last two terms can be bounded by $\varepsilon^{-2\sigma_1}$. We benefit at this point again from the scaling of the stretched mesh J_k^2,

$$\| \Delta^{-\frac{1}{2}} \frac{u^1 - u^0}{k_0} \|_{L^2}^2 \leq k_0^2 \| \Delta^{-\frac{1}{2}} d_t u^1 \|_{L^2}^2 \leq C\, \varepsilon^{-2\sigma_1} .$$

(vii) We test (3.14) with $\tilde{d}_t u^{m+1}$. In the sequel, ξ is a value between u^m and u^{m+1}.

$$\begin{aligned}\frac{1}{2} \tilde{d}_t \| \tilde{d}_t u^{m+1} \|_{L^2}^2 + \frac{k_0}{2} \| \tilde{d}_t^2 u^{m+1} \|_{L^2}^2 + \varepsilon m k_0 \| \Delta \tilde{d}_t u^{m+1} \|_{L^2}^2 \qquad (3.17) \\ + \frac{\varepsilon}{2} \tilde{d}_t \| \Delta u^{m+1} \|_{L^2}^2 + \frac{\varepsilon k_0}{2} \| \Delta \tilde{d}_t u^{m+1} \|_{L^2}^2 \\ = \frac{m k_0}{\varepsilon} \big(f'(\xi) \tilde{d}_t u^{m+1}, \Delta \tilde{d}_t u^{m+1} \big) - \frac{1}{\varepsilon} \big(f'(u^{m+1}) \nabla u^{m+1}, \nabla \tilde{d}_t u^{m+1} \big) .\end{aligned}$$

We multiply by k_0, sum over m from 1 to M, use the upper bound from (2.8) and the estimate

$$
\begin{aligned}
&\frac{1}{\varepsilon}\,\|\, f'(u^m)\,\|_{L^3}\,\|\,\nabla u^m\,\|_{L^2}\,\|\,\nabla \tilde{d}_t u^m\,\|_{L^6}\\
&\quad\le \frac{C}{\varepsilon^3 m k_0}\,\|\, f'(u^m)\,\|_{L^3}^2\,\|\,\nabla u^m\,\|_{L^2}^2 + \frac{\varepsilon m k_0}{2}\,\|\,\Delta \tilde{d}_t u^m\,\|_{L^2}^2\\
&\quad\le \frac{C}{\varepsilon^3 m k_0}\Big(\tilde{c}_2\,\|\, u^m\,\|_{L^{3(p-2)}}^{2(p-2)} + \tilde{c}_3\Big)\|\,\nabla u^m\,\|_{L^2}^2 + \frac{\varepsilon m k_0}{2}\,\|\,\Delta \tilde{d}_t u^m\,\|_{L^2}^2\\
&\quad\le \frac{C}{m k_0}\,\varepsilon^{-(2\sigma_1+1)(p-2)-3} + \frac{\varepsilon m k_0}{2}\,\|\,\Delta \tilde{d}_t u^m\,\|_{L^2}^2
\end{aligned}
$$

to get from (3.17)

$$
\begin{aligned}
&\max_{2\le m\le M}\frac{1}{2}\Big\{\|\,\tilde{d}_t u^m\,\|_{L^2}^2 + \varepsilon\,\|\,\Delta u^m\,\|_{L^2}^2\Big\} + \frac{k_0^2}{2}\sum_{m=2}^{M}\Big\{\|\,\tilde{d}_t^2 u^m\,\|_{L^2}^2\\
&\qquad\qquad + \frac{\varepsilon}{2}\,\|\,\Delta \tilde{d}_t u^m\,\|_{L^2}^2\Big\} + \frac{\varepsilon}{2}\sum_{m=2}^{M} k_m\,\|\,\Delta \tilde{d}_t u^m\,\|_{L^2}^2\\
&\quad\le \frac{1}{2}\Big\{\|\,\tilde{d}_t u^1\,\|_{L^2}^2 + \varepsilon\,\|\,\Delta u^1\,\|_{L^2}^2\Big\} + C\,\varepsilon^{-\{2\sigma_1(p-2)+p+1\}}\ln\Big(\frac{1}{k_0}\Big)\\
&\qquad\qquad + \frac{C}{\varepsilon^3}\sum_{m=2}^{M} k_m\,\|\, f'(\xi)\,\|_{L^3}^2\,\|\,\tilde{d}_t u^m\,\|_{L^6}^2\,.
\end{aligned}
\tag{3.18}
$$

The logarithmic term comes again from the bound $|\sum_{m=1}^{M}\frac{1}{m} - \ln\frac{1}{M}| < 1$. The last term in (3.18) is estimated by

$$
\begin{aligned}
&\frac{C}{\varepsilon^3}\sum_{m=1}^{M} k_m\Big(\tilde{c}_2\,\|\, u^m\,\|_{L^{3(p-2)}}^{2(p-2)} + \tilde{c}_3\Big)\|\,\nabla \tilde{d}_t u^m\,\|_{L^2}^2\\
&\quad\le C\ln\Big(\frac{1}{k_0}\Big)\,\varepsilon^{-((2\sigma_1+1)(p-2)+3)}\Big\{\varepsilon^{-(2\sigma_1+3)} + \ln\Big(\frac{1}{k_0}\Big)\,\varepsilon^{-(2\sigma_1(p-1)+p+\frac{3}{2})}\Big\}\,.
\end{aligned}
\tag{3.19}
$$

The last inequality follows from an elementary calculation.

The first two terms on the right hand side of (3.18) are bounded because of the structure of J_k^2. Therefore, we come back to (3.14), taking $m = 0$ and testing the equation with $\tilde{d}_t u^1$ lead to

$$
\begin{aligned}
\|\,\tilde{d}_t u^1\,\|_{L^2}^2 + \frac{\varepsilon k_0}{2}\,\tilde{d}_t\|\,\Delta u^1\,\|_{L^2}^2 &= -\frac{k_0}{\varepsilon}\,\big(f'(u^1)\nabla u^1, \nabla \tilde{d}_t u^1\big)\\
&\le \frac{\tilde{c}_0}{\varepsilon}\,\|\,\nabla u^1\,\|_{L^2}^2 + \frac{1}{\varepsilon}\,\|\, f'(u^1)\,\|_{L^3}\,\|\,\nabla u^1\,\|_{L^2}\,\|\,\nabla u^0\,\|_{L^6}\,.
\end{aligned}
$$

Similar to (2.8) and (2.9), using (GA$_2$) the above inequality is continued by

$$
\begin{aligned}
&\frac{2\tilde{c}_0}{\varepsilon}\,\|\,\nabla u^1\,\|_{L^2}^2 + \frac{1}{4\tilde{c}_0\varepsilon}\,\|\, f'(u^1)\,\|_{L^3}^2\,\|\,\nabla u^0\,\|_{L^6}^2\\
&\quad\le \frac{2\tilde{c}_0}{\varepsilon}\,\|\,\nabla u^1\,\|_{L^2}^2 + \frac{1}{4\tilde{c}_0\varepsilon}\Big(\|\, u^1\,\|_{L^{3(p-2)}}^{2(p-2)} + \tilde{c}_3\Big)\|\,\Delta u^0\,\|_{L^2}^2\\
&\quad\le C\Big\{\varepsilon^{-2(\sigma_1+1)} + \varepsilon^{-(2\sigma_1(p-2)+p-1)}\Big\}\Big\{\varepsilon^{-(2\sigma_1+2)} + \varepsilon^{-(2\sigma_1+1)}\Big\}\,.
\end{aligned}
\tag{3.20}
$$

Using (3.20) and (3.19), we find the following upper bound for (3.18)

$$C \ln\Big(\frac{1}{k_0}\Big)\, \varepsilon^{-((2\sigma_1+1)(p-2)+3)} \Big\{ \varepsilon^{-(2\sigma_1+3)} + \ln\Big(\frac{1}{k_0}\Big)\, \varepsilon^{-(2\sigma_1(p-1)+p+\frac{3}{2})} \Big\}$$
$$+ C\Big\{ \varepsilon^{-2((\sigma_1+\sigma_2)+1)} + \varepsilon^{-(2\sigma_1(p-2)+2\sigma_1+p+1)} \Big\}\,.$$

This concludes the proof. ☐

3.1. Verification of (GA$_3$) for the case $f(u) = u^3 - u$**.** For the reader's convenience, we verify (GA$_3$) and show the following estimates for this specific case. The results are valid for both meshes J_k^1 and J_k^2. Note that $k_m := k$ on the mesh J_k^1.

LEMMA 3.2. *For $k \le \varepsilon^3$ (resp. $k_m \le \varepsilon^3$ for J_k^2), the solution $\{(u^m, w^m)\}_{m=0}^M$ of (3.4)-(3.5) satisfies for both meshes J_k^1 and J_k^2*

$$\max_{0\le m\le M} \Big\{ \varepsilon \,\|\, \nabla u^m \,\|_{L^2}^2 + \frac{1}{\varepsilon} \,\|\, F(u^m)\,\|_{L^1} \Big\} + \sum_{m=1}^{M} k_m \Big\{ \frac{3}{4} \,\|\, \Delta^{-\frac{1}{2}} d_t u^m \,\|_{L^2}^2$$
$$+ \frac{k_m}{2}\Big(\varepsilon - \frac{k_m}{2\varepsilon^2}\Big) \|\, \nabla d_t u^m \,\|_{L^2}^2 + \frac{k_m}{2\varepsilon^2} \,\|\, d_t\big(|\, u^m\,|^2 - 1\big)\,\|_{L^2}^2 \Big\} \le \mathcal{J}_\varepsilon(u^0)\,.$$

Proof. Rewrite $f(u^{m+1})$ as follows

$$f(u^{m+1}) = \frac{1}{2}\Big(|\, u^{m+1}\,|^2 - 1\Big)\Big([u^m + u^{m+1}] + k_m d_t u^{m+1}\Big)\,. \tag{3.21}$$

Then

$$\begin{aligned} \frac{1}{2\varepsilon}(f(u^{m+1}), d_t u^{m+1}) &= \frac{1}{2\varepsilon}\Big(|\, u^{m+1}\,|^2 - 1, d_t(|\, u^{m+1}\,|^2 - 1)\Big) \\ &\quad + \frac{k_m}{2\varepsilon}\Big(|\, u^{m+1}\,|^2 - 1, |\, d_t u^{m+1}\,|^2\Big) \\ &\ge \frac{1}{2\varepsilon}\Big(|\, u^{m+1}\,|^2 - 1 \pm \big(|\, u^m\,|^2 - 1\big), d_t\big(|\, u^{m+1}\,|^2 - 1\big)\Big) - \frac{k_m}{2\varepsilon}\|\, d_t u^{m+1}\,\|_{L^2}^2 \\ &\ge \frac{1}{2\varepsilon}\, d_t \|\, |\, u^{m+1}\,|^2 - 1\,\|_{L^2}^2 + \frac{k_m}{2\varepsilon}\,\|\, d_t\big(|\, u^{m+1}\,|^2 - 1\big)\,\|_{L^2}^2 - \frac{k_m}{2\varepsilon}\|\, d_t u^{m+1}\,\|_{L^2}^2\,. \end{aligned} \tag{3.22}$$

The last contribution is decomposed as follows,

$$\begin{aligned} \frac{k_m}{2\varepsilon}\,\|\, d_t u^{m+1}\,\|_{L^2}^2 &\le \frac{k}{2\varepsilon}\,\|\, \nabla d_t u^{m+1}\,\|_{L^2}\,\|\, \Delta^{-\frac{1}{2}} d_t u^{m+1}\,\|_{L^2} \\ &\le \frac{1}{4}\,\|\, \Delta^{-\frac{1}{2}} d_t u^{m+1}\,\|_{L^2}^2 + \frac{k_m^2}{4\varepsilon^2}\,\|\, \nabla d_t u^{m+1}\,\|_{L^2}^2\,. \end{aligned}$$

Multiplying (3.7) by k_m and summing over m from 0 to M we obtain (3.7) with $\alpha_0 = 3, \gamma_3 = \frac{1}{4}$, and $\tilde{c}_4 = 2$.

