Applications of Moving Frames to Group Foliation of Differential Equations

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Peter J. Olver

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Abstract

The classical group foliation algorithm uses the continuous symmetries of a differential equation to aid in its integration. This is accomplished by transforming the differential equation into two alternative systems, called the resolving and automorphic systems. Incorporating the theory of equivariant moving frames for Lie pseudogroups, a completely symbolic and systematic version of the group foliation algorithm is introduced. In this version of the algorithm, the resolving system is derived using only knowledge of the structure of the differential invariant algebra, requiring no explicit formulae for differential invariants. Additionally, the automorphic system is replaced by an equivalent reconstruction system, again requiring only symbolic computation. The efficacy of this approach is illustrated through several examples. Further applications of aspects of group foliation are given, including the construction of Bäcklund transformations using resolving systems and a reconstruction process for an invariant submanifold flow corresponding to a given invariant signature evolution.

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Chapter 1

Introduction

This thesis is primarily concerned with an algorithm for producing explicit formulae for solutions of ordinary and partial differential equations, known as group foliation. Group foliation uses the symmetries of a differential equation to split the problem of integration into one of solving two alternative systems of differential equations, known as the resolving system and the automorphic system. Group foliation is not a new idea by any means; it was first proposed by S. Lie through suggestive examples in a paper of 1895, [**46**], and developed quite generally by E. Vessiot shortly thereafter, [**78**]. What makes group foliation worth reconsideration more than a century later is a series of new developments in the basic machinery used for the algorithm.

Beginning in 1998, P. Olver and M. Fels proposed a systematic approach to the study of the invariants of Lie groups, known as the method of equivariant moving frames, [**21**, **22**]. This approach has since proved incredibly fruitful, yielding diverse adaptations and applications to classical invariant theory, [**35**, **54**], invariant variational calculus, [**27**, **37**, **38**, **73**], invariant submanifold flows, [**58**], computer vision, [**10**, **11**], numerical methods for differential equations, [**34**, **70**], and equivalence problems, [**80**], among others. The newest development prompting the reconsideration of group foliation is the adaptation of equivariant moving frames to infinite dimensional Lie pseudogroups. The foundations of this theory appear a sequence of papers by P. Olver and J. Pohjanpelto, [**60**–**63**]. By constructively elucidating

the structure of the differential invariant algebra of an infinite dimensional Lie pseudogroup admitting a moving frame, Olver and Pohjanpelto supplied the tools necessary to carry out the group foliation algorithm via a systematic and completely symbolic procedure.

Group foliation is a process of reduction and reconstruction. The reduction step uses differential invariants for variables and seeks to characterize relations among these variables and their invariant derivatives through a system of differential equations; these equations constitute the resolving system. As such, this step requires knowledge of the structure of the differential invariant algebra, particularly the relations, or syzygies, among the invariant derivatives of the differential invariants of a Lie pseudogroup. Through the application of invariant recurrence relations, these syzygies may be determined without the need for explicit coordinate formulae for invariants. With the additional benefit of Gröbner basis techniques in differential algebra, the determination is completely algorithmic.

Unsurprisingly, the reconstruction process also benefits from the machinery of moving frames. From a geometric perspective, a moving frame facilitates reduction by providing a projection onto a space of invariant coordinates. The inverse of the moving frame reverses this projection and provides a means for reconstructing solutions of the original problem from those of the reduced problem. The end result is a system of reconstruction equations that may be derived symbolically, relying on Olver and Pohjanpelto's characterization of Maurer–Cartan forms for a Lie pseudogroup together with the invariant recurrence relations. This procedure is an alternative to reconstruction via the classical automorphic system, which requires explicit formulae for the differential invariants and often results in reconstruction equations of greater complexity.

The outline of this thesis is as follows. Chapter 2 is dedicated to prerequisite material, including a pedestrian introduction to Lie pseudogroups and basic constructions and computations from the theory of moving frames. We assume basic familiarity with the language of jets and symmetry methods for differential equations as can be found in introductory texts such as [**52**, **82**].

In Chapter 3 we present the group foliation algorithm. Section 3.2 revisits the classical algorithm essentially as Lie envisioned it, with the added benefit of the language of moving frames. Some ideas not explored in the classical theory are introduced, such as the notion of rank of a resolving system. After illustrating the possible shortcomings of reconstruction via classical automorphic systems, reconstruction equations are introduced in Section 3.3. Because the algorithm presented should ostensibly be useful for actually solving equations, we present four detailed examples in Section 3.4, in addition to two simple running examples illustrating the constructions. In these examples it is also shown how group foliation provides a perspective which subsumes the theory of group invariant and partially invariant solutions. In the last section of Chapter 3 we give a detailed historical overview of group foliation, presenting the original ideas of Lie and Vessiot together with discussion of a largely overlooked paper of H. H. Johnson providing a link between classical and modern perspectives on the algorithm.

Chapter 4 contains two applications of the ideas introduced in Chapter 3. The first is to the construction of Bäcklund transformations, given in Section 4.2. These constructions utilize a new theory of inductive moving frames; the basics of this theory are first presented in Section 4.1. The second application is to the reconstruction of invariant submanifold flows. We first present in Section 4.3 prerequisite ideas from the theory of invariant submanifold flows, including formulae for the evolution of a differential invariant signature concurrent with an invariant flow. Section 4.4 presents a derivation of submanifold flow reconstruction equations. These equations govern the evolution of reconstruction parameters accompanying a differential invariant signature flow and allow for the decomposition of an invariant submanifold flow into two systems of evolutionary partial differential equations: a signature evolution and a reconstruction evolution. For simplicity, we restrict our presentation to curve flows; generalization is straightforward. Finally, the last chapter outlines several directions for future research.

Chapter 2

Preliminaries

2.1 Lie pseudogroups

Introduction and formal definitions

Lie pseudogroups are collections of transformations of a manifold. The slight of "pseudo" comes from the fact that composition of these transformations is not always defined, and the eponym "Lie" indicates the essential property that the transformations are defined by certain differential equations, called *defining equations*.

Because we will consider infinite dimensional Lie pseudogroups, all transformations and underlying manifolds will be assumed analytic to avoid technical complications. Let M be an analytic manifold and consider the collection $\mathcal{D}(M)$ of all local diffeomorphisms $\varphi : M \to M$. By *local diffeomorphism* we mean that φ is a bijective analytic map between open subsets of M, with analytic inverse. Before proceeding with generalities, we give a few instructive examples.

Example 2.1. The full collection $\mathcal{D}(M)$ of all local diffeomorphisms of an analytic manifold is a Lie pseudogroup. Its defining equations are empty. All pseudogroups considered will be subpseudogroups of this collection.

Example 2.2. Consider transformations of $\mathbb{R}^2$ given by

$$X = \lambda x + a, \qquad U = \lambda u + b, \qquad \lambda \in \mathbb{R}^+ \quad a, b \in \mathbb{R}^2$$

Mappings of this form are clearly local diffeomorphisms, with inverse transformations

$$x = \overline{\lambda} X + \overline{a}, \qquad u = \overline{\lambda} U + \overline{b},$$

where $\overline{\lambda} = 1/\lambda$, $\overline{a} = -a/\lambda$ and $\overline{b} = -b/\lambda$. This Lie pseudogroup is said to be of *finite type*, with transformations corresponding to an action of the Lie group $\mathbb{R}^+ \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$. These transformations may also be specified as the solutions to a set of *determining equations*

$$X_{xx} = 0, \qquad X_u = 0, \qquad U_{uu} = 0, \qquad U_x = 0, \qquad X_x = U_u.$$

Example 2.3. Consider transformations of $\mathbb{R}^2$ of the form

$$X = f(x), \qquad U = \frac{u}{f'(x)},$$

where f is a local diffeomorphism of $\mathbb{R}$. The determining equations for these transformations are

$$X_u = 0, \qquad U = \frac{u}{X_x}.$$

This Lie pseudogroup is said to be of *infinite type* and can be considered an infinite dimensional collection of transformations, in contrast with the finite dimensional collection from Example 2.2.

Example 2.4. It is interesting to notice that transformations of $\mathbb{R}^2$ given by

$$X = f(x), \qquad U = f(u) \qquad (\textit{note: same } f) \tag{2.1}$$

cannot be specified by a set of determining equations. Intuitively, we might try

$$X_u = 0, \qquad U_x = 0, \tag{2.2}$$

but these in fact determine the larger pseudogroup

$$X = f(x), \qquad U = g(u), \tag{2.3}$$

and it is not possible to specify through determining equations that $f = g$. The transformations (2.2) constitute a pseudogroup that is not a Lie pseudogroup. The larger pseudogroup (2.3) *is* a Lie pseudogroup, known as the Lie completion of (2.1), [**30**].

We are now better prepared to digest the full definition of a pseudogroup. The purpose of this definition is to combine the essential properties of a group–or more accurately a groupoid–(items three and four) with common operations of restriction and "gluing" performed with transformations (items one and two).

Definition 2.5. A collection $\mathcal{G}$ of local diffeomorphisms of a manifold M is called a *pseudogroup* if it satisfies the following properties:

- if U is an open subset of M and $\varphi : U \to M$ is in $\mathcal{G}$, then $\varphi|_V \in \mathcal{G}$ for all open $V \subset U$.

- if U_ν are open subsets of M with $U = \bigcup_\nu U_\nu$ and $\varphi : U \to M$ is a local diffeomorphism with $\varphi|U_\nu \in \mathcal{G}$ for all ν, then $\varphi \in \mathcal{G}$.

- if $\varphi : U \to M$ and $\psi : V \to M$ both belong to $\mathcal{G}$ with $\varphi(U) \subset V$, then $\psi \circ \varphi \in \mathcal{G}$.

- if $\varphi : U \to M$ is in $\mathcal{G}$ and $V = \varphi(U)$, then $\varphi^{-1} : V \to M$ is in $\mathcal{G}$.

Note that these axioms imply that the identity diffeomorphism $\mathbb{1} : M \to M$ belongs to $\mathcal{G}$.

Pseudogroups are very general objects and, as illustrated by Example 2.4, may not be defined by properties that can be verified locally. We restrict our considerations to Lie pseudogroups, a class of pseudogroups whose transformations are defined by an involutive system of partial differential equations. We use the language of jets to facilitate discussion of Lie pseudogroups. Interestingly, jet spaces were invented by Ehressman for this very purpose, [**19**]. See [**52**, **82**] for a modern presentation of jet spaces.