The proof is completed by testing (3.4) with w^{m+1}, (3.5) with $d_t u^{m+1}$, and applying (3.22). ☐

The above derivation also suggests to consider the following variant scheme of (3.4)-(3.5):

$$(d_t u^{m+1}, \eta) + (\nabla w^{m+1}, \nabla \eta) = 0\,, \tag{3.23}$$

$$\varepsilon\,(\nabla u^{m+1}, \nabla v) + \frac{1}{4\varepsilon}\,\big(\big\{|\, u^m\,|^2 + |\, u^{m+1}\,|^2 - 2\big\}\big\{u^m + u^{m+1}\big\}, v\big) = (w^{m+1}, v) \quad (3{,}24)$$

for all tuple $(\eta, v) \in \big[H^1(\Omega)\big]$. It turns out that this new scheme has better stability properties than the scheme (3.4)-(3.5) does, as shown by the next lemma.

LEMMA 3.3. *The solution $\{u^m\}_{m=0}^M$ of (3.23) satisfies for any $k>0$ ($k_0>0$)*

$$\sum_{m=1}^{M} k_m \Big\{ \| \Delta^{-\frac{1}{2}} d_t u^m \|_{L^2}^2 + \frac{\varepsilon k_m}{2} \| \nabla d_t u^m \|_{L^2}^2 \Big\} + \max_{0\le m\le M} \Big\{ \frac{\varepsilon}{2} \| \nabla u^m \|_{L^2}^2 + \frac{1}{\varepsilon} \| F(u^m) \|_{L^1} \Big\} = \mathcal{J}_\varepsilon(u^0)\,.$$

Proof. After testing (3.23) with w^{m+1}, (3.24) with $d_t u^{m+1}$, the only term which needs special care is

$$\mathcal{A} \equiv \frac{1}{4\varepsilon} \big(\{ | u^m |^2 + | u^{m+1} |^2 - 2 \} \{ u^m + u^{m+1} \}, d_t u^{m+1} \big)$$

appearing on the left hand side of the equation. We apply a binomial formula twice to reformulate this term as

$$\begin{aligned} \mathcal{A} &= \frac{1}{4\varepsilon} \big(\{ | u^m |^2 + | u^{m+1} |^2 - 2 \}, d_t | u^{m+1} |^2 \big) \\ &= \frac{1}{4\varepsilon} \Big(\{ | u^m |^2 + | u^{m+1} |^2 - 2 \}, d_t \big[| u^{m+1} |^2 - 1 \big] \Big) \\ &= \frac{1}{4\varepsilon}\, d_t \| \, | u^{m+1} |^2 - 1 \, \|_{L^2}^2\,. \end{aligned}$$

The proof then is completed by taking summation over m. □

Remark: Note that Lemma 3.3 holds under no constraint onto choices of k. However, despite of this advantage of scheme (3.23)-(3.24), we prefer scheme (3.4)-(3.5) for its simpler structure and generality for different f. On the other hand, it would be interesting to also analyze the scheme (3.23)-(3.24) and compare the two schemes numerically.

3.2. Error estimates for the scheme (3.4)-(3.5). In this subsection, we present the error analysis for (3.4)-(3.5) under the assumptions (GA_1)-(GA_3), starting with the mesh J_k^1. As is shown in Subsection 3.1, the stability result of the time-discrete scheme imposes some constraint on the time step size k. In fact, in order to establish convergence of the discrete scheme (3.4)-(3.5), this constraint needs to be strengthened according to the following convergence theorem, which is the first of two main theorems in this subsection.

THEOREM 3.4. *Let $\{(u^m, w^m)\}_{m=0}^M$ solve (3.4)-(3.5) on an equidistant mesh $J_k^1 = \{t_m\}_{m=0}^M$ of mesh size $O(k)$, and $u_0 \in H^2(\Omega)$. Suppose (GA_1)-(GA_3) hold, and $1 < \delta < \frac{8}{4-N}$. Let $\rho_1(\varepsilon)$ and $\rho_2(\varepsilon)$ be same as in Lemma 2.1, and*

$$\begin{aligned} \rho_3(\varepsilon) &= \rho_2(\varepsilon) \big[\rho_1(\varepsilon)\big]^{-\frac{2N\delta}{8-(4-N)\delta}} \varepsilon^{\frac{(2\sigma_1+1)(4-N)\delta}{8-(4-N)\delta} + 2(\sigma_1+2)}, \\ \rho_4(N,\beta) &= \Big(1 + \beta + \frac{4(4-N)\delta}{8-(4-N)\delta} \Big)^{-1}, \\ \rho_5(\varepsilon, N) &= \big[\rho_1(\varepsilon)\big]^{-2N} \big[\rho_2(\varepsilon)\big]^{N-4}, \\ \rho_6(N,\beta) &= \delta \Big[(4-N)\delta - \beta(8 + 2(4-N)\delta \Big]^{-1}. \end{aligned}$$

For fixed positive values $0 < \beta < \frac{1}{2}$, let k satisfy the following constraint

$$k \leq \tilde{C}\, \min\Big\{\varepsilon^{\frac{3}{\beta}}, \varepsilon^{\alpha_0}, \big[\rho_3(\varepsilon, N)\big]^{\rho_4(N,\beta)}, \big[\rho_5(\varepsilon, N)\big]^{\rho_6(N,\beta)}\Big\}. \tag{3.25}$$

Then there exists a positive constant $\tilde{C} = \tilde{C}(u_0; \gamma_1, \gamma_2, C_0, T, \Omega)$ such that the solution of (3.4)-(3.5) satisfies the following error estimate

$$\max_{0\leq m\leq M} \| u(t_m) - u^m \|_{H^{-1}} + \Big(k \sum_{m=1}^{M} \Big\{k \| d_t(u(t_m) - u^m) \|^2_{H^{-1}} + k^\beta \| \nabla(u(t_m) - u^m) \|^2_{L^2}\Big\}\Big)^{\frac{1}{2}} \leq \tilde{C}\, k^{\frac{1-\beta}{2}} \big[\rho_2(\varepsilon)\big]^{\frac{1}{2}}.$$

Proof. The proof is split into four steps: the first step deals with consistency error and shows the relevancy of the condition $(\mathrm{GA}_1)_3$ imposed on f. Steps two and three use Proposition 2.3 and stability properties of the implicit Euler-method to avoid exponential blow-up in ε^{-1} of the error constant. In the last step, an inductive argument is used to handle the difficult caused by the super-quadratic term in $(\mathrm{GA}_1)_3$.

Step 1: Let $e^m := u(t_m) - u^m \in L^2_0(\Omega)$ and $g^m := w(t_m) - w^m$ denote the error functions. Subtracting (3.4)-(3.5) from (3.1)-(3.2) respectively, we obtain the error equations

$$(d_t e^{m+1}, \eta) + (\nabla g^{m+1}, \nabla \eta) = (\mathcal{R}(u_{tt}; m), \eta)\,, \tag{3.26}$$

$$\varepsilon(\nabla e^{m+1}, \nabla v) + \frac{1}{\varepsilon}\big(f(u(t_m)) - f(u^{m+1}), v\big) = (g^{m+1}, v)\,, \tag{3.27}$$

which are valid for all $(\eta, v) \in \big[H^1(\Omega)\big]^2$, and

$$\mathcal{R}(u_{tt}; m) = -\frac{1}{k}\int_{t_{m-1}}^{t_m} (s - t_{m-1})\, u_{tt}(s)\, \mathrm{d}s\,. \tag{3.28}$$

We choose $(\eta, v) = (-\Delta^{-1} e^{m+1}, e^{m+1})$ and find

$$\begin{aligned}\frac{1}{2}\, d_t\| \Delta^{-\frac{1}{2}} e^{m+1} \|^2_{L^2} + \frac{k}{2} \| \Delta^{-\frac{1}{2}} d_t e^{m+1} \|^2_{L^2} + \varepsilon \| \nabla e^{m+1} \|^2_{L^2} \\ + \frac{1}{\varepsilon}\big(f(u(t_{m+1})) - f(u^{m+1}), e^{m+1}\big) = \big(-\Delta^{-1}\mathcal{R}(u_{tt}; m), e^{m+1}\big).\end{aligned} \tag{3.29}$$

From (ix) of Lemma 2.1,

$$\begin{aligned}&k \sum_{m=0}^{M} \| \Delta^{-1}\mathcal{R}(u_{tt}; m) \|^2_{H^{-1}} \\ &\leq \frac{1}{k}\sum_{m=0}^{M}\Big[\int_{t_m}^{t_{m+1}} \frac{(s - t_m)^2}{\tau(s)}\, \mathrm{d}s\Big]\Big[\int_{t_m}^{t_{m+1}} \tau(s) \| \Delta^{-1} u_{tt}(s) \|^2_{H^{-1}}\, \mathrm{d}s\Big] \\ &\leq C\, k\, \rho_2(\varepsilon)\,.\end{aligned} \tag{3.30}$$

To control the last term on the left hand side of (3.29), we use $(\mathrm{GA}_1)_3$,

$$\begin{aligned}&\frac{1}{\varepsilon}\Big(f(u(t_{m+1})) - f(u^{m+1}), e^{m+1}\Big) \\ &\qquad \geq \frac{\gamma_1}{\varepsilon}\Big(f'\big(u(t_{m+1})\big)e^{m+1}, e^{m+1}\Big) - \frac{\gamma_2}{\varepsilon}\, \| e^{m+1}\|^{2+\delta}_{L^{2+\delta}}\,.\end{aligned} \tag{3.31}$$

Step 2: We want to use the following spectrum estimate result (see Proposition 2.3) to bound from below the first term on the right hand side of (3.31)

$$\varepsilon \| \nabla\phi \|_{L^2}^2 + \frac{1}{\varepsilon}\left(f'(u)\phi,\phi\right) \ge -C_0 \| \Delta^{-\frac12}\phi \|_{L^2}^2 , \quad \forall\, \phi \in H^1(\Omega)\,, \tag{3.32}$$

where $C_0 > 0$ is independent of ε. At the same time, we want to make use of the $H^{-1}(\Omega)$ norm of $-\Delta^{-1}\mathcal{R}(u_{tt}; m)$ in order to keep the power of $\frac{1}{\varepsilon}$ as low as possible in the error constant. The latter requires to keep portions of $\| \nabla e^{m+1} \|_{L^2}^2$ on the left hand side of the error equation (3.29). To this end, we apply (3.32) with a scaling factor $\gamma_1(1-\frac{k^\beta}{2})$, which together with (3.31) and (3.29) gives

$$\begin{aligned}
&\frac12 d_t\| \Delta^{-\frac12} e^{m+1} \|_{L^2}^2 + \frac{k}{2}\| \Delta^{-\frac12} d_t e^{m+1} \|_{L^2}^2 + \frac{\varepsilon}{2}\big[1-\gamma_1(1-\frac{k^\beta}{2})\big] \| \nabla e^{m+1} \|_{L^2}^2 \\
&\quad\le C_0\gamma_1(1-\frac{k^\beta}{2}) \| \Delta^{-\frac12} e^{m+1} \|_{L^2}^2 - \frac{\gamma_1 k^\beta}{2\varepsilon}\left(f'\big(u(t_{m+1})\big)e^{m+1}, e^{m+1}\right) \\
&\qquad + \frac{C}{\varepsilon}\| e^{m+1} \|_{L^{2+\delta}}^{2+\delta} + C\,\frac{k^{-\beta}}{\varepsilon}\| \Delta^{-\frac12}\mathcal{R}(u_{tt}; m) \|_{H^{-1}}^2 .
\end{aligned} \tag{3.33}$$

From (2.1), the second term on the right hand side can be bounded as

$$\begin{aligned}
-\frac{\gamma_1 k^\beta}{2\varepsilon}\Big(f'\big(u(t_{m+1})\big)e^{m+1}, e^{m+1}\Big) &\le \tilde{c}_0\frac{\gamma_1 k^\beta}{2\varepsilon}\| e^{m+1} \|_{L^2}^2 \\
&\le \tilde{c}_0\frac{\gamma_1 k^\beta}{8\varepsilon^3}\| \Delta^{-\frac12} e^{m+1} \|_{L^2}^2 + \frac{\gamma_1\varepsilon k^\beta}{4}\| \nabla e^{m+1} \|_{L^2}^2 .
\end{aligned} \tag{3.34}$$

Then, we obtain from (3.30) and (3.34) after summing (3.33) over m from 0 to $\ell\,(\le M)$

$$\begin{aligned}
&\frac12 \| \Delta^{-\frac12} e^{\ell+1} \|_{L^2}^2 + k\sum_{m=0}^{\ell}\Big\{\frac{k}{2}\| \Delta^{-\frac12} d_t e^{m+1} \|_{L^2}^2 + \frac{\varepsilon}{4}\big[1-\gamma_1(1-\frac{k^\beta}{2})\big] \| \nabla e^{m+1} \|_{L^2}^2\Big\} \\
&\quad\le \Big(C_0\gamma_1 + \tilde{c}_0\gamma_1 k^\beta\varepsilon^{-3}\Big)\, k\sum_{m=0}^{\ell}\| \Delta^{-\frac12} e^{m+1} \|_{L^2}^2 + C\,k^{1-\beta}\,\rho_2(\varepsilon) \\
&\qquad + \frac{Ck}{\varepsilon}\sum_{m=0}^{\ell}\| e^{m+1} \|_{L^{2+\delta}}^{2+\delta} .
\end{aligned} \tag{3.35}$$

Note that $k = O(\varepsilon^{\frac{3}{\beta}})$ in the coefficient of the first term on the right hand side in order to avoid exponential growth in $\frac{1}{\varepsilon}$ of the stability constraint arising from discrete Gronwall's inequality.

Step 3: We now need to bound the super-quadratic term at the end of the inequality (3.35). First, a shift in the super-index leads to

$$\frac{1}{\varepsilon}\| e^{m+1} \|_{L^{2+\delta}}^{2+\delta} \le \frac{C}{\varepsilon}\Big(\| e^m \|_{L^{2+\delta}}^{2+\delta} + k^{2+\delta}\| d_t e^{m+1} \|_{L^{2+\delta}}^{2+\delta}\Big) . \tag{3.36}$$

For the first term on the right hand side, we interpolate $L^{2+\delta}$ between L^2 and H^2, and using (v) of Lemma 2.1 and (v) of Lemma 3.1, we infer

$$\begin{aligned}
\frac{1}{\varepsilon}\| e^m \|_{L^{2+\delta}}^{2+\delta} &\le C\Big(\| \Delta e^m \|_{L^2}^{\frac{N\delta}{4}}\| e^m \|_{L^2}^{\frac{8+(4-N)\delta}{4}} + \| e^m \|_{L^2}^{2+\delta}\Big) \\
&\le C\| e^m \|_{L^2}^{\frac{8+(4-N)\delta}{4}}\Big(\| \Delta e^m \|_{L^2}^{\frac{N\delta}{4}} + \| e^m \|_{L^2}^{\frac{N\delta}{4}}\Big) \\
&\le C\,\rho_1(\varepsilon)^{\frac{N\delta}{8}}\| e^m \|_{L^2}^{2\frac{8+(4-N)\delta}{8}} .
\end{aligned} \tag{3.37}$$

Since $\int_\Omega e^m \,\mathrm{d}x = 0$, the above inequality is continued by

$$\frac{1}{\varepsilon} \| e^m \|_{L^{2+\delta}}^{2+\delta} \le C \Big[\frac{1}{\gamma} \rho_1(\varepsilon)^{-\frac{N\delta}{8}} \| \Delta^{-\frac{1}{2}} e^m \|_{L^2}^{\frac{8+(4-N)\delta}{8}} \Big] \Big[\gamma \| \nabla e^m \|_{L^2}^{\frac{8+(4-N)\delta}{8}} \Big] \tag{3.38}$$

for some $\gamma > 0$ to be fixed in the sequel.

The subsequent analysis deals with $0 < \delta < \frac{8}{4-N}$, which is the more involved case. It is crucial to recover a super-quadratic exponent for $\| \Delta^{-\frac{1}{2}} e^m \|_{L^2}$ in Step 4.

We come back to (3.38) and to look for $\alpha > 0$ such that

$$\gamma^\alpha \| \nabla e^m \|_{L^2}^{\frac{8+(4-N)\delta}{8}\alpha} \le \frac{\varepsilon}{4} \Big[1 - \gamma_1 \big(1 - \frac{k^\beta}{2} \big) \Big] \| \nabla e^m \|_{L^2}^2 ,$$

which implies

$$\frac{8+(4-N)\delta}{8} \alpha = 2 \quad \text{or} \quad \alpha = \frac{16}{8+(4-N)\delta} ,$$

and then set

$$\gamma^{-1} = C \, \varepsilon^{-\frac{1}{\alpha}} \, k^{-\frac{\beta}{\alpha}} .$$

We use these choices in (3.38), together with Young's inequality, to find after a short calculation that (3.38) is continued by

$$\begin{aligned} \frac{1}{\varepsilon} \| e^m \|_{L^{2+\delta}}^{2+\delta} \le C \big[\varepsilon k^\beta \big]^{-\frac{8+(4-N)\delta}{8-(4-N)\delta}} \rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}} \| \Delta^{-\frac{1}{2}} e^m \|_{L^2}^{2(1+\frac{(4-N)\delta}{8-(4-N)\delta})} \\ + \frac{\varepsilon}{8} \big[1 - \gamma_1 (1 - \frac{k^\beta}{2}) \big] \| \nabla e^m \|_{L^2}^2 . \end{aligned} \tag{3.39}$$

Similarly, the second term on the right hand side of (3.36) can be bounded as

$$\begin{aligned} &\frac{k^{2+\delta}}{\varepsilon} \| d_t e^{m+1} \|_{L^{2+\delta}}^{2+\delta} \\ &\le \frac{C k^{2+\delta}}{\varepsilon} \Big(\| \Delta d_t e^{m+1} \|_{L^2}^{\frac{N\delta}{4}} \| d_t e^{m+1} \|_{L^2}^{\frac{8+(4-N)\delta}{4}} + \| d_t e^{m+1} \|_{L^2}^{2+\delta} \Big) \\ &\le \frac{C k^{2+\delta}}{\varepsilon} \| d_t e^{m+1} \|_{L^2}^{\frac{8+(4-N)\delta}{4}} \Big(\| \Delta d_t e^{m+1} \|_{L^2}^{\frac{N\delta}{4}} + \| d_t e^{m+1} \|_{L^2}^{\frac{N\delta}{4}} \Big) \\ &\le C \, k^{2+\frac{(4-N)\delta}{4}} \rho_1(\varepsilon)^{\frac{N\delta}{8}} \| d_t e^{m+1} \|_{L^2}^{2\frac{8+(4-N)\delta}{8}} \\ &\le C \Big[\frac{1}{\gamma} k^{2+\frac{(4-N)\delta}{4}} \rho_1(\varepsilon)^{\frac{N\delta}{8}} \| \nabla d_t e^{m+1} \|_{L^2}^{\frac{8+(4-N)\delta}{8}} \Big] \Big[\gamma \| \Delta^{-\frac{1}{2}} d_t e^{m+1} \|_{L^2}^{\frac{8+(4-N)\delta}{8}} \Big] . \end{aligned} \tag{3.40}$$

We look for $\alpha > 0$ such that

$$\gamma^\alpha \| \Delta^{-\frac{1}{2}} d_t e^{m+1} \|_{L^2}^{\frac{8+(4-N)\delta}{8}\alpha} \le \frac{k}{4} \| \Delta^{-\frac{1}{2}} d_t e^{m+1} \|_{L^2}^2 .$$

This implies

$$\frac{8+(4-N)\delta}{8} \alpha = 2 \quad \text{or} \quad \alpha = \frac{16}{8+(4-N)\delta} ,$$

and

$$\gamma = k^{\frac{8+(4-N)\delta}{16}} .$$

Hence, an upper bound for (3.40) is

$$
\begin{aligned}
&\frac{k}{4}\,\|\,\Delta^{-\frac{1}{2}}d_te^{m+1}\,\|_{L^2}^2 + Ck^{\left(\frac{8+(4-N)\delta}{4}-\frac{8-(4-N)\delta}{16}\right)\frac{16}{8-(4-N)\delta}}\\
&\qquad\times\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\,\|\,\nabla d_te^{m+1}\,\|_{L^2}^{2\frac{8+(4-N)\delta}{8-(4-N)\delta}}\\
&= C\,k^{3\frac{8+(4-N)\delta}{8-(4-N)\delta}}\,\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\,\|\,\nabla d_te^{m+1}\,\|_{L^2}^{2\frac{8+(4-N)\delta}{8-(4-N)\delta}} + \frac{k}{4}\,\|\,\Delta^{-\frac{1}{2}}d_te^{m+1}\,\|_{L^2}^2\\
&\le C\,k^{3\frac{8+(4-N)\delta}{8-(4-N)\delta}-2\frac{(4-N)\delta}{8-(4-N)\delta}}\,\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\\
&\qquad\times\max_{i=0,1}\Big\{\|\,\nabla e^{m+i}\,\|_{L^2}^{2\frac{(4-N)\delta}{8-(4-N)\delta}}\Big\}\,\|\,\nabla d_te^{m+1}\,\|_{L^2}^2 + \frac{k}{4}\,\|\,\Delta^{-\frac{1}{2}}d_te^{m+1}\,\|_{L^2}^2\\
&\le C\,k^{3+4\frac{(4-N)\delta}{8-(4-N)\delta}}\,\varepsilon^{-\frac{(2\sigma_1+1)(4-N)\delta}{8-(4-N)\delta}}\,\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\,\|\,\nabla d_te^{m+1}\,\|_{L^2}^2 + \frac{k}{4}\,\|\,\Delta^{-\frac{1}{2}}d_te^{m+1}\,\|_{L^2}^2\,.
\end{aligned}
$$

Finally, combining these results with (3.35), and using (vi) of Lemma 2.1 and (ii) of 3.1 we get

$$
\begin{aligned}
&\frac{1}{2}\,\|\,\Delta^{-\frac{1}{2}}e^{\ell+1}\,\|_{L^2}^2 + k\sum_{m=0}^{\ell}\Big\{\frac{k}{2}\,\|\,\Delta^{-\frac{1}{2}}d_te^{m+1}\,\|_{L^2}^2 + \frac{\varepsilon}{8}\big[1-\gamma_1(1-\frac{k^\beta}{2})\big]\,\|\,\nabla e^{m+1}\,\|_{L^2}^2\Big\}\\
&\quad\le \Big(C_0\gamma_1 + \tilde{c}_0\gamma_1 k^\beta\varepsilon^{-3}\Big)\,k\sum_{m=0}^{\ell}\|\,\Delta^{-\frac{1}{2}}e^{m+1}\,\|_{L^2}^2 + C\,k^{1-\beta}\,\rho_2(\varepsilon) \qquad (3.41)\\
&\qquad + C\,\big[\varepsilon k^\beta\big]^{-\frac{8+(4-N)\delta}{8-(4-N)\delta}}\,\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\,k\sum_{m=0}^{\ell}\|\,\Delta^{-\frac{1}{2}}e^m\,\|_{L^2}^{2(1+\frac{(4-N)\delta}{8-(4-N)\delta})}\\
&\qquad + C\,k^{2+4\frac{(4-N)\delta}{8-(4-N)\delta}}\,\varepsilon^{-\{\frac{(2\sigma_1+1)(4-N)\delta}{8-(4-N)\delta}+2(\sigma_1+2)\}}\,\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\,.
\end{aligned}
$$

Step 4: We now conclude the proof by the following induction argument which is based on the results from Steps 1 to 3. Suppose that for sufficiently small time steps satisfying

$$
k \le \tilde{C}\,\min\Big\{\varepsilon^{\frac{3}{\beta}}, \varepsilon^{\alpha_0}, \big[\rho_3(\varepsilon)\big]^{\rho_4(N,\beta)}, \big[\rho_5(\varepsilon,N)\big]^{\rho_6(N,\beta)}\Big\}\,, \qquad (3.42)
$$

and $0<\beta<\frac{1}{2}$, there exist two positive constants

$$
c_1 = c_1(t_\ell,\Omega,u_0,\sigma_i,p)\,,\quad c_2 = c_2(t_\ell,\Omega,u_0,\sigma_i,p;C_0)\,,
$$

independent of k and ε, such that the following inequality holds

$$
\begin{aligned}
\max_{0\le m\le\ell}\frac{1}{2}\,\|\,\Delta^{-\frac{1}{2}}e^m\,\|_{L^2}^2 + k\sum_{m=1}^{\ell}\Big(\frac{k}{2}\,\|\,\Delta^{-\frac{1}{2}}d_te^m\,\|_{L^2}^2 + \frac{\gamma_1\varepsilon k^\beta}{2}\,\|\,\nabla e^m\,\|_{L^2}^2\Big)&\\
\le c_1\,k^{1-\beta}\,\rho_2(\varepsilon)\exp\big(c_2t_\ell\big)\,.& \qquad (3.43)
\end{aligned}
$$