For $n \geq 0$ let $\mathcal{D}^{(n)} \subset J^n(M, M)$ be the bundle formed by the nth order jets $\mathrm{j}_n\varphi$ of local diffeomorphisms φ. Local coordinates $z = (z^1, \ldots, z^m)$ and $\varphi(z) = Z = (Z^1, \ldots, Z^m)$ on $\mathcal{D}(M)$ induce coordinates $\mathrm{j}_n\varphi = (z, Z^{(n)})$ on $\mathcal{D}^{(n)}$, where $Z^{(n)}$ indicates the derivatives

$$Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \cdots \partial z^{a_k}}, \qquad b = 1, \ldots, m, \quad A = (a_1, \ldots, a_k),$$

of order $0 \le k \le n$. The coordinates $z = \boldsymbol{\sigma}(\mathrm{j}_n\varphi)$ and $Z = \boldsymbol{\tau}(\mathrm{j}_n\varphi)$ are called the source and target coordinates of φ. A local diffeomorphism $\psi \in \mathcal{D}$ acts on $\mathcal{D}^{(n)}$ by either left composition or right composition with the inverse, when defined:

$$L_\psi(\mathrm{j}_n\varphi) = \mathrm{j}_n(\psi \circ \varphi) \qquad \text{or} \qquad R_\psi(\mathrm{j}_n\varphi) = \mathrm{j}_n(\varphi \circ \psi^{-1}). \tag{2.4}$$

Example 2.6. Consider $M = \mathbb{R}$. Concretely, the diffeomorphism jet bundle $J^2(\mathbb{R}, \mathbb{R})$ is an open subset of $\mathbb{R}^4$, with local coordinates (x, X, X_x, X_{xx}), where $X_x \ne 0$ as required by the inverse function theorem. Composition in the diffeomorphism jet bundle is simply composition of Taylor series: if $X = \varphi(x)$ and $\boldsymbol{X} = \psi(X)$, then

$$\begin{aligned}
\mathrm{j}_2(\psi \circ \varphi)(x) &= (X, \boldsymbol{X}, \boldsymbol{X}_X, \boldsymbol{X}_{XX}) \cdot (x, X, X_x, X_{xx}) \\
&= (x, \boldsymbol{X}, \boldsymbol{X}_X X_x, \boldsymbol{X}_X X_{xx} + \boldsymbol{X}_{XX} X_x^2).
\end{aligned}$$

There are natural projections $\pi_k^n : \mathcal{D}^{(n)} \to \mathcal{D}^{(k)}$ corresponding to truncation of Taylor series. These projections constitute an inverse system, and we define the inverse limit

$$\mathcal{D}^{(\infty)} = \varprojlim \mathcal{D}^{(n)}.$$

In practice, a Lie pseudogroup is distinguished by the fact that its transformations are specified by a system of differential equations. These differential equations are characterized by the following definition.

Definition 2.7. Let $\mathcal{G} \subset \mathcal{D}$ be a pseudogroup and $\mathcal{G}^{(n)}$ the collection of n-th order jets of $\mathcal{G}$. $\mathcal{G}$ is called a *Lie pseudogroup* of order $n^\star$ if for all $n \ge n^\star$

- $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ forms a smooth, embedded subbundle

- $\pi_n^{n+1} \colon \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a bundle projection,

- if $\varphi \in \mathcal{D}$ satisfies $\mathrm{j}_{n^\star}\varphi \subset \mathcal{G}^{(n^\star)}$ then $\varphi \in \mathcal{G}$.

The conditions of Definition 2.7 imply that $\mathcal{G}^{(n^\star)}$ is described by a system of $n^{\star th}$ order differential equations

$$F^{(n^\star)}(z, Z^{(n^\star)}) = 0, \tag{2.5}$$

called the *determining system* of $\mathcal{G}$. For $n \geq n^\star$, $\mathcal{G}^{(n)}$ is described by the prolongation of the determining equations (2.5). This prolongation is obtained by repeated application of the total differential operators

$$\mathbb{D}_{z^b} = \frac{\partial}{\partial z^b} + \sum_{\#A \geq 0} Z_{A,b} \frac{\partial}{\partial Z_A}, \qquad b = 1, \ldots, m, \qquad (2.6)$$

called the *total derivative operators* on $\mathcal{D}^{(\infty)}$. These total derivative operators are simply a formalized way of differentiating while treating the target variables Z as functions of the source variables z.

Example 2.8. The following Lie pseudogroup of transformations of $\mathbb{R}^3$ will be used throughout the thesis as a running example:

$$X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}, \qquad (2.7)$$

where f is a local diffeomorphism of $\mathbb{R}$. This Lie pseudogroup has determining equations

$$X_y = X_u = 0, \qquad Y = y, \qquad U = \frac{u}{X_x}. \qquad (2.8)$$

The pseudogroup jets are obtained via differentiation:

$$\begin{aligned}
X = f, \qquad Y = y, \qquad U = \frac{u}{f_x}, \qquad X_x = f_x, \\
X_u = 0, \qquad U_x = \frac{u f_{xx}}{f_x^2}, \qquad U_u = \frac{1}{f_x}, \qquad X_{xx} = f_{xx}, \qquad \ldots
\end{aligned} \qquad (2.9)$$

where $f, f_x, f_{xx}, \ldots$ are the Taylor coefficients of $f(x)$ at the source point z and can be treated as parameters for an infinite dimensional Lie pseudogroup transformation analogous to the parameters of a finite dimensional Lie group action. Practically speaking, the pseudogroup jets are determined by choice of a source point (x, y, u) and the pseudogroup parameters $f, f_x, f_{xx}, \ldots$.

Example 2.9. Transformations given by action of a Lie group G comprise a Lie pseudogroup. Consider the set of invariant differential forms μ^i on M dual to the infinitesimal generators of the Lie group action. The μ^i are a coframe along the orbits of the action, and a local diffeomorphism φ corresponds to the action of an element of G if and only if $\varphi^*(\mu^i) = \mu^i$ for all i. This yields a set of first order determining equations for φ.

Definition 2.10. A Lie pseudogroup $\mathcal{G}$ is said to be of *finite type* if there is a finite k such all transformations of $\mathcal{G}$ are determined by their k-jet. Otherwise $\mathcal{G}$ is said to be of *infinite type*.

Lie pseudogroups of finite type correspond to actions of finite dimensional Lie groups. Thus all of the theory herein also applies to Lie group actions.

The infinitesimal approach

The true utility of Lie pseudogroups comes from the ability to work with objects and transformations at an infinitesimal level. Roughly, one can think of passing to the infinitesimal as a process of linearization. We now describe the characterization of a Lie pseudogroup in terms of its Lie algebra of infinitesimal generators, and the passage from the (possibly nonlinear) determining system (2.5) for $\mathcal{G}$ to a linear determining system (2.11) for the infinitesimal generators of $\mathcal{G}$.

Let $\mathcal{X}(M)$ be the space of locally defined vector fields on M. In local coordinates $z = (z^1, \ldots, z^m)$ such a vector field $\mathbf{v}$ may be expressed as

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a}. \tag{2.10}$$

Let $\mathrm{J}^n TM$ be the nth order jet bundle of the tangent bundle of M. A jet of a vector field $\mathbf{v}$ (i.e a jet of a section of this bundle) will have a local coordinate expression $\mathrm{j}_n \mathbf{v} = (z, \zeta^{(n)})$, where $\zeta^{(n)}$ denotes the collection of derivatives ζ_A^a, obtained by applying the total derivative operators (2.6):

$$\zeta_A^a = \mathbb{D}_A \zeta^a \qquad a = 1, \ldots, m, \quad 0 \leq \#A \leq n.$$

Given a Lie pseudogroup $\mathcal{G}$, a (locally defined) Lie algebra $\mathfrak{g} \subset \mathcal{X}(M)$ is determined by the condition that the flow of a local vector field is a transformation belonging to the pseudogroup $\mathcal{G}$. In fact, the Lie algebra $\mathfrak{g}$ is characterized by the condition that a vector field (2.10) is in $\mathfrak{g}$ if its $n^\star$-jet is a solution of a certain linear system of partial

differential equations

$$L^{(n^\star)}(z, \zeta^{(n^\star)}) = 0. \tag{2.11}$$

These equations are called the *infinitesimal determining system* of $\mathfrak{g}$, and are obtained by linearizing the determining system (2.5) at the identity transformation.

Example 2.11. The infinitesimal generators of the pseudogroup action (2.7) are local vector fields

$$\mathbf{v} = \xi(x, y, u)\frac{\partial}{\partial x} + \eta(x, y, u)\frac{\partial}{\partial y} + \phi(x, y, u)\frac{\partial}{\partial u} \tag{2.12}$$

whose coefficients ξ, η, ϕ are solutions to the infinitesimal determining system

$$\xi_y = \xi_u = 0, \qquad \eta = 0, \qquad \phi = -u\,\xi_x. \tag{2.13}$$

This system is obtained by linearizing the determining equations (2.8) at the jet of the identity transformation $\mathbb{1}^{(1)}$. Alternatively we can solve the determining equations and express the infinitesimal generators directly in terms of the solution:

$$\mathbf{v} = a(x)\frac{\partial}{\partial x} - u\,a_x(x)\frac{\partial}{\partial u},$$

where $a(x)$ is an arbitrary analytic function. Relations among higher order vector field jets are obtained by prolongation of (2.13).

Maurer–Cartan forms

Dual to the infinitesimal picture of Lie pseudogroups is the theory of Maurer–Cartan forms. We now recall the construction of Maurer–Cartan forms for Lie pseudogroups, following [**60**].

Because $\mathcal{D}^{(\infty)}(M) \subset J^\infty(M, M)$, there is a natural splitting of the cotangent bundle of $\mathcal{D}^{(\infty)}(M)$ induced by source and target coordinates. As a result, the exterior derivative splits into horizontal and vertical (or pseudogroup) components

$$d = d_M + d_{\mathcal{G}}.$$

This splitting should be interpreted as a separation of d into exterior differentiation with respect to manifold coordinates, d_M, and pseudogroup parameters, $d_{\mathcal{G}}$. Also observe that the splitting is invariant under pseudogroup multiplication (2.4), meaning d_M and $d_{\mathcal{G}}$ preserve invariance of differential forms.

The *zero order right Maurer–Cartan forms* are the right invariant contact forms

$$\mu^a = d_{\mathcal{G}} Z^a = dZ^a - \sum_{b=1}^m Z_{z^b}^a dz^b.$$

Because the target coordinates Z^a are right invariant, the right Maurer–Cartan forms are also right invariant. Higher order right Maurer–Cartan forms

$$\mu_A^a = \mathbb{D}_Z^A \mu^a \tag{2.14}$$

are obtained via Lie differentiation with respect to the implicit differentiation operators

$$\mathbb{D}_{Z^a} = \sum_{b=1}^m z_{Z^a}^b \, \mathbb{D}_{z^b}, \qquad \text{where} \qquad (z_{Z^a}^b) = (Z_{z^b}^a)^{-1}. \tag{2.15}$$

The expressions $z_{Z^a}^b$ provide an alternative collection of pseudogroup jet coordinates and will be used extensively later in the thesis. We denote by $\mu^{(n)}$ the set of right Maurer–Cartan forms of order $\leq n$.

Example 2.12. For the diffeomorphism pseudogroup $\mathcal{D}(\mathbb{R}^3)$ with local coordinates

$$X = f(x, y, u), \qquad Y = g(x, y, u), \qquad U = h(x, y, u),$$

the zero order right Maurer–Cartan forms are

$$\begin{aligned}
\mu^x &= dX - X_x \, dx - X_y \, dy - X_u \, du \\
\mu^y &= dY - Y_x \, dx - Y_y \, dy - Y_u \, du \\
\mu^u &= dU - U_x \, dx - U_y \, dy - U_u \, du.
\end{aligned} \tag{2.16}$$

The implicit differentiation operators (2.15) are

$$\begin{bmatrix} \mathbb{D}_X \\ \mathbb{D}_Y \\ \mathbb{D}_U \end{bmatrix} = \begin{bmatrix} X_x & X_y & X_u \\ Y_x & Y_y & Y_u \\ U_x & U_y & U_u \end{bmatrix}^{-T} \begin{bmatrix} \mathbb{D}_x \\ \mathbb{D}_y \\ \mathbb{D}_u \end{bmatrix}. \tag{2.17}$$

and the higher order right Maurer–Cartan forms are obtained by Lie differentiating (2.16) with respect to (2.17).

Example 2.13. Zero order right Maurer–Cartan forms for the Lie pseudogroup (2.7) may be obtained by applying the determining system (2.8) to (2.16)

$$
\begin{aligned}
\mu^x &= dX - X_x\, dx \\
\mu^y &= 0 \\
\mu^u &= d\left(\frac{u}{X_x}\right) - \frac{uX_{xx}}{X_x^2}\, dx - \frac{1}{X_x}\, du \\
&= \frac{u}{X_x^2}\, dX_x - \frac{uX_{xx}}{X_x^2}\, dx.
\end{aligned}
\tag{2.18}
$$

Also applying (2.8) to (2.17) gives the implicit differentiation operators for $\mathcal{G}$:

$$
\begin{bmatrix} \mathbb{D}_X \\ \mathbb{D}_Y \\ \mathbb{D}_U \end{bmatrix}
= \begin{bmatrix} X_x & 0 & 0 \\ 0 & 1 & 0 \\ \frac{uX_{xx}}{X_x^2} & 0 & \frac{1}{X_x} \end{bmatrix}^{-T}
\begin{bmatrix} \mathbb{D}_x \\ \mathbb{D}_y \\ \mathbb{D}_u \end{bmatrix}
= \begin{bmatrix} \frac{1}{X_x}\mathbb{D}_x + \frac{UX_{xx}}{X_x}\mathbb{D}_u \\ \mathbb{D}_y \\ X_x\mathbb{D}_u \end{bmatrix}.
\tag{2.19}
$$

Lie differentiating the zero order Maurer–Cartan forms with respect to (2.19) yields the higher order Maurer–Cartan forms. For example,

$$
\begin{aligned}
\mu_X^x &= \mathbb{D}_X\mu^x = \frac{1}{X_x}dX_x - \frac{X_{xx}}{X_x}dx, \\
\mu_{XX}^x &= \mathbb{D}_X\mu_X^x = \frac{1}{X_x^2}\, dX_{xx} - \frac{X_{xx}}{X_x^3}\, dX_x - \frac{X_{xxx}X_x - X_{xx}^2}{X_x^3}\, dx,
\end{aligned}
\tag{2.20}
$$

from which it might be observed that $\mu^u = -u\mu_X^x$. In fact, all Maurer–Cartan forms may be written in terms of $\mu_{X^k}^x, k \geq 0$, and this information may be gleaned without explicit knowledge of the Maurer–Cartan forms, as we now explain.

Application of the determining equations, as in Example 2.13, amounts to pulling back by the inclusion map $i\colon \mathcal{G}^{(\infty)} \hookrightarrow \mathcal{D}^{(\infty)}$. The pulled-back Maurer–Cartan forms $i^*(\mu_A^a)$ are no longer linearly independent, but instead satisfy linear relations. These relations are identical to those of the infinitesimal determining system.

Proposition 2.14. Let $\mathcal{G}$ be a Lie pseudogroup of order $n^\star$. Then for all $n \geq n^\star$, the restricted Maurer–Cartan forms $\mu^{(n)}|_{\mathcal{G}}$ satisfy the nth *order lifted determining system*

$$
L^{(n)}(Z, \mu^{(n)}) = 0,
\tag{2.21}
$$

obtained from the infinitesimal determining system (2.11) and its prolongation by making the substitutions $z^a \to Z^a$ and $\zeta_A^a \to \mu_A^a$.

Example 2.15. Continuing Example 2.11, the right-invariant Maurer–Cartan forms of the Lie pseudogroup (2.7) satisfy the lifted determining system

$$\mu_Y^x = \mu_U^x = 0, \qquad \mu^y = 0, \qquad \mu^u = -U\mu_X^x, \tag{2.22}$$

obtained from the infinitesimal determining equations (2.13) by making the substitutions

$$\xi_A \to \mu_A^x, \quad \eta_A \to \mu_A^y, \quad \phi_A \to \mu_A^u \qquad \text{and} \qquad x \to X, \quad y \to Y, \quad u \to U.$$

Linear relations among the higher order Maurer–Cartan forms are obtained by Lie differentiating (2.22) by $\mathbb{D}_X, \mathbb{D}_Y, \mathbb{D}_U$. It follows that a basis of right-invariant Maurer–Cartan forms is given by $\mu_{X^k}^x$, $k \geq 0$.

Relation between right and left Maurer–Cartan forms

In a sleight of notation, the preceding discussion of right Maurer–Cartan forms may be applied to left Maurer–Cartan forms by making the substitution $z^a \leftrightarrow Z^a$. Denoting the inverse of $Z = \varphi(z)$ by $z = \varphi^{-1}(Z)$, the zero order left Maurer–Cartan forms for $\mathcal{D}(M)$ are

$$\overline{\mu}^a = dz^a - \sum_{b=1}^m z_{Z^b}^a \, dZ^b.$$

Higher order left Maurer–Cartan forms

$$\overline{\mu}_A^a = \mathbb{D}_z^A \overline{\mu}^a$$

are obtained by Lie differentiation with respect to

$$\mathbb{D}_{z^a} = \sum_{b=1}^m Z_{z^a}^b \, \mathbb{D}_{Z^b}, \qquad \text{where} \qquad (Z_{z^a}^b) = (z_{Z^b}^a)^{-1}. \tag{2.23}$$

The collection of all left Maurer–Cartan forms up to order n will be denoted $\overline{\mu}^{(n)}$. For the left Maurer–Cartan forms the infinitesimal determining equations (2.21) remain unchanged, apart from the interchange $z \leftrightarrow Z$ and $\mu \leftrightarrow \overline{\mu}$:

$$L^{(n)}(z, \overline{\mu}^{(n)}) = 0.$$

For applications to follow it will be useful to know the relation between left and right invariant Maurer–Cartan forms. For the order zero Maurer–Cartan forms we have

$$\overline{\mu}^a = dz^a - \sum_{b=1}^{m} z_{Z^b}^a \, dZ^b = -\sum_{b=1}^{m} z_{Z^b}^a (dZ^b - \sum_{c=1}^{m} Z_{z^c}^b \, dz^c) = -\sum_{b=1}^{m} z_{Z^b}^a \, \mu^b. \qquad (2.24)$$

Relations among the higher order Maurer–Cartan forms may be found by Lie differentiating (2.24) with respect to (2.23)

$$\overline{\mu}_A^a = -\sum_{b=1}^{m} \sum_{B \leq A} \binom{A}{B} \mathbb{D}_z^B(z_{Z^b}^a) \cdot \mathbb{D}_z^{A-B}(\mu^b). \qquad (2.25)$$

For example, the first order left and right Maurer–Cartan forms are related by

$$\overline{\mu}_b^a = \mathbb{D}_{z^b}(\overline{\mu}^a) = -\sum_{b,c=1}^{m} Z_{z^b}^c (z_{Z^b Z^c}^a \, \mu^b + z_{Z^b}^a \, \mu_c^b). \qquad (2.26)$$

For a Lie pseudogroup $\mathcal{G} \subset \mathcal{D}$, the relations between the left and right invariant Maurer–Cartan forms are obtain by restricting (2.24), (2.25) to the determining system (2.5) and the lifted determining equations (2.21).

Example 2.16. Continuing Example 2.13, formula (2.24) reduces to

$$\begin{aligned}
\overline{\mu}^X &= -[x_X \, \mu^x + x_Y \, \mu^y + x_U \, \mu^u] = -x_X \, \mu^x, \\
\overline{\mu}^Y &= -[y_X \, \mu^x + y_Y \, \mu^y + y_U \, \mu^u] = -\mu^y = 0, \\
\overline{\mu}^U &= -[u_X \, \mu^x + u_Y \, \mu^y + u_U \, \mu^u] = \frac{u \, x_{XX}}{x_X} \mu^x - \frac{1}{x_X} \mu^u = \frac{u \, x_{XX}}{x_X} \mu^x - u \, \mu_X^x,
\end{aligned} \qquad (2.27)$$

where we used (2.22) and the determining equations

$$x_Y = x_U = 0, \qquad y = Y, \qquad u = \frac{U}{x_X}.$$

Lie differentiating (2.27) with respect to

$$\mathbb{D}_x = \frac{1}{x_X} \mathbb{D}_X + \frac{u \, x_{XX}}{x_X} \mathbb{D}_U, \qquad \mathbb{D}_y = \mathbb{D}_Y, \qquad \mathbb{D}_u = x_X \mathbb{D}_U$$

yields the relations among the higher order Maurer–Cartan forms. For example,

$$\begin{aligned}
\overline{\mu}_x^X &= \mathbb{D}_x(\overline{\mu}^X) = -\frac{x_{XX}}{x_X} \mu^x - \mu_X^x, \\
\overline{\mu}_{xx}^X &= \mathbb{D}_x(\overline{\mu}_x^X) = \left(\frac{x_{XX}^2}{x_X^3} - \frac{x_{XXX}}{x_X^2}\right) \mu^x - \frac{x_{XX}}{x_X^2} \mu_X^x - \frac{1}{x_X} \mu_{XX}^x.
\end{aligned}$$

2.2 Moving frames

The other jet bundle

It has been said that the theory of moving frames for Lie pseudogroups is a tale of two jet bundles. The first jet bundle of our story, $\mathcal{D}^{(\infty)}$, has already been introduced. We are now interested in the action of a Lie pseudogroup $\mathcal{G}$ on submanifolds of M, and thus the jet bundle of contact equivalence classes of p-dimensional submanifolds, $\mathrm{J}^{\infty}(M, p)$, makes its entrance. Again, see [**52**] for background material on the language of jets.

We begin with a discussion of the induced action of a Lie pseudogroup $\mathcal{G}$ on $\mathrm{J}^n(M, p)$. Since all considerations will be local, we identify M with a Cartesian product $X \times U$ of submanifolds X and U with local coordinates $z = (x, u)$. The coordinates $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ are treated as independent and dependent variables, respectively. This induces the local coordinates $z^{(n)} = (x, u^{(n)})$ on $\mathrm{J}^n(M, p)$, where $u^{(n)}$ denotes the collection of derivatives u_J^{α}, with $\alpha = 1, \ldots, q$ and $0 \leq \#J \leq n$.