The last two constraints in (3.42) arise as follows. The first of them comes from controlling the last error term in (3.41)

$$
k^{2+4\frac{(4-N)\delta}{8-(4-N)\delta}}\,\varepsilon^{-\{\frac{(2\sigma_1+1)(4-N)\delta}{8-(4-N)\delta}+2(\sigma_1+2)\}}\,\big[\rho_1(\varepsilon)\big]^{\frac{2N\delta}{8-(4-N)\delta}} \le \frac{c_1}{2}\,k^{1-\beta}\,\rho_2(\varepsilon)\exp\big(c_2t_\ell\big)\,.
$$

Note that the exponent in the second sum on the right hand side of (3.41) is bigger than 2, hence we can recover (3.43) at the $(\ell+1)$th time step by using the discrete Gronwall's inequality, provided that

$$\big[\varepsilon k^{\beta}\big]^{-\frac{8+(4-N)\delta}{8-(4-N)\delta}}\,\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\,\big[k^{1-\beta}\rho_2(\varepsilon)\big]^{1+\frac{(4-N)\delta}{8-(4-N)\delta}} \leq \frac{c_1}{2}k^{1-\beta}\,\rho_2(\varepsilon)\exp\big(c_2 t_{\ell+1}\big)\,,$$

which gives the last constraint in (3.42). The proof is complete. $\square$

Remark: (a). Theorem 3.4 is stated for $0<\delta<\frac{8}{4-N}$, which covers assumption (GA_1) for the case $N=2,3$. For $N=2$, the error estimate is valid for all $0<\delta<\infty$, the above proof can be adapted and simplified for the case $\delta>\frac{8}{4-N}$. Note that in this case the crucial requirement of super-quadratical growth is already met in (3.38), then we can immediately jump to Step 4 to proceed. Finally, the case $\delta=\frac{8}{4-N}$ is again easy to take care, thanks to Lemma 2.1 and 3.1.

(b). In addition to the spectrum estimate of Proposition 2.3, the stability estimate (v) of Lemma 3.1 is critical to the analysis.

(c). It is natural to estimate the error of (3.4)-(3.5) in the norm of $\ell^\infty\big(J_k^1;H^{-1}(\Omega)\big)$ $\cap\ell^2\big(J_k^1;H^1(\Omega)\big)$, its structure allows to test with $-\Delta^{-1}e^{m+1}$, which then interferes with limited regularity property of u_{tt} and cuts convergence rate in (3.30) down to sub-optimal order. As to be demonstrated in the sequel, using stretched meshes J_k^2 will help to attain a quasi-optimal order for the Euler method (3.4)-(3.5).

(d). A straightforward idea to benefit from the damping property of (1.1) is to multiply (3.29) by a time-weight $\tau_{m+1}:=\min\{t_{m+1},1\}$ before summation; this would give an optimal convergence rate $Ck^2\rho_2(\varepsilon)$ in (3.30) thanks to (ix) of Lemma 2.1. On the other hand, this would require to control the error $\{e^m\}_{m=0}^M$ in the norm of $\ell^2\big(J_k^1;H^{-1}(\Omega)\big)$ by using a (parabolic) duality argument. This strategy does not seem to be successful in the present analysis where we focus on non-exponential dependencies on $\frac{1}{\epsilon}$ of involved stability constants.

(e). It is clear that the smaller β, the better the error bound, since the exponent of k is closer to $\frac12$. Small values of β, however, restrict admissible time steps to small sizes.

(f). The proof of Theorem 3.4 suggests the following numerical stabilization technique for the Cahn-Hilliard equation (3.4)-(3.5)

$$(d_t u^{m+1},\eta)+(\nabla w^{m+1},\nabla\eta)=0\quad\forall\,\eta\in H^1(\Omega)\,, \tag{3.44}$$

$$\varepsilon(1+\frac{k^{\zeta_1}}{\varepsilon^{\zeta_2}})\,(\nabla u^{m+1},\nabla v)+\frac{1}{\varepsilon}\big(f(u^{m+1}),v\big)=(w^{m+1},v)\quad\forall\,v\in H^1(\Omega)\,, \tag{3.45}$$

where $\zeta_i\geq 0$ for $i=1,2$. We will not go into further discussion of these methods here.

For given more regular initial data $u_0\in H^3(\Omega)$ and domains Ω (see assumptions in Lemma 2.2), the convergence rate can be improved to $Ck^{1-\beta}\rho_2(\varepsilon)$. The key ingredient for proving that is to use (v) of Lemma 2.2 to improve on the estimate (3.30).

Corollary 3.5. *Let $\{(u^m,w^m\}_{m=0}^M$ solve (3.4)-(3.5) on an equidistant mesh $J_k^1=\{t_m\}_{m=0}^M$ of mesh size $O(k)$, for $u_0\in H^3(\Omega)$, and $\partial\Omega$ of class $C^{2,1}$ (or a convex polygonal domain when $N=2$). Suppose (GA_1)-(GA_3) hold, and $1<\delta<\frac{8}{4-N}$. Let ρ_j be same as in Theorem 3.4. For fixed positive values $0<\beta<\frac12$, let k satisfy the following constraint*

$$k\leq\tilde C\,\min\Big\{\varepsilon^{\frac{3}{\beta}},\varepsilon^{\alpha_0},\big[\rho_3(\varepsilon,N)\big]^{\rho_4(N,\beta)},\big[\rho_5(\varepsilon,N)\big]^{\rho_6(N,\beta)}\Big\}\,, \tag{3.46}$$

Then there exists a positive constant $\tilde{C}=\tilde{C}(u_0;\gamma_1,\gamma_2,C_0,T,\Omega)$ such that the solution of (3.4)-(3.5) satisfies the following error estimate

$$\max_{0\le m\le M}\|u(t_m)-u^m\|_{H^{-1}}+\Big(k\sum_{m=1}^{M}\Big\{k\,\|d_t(u(t_m)-u^m)\|_{H^{-1}}^2 +k^{\beta}\,\|\nabla(u(t_m)-u^m)\|_{L^2}^2\Big\}\Big)^{\frac12}\le \tilde{C}\,k^{\frac{2-\beta}{2}}\,\big[\rho_2(\varepsilon)\big]^{\frac12}.$$

For $u_0\in H^2(\Omega)$, the error bound given in Theorem 3.4 is not optimal, the crucial step where we loose accuracy by one order of magnitude on the time step k is (3.30), since we are only provided with a bound for $\sqrt{\tau}(\Delta)^{-1}u_{tt}\in L^2\big(J;H^{-1}(\Omega)\big)$; see (ix) of Lemma 2.1. The following result reflects the stabilizing effect of the mesh J_k^2 in this respect. Note that the proof of Theorem 3.4 only requires (iv)-(v) of Lemma 3.1 which are replaced by (vii) in the case of the mesh J_k^2.

THEOREM 3.6. *Suppose that the assumptions and shorthand notation of Theorem 3.4 hold. Define*

$$\tilde{\rho}_4(N,\beta)=\Big[\beta+\frac{4(4-N)\delta}{8-(4-N)\delta}\Big]^{-1},$$
$$\tilde{\rho}_6(N,\beta)=\delta\Big[2(4-N)\delta-\beta(16+3(4-N)\delta\Big]^{-1}.$$

For fixed positive values $0<\beta<\frac12$, let k_0 satisfy the following constraint

$$k_0\le \tilde{C}\,\min\Big\{\varepsilon^{\frac{3}{\beta}},\varepsilon^{\frac{\alpha_0}{2}},\big[\rho_3(\varepsilon,N)\big]^{\tilde{\rho}_4(N,\beta)},\big[\rho_5(\varepsilon,N)\big]^{\tilde{\rho}_6(N,\beta)}\Big\}. \tag{3.47}$$

Let $\{(u^m,w^m)\}_{m=0}^M$ be the solution to (3.4)-(3.5) on the mesh J_k^2 defined in (3.6). Then there holds the following improved error estimate

$$\max_{0\le m\le M}\|u(t_m)-u^m\|_{H^{-1}}+\Big(\sum_{m=1}^{M}k_m\Big\{k_m\,\|d_t\{u(t_m)-u^m\}\|_{H^{-1}}^2 +k_m^{\beta}\,\|\nabla\big\{u(t_m)-u^m\big\}\|_{L^2}^2\Big\}\Big)^{\frac12}\le \tilde{C}\,k_0^{\frac{2-\beta}{2}}\,\big[\rho_2(\varepsilon)\big]^{\frac12}.$$

Proof. The proof follows the steps of that of Theorem 3.4. We only sketch the necessary modifications in the following.

Step 1: On the stretched mesh J_k^2, the residual $\mathcal{R}(u_{tt},m)$ can be bounded as follows

$$\begin{aligned}\sum_{m=1}^{M}k_{m+1}\|\Delta^{-1}\mathcal{R}(u_{tt},m)\|_{H^{-1}}^2&=\sum_{m=1}^{M}\frac{1}{k_{m+1}}\Big\|\int_{t_m}^{t_{m+1}}(s-t_m)\,\Delta^{-1}u_{tt}(s)\,\mathrm{d}s\Big\|_{H^{-1}}^2\\
&\le\sum_{m=1}^{M}\frac{1}{k_{m+1}}\int_{t_m}^{t_{m+1}}\frac1s(s-t_m)^2\,\mathrm{d}s\int_{t_m}^{t_{m+1}}\tau(s)\,\|\Delta^{-1}u_{tt}(s)\|_{H^{-1}}^2\,\mathrm{d}s\\
&\le C\max_{0\le m\le M}\Big\{\frac{1}{k_{m+1}}\int_{t_m}^{t_{m+1}}\frac1s(s-t_m)^2\,\mathrm{d}s\Big\}\rho_2(\varepsilon)\\
&\le C\max_{1\le m\le M}\frac{k_m^2}{t_m}\,\rho_2(\varepsilon)\\
&\le Ck_0^4\max_{1\le m\le M}\frac{(m+1)^2}{t_m}\,\rho_2(\varepsilon)\le C\,k_0^2\,\rho_2(\varepsilon),\end{aligned}$$

thanks to (ix) of Lemma 2.1. This improved upper bound for the residual replaces (3.30) in the proof of Theorem 3.4.

Step 2: This step is the same.