Because the action of a Lie pseudogroup $\mathcal{G}$ on M preserves n^{th} order contact of submanifolds (see Figure 2.1), there is an induced action of $\mathcal{G}$ on $\mathrm{J}^n(M, p)$ we will write as

$$(x, u^{(n)}) \mapsto (X, U^{(n)}).$$

Here, $U^{(n)}$ denotes the collection of all derivatives up to order n of the target dependent variables with respect to the target independent variables:
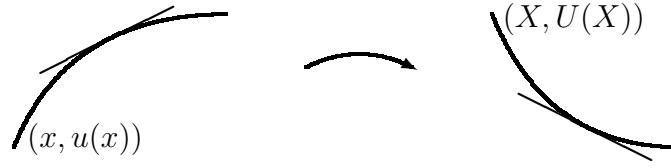
$$U_{X^J}^{\alpha} = D_X^J U^{\alpha}, \qquad \alpha = 1, \ldots, q, \quad \#J \geq 0,$$

which are obtained by applying the *lifted total derivative operators*

$$D_{X^i} = \sum_{j=1}^{p} B_i^j D_{x^j}, \qquad \text{where} \qquad (B_i^j) = (D_{x^i} X^j)^{-1}.$$

These differential operators simply codify the process of implicit differentiation.

Example 2.17. Consider again the Lie pseudogroup of Example 2.7, now assumed to act on surfaces in $\mathbb{R}^3$ given locally as graphs $\big(x, y, u(x, y)\big)$. In this setting, the

Figure 2.1: The action of $\mathcal{G}$ preserves contact.

prolonged action is obtained by applying to U the lifted total derivative operators

$$D_X = \frac{1}{X_x} D_x, \qquad D_Y = D_y.$$

For example, the second order prolonged action is

$$X = f, \qquad Y = y, \qquad U = \frac{u}{f_x} \qquad U_Y = \frac{u_y}{f_x}, \qquad U_X = \frac{u_x f_x - u f_{xx}}{f_x^3},$$

$$U_{YY} = \frac{u_{yy}}{f_x}, \qquad U_{XY} = \frac{u_{xy} - U_Y f_{xx}}{f_x^2}, \qquad U_{XX} = \frac{u_{xx} f_x - u f_{xxx}}{f_x^4} - 3 \frac{U_X f_{xx}}{f_x^2}.$$

Remark 2.18. The pseudogroup parameters $f, f_x, f_{xx}, \ldots$ serve as coordinates for the fiber $\mathcal{G}^{(n)}|_z$ over a fixed source point. As previously explained, one can think of them as analogous to group parameters associated with the action of a finite dimensional Lie group. As such we may refer to a particular Lie pseudogroup transformation as we would the action of a group element:

$$g^{(n)} \cdot (x, u^{(n)}) = (X, U^{(n)}).$$

In the case that there are only a finite number of pseudogroup parameters, it is indeed the prolonged action of a finite dimensional Lie group.

For the construction of a moving frame for a Lie pseudogroup action, we will require that the Lie pseudogroup act regularly and freely. Regularity is a topological condition, requiring that the orbits of the Lie pseudogroup form a regular foliation. Freeness requires that any pseudogroup transformation that fixes a *submanifold n*-jet must have the same *pseudogroup n*-jet as the identity transformation.

Definition 2.19. A Lie pseudogroup is said to act *regularly* on J^n if the orbits of the action form a regular foliation with leaves intersecting arbitrarily small open sets in pathwise connected subsets.

Definition 2.20. Let

$$\mathcal{G}_{z^{(n)}}^{(n)} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}|_z : g^{(n)} \cdot z^{(n)} = z^{(n)} \right\}$$

be the isotropy subgroup of $z^{(n)}$. The pseudogroup $\mathcal{G}$ acts *freely* at $z^{(n)}$ if $\mathcal{G}_{z^{(n)}}^{(n)} = \{\mathbb{1}_z^{(n)}\}$.
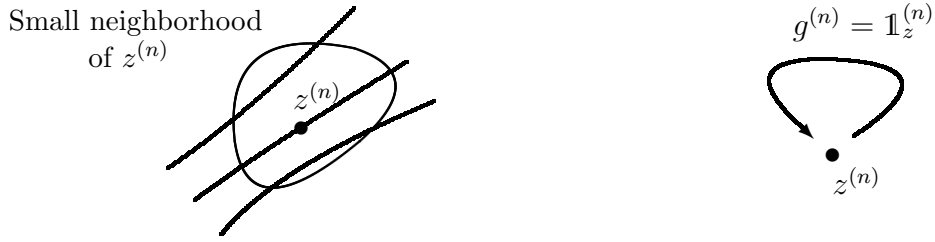

Figure 2.2: Regularity and freeness.

The pseudogroup $\mathcal{G}$ is said to act *locally freely at* $z^{(n)}$ if it acts freely on an open subset of J^n containing $z^{(n)}$. In the analytic category, a locally free action is then free on a dense open subset $\mathcal{V}^n \subset \mathrm{J}^n$, called the set of regular n-jets.

Remark 2.21. This notion of freeness is less restrictive than the usual notion of freeness for a group action. First, any pseudogroup action is trivially free at order zero, so we need only be concerned with freeness at orders $n \geq 1$. But, the two notions of freeness may disagree up to any order: for example, the Lie group action

$$(x, u) \mapsto (x + a, u + bx^2 + cx + d)$$

is not free in the traditional sense until second prolongation

$$(x, u, u_x, u_{xx}) \mapsto (x + a, u + bx^2 + cx + d, u_x + 2bx + c, u_{xx} + 2b),$$

even though it is free in the sense of Definition 2.20 at all orders. One can generalize this example in an obvious manner.

Because freeness will be an essential ingredient of all constructions to follow, it is important to know that freeness persists under prolongation. Proof of the following theorem and further discussion appears in [**63**].

Theorem 2.22 (Persistence of freeness)**.** If a pseudogroup acts locally freely at $z^{(n)}$ for some $n \geq 0$, then it acts locally freely at $z^{(k)}$ for all $k > n$.

Moving frames and their construction

We are now prepared to define and describe the construction of a moving frame. For this purpose we introduce the nth *order lifted bundle*

$$\mathcal{E}^n = \mathrm{J}^n \times \mathcal{G}^{(n)} \to \mathrm{J}^n$$

whose local coordinates are given by $(z^{(n)}, g^{(n)})$, where as before $z^{(n)}$ are the nth order submanifold jet coordinates and $g^{(n)}$ are the coordinates on the fiber $\mathcal{G}^{(n)}|_z$. The bundle projection is simply projection onto the first factor.

Definition 2.23. A *right moving frame* is a right equivariant section of the bundle $\mathcal{E}^n$. Specifically, a section $\rho^{(n)} : \mathrm{J}^n \to \mathcal{E}^n$ is a right moving frame if it satisfies

$$\rho^{(n)}(g^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot (g^{(n)})^{-1}$$

for all $g^{(n)} \in \mathcal{G}^{(n)}|_z$.

The construction of a moving frame relies on the choice of a submanifold $\mathcal{K}^n$ of J^n intersecting each of the pseudogroup orbits transversally, called a cross-section. The existence of a cross-section is guaranteed by the assumption that the pseudogroup acts regularly. Once a cross-section is chosen, the freeness of the pseudogroup action may be used to determine a unique pseudogroup jet mapping each submanifold jet to the cross-section (see Figure 2.3). This is our moving frame.

Proposition 2.24. Suppose that $\mathcal{G}$ acts regularly and freely on $\mathcal{V}^n \subset \mathrm{J}^n$. Let $\mathcal{K}^n \subset \mathcal{V}^n$ be a cross-section to the orbits of $\mathcal{G}$. Given $z^{(n)} \in \mathcal{V}^n$, define $\rho^{(n)}(z^{(n)}) = (z^{(n)}, g^{(n)}) \in \mathcal{E}^n$ by letting $g^{(n)}$ be the unique pseudogroup jet such that that $g^{(n)} \cdot z^{(n)} \in \mathcal{K}^n$. Then $\rho^{(n)}$ is a right moving frame.

Remark 2.25. To each right moving frame corresponds a unique *left moving frame* given by pseudogroup inversion, denoted $\overline{\rho}^{(n)}$:

$$\overline{\rho}^{(n)}(\rho^{(n)}(z^{(n)}) \cdot z^{(n)}) = (\rho^{(n)}(z^{(n)}))^{-1}.$$

Geometrically, the left moving frame maps off of the cross-section to the submanifold jet $z^{(n)}$.
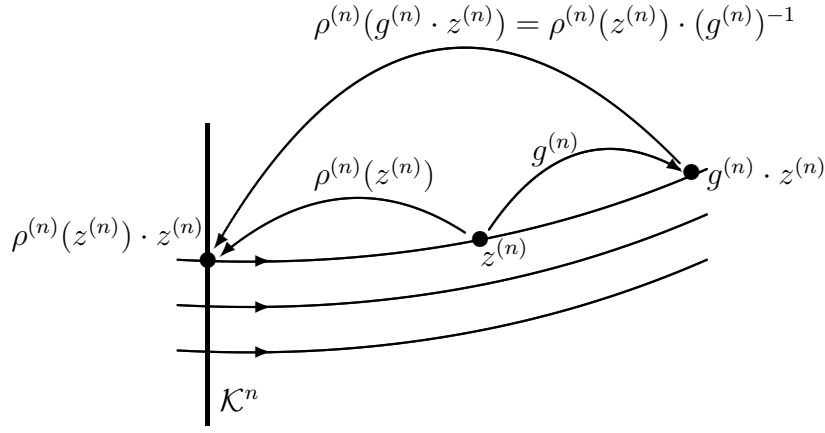
$$\rho^{(n)}\big(g^{(n)} \cdot z^{(n)}\big) = \rho^{(n)}\big(z^{(n)}\big) \cdot \big(g^{(n)}\big)^{-1}$$

$$\rho^{(n)}\big(z^{(n)}\big) \qquad g^{(n)}$$

$$g^{(n)} \cdot z^{(n)}$$

$$\rho^{(n)}\big(z^{(n)}\big) \cdot z^{(n)} \qquad z^{(n)}$$

$$\mathcal{K}^n$$

Figure 2.3: The construction of a moving frame

To simplify our constructions we will usually choose our cross-section $\mathcal{K}^n$ to be a *coordinate cross-section*. This means that $\mathcal{K}^n$ will have the the form

$$x^{i_1} = c_1, \quad \ldots, \quad x^{i_l} = c_l, \qquad u_{J_{l+1}}^{\alpha_{l+1}} = c_{l+1}, \quad \ldots, \quad u_{J_{r_n}}^{\alpha_{r_n}} = c_{r_n}, \tag{2.28}$$

where $r_n = \dim \mathcal{G}^{(n)}|_z$. Our right moving frame is then obtained by solving the resulting *normalization equations*

$$X^{i_1}\big(z^{(n)}, g^{(n)}\big) = c_1, \quad \ldots, \quad X^{i_l}\big(z^{(n)}, g^{(n)}\big) = c_l,$$
$$U_{X^{J_{l+1}}}^{\alpha_{l+1}}\big(z^{(n)}, g^{(n)}\big) = c_{l+1}, \quad \ldots, \quad U_{X^{J_{r_n}}}^{\alpha_{r_n}}\big(z^{(n)}, g^{(n)}\big) = c_{r_n},$$

for the pseudogroup jets $g^{(n)}$. This is possible by virtue of the implicit function theorem.

Example 2.26. We now construct a moving frame for our favorite Lie pseudogroup action (2.7). A standard cross-section to the pseudogroup orbits is

$$x = 0, \qquad u = 1, \qquad u_{x^k} = 0, \qquad 1 \le k \le n. \tag{2.29}$$

To determine the pseudogroup jet mapping a generic point $(x, u^{(n)})$ to $\mathcal{K}^n$ we solve the *normalization equations*

$$X = f = 0 \qquad U = \frac{u}{f_x} = 1, \qquad U_{X^k} = \frac{u_{x^k}}{f_{x^{k+1}}} = 0, \qquad 1 \le k \le n,$$

for the pseudogroup parameters $f, f_x, f_{xx}, \ldots$. This yields the right moving frame

$$f = 0, \qquad f_x = u, \qquad f_{x^{k+1}} = u_{x^k}, \qquad 1 \le k \le n. \tag{2.30}$$

The left moving frame is obtained by inverting (2.30):

$$\bar{f} = x, \qquad \bar{f}_X = \frac{1}{f_x} = \frac{1}{u}, \qquad \bar{f}_{XX} = -\frac{f_{xx}}{f_x^3} = -\frac{u_x}{u^3}, \qquad \cdots, \tag{2.31}$$

where we use the convention that inverse parameters will be distinguished with a top bar.

All constructions thus far have been for a finite n, but may be extended to infinite order through the projective or injective limit. In particular, given a free and regular prolonged action of $\mathcal{G}$ on J^n we have, by Theorem 2.22, a free action on the projective limit J^∞. We can construct a limiting moving frame $\rho^{(\infty)} : \mathrm{J}^\infty \to \mathcal{E}^\infty$ given a sequence of moving frames $\rho^{(k)} : \mathrm{J}^k \to \mathcal{E}^k$, $k \ge n$, mutually consistent with the projections onto lower order jet spaces. This consistency may be achieved through a choice of compatible cross-sections, in which $\mathcal{K}^n$ is simply the projection of $\mathcal{K}^k$ to J^n for $k > n$.

Invariantization and the recurrence relation

The most important function of the moving frame is to provide an algorithmic process for creating and computing with invariant functions, differential operators and differential forms. The key to this algorithmic process is *invariantization* and the interaction of invariantization with differentiation, given by the *recurrence relation*. We first describe the lift of a differential form on J^∞ to $\mathcal{E}^\infty$ and the resulting lifted recurrence relation for differential forms on $\mathcal{E}^\infty$.

Supplementing the standard coframe on J^∞

$$dx^1, \ldots, dx^p, \qquad \theta_J^\alpha = du_J^\alpha - \sum_{j=1}^p u_{J,j}^\alpha \, dx^j \qquad \alpha = 1, \ldots, q, \quad \#J \ge 0 \tag{2.32}$$

with the Maurer–Cartan forms μ_A^a from (2.14) yields a coframe for the lifted bundle $\mathcal{E}^\infty$. This creates a bigrading of the differential forms on $\mathcal{E}^\infty$ via their jet degree and

group degree. In particular, we can define the jet projection π_J, which projects a differential form on $\mathcal{E}^\infty$ onto its pure jet components in the bigrading. Symbolically, this amounts to setting all Maurer–Cartan forms equal to 0.

Definition 2.27. Let ω be a differential form on J^∞. Its *lift* is the invariant differential form on $\mathcal{E}^\infty$ given by

$$\boldsymbol{\lambda}(\omega) = \pi_J[(g^{(\infty)})^*(\omega)]. \tag{2.33}$$

Note that if ω is a submanifold jet coordinate, its lift is simply the prolonged action:

$$X^i = \boldsymbol{\lambda}(x^i), \qquad U^\alpha_{X^J} = \boldsymbol{\lambda}(u^\alpha_J).$$

Definition 2.28. Let $\rho\colon J^\infty \to \mathcal{E}^\infty$ be a right moving frame. The *invariantizaton map* is

$$\iota = \rho^* \circ \boldsymbol{\lambda}, \tag{2.34}$$

which projects differential forms on J^∞ onto invariant differential forms on J^∞.

Example 2.29. Continuing with our running example, invariantization of the submanifold jet coordinates produces the *normalized invariants*

$$H^y = \iota(y) = y, \qquad I_{01} = \iota(u_y) = \frac{u_y}{u},$$
$$I_{11} = \iota(u_{xy}) = \frac{uu_{xy} - u_x u_y}{u^3}, \qquad I_{02} = \iota(u_{yy}) = \frac{u_{yy}}{u}. \tag{2.35}$$

Invariantization of the J^∞ coframe produces the invariant one-forms:

$$\varpi^x = \iota(dx) = u\,dx, \qquad \varpi^y = \iota(dy) = dy$$

$$\vartheta = \iota(\theta) = \frac{\theta}{u}, \qquad \vartheta_x = \iota(\theta_x) = \frac{\theta_x}{u^2} - \frac{u_x \theta}{u^3}, \qquad \vartheta_y = \iota(\theta_y) = \frac{\theta_y}{u},$$

and so on.

In general we denote by

$$\varpi^i = \iota(dx^i) \qquad i = 1, \dots, p,$$
$$\vartheta^\alpha_J = \iota(\theta^\alpha_J) \qquad \alpha = 1, \dots, q, \quad \#J \geq 0, \tag{2.36}$$

the invariantization of the horizontal coframe and basic contact one-forms, and by

$$H^i = \iota(x^i), \qquad I_J^\alpha = \iota(u_J^\alpha), \tag{2.37}$$

the invariantization of the submanifold jet coordinates. The invariants H^i, I_J^α will be called *normalized invariants*. Invariantization of the jet coordinates (2.28) defining the (coordinate) cross-section $\mathcal{K}^\infty$ are constant by construction, and are called *phantom invariants*.

Remark 2.30. It will often be convenient to work modulo contact forms in order to study only the structure of differential invariants. For this purpose we will use the notation $\equiv$ to mean equality of differential forms up to addition of a lifted or invariant contact form. By projecting onto the invariantized horizontal coframe $\varpi^1, \ldots, \varpi^p$, we obtain information about the invariant derivatives of a differential function, defined to be the operators $\mathcal{D}_i$ dual to the invariant forms ϖ^i:

$$dF \equiv \sum_{i=1}^{p} \mathcal{D}_i(F)\,\varpi^i \qquad \text{for any differential function} \quad F(x, u^{(\infty)}). \tag{2.38}$$

With an algorithmic method for creating invariants in hand, we now introduce a method for determining the calculus of these manufactured invariants, known as the *universal recurrence relation*. As with the process of invariantization, we first introduce a *lifted recurrence relation* for differential forms on $\mathcal{E}^\infty$, then pull back by a choice of moving frame to produce a recurrence relation for differential forms on J^∞.

Before introducing the lifted recurrence relation, we extend the lift map (2.33) to act on vector field jets $\zeta^{(n)}$ by defining

$$\boldsymbol{\lambda}(\zeta_A^a) := \mu_A^a$$

to be the corresponding right-invariant Maurer–Cartan form. With this extended notion of lift, we have the following identity:

Theorem 2.31. Let ω be a differential form on J^∞. Then

$$d[\boldsymbol{\lambda}(\omega)] = \boldsymbol{\lambda}[d\omega + \mathbf{v}^{(\infty)}(\omega)]. \tag{2.39}$$

This identity is known as the *lifted recurrence relation*.

Given a choice of moving frame ρ, pulling back (2.39) by ρ immediately produces

$$d[\iota(\omega)] = \iota[d\omega + \mathbf{v}^{(\infty)}(\omega)]. \tag{2.40}$$

Equation (2.40) is called the *universal recurrence relation*. We will be particularly interested in the case where ω is a differential function, often simply one of the submanifold jet coordinate functions.

Remark 2.32. As will be seen in many examples to follow, the universal recurrence relation (2.40) possesses two powerful features: first, the "starting values" for the recurrence may be computed using only knowledge of the cross-section and the infinitesimal generators; second, all computations may be done symbolically, with no need for coordinate expressions for the moving frame or invariantized functions or forms.

Example 2.33. In this example we compute the invariant recurrence relations (2.40) for the normalized invariants (2.35) of Example 2.29. We begin by applying the lifted recurrence relation (2.39), for which we will need the prolongation of the infinitesimal generator (2.12):

$$\begin{aligned}
\mathbf{v}^{(\infty)} =& a(x)\frac{\partial}{\partial x} - u\,a_x\frac{\partial}{\partial u} - (u\,a_{xx} + 2u_x\,a_x)\frac{\partial}{\partial u_x} - u_y\,a_x\frac{\partial}{\partial u_y} - u_{yy}\,a_x\frac{\partial}{\partial u_{yy}} \\
& - (u_y\,a_{xx} + 2u_{xy}\,a_x)\frac{\partial}{\partial u_{xy}} - (u\,a_{xxx} + 3u_x\,a_{xx} + 3u_{xx}\,a_x)\frac{\partial}{\partial u_{xx}} - \cdots .
\end{aligned}$$

Substituting the jet coordinate functions x, y, u, u_x, u_y, ... for ω in (2.39) we obtain (working modulo contact forms):

$$\begin{aligned}
&dX = \Omega^x + \mu, \qquad dY = \Omega^y, \\
&dU \equiv U_X\,\Omega^x + U_Y\,\Omega^y - U\,\mu_X^x, \\
&dU_X \equiv U_{XX}\,\Omega^x + U_{XY}\,\Omega^y - U\,\mu_{XX}^x - 2U_X\,\mu_X^x, \\
&dU_Y \equiv U_{XY}\,\Omega^x + U_{YY}\,\Omega^y - U_Y\,\mu_X^x, \\
&dU_{XX} \equiv U_{XXX}\,\Omega^x + U_{XXY}\,\Omega^y - U\,\mu_{XXX}^x - 3U_X\,\mu_{XX}^x - 3U_{XX}\,\mu_x^x, \\
&dU_{XY} \equiv U_{XXY}\,\Omega^x + U_{XYY}\,\Omega^y - U_Y\,\mu_{XX}^x - 2U_{XY}\,\mu_X^x, \\
&dU_{YY} \equiv U_{XYY}\,\Omega^x + U_{YYY}\,\Omega^y - U_{YY}\,\mu_X^x, \qquad \cdots .
\end{aligned} \tag{2.41}$$

where

$$\Omega^x = \boldsymbol{\lambda}(dx), \qquad \Omega^y = \boldsymbol{\lambda}(dy)$$

denotes the lift of the horizontal coframe and $\mu^x, \mu^x_X, \mu^x_{XX}, \dots$ are the Maurer–Cartan forms (2.18), (2.20). Taking the pull back by the moving frame ρ of Example 2.26, (2.41) becomes

$$
\begin{aligned}
0 &= \varpi^x + \mu, \qquad dH^y = \varpi^y, \\
0 &\equiv I_{01}\,\varpi^y - \mu_X, \\
0 &\equiv I_{11}\,\varpi^y - \mu^x_{XX}, \\
dI_{01} &\equiv I_{11}\,\varpi^x + I_{02}\,\varpi^y - I_{01}\,\mu^x_X, \\
0 &\equiv I_{21}\,\varpi^y - \mu^x_{XXX}, \\
dI_{11} &\equiv I_{21}\,\varpi^x + I_{12}\,\varpi^y - I_{01}\,\mu^x_{XX} - 2I_{11}\,\mu^x_X, \\
dI_{02} &\equiv I_{12}\,\varpi^x + I_{03}\,\varpi^y - I_{02}\,\mu^x_X, \qquad \dots.
\end{aligned}
\tag{2.42}
$$