Step 3: We use (vii) of Lemma 3.1, instead of (v) to bound $\max_{0\le m\le M}\|\Delta u^m\|_{L^2}$. Then (3.38) and (3.39) are replaced by

$$\frac{1}{\varepsilon}\|e^m\|_{L^{2+\delta}}^{2+\delta} \le C\left[\varepsilon k_m^\beta\right]^{-\frac{8+(4-N)\delta}{8-(4-N)\delta}}\tilde\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\|\Delta^{-\frac12}e^m\|_{L^2}^{2(1+\frac{(4-N)\delta}{8-(4-N)\delta})} + \frac{\varepsilon}{8}\left[1-\gamma_1(1-\frac{k_m^\beta}{2})\right]\|\nabla e^m\|_{L^2}^2 . \tag{3.48}$$

Again, the argument applies for values $\delta < \frac{8}{4-N}$. Instead of (3.40), we now have

$$\begin{aligned}\frac{k_{m+1}^{2+\delta}}{\varepsilon}\|d_t e^{m+1}\|_{L^{2+\delta}}^{2+\delta} &\le \frac{Ck_{m+1}^{2+\delta}}{\varepsilon}\left(\|\Delta d_t e^{m+1}\|_{L^2}^{\frac{N\delta}{4}}\|d_t e^{m+1}\|_{L^2}^{\frac{8+(4-N)\delta}{4}} + \|d_t e^{m+1}\|_{L^2}^{2+\delta}\right)\\ &\le \frac{Ck_{m+1}^{2+\delta}}{\varepsilon}\|d_t e^{m+1}\|_{L^2}^{\frac{8+(4-N)\delta}{4}}\left(\|\Delta d_t e^{m+1}\|_{L^2}^{\frac{N\delta}{4}} + \|d_t e^{m+1}\|_{L^2}^{\frac{N\delta}{4}}\right)\\ &\le C\,k_{m+1}^{3+4\frac{(4-N)\delta}{8-(4-N)\delta}}\varepsilon^{-\frac{(2\sigma_1+1)(4-N)\delta}{8-(4-N)\delta}}\rho_1(\varepsilon)^{\frac{2N\delta}{8-(4-N)\delta}}\|\nabla d_t e^{m+1}\|_{L^2}^2\\ &\qquad + \frac{k_{m+1}}{4}\|\Delta^{-\frac12}d_t e^{m+1}\|_{L^2}^2 .\end{aligned}$$

Finally, (3.41) is replaced by

$$\begin{aligned}&\frac12\|\Delta^{-\frac12}e^\ell\|_{L^2}^2 + \sum_{m=1}^{\ell}\Big\{\frac{k_m^2}{2}\|\Delta^{-\frac12}d_t e^m\|_{L^2}^2 + \frac{\varepsilon k_m}{8}\left[1-\gamma_1(1-\frac{k_m^\beta}{2})\right]\|\nabla e^m\|_{L^2}^2\Big\}\\ &\quad\le \Big(C_0\gamma_1 + \tilde c_0\gamma_1 k_0^\beta\varepsilon^{-3}\Big)\sum_{m=1}^{\ell}k_m\|\Delta^{-\frac12}e^m\|_{L^2}^2 + C\,k_0^{2-\beta}\,\rho_2(\varepsilon)\\ &\quad d + C\left[\tilde\rho_1(\varepsilon)\right]^{\frac{2N\delta}{8-(4-N)\delta}}\sum_{m=1}^{\ell}k_m\left[\varepsilon k_m^\beta\right]^{-\frac{8+(4-N)\delta}{8-(4-N)\delta}}\|\Delta^{-\frac12}e^m\|_{L^2}^{2(1+\frac{(4-N)\delta}{8-(4-N)\delta})}\\ &\quad + C\,k_0^{2+4\frac{(4-N)\delta}{8-(4-N)\delta}}\,\varepsilon^{-\{\frac{(2\sigma_1+1)(4-N)\delta}{8-(4-N)\delta}+2(\sigma_1+2)\}}\left[\tilde\rho_1(\varepsilon)\right]^{\frac{2N\delta}{8-(4-N)\delta}} .\end{aligned}\tag{3.49}$$

Step 4: The inductive argument now reads: Suppose that for sufficiently small basic time step k_0 satisfying

$$k_0 \le \tilde C\,\min\Big\{\varepsilon^{\frac{3}{\beta}}, \varepsilon^{\alpha_0}, \left[\rho_3(\varepsilon)\right]^{\tilde\rho_4(N,\beta)}, \left[\rho_5(\varepsilon,N)\right]^{\tilde\rho_6(N,\beta)}\Big\}, \tag{3.50}$$

and $0<\beta<\frac12$, there exist two positive constants

$$\tilde c_1 = \tilde c_1\big(t_\ell,\Omega,u_0,\sigma_i,p\big), \qquad \tilde c_2 = \tilde c_2\big(t_\ell,\Omega,u_0,\sigma_i,p;C_0\big),$$

independent of k_0 and ε, such that the following inequality holds,

$$\max_{0\le m\le\ell}\|\Delta^{-\frac12}e^m\|_{L^2}^2 + \sum_{m=1}^{\ell}k_m\Big(\frac{k_m}{2}\|\Delta^{-\frac12}d_t e^m\|_{L^2}^2 + \frac{\gamma_1\varepsilon k_m^\beta}{2}\|\nabla e^m\|_{L^2}^2\Big) \le \tilde c_1\,k_0^{2-\beta}\,\rho_2(\varepsilon)\exp\big(\tilde c_2 t_\ell\big) . \tag{3.51}$$

We employ the following necessary criterion

$$k_0^{2+4\frac{(4-N)\delta}{8-(4-N)\delta}}\varepsilon^{-\{\frac{(2\sigma_1+1)(4-N)\delta}{8-(4-N)\delta}+2(\sigma_1+2)\}}\big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{2N\delta}{8-(4-N)\delta}} \le \frac{\tilde{c}_1}{2}\,k_0^{2-\beta}\,\rho_2(\varepsilon)\,, \tag{3.52}$$

which implies the third condition in (3.47).

Finally, we need to make sure that

$$\sum_{m=1}^{\ell} k_m\big[\varepsilon k_m^{\beta}\big]^{-\frac{8+(4-N)\delta}{8-(4-N)\delta}}\,\big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{2N\delta}{8-(4-N)\delta}}\,\big[k_0^{2-\beta}\rho_2(\varepsilon)\big]^{1+\frac{(4-N)\delta}{8-(4-N)\delta}}$$
$$\le \frac{\tilde{c}_1}{2}k_0^{2-\beta}\,\rho_2(\varepsilon)\exp\big(\tilde{c}_2 t_{\ell+1}\big)\,.$$

This completes the induction argument and the proof, too. □

4. Error analysis for a fully discrete mixed finite element approximation. In this section we propose and analyze a fully discrete mixed finite element method for (3.4)-(3.5) on the meshes J_k^1 and J_k^2. The lowest order Ciarlet-Raviart mixed finite element method (cf. Chapter 7 of [17] and [34]) for the biharmonic problem is used for spatial discretization. Like in Section 3, special emphasis is given to analyze the dependence of the error bounds on $\frac{1}{\varepsilon}$. Throughout this section, we assume that $u_0 \in H^2(\Omega)$ and $\partial\Omega$ is of class $C^{1,1}$, and that (GA$_1$)-(GA$_3$) are satisfied. Sub-optimal error estimates for the fully discrete scheme on J_k^1 and improved quasi-optimal estimates (for $N=2$; with slightly deteriorated rate for $N=3$) on J_k^2 are established, see Theorem 4.3 and Corollary 4.4.

We recall that the fully discrete mixed finite element discretization of (3.4)-(3.5) is defined as: Find $\big\{(U^m,W^m)\big\}_{m=1}^M \in \big[\mathcal{S}_h\big]^2$ such that

$$(d_tU^{m+1},\eta_h) + (\nabla W^{m+1},\nabla\eta_h) = 0 \qquad \forall\,\eta_h\in\mathcal{S}_h\,, \tag{4.1}$$

$$\varepsilon\,(\nabla U^{m+1},\nabla v_h) + \frac{1}{\varepsilon}\,(f(U^{m+1}),v_h) = (W^{m+1},v_h) \quad \forall\,v_h\in\mathcal{S}_h\,, \tag{4.2}$$

with some suitable starting value U^0, and a quasi-uniform "triangulation" $\mathcal{T}_h$ of Ω. Where $\mathcal{S}_h$ denotes the P_1 conforming finite element space defined by

$$\mathcal{S}_h := \Big\{v_h \in C(\overline{\Omega})\,; v_h|_K \in P_1(K),\,\forall\,K\in\mathcal{T}_h\Big\}\,.$$

The mixed finite element space $\mathcal{S}_h\times\mathcal{S}_h$ is the lowest order element among a family of stable mixed finite element spaces known as the Ciarlet-Raviart mixed finite elements for the biharmonic problem (cf. [17, 34]). In fact, it is not hard to check that the following *inf-sup* condition holds

$$\inf_{0\not\equiv\eta_h\in\mathcal{S}_h}\ \sup_{0\not\equiv\psi_h\in\mathcal{S}_h}\frac{(\nabla\psi_h,\nabla\eta_h)}{\|\,\psi_h\,\|_{H^1}\,\|\,\eta_h\,\|_{H^1}} \ge c_0$$

for some $c_0>0$.

Also, we note that $(d_tU^{m+1},1)=0$, which implies that $(U^{m+1},1)=(U^0,1)$ for $m=0,1,\cdot,M-1$. Hence, the mass is also conserved by the fully discrete solution at each time step.

We define the $L^2(\Omega)$-projection $Q_h : L^2(\Omega)\to\mathcal{S}_h$ by

$$(Q_hv-v,\eta_h)=0 \quad \forall\,\eta\in\mathcal{S}_h\,, \tag{4.3}$$

and the elliptic projection $P_h : H^1(\Omega) \to \mathcal{S}_h$ by

$$\big(\nabla[P_h v - v], \nabla \eta_h\big) = 0 \quad \forall\, \eta_h \in \mathcal{S}_h\,, \tag{4.4}$$

$$(P_h v - v, 1) = 0\,. \tag{4.5}$$

We refer to Section 4 of [25] for a list of approximation properties of P_h. In the sequel, we confine to meshes $\mathcal{T}_h$ that allow for H^1-stability of Q_h, see [14] for their further characterization.

We also introduce space notation

$$\mathring{\mathcal{S}}_h := \Big\{ v_h \in \mathcal{S}_h\,; (v_h, 1) = 0 \Big\}\,,$$

and define the discrete inverse Laplace operator: $-\Delta_h^{-1} : L_0^2(\Omega) \to \mathring{\mathcal{S}}_h$ such that

$$\big(\nabla(-\Delta_h^{-1} v), \nabla \eta_h\big) = (v, \eta_h) \quad \forall\, \eta_h \in \mathcal{S}_h\,. \tag{4.6}$$

Lemma 4.1. *For $J_k = J_k^1$ or J_k^2, the solution $\big\{(U^m, W^m)\big\}_{m=0}^M$ of (4.1)-(4.2) satisfies*

(i) $$\frac{1}{|\Omega|}\int_\Omega U^m \,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega U^0 \,\mathrm{d}x\,, \quad m = 1, 2, \cdots, M,$$

(ii) $$\|\, d_t U^m \,\|_{H^{-1}} \le C \,\|\, \nabla W^m \,\|_{L^2}, \quad m = 1, 2, \cdots, M,$$

(iii) $$\max_{0\le m\le M} \Big\{ \varepsilon \,\|\, \nabla U^m \,\|_{L^2}^2 + \frac{1}{\varepsilon} \,\|\, F(U^m) \,\|_{L^1} \Big\} + \sum_{m=1}^M k_m \Big(\|\, \nabla W^m \,\|_{L^2}^2 + \varepsilon k_m \,\|\, \nabla d_t U^m \,\|_{L^2}^2 \Big) \le C \mathcal{J}_\varepsilon(U^0)\,,$$

(iv) $$\sum_{m=1}^M k_m \|\, d_t U^m \,\|_{H^{-1}}^2 \le C \mathcal{J}_\varepsilon(U^0)\,.$$

Proof. The assertion (i) is an immediate consequence of setting $\eta_h = 1$ in (4.1).

For any $\phi \in H^1(\Omega)$, from (4.1), (4.3), and the stability of Q_h in $H^1(\Omega)$ (cf. [14] and references therein) we have

$$\begin{aligned}(d_t U^m, \phi) &= (d_t U^m, Q_h \phi) + (d_t U^m, \phi - Q_h \phi) \\ &= (\nabla W^m, \nabla Q_h \phi) \le C \,\|\, \nabla W^m \,\|_{L^2} \|\, \nabla \phi \,\|_{L^2}\,.\end{aligned}$$

Assertion (ii) then follows from

$$\|\, d_t U^m \,\|_{H^{-1}} = \sup_{0 \not\equiv \phi \in H^1} \frac{(d_t U^m, \phi)}{\|\, \phi \,\|_{H^1}} \le C \,\|\, \nabla W^m \,\|_{L^2}\,.$$

To show assertion (iii), setting $\eta_h = W^{m+1}$ in (4.1) and $v_h = d_t U^{m+1}$ in (4.2) and adding the resulting equations give

$$\|\, \nabla W^{m+1} \,\|_{L^2}^2 + \frac{\varepsilon}{2}\frac{d}{dt} \|\, \nabla U^{m+1} \,\|_{L^2}^2 + \frac{\varepsilon k_{m+1}}{2} \|\, \nabla d_t U^{m+1} \,\|_{L^2}^2 + \frac{1}{\varepsilon}\big(f(U^{m+1}), d_t U^{m+1}\big) = 0\,. \tag{4.7}$$

The statement then follows from (GA_3) and (ii) after multiplying (4.7) by k_{m+1} and taking sum over m from 0 to $\ell\,(\leq M-1)$. Assertion (iv) is an immediate consequence of (ii) and (iii). □

Remark: In view of (i) of Lemma 2.1 and (i) of Lemma 4.1, in order for the scheme (4.1)-(4.2) to conserve the mass of the underlying physical problem, it is necessary to require $(U^0 - u_0^\varepsilon, 1) = 0$ for the starting value U^0. This condition will be assumed in the rest of this section.