The notation μ^a_A has been reused for the pullbacks $\rho^*(\mu^a_A)$, hopefully without confusion. Solving these equations for μ^a_A, we find

$$\mu^x = -\varpi^x, \qquad \mu^x_{X^k} \equiv I_{k-1,1}\,\varpi^y, \qquad k \geq 1. \tag{2.43}$$

Substituting the expressions (2.43) into the remaining recurrence relations (2.41) yields the recurrence relations for the normalized invariants

$$
\begin{aligned}
dH^y &= \varpi^y, \qquad dI_{01} \equiv I_{11}\,\varpi^x + (I_{02} - I_{01}^2)\,\varpi^y, \\
dI_{11} \equiv I_{21}\,\varpi^x + (I_{12} - 3I_{01}I_{11})\,\varpi^y, &\qquad dI_{02} \equiv I_{12}\,\varpi^x + (I_{03} - I_{01}I_{02})\,\varpi^y,
\end{aligned}
\tag{2.44}
$$

and so on. Using (2.38) we deduce from (2.44) the relations

$$
\begin{aligned}
\mathcal{D}_x H^y &= 0, & \mathcal{D}_y H^y &= 1, \\
\mathcal{D}_x I_{01} &= I_{11}, & \mathcal{D}_y I_{01} &= I_{02} - I_{01}^2, \\
\mathcal{D}_x I_{11} &= I_{21}, & \mathcal{D}_y I_{11} &= I_{12} - 3I_{01}I_{11}, \\
\mathcal{D}_x I_{02} &= I_{12}, & \mathcal{D}_y I_{02} &= I_{03} - I_{01}I_{02}.
\end{aligned}
\tag{2.45}
$$

It is important to note that in this example the explicit expressions for the moving frame (2.30) were never required; knowledge of the cross-section was sufficient to provide expressions for the pulled-back Maurer–Cartan forms and hence the general structure of the recurrence relations.

The algebra of differential invariants

The collection of differential invariants forms an algebra and the structure of this algebra will be important in applications to follow. We now explain the rudiments of generating this algebra using invariant differential operators and the functional relationships within it, known as syzygies. A deeper discussion of differential invariant algebras for Lie pseudogroups may be found in [**63**].

It is first worth noticing that the process of invariantization is sufficient to give a complete set of functionally independent invariants: any differential invariant may be written as a function of the the normalized invariants, and the normalized invariants themselves are functionally independent. Indeed, given any differential invariant, $F(x, u^{(\infty)})$, application of the invariantization map gives F as a function of the normalized invariants:

$$F(x, u^{(\infty)}) = \iota(F(x, u^{(\infty)})) = F(H, I^{(\infty)}).$$

This simple and useful fact is sometimes called the *replacement theorem*.

Because invariant differential operators create new differential invariants, it is natural to try to generate a complete set of differential invariants using these operators.

Definition 2.34. A set of invariants $\mathcal{I}$ is said to generate the algebra of differential invariants if all differential invariants can be expressed as some function of the invariants $I \in \mathcal{I}$ and their invariant derivatives $\mathcal{D}_J I$.

Theorem 2.35. The algebra of differential invariants for a Lie pseudogroup is always generated by a finite set.

This theorem was first proved by Lie for Lie group actions, [**45**, p. 760], and later extended to infinite dimensional Lie pseudogroups by Tresse, [**75**]. Many modern proofs have since appeared, [**41, 42, 66**], including some based on moving frames, [**28, 57, 63**]. The proof of the Lie–Tresse theorem presented in [**63**] is constructive and identifies a generating set using Gröbner basis methods. For our purposes it will suffice to find a generating set by inspection of the normalized invariants and their recurrence relations.

Example 2.36. By inspection of the recurrence relations (2.45), one finds the generating set

$$\mathcal{I} = \{H^y, I_{01}, I_{11}\}$$

for the differential invariants for the Lie pseudogroup action (2.7). All other normalized invariants (and hence all differential invariants) may be expressed as functions of these three invariants and their invariant derivatives:

$$I_{02} = \mathcal{D}_y I_{01} + I_{01}^2,$$
$$I_{03} = \mathcal{D}_y I_{02} + I_{01} I_{02},$$
$$I_{12} = \mathcal{D}_y I_{11} + 3 I_{01} I_{11},$$
$$I_{21} = \mathcal{D}_x I_{11},$$

and so on.

Although the normalized invariants are functionally independent, the differentiated invariants will satisfy functional relationships, known as syzygies.

Definition 2.37. A *syzygy* is a nontrivial functional relationship

$$S(\ldots, \mathcal{D}_J I^\sigma, \ldots) = 0, \qquad \#J \geq 0, \quad \sigma = 1, \ldots, k,$$

among the generating invariants $I^1, \ldots I^k$ and their invariant derivatives.

Example 2.38. The invariant I_{12} appears in two different equations in (2.45):

$$\mathcal{D}_y I_{11} = I_{12} - 3 I_{01} I_{11} \qquad \text{and} \qquad \mathcal{D}_x I_{02} = I_{12}.$$

Equating these two expressions for I_{12} we obtain the syzygy

$$\mathcal{D}_x I_{02} = \mathcal{D}_y I_{11} + 3 I_{01} I_{11}. \tag{2.46}$$

Any collection of syzygies may be used to generate more through linear combinations and invariant differentiation. It is natural then to seek a generating set of syzygies, and to hope that this set may be finite. Unfortunately it is not generally finite, but this is remedied by distinguishing certain syzygies resulting from the noncommutativity of invariant differentiation.

Definition 2.39. Syzygies resulting from the commutators of invariant differential operators:

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{p} Y_{i,j}^k \mathcal{D}_k$$

are called *commutator syzygies*.

Example 2.40. Returning to Example 2.33, setting $\omega = dx$ and $\omega = dy$ in the recurrence relation (2.40) yields

$$d\varpi^x = I_{01}\,\omega^y \wedge \varpi^x, \qquad d\varpi^y = 0.$$

By duality we deduce the commutator relation

$$[\mathcal{D}_x, \mathcal{D}_y] = I_{01}\,\mathcal{D}_x.$$

Commutator syzygies then arise as direct consequences of this relation; for example,

$$\mathcal{D}_y\mathcal{D}_x^2 I = (\mathcal{D}_x\,\mathcal{D}_y - I_{01}\,\mathcal{D}_x)\mathcal{D}_x I = \mathcal{D}_x^2\mathcal{D}_y I - (\mathcal{D}_x I_{0,1})\mathcal{D}_x I - 2I_{01}\,\mathcal{D}_x^2 I.$$

Definition 2.41. A collection $\mathcal{S}$ of syzygies is said to form a generating system if every syzygy can be written as a linear combination of members of $\mathcal{S}$ and finitely many of their derivatives, modulo the commutator syzygies.

Theorem 2.42. Every Lie pseudogroup action admitting a moving frame has a finite generating system of syzygies.

Example 2.43. The equation (2.46) is a generating syzygy for our running example.

A proof of Theorem 2.42 appears in [**63**]. As with the generators of the differential invariant algebra, the generating syzygies may be constructed using Gröbner basis methods. As previously noted, it will be sufficient for our purposes to determine generating sets $\mathcal{I}, \mathcal{S}$ of differential invariants and syzygies by direct observation.

Equivalence of submanifolds and signatures

We ask: given two p-dimensional submanifolds $N, \overline{N} \subset M$, when is there a transformation $g \in \mathcal{G}$ such that $g \cdot N = \overline{N}$? For such a transformation to exist, these submanifolds must have the same differential invariants. Unfortunately a direct comparison

of invariants is not useful in practice because of the dependence of the invariants on parametrization of the submanifolds. To remedy this problem, we compare functional relationships among the differential invariants, which are intrinsic.

Theorem 2.44. Two submanifolds are locally equivalent if and only if they have the same functional relationships among their differential invariants.

Remark 2.45. Functional relationships among invariants arising from the restriction to a submanifold are sometimes also called syzygies, or distinguishing syzygies. We will reserve the term syzygy for those relationships satisfied universally by differential invariants, as given in Definition 2.37.

A way to characterize these functional relationships is through the notion of a *differential invariant signature* (also called in older references a *classifying manifold*), [**53**]. The idea is to compare the projections of the submanifolds $N, \overline{N}$ onto a space coordinatized by a complete set of differential invariants. If these projections are overlapping and equal in dimension, then all functional relationships among the invariants must be the same, and the submanifolds must be locally equivalent. Because normalized invariants provide a complete set of invariants and act as coordinates on the cross-section $\mathcal{K}^\infty$, we use the moving frame for our projection. To compare submanifolds $N, \overline{N}$, we must assume that the signatures have constant dimension and that these submanifolds lie in the domain of definition of the moving frame.

Definition 2.46. Let $N \subset M$ be a p-dimensional submanifold, $\mathcal{G}$ a Lie pseudogroup acting on M and ρ a moving frame for $\mathcal{G}$. The *differential invariant signature* of N is the projection $\rho(\mathrm{j}_\infty N) \subset \mathcal{K}^\infty$.

Theorem 2.47. Two submanifolds are locally equivalent if and only if their differential invariant signatures overlap.

See [**53**, **80**] for proof of this theorem and further discussion of signatures.

Chapter 3

Group foliation of differential equations

3.1 Introduction

Group foliation is, at its core, a technique that uses symmetry to construct explicit solutions to differential equations. The idea is as follows: suppose that we are given a differential equation $\Delta = 0$ admitting a Lie pseudogroup $\mathcal{G}$ of symmetries. We split the problem of solving $\Delta = 0$ into one of solving two associated systems of differential equations called the *resolving system* and *automorphic system*. More precisely, it is an associated family of equations; each solution to the resolving system determines a particular $\mathcal{G}$-automorphic system, which in turn yields solutions to the original equation $\Delta = 0$.

Definition 3.1. A system of differential equations is called $\mathcal{G}$-*automorphic* if all of its solutions can be obtained from a single fixed solution via transformations belonging to $\mathcal{G}$.

The method can loosely be interpreted as a process of *reduction and reconstruction*: the resolving equations represent a reduction of $\Delta = 0$ obtained by "removing" symmetries. The automorphic system in turn possesses "maximal" symmetry, and

facilitates reconstruction of the original solutions from the reduced problem using the symmetry transformations. Before further discussion of group foliation, we give a simple example to illustrate the basic objects and constructions.

Example 3.2 (Solving a nonlinear differential equation using group foliation)**.** In this example, we solve the nonlinear diffusion equation

$$u_t = u_{xx} - \frac{u_x^2}{u} \tag{3.1}$$

using the method of group foliation. Observe that (3.1) admits the symmetry transformations

$$X = x, \qquad T = t, \qquad U = \lambda u, \qquad \lambda \in \mathbb{R}^+. \tag{3.2}$$

We begin by studying the structure of the differential invariant algebra for (3.2). A generating set of differential invariants for this action consists of

$$x, \qquad t, \qquad I = \frac{u_x}{u}, \qquad J = \frac{u_t}{u};$$

all differential invariants are functions of these and their derivatives with respect to the invariant differential operators

$$\mathcal{D}_x = D_x, \qquad \mathcal{D}_t = D_t.$$

Because the independent variables x, t are not affected by the scaling transformations (3.2), the invariant differential operators are just the usual total derivatives. It can be seen by direct calculation that there is a syzygy

$$\mathcal{D}_x J = \mathcal{D}_t I \tag{3.3}$$

among the generating invariants. This syzygy is generating in the sense of Definition 2.41; all syzygies among higher order invariants are obtained through differentiation of this relation and there are no commutator syzygies.

   Now, because the equation (3.1) is invariant under (3.2), and has a solution space contained in the set $\mathcal{V}^n$ of regular jets, it may be written in terms of the differential invariants, [**52**, Proposition 2.56]. In particular, the differential invariant u_{xx}/u satisfies

$$\frac{u_{xx}}{u} = \mathcal{D}_x I + I^2,$$

and thus (3.1) takes the form

$$J = \mathcal{D}_x I \qquad \Longleftrightarrow \qquad u_t = u_{xx} - \frac{u_x^2}{u}. \tag{3.4}$$

One can think of this relation as a syzygy valid on solutions of the equation (3.1); to distinguish this syzygy from the universal syzygy (3.3), equation (3.4) will be called a *constraint syzygy*. Thus, on solutions to (3.1), the invariants I, J satisfy

$$J = \mathcal{D}_x I, \qquad \mathcal{D}_x J = \mathcal{D}_t I. \tag{3.5}$$

We can interpret (3.5) as comprising a system of differential equations for I, J as functions of x, t. This is the *resolving system* for equation (3.1) with respect to the symmetry (3.2) and chosen invariants. More explicitly, we may write

$$I = F^1(x,t), \qquad J = F^2(x,t), \tag{3.6}$$

and the resolving system takes the form

$$F^2 = \frac{\partial F^1}{\partial x}, \qquad \frac{\partial F^2}{\partial x} = \frac{\partial F^1}{\partial t}. \tag{3.7}$$

Suppose now that we have a particular solution $F^1(x,t)$, $F^2(x,t)$ to (3.7). Using the explicit expressions for I and J as differential invariants, we form the system

$$I = \frac{u_x}{u} = F^1(x,t), \qquad J = \frac{u_t}{u} = F^2(x,t), \tag{3.8}$$

resulting in a set of differential equations for u based on the chosen resolving system solution. This is the corresponding *automorphic system*, all of whose solutions will satisfy the original equation (3.1).

For example, if we choose the resolving system solution

$$F^1(x,t) = x, \qquad F^2(x,t) = 1$$

we arrive at the automorphic system

$$\frac{u_x}{u} = x, \qquad \frac{u_t}{u} = 1,$$

whose solutions

$$u(x,t) = \mu\, e^{\frac{1}{2}x^2 + t}, \qquad \mu > 0,$$

may be obtained from the single fixed solution $e^{\frac{1}{2}x^2+t}$ via the transformations (3.2). In fact, we find by identical reasoning the family of solutions

$$u(x,t) = \mu \, e^{v(x,t)}, \qquad \mu > 0,$$

to (3.1), where $v(x,t)$ is any solution to the linear diffusion equation $v_t = v_{xx}$.

3.2 The group foliation algorithm

Suppose that

$$\Delta(x, u^{(n)}) = 0 \tag{3.9}$$

is an n-th order differential equation admitting a Lie pseudogroup $\mathcal{G}$ of symmetries. By definition of invariance, $\mathcal{G}$ must map solutions of $\Delta = 0$ to other solutions. Thus there is an induced action of $\mathcal{G}$ on the solution set $\mathcal{S}$, partitioning this set into orbits. If the submanifold lies within the set of regular jets $\mathcal{V}^k \subset \mathrm{J}^k$, these orbits determine invariant submanifolds in J^k, $k \geq 0$, traced out by the action of $\mathcal{G}$ on the prolonged graph of a given solution. Our starting point will be a description of these invariant submanifolds using the differential invariants of $\mathcal{G}$, which will lead to the main idea of the group foliation.

Automorphic systems

Let $\mathcal{K}^k$ be a cross-section to the prolonged action of $\mathcal{G}$ on J^k and let $u_0 : X \to U$ be an arbitrary function whose prolonged graph $(x, u^{(k)}(x))$ lies in a neighborhood of $\mathcal{K}^k$. We will soon be concerned with $u_0(x)$ a solution to a differential equation $\Delta = 0$, but the immediate discussion does not rely on this assumption. Consider the orbit under $\mathcal{G}$ of the k-th prolongation of the graph $(x, u_0^{(k)}(x))$:

$$\mathcal{A}(u_0^{(k)}) = \{g^{(k)} \cdot (x, u_0^{(k)}(x)) \, : \, g^{(k)} \in \mathcal{G}^{(k)}\} \subset \mathrm{J}^k.$$

Let ϱ_k be the dimension of the intersection of $\mathcal{A}(u_0^{(k)})$ with $\mathcal{K}^k$. We assume that the dimension of this intersection is constant. Increasing the order of prolongation, we

have the non-decreasing sequence

$$0 \leq \varrho_0 \leq \varrho_1 \leq \cdots \leq p.$$

Since we consider infinite dimensional Lie pseudogroups, it is possible that the dimension of the orbit $\mathcal{A}(u_0^{(k)})$ may increase without bound as k increases, but ϱ_k is bounded above by the dimension p of the graph of u_0 and must stabilize.

Definition 3.3. The smallest order s such that $\varrho_s = \varrho_{s+i}$ for all $i \geq 1$ will be called the *order* and $\varrho = \varrho_s$ will be called the *invariant rank of the function* u_0.
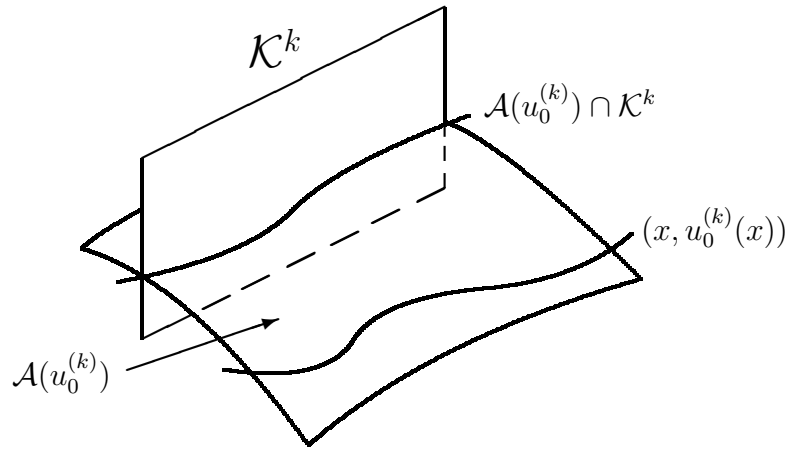

Figure 3.1: The orbit of a graph and intersection with cross-section

As guaranteed by Theorem 2.35, we may choose $k \geq s$ so that there is a functionally independent generating set $\mathcal{I}$ of differential invariants of order k or less. These invariants provide coordinates for the cross-section $\mathcal{K}^k$, and will thus allow us to write the intersection $\mathcal{A}(u_0^{(k)}) \cap \mathcal{K}^k$ as a parametrized submanifold of $\mathcal{K}^k$. For this purpose, distinguish a set of functionally independent differential invariants $\{I^1, \ldots, I^\varrho\} \subset \mathcal{I}$, to be used as parametric variables. We may then use the remaining invariants as dependent variables for the parametrization, writing $\mathcal{A}(u_0^{(k)}) \cap \mathcal{K}^k$ as a graph in $\mathcal{K}^k$:

$$\mathcal{A}_\varrho : \qquad J^1 = F^1(I^1, \ldots, I^\varrho), \qquad \ldots, \qquad J^\nu = F^\nu(I^1, \ldots, I^\varrho), \tag{3.10}$$

where $\mathcal{I} = \{I^1, \ldots, I^\varrho, J^1 \ldots, J^\nu\}$ is the full generating set of invariants. The system (3.10) is automorphic by construction, and so will be called an automorphic system $\mathcal{A}_\varrho$ of rank ϱ, dropping reference to u_0. In fact, every system of the form (3.10) is automorphic, as we will later show in Proposition 3.14.

Remark 3.4. In practice we may distinguish the invariants $I^1, \ldots, I^\varrho$ by verifying the independence condition

$$dI^1 \wedge \cdots \wedge dI^\varrho \not\equiv 0$$

on $\mathcal{A}_\varrho$. This may be done symbolically, without the need for explicit formulae for the invariants.

Example 3.5. Returning to the introductory Example 3.2, distinguishing the invariants x, t from the generating set

$$\mathcal{I} = \{x, t, I = \frac{u_x}{u}, J = \frac{u_t}{u}\},$$

resulted in rank 2 automorphic systems of the form

$$I = F^1(x, t), \qquad J = F^2(x, t).$$

Example 3.6. We now obtain the automorphic systems of rank 1 and 2 for (3.10) for our running example Lie pseudogroup (2.7). Recall the differential invariants (2.35); to simplify the notation, let

$$H = H^y, \qquad J = I_{01}, \qquad K = I_{11}, \qquad L = I_{02}. \tag{3.11}$$

Also recall from Example 2.36 that (3.11) is a generating set. Distinguishing the invariants H, J as parameters, we have the independence condition

$$dH \wedge dJ \equiv K\, \varpi^y \wedge \varpi^x. \tag{3.12}$$

Thus if $K \neq 0$ the invariants H, J are independent, and automorphic systems of rank 2 have the form

$$
\begin{aligned}
K &= F^1(H, J) \\
L &= F^2(H, J)
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\frac{uu_{xy} - u_x u_y}{u^3} &= F^1\left(y, \frac{u_y}{u}\right) \\
\frac{u_{yy}}{u} &= F^2\left(y, \frac{u_y}{u}\right).
\end{aligned}
\tag{3.13}
$$

When $K = 0$ we may choose H as a parameter to obtain the rank 1 automorphic

systems

$$J = F^1(H)$$
$$K = 0 \qquad \Rightarrow$$
$$L = F^2(H)$$

$$\frac{u_y}{u} = F^1(y)$$
$$\frac{uu_{xy} - u_x u_y}{u^3} = 0 \qquad (3.14)$$
$$\frac{u_{yy}}{u} = F^2(y).$$

Syzygies and resolving systems

The choice of $F^1, \ldots, F^\nu$ in (3.10) may not be arbitrary. For instance, in Example 3.5 if the functions F^1, F^2 do not satisfy the integrability condition

$$\frac{\partial F^1}{\partial t} = \frac{\partial F^2}{\partial x},$$

there will be no hope of solving the resulting equations. Because (3.10) is expressed in terms of differential invariants, application of syzygies among the invariants will lead to integrability conditions. Consideration of these syzygies leads us to a system of differential equations for the functions $F^1, \ldots, F^\nu$ in the automorphic system $\mathcal{A}_\varrho$ that we will call the *syzygy system*.