As is shown in [25] in the context of the Allen-Cahn equation, in order to establish error bounds that depend on low order polynomials of $\frac{1}{\varepsilon}$, the crucial idea is to utilize the spectrum estimate result of Proposition 2.3 for the linearized Cahn-Hilliard operator. In the following we show that the spectrum estimate still holds if the value $u(t_m)$, which is the solution of (1.1)-(1.3) at t_m, is replaced by the elliptic projection $P_h u^m$ of the solution u^m to (3.4)-(3.5) at t_m, and the nonlinear term is scaled by a factor $1-\varepsilon$, provided that the mesh sizes k_0 and h are small enough. As expected, this result plays a critical role in our error analysis for the fully discrete finite element discretization.

We define for $\varphi^m \in L_0^2(\Omega)\cap H^2(\Omega)$, $0 \leq m \leq M$, and $c = c(\Omega)$,

$$\hat{C}_1\big(\varepsilon, \{\varphi^m\}_{m=0}^M\big) \equiv \max_{J_k^2} \|\,\varphi^m\,\|_{L^\infty} \leq c\,\max_{J_k^2}\Big(\|\,\nabla\varphi^m\,\|^{\frac{5-N}{4}}\,\|\,\Delta\varphi^m\,\|_{L^2}^{\frac{N-1}{4}} + \|\,\varphi^m\,\|_{L^2}\Big)\,,$$

$$\begin{aligned} C_1(\varepsilon) &\equiv \max\Big\{\hat{C}_1\big(\varepsilon, \{u^m\}_{m=0}^M\big), \hat{C}_1\big(\varepsilon, \{u(t_m)\}_{m=0}^M\big)\Big\} \\ &\leq \rho_7(\varepsilon, N) := c\,\varepsilon^{-(2\sigma_1+1)\frac{5-N}{8}}\,\big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{N-1}{8}}\,, \end{aligned} \tag{4.8}$$

$$C_2(\varepsilon) = \max_{|v|\leq 2C_1(\varepsilon)} |f''(v)|\,, \tag{4.9}$$

and C_3 be the smallest positive ε-independent constant such that (cf. Chapter 7 of [10])

$$\begin{aligned} \max_{J_k^2}\|\,u^m - P_h u^m\,\|_{L^\infty(J;L^\infty)} &\leq C_3\,h^{\frac{4-N}{2}}|\ln h|^{\frac{3-N}{2}}\,\max_{J_k^2}\|\,u^m\,\|_{H^2} \\ &\leq C_3\,h^{\frac{4-N}{2}}|\ln h|^{\frac{3-N}{2}}\,\big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{1}{2}}\,. \end{aligned} \tag{4.10}$$

Proposition 4.2. *Let the assumptions of Proposition 2.3 hold and C_0 be same as there. Let $\{u^m\}_{m=0}^M$ be the solution of (3.4)-(3.5) and $\{P_h u^m\}_{m=0}^M$ be its elliptic projection. Suppose that the assumptions of Theorem 3.6 are valid. Then there holds for small $\varepsilon > 0$ and any $0 \leq m \leq M$*

$$\lambda_{CH}^{h,k_0} \equiv \inf_{\substack{0\not\equiv\psi\in L_0^2(\Omega)\\ \Delta w=\psi,\,\frac{\partial w}{\partial n}=0}} \frac{\varepsilon\,\|\,\nabla\psi\,\|_{L^2}^2 + \frac{1-\varepsilon}{\varepsilon}\,(f'(P_h u^m)\psi, \psi)}{\|\,\nabla w\,\|_{L^2}^2} \geq -\,(1-\varepsilon)(C_0+1),$$

provided that k_0 and h satisfy for some $c = c(\Omega)$

$$h^{\frac{4-N}{2}}\,|\ln h|^{\frac{3-N}{2}} \leq \Big(C_2(\varepsilon)\,C_3\,\big[\rho_1(\varepsilon)\big]^{\frac{1}{2}}\Big)^{-1}\,\varepsilon^2\,, \tag{4.11}$$

$$k_0 \leq \Big\{\varepsilon^{2(\sigma_1+1)}\,\big[\rho_2(\varepsilon)\big]^{-1}\Big(\frac{\varepsilon^2}{c}\,\big[\tilde{\rho}_1(\varepsilon)\big]^{-\frac{N}{8}}\Big)^{\frac{16}{4-N}}\Big\}^{\frac{1}{2-\beta}}\,. \tag{4.12}$$

Proof. Step 1: From the definitions of $C_2(\varepsilon)$ and C_3, we immediately have

$$\max_{J_k^2} \| P_h u^m \|_{L^\infty} \le \max_{J_k^2} \Big\{ \| u^m \|_{L^\infty} + \| P_h u^m - u^m \|_{L^\infty} \Big\} \le 2 \max_{J_k^2} \| u^m \|_{L^\infty}$$

if h satisfies (4.11).

It then follows from the Mean Value Theorem that

$$\begin{aligned} \max_{J_k^2} \| f'(P_h u^m) - f'(u^m) \|_{L^\infty} &\le \sup_{|\xi| \le 2C_1(\varepsilon)} |f''(\xi)| \max_{J_k^2} \| P_h u^m - u^m \|_{L^\infty} \\ &\le C_2(\varepsilon)\, C_3\, h^{\frac{4-N}{2}}\, |\ln h|^{\frac{3-N}{2}} \big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{1}{2}} \qquad (4.13) \\ &\le \varepsilon^2 . \end{aligned}$$

Using the inequality $a \ge b - |a-b|$ and (4.13) we get

$$\begin{aligned} f'(P_h u^m) &\ge f'(u^m) - \| f'(P_h u^m) - f'(u^m) \|_{L^\infty} \qquad (4.14) \\ &\ge f'(u^m) - \varepsilon^2 . \end{aligned}$$

Step 2: By a Gagliardo-Nirenberg's inequality we get for any $0 \le m \le M$

$$\begin{aligned} \| u(t_m) - u^m \|_{L^\infty} &\le c \| u(t_m) - u^m \|_{L^2}^{\frac{4-N}{4}} \Big\{ \| \Delta u(t_m) \|_{L^2} + \| \Delta u^m \|_{L^2} \Big\}^{\frac{N}{4}} \qquad (4.15) \\ &\le c \| u(t_m) - u^m \|_{H^{-1}}^{\frac{4-N}{8}} \Big\{ \| \nabla u(t_m) \|_{L^2} + \| \nabla u^m \|_{L^2} \Big\}^{\frac{4-N}{8}} \big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{N}{8}} \\ &\le c \big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{N}{8}} \Big[\varepsilon^{-(2\sigma_1+1)} \rho_2(\varepsilon)\, k_0^{2-\beta} \Big]^{\frac{4-N}{16}} , \end{aligned}$$

thanks to Theorem 3.6.

If k_0 satisfies (4.12), we find by the Mean Value Theorem and (4.15) that

$$\begin{aligned} \max_{J_k^2} \| f'(u^m) - f'(u(t_m)) \|_{L^\infty} &\le \sup_{|\tilde{\xi}| \le 2C_1(\varepsilon)} |f''(\tilde{\xi})| \max_{J_k^2} \| u^m - u(t_m) \|_{L^\infty} \\ &\le c\, C_2(\varepsilon) \big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{N}{8}} \Big[\varepsilon^{-(2\sigma_1+1)} \rho_2(\varepsilon)\, k_0^{2-\beta} \Big]^{\frac{4-N}{16}} \\ &\le \varepsilon^2 , \end{aligned}$$

which implies that

$$f'(u^m) \ge f'\big(u(t_m)\big) - \varepsilon^2 . \qquad (4.16)$$

It follows from (4.14) and (4.16) that

$$f'(P_h u^m) \ge f'\big(u(t_m)\big) - 2\varepsilon^2 . \qquad (4.17)$$

In addition, since $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$ we have

$$\| \psi \|_{L^2}^2 = (\nabla \psi, \nabla w) \le \frac{1}{2} \Big(\frac{\varepsilon}{1-\varepsilon} \| \nabla \psi \|_{L^2}^2 + \frac{1-\varepsilon}{\varepsilon} \| \nabla w \|_{L^2}^2 \Big) . \qquad (4.18)$$

Substituting (4.17) and (4.18) into the definition of λ_{CH}^{h,k_0} we get

$$\lambda_{CH}^{h,k_0} \ge \inf_{\substack{0 \not\equiv \psi \in L_0^2(\Omega) \\ \Delta w = \psi,\, \frac{\partial w}{\partial n} = 0}} \frac{(1-\varepsilon)\big[\varepsilon \| \nabla \psi \|_{L^2}^2 + \frac{1}{\varepsilon} (f'(u^m)\psi, \psi)\big]}{\| \nabla w \|_{L^2}^2} - (1-\varepsilon)^2 .$$

The proof is completed by applying Proposition 2.3. □

The main result in this section is stated in the following theorem.

THEOREM 4.3. *Let* $\{U^m, W^m\}_{m=0}^M$ *solve (4.1)-(4.2) on* $J_k = J_k^2$ *and on a quasi-uniform triangulation* $\mathcal{T}_h$ *of* Ω*, allowing for inverse inequalities and* H^1*-stability of the* L^2*-projection in the continuous linear finite element space. Suppose that the assumptions and notation of Theorem 3.6 hold. For* $0 < \beta < \frac{1}{2}$*,* $N = 2, 3$*, and* $\nu > 0$*, define*

$$\rho_7(\varepsilon, N) := c\,\varepsilon^{-(2\sigma_1+1)\frac{5-N}{8}} \big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{N-1}{8}}$$

$$\gamma_1^* := \max\big\{1-\varepsilon^r, \gamma_1\big\}, \quad \text{for any } r > 1\,,$$

$$\pi(h, k_0, \varepsilon, N) := h^4\Big\{h^{2\nu} + \ln\Big(\frac{1}{k_0}\Big)\tilde{\rho}_1(\varepsilon)\, h^2 + \frac{2(1-\varepsilon)}{1-\varepsilon-\gamma_1^\star}\, \varepsilon^{-(2\sigma_1+6)} \big[\rho_7(\varepsilon, N)\big]^{2(p-2)}\Big\}.$$

Let k_0*,* h *and* U^0 *satisfy the following constraints*

1). $k_0 \le \tilde{C}\, \min\Big\{\varepsilon^{\frac{3}{\beta}}, \varepsilon^{\frac{\alpha_0}{2}}, \big[\rho_3(\varepsilon, N)\big]^{\tilde{\rho}_4(N,\beta)}, \big[\rho_5(\varepsilon, N)\big]^{\tilde{\rho}_6(N,\beta)}\Big\}$,

2). $k_0 \le \Big\{\varepsilon^{2(\sigma_1+1)} \big[\rho_2(\varepsilon)\big]^{-1} \Big(\frac{\varepsilon^2}{c} \big[\tilde{\rho}_1(\varepsilon)\big]^{-\frac{N}{8}}\Big)^{\frac{16}{4-N}}\Big\}^{\frac{1}{2-\beta}}$,

3). $h^{\frac{4-N}{2}}\, |\ln h|^{\frac{3-N}{2}} \le \big(C_2(\varepsilon)\, C_3\, \big[\rho_1(\varepsilon)\big]^{\frac{1}{2}}\big)^{-1} \varepsilon^2$,

4). $k_0^{-2\beta\frac{8+(4-N)\delta}{8-(4-N)\delta}} \Big(\pi(h, k_0, \varepsilon, N) + k_0^{2-\beta}\rho_2(\varepsilon)\Big)^{\frac{(4-N)\delta}{8-(4-N)\delta}} \le \Big(\varepsilon\big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{2N\delta}{8-(4-N)\delta}}\Big)^{-1}$,

5). $(U^0, 1) = (u_0, 1)$ *and* $\|\, U^0 - u_0 \,\|_{H^{-1}} \le C h^{2+\nu} \|\, u_0 \,\|_{H^2}$.

Then the solution of (4.1)-(4.2) satisfies the error estimates

$$\text{(i)} \quad \max_{0\le m\le M} \|\, u(t_m) - U^m \,\|_{H^{-1}} + \Big(\sum_{m=1}^M k_m^2 \|\, d_t\big(u(t_m) - U^m\big) \,\|_{H^{-1}}^2\Big)^{\frac{1}{2}} + \Big(\sum_{m=1}^M k_m \|\, u(t_m) - U^m \,\|_{L^2}^2\Big)^{\frac{1}{2}}$$
$$\le C\,\Big(h^4 \varepsilon^{-(2\sigma_1+3)} + \pi(h, k_0, \varepsilon, N) + k_0^{(2-\beta)}\rho_2(\varepsilon)\Big)^{\frac{1}{2}},$$

$$\text{(ii)} \quad \Big(\sum_{m=1}^M k_m \|\, \nabla(u(t_m) - U^m) \,\|_{L^2}^2\Big)^{\frac{1}{2}}$$
$$\le C\,\Big(h^2 \varepsilon^{-(2\sigma_1+4)} + \pi(h, k_0, \varepsilon, N) + k_0^{(2-\beta)}\rho_2(\varepsilon)\Big)^{\frac{1}{2}}.$$

Proof. As in the proof of Theorem 3.6, here we divide the proof into four steps. Also, in the proof we make use of the facts $\mathcal{S}_h \subset H^1(\Omega)$ and $u^m - U^m \in L_0^2(\Omega)$. New difficulties that enter the analysis are (i) a different treatment of the super-quadratic error term is used: instead of interpolating $L^{2+\delta}(\Omega)$ between $L^2(\Omega)$ and $H^2(\Omega)$, we carry this out elementwise. (ii) the discrete spectrum result of Proposition 4.2 is in place of that of Proposition 2.3.

Steps 1 & 2: Let $E^m := u^m - U^m$ and $G^m := w^m - W^m$. We subtract (4.1)-(4.2) from (3.4)-(3.5) to get the error equations

$$(d_t E^m, \eta_h) + (\nabla G^m, \nabla \eta_h) = 0 \quad \forall\, \eta_h \in \mathcal{S}_h\,, \tag{4.19}$$

$$\varepsilon\,(\nabla E^m, \nabla v_h) + \frac{1}{\varepsilon}\,(f(u^m) - f(U^m), v_h) = (G^m, v_h) \quad \forall\, v_h \in \mathcal{S}_h\,. \tag{4.20}$$

Introduce the decompositions: $E^m := \Theta^m + \Phi^m$ and $G^m := \Lambda^m + \Psi^m$, where

$$\begin{aligned} \Theta^m &:= u^m - P_h u^m\,, &\qquad \Phi^m &:= P_h u^m - U^m\,,\\ \Lambda^m &:= w^m - P_h w^m\,, &\qquad \Psi^m &:= P_h w^m - W^m\,. \end{aligned}$$

Then from the definition of P_h in (4.4)-(4.5) we can rewrite (4.19)-(4.20) as follows

$$(d_t \Phi^m, \eta_h) + (\nabla \Psi^m, \nabla \eta_h) = -(d_t \Theta^m, \eta_h)\,, \tag{4.21}$$

$$\begin{aligned} \varepsilon(\nabla \Phi^m, \nabla v_h) + \frac{1}{\varepsilon}\big(f(P_h u^m) - f(U^m), v_h\big) &= (\Psi^m, v_h) + (\Lambda^m, v_h) \\ &\quad -\frac{1}{\varepsilon}\big(f(u^m) - f(P_h u^m), v_h\big)\,. \end{aligned} \tag{4.22}$$

Since $E^m, \Phi^m \in L^2_0(\Omega)$ for $0 \le m \le M$, setting $\eta_h = -\Delta_h^{-1}\Phi^m$ in (4.21) and $v_h = \Phi^m$ in (4.22) and taking summation over m from 1 to ℓ ($\le M$), after adding the equations we arrive at

$$\begin{aligned} \frac{1}{2}\,\|\,\nabla \Delta_h^{-1} \Phi^\ell\,\|_{L^2}^2 &+ \sum_{m=1}^{\ell} \frac{k_m^2}{2}\,\|\,\nabla \Delta_h^{-1} d_t \Phi^m\,\|_{L^2}^2 \\ &+ \sum_{m=1}^{\ell} k_m \left(\varepsilon \|\,\nabla \Phi^m\,\|_{L^2}^2 + \frac{1}{\varepsilon}\big(f(P_h u^m) - f(U^m), \Phi^m\big)\right) \\ = \sum_{m=1}^{\ell} k_m &\Big(-(d_t \Theta^m, -\Delta_h^{-1}\Phi^m) + (\Lambda^m, \Phi^m)\Big) \\ &+ \frac{1}{\varepsilon}\sum_{m=1}^{\ell} k_m \left(f(u^m) - f(P_h u^m), \Phi^m\right) + \|\,\nabla \Delta_h^{-1} \Phi^0\,\|_{L^2}^2\,. \end{aligned} \tag{4.23}$$

The first sum on the right hand side can be bounded by taking into account the character of the stretched mesh.

$$\begin{aligned} &\sum_{m=1}^{\ell} k_m \Big\{(d_t \Theta^m, \Delta_h^{-1}\Phi^m) + (\Lambda^m, \Phi^m)\Big\} \\ &\le C \sum_{m=1}^{\ell} k_m \Big\{\ln\Big(\frac{1}{k_0}\Big)\|\,\tilde{d}_t \Theta^m\,\|_{H^{-1}}^2 + \frac{2(1-\varepsilon)}{\varepsilon(1-\varepsilon-\gamma_1^\star)}\|\,\Lambda^m\,\|_{H^{-1}}^2\Big\} \\ &\quad + \frac{\varepsilon(1-\varepsilon-\gamma_1^\star)}{2(1-\varepsilon)}\sum_{m=1}^{\ell} k_m \|\,\nabla \Phi^m\,\|_{L^2}^2 + \Big[\ln\Big(\frac{1}{k_0}\Big)\Big]^{-1} \sum_{m=1}^{\ell} \frac{1}{m}\,\|\,\nabla \Delta_h^{-1} \Phi^m\,\|_{L^2}^2 \end{aligned} \tag{4.24}$$

We make use of the following approximation properties in H^{-1} of the elliptic projection P_h, for $\kappa = 0, 1$ (cf. [20])

$$\begin{aligned} \|\,u^m - P_h u^m\,\|_{H^{-1}} &\le C\,h^{3-\kappa}\,\|\,u^m\,\|_{H^{2-\kappa}}\,,\\ \|\,d_t(u^m - P_h u^m)\,\|_{H^{-1}} &\le C\,h^{3-\kappa}\,\|\,d_t u^m\,\|_{H^{2-\kappa}}\,,\\ \|\,d_t(w^m - P_h w^m)\,\|_{H^{-1}} &\le C\,h^{3-\kappa}\|\,w^m\,\|_{H^{2-\kappa}}\,. \end{aligned}$$

Thanks to Lemma 3.1 (vii), we can bound the right hand side of (4.24) by

$$
\begin{aligned}
&C\Big\{\ln\big(\frac{1}{k_0}\big)\tilde{\rho}_1(\varepsilon)\,h^6+\frac{2(1-\varepsilon)}{\varepsilon(1-\varepsilon-\gamma_1^\star)}\,h^6\Big\} && (4.25)\\
&\quad+\frac{\varepsilon(1-\varepsilon-\gamma_1^\star)}{2(1-\varepsilon)}\sum_{m=1}^{\ell}k_m\|\,\nabla\Phi^m\,\|_{L^2}^2+\Big[\ln\Big(\frac{1}{k_0}\Big)\Big]^{-1}\sum_{m=1}^{\ell}\frac{1}{m}\,\|\,\nabla\Delta_h^{-1}\Phi^m\,\|_{L^2}^2\,.
\end{aligned}
$$

Because of the inequality at the beginning of the proof of Proposition 4.2, the second sum on the right hand side of (4.23) can be bounded by

$$
\begin{aligned}
&\frac{1}{\varepsilon}\sum_{m=1}^{\ell}k_m\big(f(u^m))-f(P_hu^m),\Phi^m\big)=\frac{1}{\varepsilon}\sum_{m=1}^{\ell}k_m\big(f'(\xi^m)\Theta^m,\Phi^m\big) && (4.26)\\
&\leq\sum_{m=1}^{\ell}k_m\Big\{\frac{\varepsilon(1-\varepsilon-\gamma_1^\star)}{2(1-\varepsilon)}\,\|\,\nabla\Phi^m\,\|_{L^2}^2+C\,\frac{2(1-\varepsilon)}{\varepsilon^3(1-\varepsilon-\gamma_1^\star)}\,\|\,\Theta^m\,\|_{L^2}^2\,\|\,f'(\xi^m)\,\|_{L^\infty}^2\Big\}\\
&\leq\frac{\varepsilon(1-\varepsilon-\gamma_1^\star)}{2(1-\varepsilon)}\sum_{m=1}^{\ell}k_m\|\,\nabla\Phi^m\,\|_{L^2}^2+C\,h^4\,\frac{2(1-\varepsilon)}{1-\varepsilon-\gamma_1^\star}\,\varepsilon^{-(2\sigma_1+6)}\,\big[\rho_7(\varepsilon,N)\big]^{2(p-2)}\,.
\end{aligned}
$$

By $(\mathrm{GA}_1)_3$, the last term on the left hand side of (4.23) is bounded from below by

$$
\begin{aligned}
&\frac{1}{\varepsilon}\sum_{m=1}^{\ell}k_m\big(f(P_hu^m)-f(U^m),\Phi^m\big) && (4.27)\\
&\qquad\geq\frac{1}{\varepsilon}\sum_{m=1}^{\ell}k_m\Big(\gamma_1\big(f'(P_hu^m)\Phi^m,\Phi^m\big)-\gamma_2\|\,\Phi^m\,\|_{L^{2+\delta}}^{2+\delta}\Big)\,.
\end{aligned}
$$

Substituting (4.24)-(4.27) into (4.23) we arrive at

$$
\begin{aligned}
&\frac{1}{2}\,\|\,\nabla\Delta_h^{-1}\Phi^\ell\,\|_{L^2}^2+\sum_{m=1}^{\ell}\frac{k_m^2}{2}\|\,\nabla\Delta_h^{-1}d_t\Phi^m\,\|_{L^2}^2 && (4.28)\\
&\quad+\frac{\gamma_1}{1-\varepsilon}\sum_{m=1}^{\ell}k_m\Big(\varepsilon\|\,\nabla\Phi^m\,\|_{L^2}^2+\frac{1-\varepsilon}{\varepsilon}\big(f'(P_hu^m)\Phi^m,\Phi^m\big)\Big)\\
&\leq C\,\pi(h,k_0,\varepsilon,N)+\frac{\gamma_2}{\varepsilon}\sum_{m=1}^{\ell}k_m\|\,\Phi^m\,\|_{L^{2+\delta}}^{2+\delta}\\
&\quad+\Big[\ln\Big(\frac{1}{k_0}\Big)\Big]^{-1}\sum_{m=1}^{\ell}\frac{1}{m}\,\|\,\nabla\Delta_h^{-1}\Phi^m\,\|_{L^2}^2\,.
\end{aligned}
$$

We could bound the last term on the left hand side from below using Proposition 4.2, however, this will consume all the contribution of $\varepsilon\|\,\nabla\Phi^m\,\|_{L^2}^2$ on the left hand side. On the other hand, in order to bound the super-quadratic term on the right hand side in Step 3 below, we do need some (small) help from this $\varepsilon\|\,\nabla\Phi^m\,\|_{L^2}^2$. For that reason, we only apply Proposition 4.2 with a scaling factor $\big(1-\frac{k^\beta}{2}\big)$,

$$
\begin{aligned}
&\big(1-\frac{k^\beta}{2}\big)\Big(\,\varepsilon\|\,\nabla\Phi^m\,\|_{L^2}^2+\frac{1-\varepsilon}{\varepsilon}\big(f'(P_hu^m)\Phi^m,\Phi^m\big)\,\Big) && (4.29)\\
&\qquad\geq-\big(1-\frac{k^\beta}{2}\big)(1-\varepsilon)(C_0+2)\|\,\nabla\Delta^{-1}\Phi^m\,\|_{L^2}^2\,,\\
&\qquad\geq-(C_0+1)\|\,\nabla\Delta^{-1}\Phi^m\,\|_{L^2}^2\,.
\end{aligned}
$$

The leftover term is controlled by

$$
\begin{aligned}
&\frac{\gamma_1 k^\beta}{\varepsilon}\big(f'(P_h u^m)\Phi^m, \Phi^m\big) \qquad (4.30)\\
&\qquad\le \frac{\gamma_1 \varepsilon k^\beta}{4}\|\nabla \Phi^m\|_{L^2}^2 + \tilde{c}_0\,\frac{\gamma_1 k^\beta}{\varepsilon^3}\,\|\nabla\Delta_h^{-1}\Phi^m\|_{L^2}^2\,.
\end{aligned}
$$

Combining (4.28)-(4.30) we finally get

$$
\begin{aligned}
&\frac{1}{2}\|\nabla\Delta_h^{-1}\Phi^\ell\|_{L^2}^2 + \sum_{m=1}^{\ell} k_m\Big[\frac{k_m}{2}\|\nabla\Delta_h^{-1}d_t\Phi^m\|_{L^2}^2 + \frac{\gamma_1\varepsilon k^\beta}{2(1-\varepsilon)}\|\nabla\Phi^m\|_{L^2}^2\Big]\\
&\le C\,\pi(h,k_0,\varepsilon,N) + \frac{\gamma_2}{\varepsilon}\sum_{m=1}^{\ell} k_m\|\Phi^m\|_{L^{2+\delta}}^{2+\delta} \qquad (4.31)\\
&\qquad + \Big\{\Big[\ln\Big(\frac{1}{k_0}\Big)\Big]^{-1} + \tilde{c}_0\gamma_1 k^\beta\varepsilon^{-3}\Big\}\sum_{m=1}^{\ell}\frac{1}{m}\|\nabla\Delta_h^{-1}\Phi^m\|_{L^2}^2\,.
\end{aligned}
$$

where we have used the fact that $0<\varepsilon<1$ and $\|\nabla\Delta^{-1}v_h\|_{L^2} = \|\nabla\Delta_h^{-1}v_h\|_{L^2}$ for any $v_h\in\mathring{\mathcal{S}}_h$.

Step 3: Notice that the structure of (4.31) is exactly the same as in (3.35). Hence, we can follow the argumentation of Step 3 in the proof of Theorem 3.4 with the following modification: the estimates (3.37) and (3.40) (up to second to the last inequality) are performed for every $K\in\mathcal{T}_h$; by convexity of the function $g(s)=s^q$, for $q\ge 1$ and $s\ge 0$ we can afterwards recover estimates for $\Omega=\bigcup K$ which involves a uniform constant C. We refer to Section 3 of [26] for the details of a similar type argument. Finally, we make use of the properties of $-\Delta_h^{-1}$ and $\Phi^m\in\mathcal{S}_h$, for $0\le m\le M$. Consequently, we can reduce (4.31) into the following form which corresponds to (3.41),

$$
\begin{aligned}
&\frac{1}{2}\|\nabla\Delta_h^{-1}\Phi^\ell\|_{L^2}^2 + \sum_{m=1}^{\ell} k_m\Big\{\frac{k_m}{2}\|\nabla\Delta_h^{-1}d_t\Phi^m\|_{L^2}^2 + \frac{\gamma_1\varepsilon k^\beta}{2(1-\varepsilon)}\|\nabla\Phi^m\|_{L^2}^2\Big\}\\
&\le C\,\pi(h,k_0,\varepsilon,N) + \Big\{\Big[\ln\Big(\frac{1}{k_0}\Big)\Big]^{-1} + \tilde{c}_0\gamma_1 k^\beta\varepsilon^{-3}\Big\}\sum_{m=1}^{\ell}\frac{1}{m}\|\nabla\Delta_h^{-1}\Phi^m\|_{L^2}^2\\
&\qquad + C\,\big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{2N\delta}{8-(4-N)\delta}}\sum_{m=0}^{\ell} k_m\big[\varepsilon k_m^\beta\big]^{-\frac{8+(4-N)\delta}{8-(4-N)\delta}}\|\nabla\Delta_h^{-1}\Phi^m\|_{L^2}^{2(1+\frac{(4-N)\delta}{8-(4-N)\delta})}\\
&\qquad + C\,k_0^{2+4\frac{(4-N)\delta}{8-(4-N)\delta}}\,\varepsilon^{-\{\frac{(2\sigma_1+1)(4-N)\delta}{8-(4-N)\delta}+2(\sigma_1+2)\}}\,\big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{2N\delta}{8-(4-N)\delta}}\,. \qquad (4.32)
\end{aligned}
$$

Step 4: We conclude the proof by an inductive argument based on (4.32). Suppose that for

$$
k_0 \le \tilde{C}\,\min\Big\{\varepsilon^{\frac{3}{\beta}}, \varepsilon^{\frac{\alpha_0}{2}}, \big[\rho_3(\varepsilon)\big]^{\tilde{\rho}_4(N,\beta)}, \big[\rho_5(\varepsilon,N)\big]^{\tilde{\rho}_6(N,\beta)}\Big\} \qquad (4.33)
$$

and $0<\beta<\frac{1}{2}$, there exist two positive constants

$$
\tilde{c}_1 = \tilde{c}_1(t_\ell,\Omega,u_0,\sigma_i,p)\,,\quad \tilde{c}_2 = \tilde{c}_2(t_\ell,\Omega,u_0,\sigma_i,p;C_0)\,,
$$

independent of k and ε, such that the following inequality holds

$$\max_{0\leq m\leq \ell} \frac{1}{2} \| \nabla\Delta_h^{-1}\Phi^m \|_{L^2}^2 + \sum_{m=1}^{\ell} k_m \Big\{ \frac{k_m}{2} \| \nabla\Delta_h^{-1} d_t\Phi^m \|_{L^2}^2 + \frac{\gamma_1 \varepsilon k_m^{\beta}}{2} \| \nabla\Phi^m \|_{L^2}^2 \Big\} \\ \leq \tilde{c}_1 \Big(\pi(h,k_0,\varepsilon,N) k_0^{2-\beta} \rho_2(\varepsilon) \Big) \exp\big(c_2 t_\ell\big) . \tag{4.34}$$

Criterion (3.52) then gives the third condition in (4.34).

We also need to make sure that

$$\sum_{m=1}^{\ell+1} k_m \big[\varepsilon k_m^{\beta}\big]^{-\frac{8+(4-N)\delta}{8-(4-N)\delta}} \big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{2N\delta}{8-(4-N)\delta}} \Big(\pi(h,k_0,\varepsilon,N) + k_0^{2-\beta}\rho_2(\varepsilon)\Big)^{1+\frac{(4-N)\delta}{8-(4-N)\delta}} \\ \leq \frac{\tilde{c}_1}{2} \Big(\pi(h,k_0,\varepsilon,N) + k_0^{2-\beta}\rho_2(\varepsilon)\Big) \exp\big(c_2 t_{\ell+1}\big) . \tag{4.35}$$

This completes the induction proof of (4.34).

Finally, the assertion (i) follows from applying the triangle inequality on $E^m = \Theta^m + \Phi^m$ and Theorem 3.6. The assertion (ii) follows in the same way. The proof is complete. □

Remark: (a). The proof clearly shows how the three mesh conditions arise. The first entry in condition 1) reflects the subtle interplay between accuracy requirements and smallness of the time-steps k_0; the second entry is for the stability of the time discretization (see $(\mathrm{GA})_3$). The remaining entries in 1) account for the super-quadratic nonlinearity of f (see $(\mathrm{GA}_1)_3$). The conditions 2) and 3) are to ensure the discrete spectrum estimate (see Proposition 4.2). A slight coupling of temporal and spatial discretization parameters is given by 4), which deteriorates as β gets smaller.

(b). Clearly, both $U^0 = Q_h u_0$ and $U^0 = P_h u_0$ satisfy the condition 5) with $\nu = 1$. The L^2 projection $Q_h u_0$ has the advantage of being cheaper to be obtained compared to the elliptic projection $P_h u_0$. Also, we note that the first identity in 5) is necessary in oder for the fully discrete scheme (4.1)-(4.2) to conserve the mass.

We conclude this section and the paper with a corollary which addresses the case $u_0 \in H^3(\Omega)$ and $\partial\Omega \in C^{2,1}$. The subsequent corollary extends Corollary 3.5 to the fully discrete case (4.1)-(4.2), allowing for the equidistant mesh J_k^1. Its proof follows the lines of that for Theorem 4.3; the only exception is that the estimates (4.24) and (4.25) become easier since the estimates (v) and (viii) of Lemma 3.1 can be utilized. We leave the details of the verification to the interested readers.

Corollary 4.4. *Let $\big\{U^m, W^m\big\}_{m=0}^M$ solve (4.1)-(4.2) on a quasi-uniform mesh J_k^1 and a quasi-uniform triangulation $\mathcal{T}_h$ of Ω. Let the assumptions of Corollary 3.5 hold and the notation of Theorem 4.3 be valid. For $0 < \beta < \frac{1}{2}$, $\nu > 0$ and $N = 2, 3$, let*

$$\hat{\pi}(h,\varepsilon,N) := h^4 \Big\{ h^{2\nu} + \rho_1(\varepsilon) + \frac{2(1-\varepsilon)}{1-\varepsilon-\gamma_1^{\star}} \varepsilon^{-(2\sigma_1+6)} \big[\rho_7(\varepsilon,N)\big]^{2(p-2)} \Big\} ,$$

suppose that k, h and U^0 satisfy the following constraints

1). $k \le \tilde{C}\, \min\Big\{\varepsilon^{\frac{3}{\beta}}, \varepsilon^{\alpha_0}, \big[\rho_3(\varepsilon,N)\big]^{\tilde{\rho}_4(N,\beta)}, \big[\rho_5(\varepsilon,N)\big]^{\tilde{\rho}_6(N,\beta)}\Big\}$,

2). $k \le \Big\{\varepsilon^{2(\sigma_1+1)}\,\big[\rho_2(\varepsilon)\big]^{-1}\Big(\frac{\varepsilon^2}{c}\,\big[\tilde{\rho}_1(\varepsilon)\big]^{-\frac{N}{8}}\Big)^{\frac{16}{4-N}}\Big\}^{\frac{1}{2-\beta}}$,

3). $h^{\frac{4-N}{2}}\,|\ln h|^{\frac{3-N}{2}} \le \Big(C_2(\varepsilon)\,C_3\,\big[\rho_1(\varepsilon)\big]^{\frac{1}{2}}\Big)^{-1}\varepsilon^2$,

4). $k^{-\beta\frac{8+(4-N)\delta}{8-(4-N)\delta}}\Big(\hat{\pi}(h,\varepsilon,N)+k^{2-\beta}\rho_2(\varepsilon)\Big)^{\frac{(4-N)\delta}{8-(4-N)\delta}} \le \Big(\varepsilon\big[\tilde{\rho}_1(\varepsilon)\big]^{\frac{2N\delta}{8-(4-N)\delta}}\Big)^{-1}$,

5). $(U^0,1)=(u_0,1)$ *and* $\|\, U^0-u_0\,\|_{H^{-1}} \le Ch^{2+\nu}\|\, u_0\,\|_{H^2}$.

Then the solution of (4.1)-(4.2) satisfies the error estimates

$$
\begin{aligned}
\text{(i)}\quad & \max_{0\le m\le M}\|\, u(t_m)-U^m\,\|_{H^{-1}} + \Big(k\sum_{m=1}^{M} k\|\, d_t\big(u(t_m)-U^m\big)\,\|_{H^{-1}}^2\Big)^{\frac{1}{2}} \\
& \qquad + \Big(k\sum_{m=1}^{M}\|\, u(t_m)-U^m\,\|_{L^2}^2\Big)^{\frac{1}{2}} \\
& \qquad \le C\left\{h^4\varepsilon^{-(2\sigma_1+3)} + \hat{\pi}(h,\varepsilon,N) + k_0^{(2-\beta)}\rho_2(\varepsilon)\right\}^{\frac{1}{2}}, \\
\text{(ii)}\quad & \Big(k\sum_{m=1}^{M}\|\, \nabla(u(t_m)-U^m)\,\|_{L^2}^2\Big)^{\frac{1}{2}} \\
& \qquad \le C\left\{h^2\varepsilon^{-(2\sigma_1+4)} + \hat{\pi}(h,\varepsilon,N) + k_0^{(2-\beta)}\rho_2(\varepsilon)\right\}^{\frac{1}{2}}.
\end{aligned}
$$

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