We first discuss syzygy systems for full rank automorphic systems, so suppose for the moment that $\varrho = p$. Let $\mathcal{S}$ be the set of fundamental syzygies among the invariants $I^1, \ldots, I^\varrho, J^1, \ldots, J^\nu$. Making the chain rule substitutions

$$\mathcal{D}_i = \sum_{j=1}^{\varrho} \left(\mathcal{D}_i I^j\right) D_{I^j}, \qquad (3.15)$$

we may write the invariant differential operators appearing in each syzygy in terms of the derivatives D_{I^j} with respect to the invariant parameters $I^1, \ldots, I^\varrho$. Without loss of generality, we will assume $\mathcal{D}_i I^j$, $i = 1, \ldots, p$, $j = 1, \ldots, \varrho$, are again functions of the generating invariants $\mathcal{I}$ by increasing the order of prolongation and adding more invariants to $\mathcal{I}$ if necessary (we do not require $\mathcal{I}$ to be minimal, of course). Application of the fundamental syzygies to the system (3.10) results in a system of partial differential equations for $F^1, \ldots, F^\nu$. This system is called the *syzygy system* for $\mathcal{A}_\varrho$.

Example 3.7. We now obtain the syzygy system associated to the rank 2 automorphic system (3.13). By the chain rule we express the invariant total derivative operators $\mathcal{D}_x$, $\mathcal{D}_y$ in terms of D_H, D_J:

$$
\begin{aligned}
\mathcal{D}_x &= \mathcal{D}_x H \cdot D_H + \mathcal{D}_x J \cdot D_J = K D_J, \\
\mathcal{D}_y &= \mathcal{D}_y H \cdot D_H + \mathcal{D}_y J \cdot D_J = D_H + (L - J^2) D_J.
\end{aligned}
\tag{3.16}
$$

There is a single fundamental syzygy from Example 2.38, which may be written in terms of the operators (3.16):

$$
\mathcal{D}_x L = \mathcal{D}_y K + 3JK \qquad \Longleftrightarrow \qquad K D_J L = D_H K + (L - J^2) D_J K + 3JK.
$$

Written explicitly as a partial differential equation for the functions $F^i(H, J)$, (3.13) yields

$$
F^1 \frac{\partial F^2}{\partial J} = \frac{\partial F^1}{\partial H} + (F^2 - J^2) \frac{\partial F^1}{\partial J} + 3J F^1.
\tag{3.17}
$$

This is the syzygy system for the rank 2 automorphic system (3.13).

Remark 3.8. As can be seen in this example, the substitution (3.15) may be made symbolically, without explicit formulae for the invariants. The formulae (3.16) follow directly from the recurrence relations.

We now address the case when the automorphic systems considered have less than full rank, i.e. $\varrho < p$. In this instance, the substitution (3.15) may introduce new dependencies among the differentiated invariants in addition to the fundamental syzygies and their consequences. We will call these dependencies *restriction syzygies* since they arise from restricting the differential operators to submanifolds parametrized by $I^1, \ldots, I^\varrho$.

Example 3.9. For the rank 1 automorphic system (3.14), the fundamental syzygy (2.38) is trivial. To see this, express the invariant total derivative operators $\mathcal{D}_x$, $\mathcal{D}_y$ in terms of the single operator D_H:

$$
\begin{aligned}
\mathcal{D}_x &= \mathcal{D}_x H \cdot D_H = 0 \\
\mathcal{D}_y &= \mathcal{D}_y H \cdot D_H = D_H.
\end{aligned}
\tag{3.18}
$$

On the other hand, by substitution of (3.18) into the recurrence relations (2.45) we find

$$D_H J = L - J^2$$
$$I_{21} = 0$$
$$I_{12} = 0 \tag{3.19}$$
$$I_{03} = D_H L - JL,$$

and so on. Thus, there is a new *restriction syzygy*

$$D_H J = L - J^2$$

among the generating invariants, arising from the restriction of the invariants and invariant differential operators to submanifolds of the form (3.14). It can be seen by inspection that this restriction syzygy is generating. Thus we arrive at the rank 1 syzygy system for the functions $F^1(H), F^2(H)$:

$$\frac{\partial F^1}{\partial H} = F^2 - (F^1)^2.$$

Remark 3.10. We will henceforth refrain from referencing the functions F^i in our examples when it is understood that each J^i is a function $J^i(I^1, \ldots, I^\varrho)$.

Analogous to Theorem 2.42 in the full rank case $\varrho = p$ (where the restriction syzygies are identical to the usual syzygies), the restriction syzygies for $\varrho < p$ are also finitely generated.

Proposition 3.11. Suppose that the Lie pseudogroup $\mathcal{G}$ admits a moving frame. For any choice of distinguished invariants $I^1, \ldots, I^\varrho$, the set of restriction syzygies resulting from substitution of the relations (3.15) into the recurrence relations is finitely generated. A finite generating set of restriction syzygies will be called *fundamental restriction syzygies*.

We are now prepared to define the syzygy system for all ranks $\varrho \leq p$.

Definition 3.12. The *syzygy system* $\mathcal{S}_\varrho$ for a rank ϱ automorphic system $\mathcal{A}_\varrho$ is the finite system of partial differential equations for $F^1, \ldots, F^\nu$ as functions of the invariant parameters $I^1, \ldots, I^\varrho$ obtained by applying to $\mathcal{A}_\varrho$ the fundamental restriction syzygies.

Remark 3.13. It is important to note that the syzygy system does not impose extra conditions on the solutions of the system $\mathcal{A}_\varrho$; $\mathcal{S}_\varrho$ is a collection of integrability conditions on the functions F^j. Since our primary concern throughout is algorithmic, we will not be concerned here with a discussion of formal integrability or involutivity, [**72**].

With a better understanding of the structure of differential invariants restricted to $\mathcal{A}_\varrho$, it is now easy to see why a system of the form (3.10) must be automorphic.

Proposition 3.14. Any system $\mathcal{A}_\varrho$ of the form (3.10) is automorphic.

Proof. If $F^1, \ldots, F^\nu$ are chosen in such a way that the system does not admit any solutions, the proposition is vacuous. Thus, suppose that $\mathcal{A}_\varrho$ admits solutions and two solutions are given. To show that these solutions are related by a pseudogroup transformation, by Theorem 2.47 it suffices to show that their signatures overlap. This follows from the following consideration: since both solutions have the same invariant rank ϱ, the images of the distinguished differential invariants $I^1, \ldots, I^\varrho$ overlap. Since all differential invariants can be determined as functions of these distinguished invariants, the images of all differential invariants, and hence the signatures, must overlap. $\square$

Let us return now to the context in which our automorphic systems (3.10) arise as orbits of solutions $u_0(x)$ to a $\mathcal{G}$-invariant differential equation $\Delta = 0$, and discuss how to apply these systems to the problem of finding solutions to $\Delta = 0$.

Starting with a solution $u_0(x)$ to a $\mathcal{G}$-invariant equation $\Delta = 0$, solutions to the $\mathcal{G}$-automorphic system $\mathcal{A}(u_0^{(k)})$ will again satisfy $\Delta = 0$ by invariance. Unfortunately, this observation does not offer obvious practical value for finding solutions to $\Delta = 0$; indeed, if a "seed" solution u_0 is known, one can simply apply the pseudogroup transformations to u_0 and avoid automorphic systems altogether. The preceding construction of syzygy systems suggests an alternative approach: append to the syzygy system the condition $\Delta = 0$. By adding this condition, we ensure that the automorphic systems determined by solving the syzygy system are those generated by

solutions to $\Delta = 0$. Note that, by the invariance of $\Delta = 0$, this amounts to adding new relations among the generating invariants; these relations will be called *constraint syzygies*. The constraint syzygies together with the restriction syzygies give a set of differential equations, called the *resolving system*, whose solutions determine automorphic systems generated by solutions of $\Delta = 0$.

Definition 3.15. The rank ϱ *resolving system* $\mathcal{R}_\varrho(\Delta)$ of a differential equation $\Delta = 0$ foliated by $\mathcal{G}$ is the system of differential equations obtained by appending to the syzygy system $\mathcal{S}_\varrho$ the constraint syzygy $\Delta = 0$ and its differential consequences.

Example 3.16. We now obtain the rank 2 resolving system for the nonlinear wave equation

$$uu_{xy} - u_x u_y = u^3, \tag{3.20}$$

foliated by the Lie pseudogroup (2.7). First observe that this Lie pseudogroup is a symmetry group of (3.20). Invariantization of (3.20) gives the constraint syzygy

$$K = 1. \tag{3.21}$$

Appending the constraint syzygy to the syzygy system (3.17) yields the resolving system

$$
\begin{aligned}
K &= 1 \\
D_J L &= 3J.
\end{aligned}
\tag{3.22}
$$

Note that there is no rank 1 resolving system because the constraint syzygy (3.21) is not compatible with the dependence condition $K = 0$ from (3.12), i.e. $1 \neq 0$.

Remark 3.17. The addition of the constraint syzygy may, as usual, be performed symbolically by direct invariantization of the equation $\Delta = 0$ and use of the recurrence relation to write all invariants appearing in $\iota(\Delta)$ in terms of the generating invariants. We assume that the solution space of $\Delta = 0$ lies within the set of regular jets so that the equation may be written as a level set of differential invariants; see [**52**, Proposition 2.56].

Solving differential equations

All the ingredients for the group foliation algorithm are now in place. See Figure 3.2 for the geometry behind the algorithm.

Algorithm 3.18 (Group foliation). Let $\Delta(x, u^{(n)}) = 0$ be an n-th order differential equation invariant under a Lie pseudogroup $\mathcal{G}$ and suppose that $\mathcal{G}$ admits a moving frame on the solution space of $\Delta = 0$.

• Determine a rank ϱ for which rank ϱ solutions will be sought. Prolong to order $k \geq s$, where s is the order of stabilization of a generic rank ϱ solution, so that the normalized invariants of order at most k form a generating set.

• Choose distinguished invariants $I^1, \ldots, I^\varrho$ from among the normalized invariants so that $\mathcal{D}_i I^j$ have order no greater than k. These invariants will be used as independent variables and the remaining normalized invariants $J^1, \ldots, J^\nu$ as dependent variables in the automorphic system

$$A_\varrho: \qquad J^1 = F^1(I^1, \ldots, I^\varrho), \quad \ldots, \quad J^\nu = F^\nu(I^1, \ldots, I^\varrho).$$

• Compute the order ϱ resolving system $\mathcal{R}_\varrho(\Delta)$ by applying the restriction syzygies and the constraint syzygy $\iota(\Delta) = 0$ to $\mathcal{A}_\varrho$.

• Find a solution $F^1(I^1, \ldots, I^\varrho), \ldots, F^\nu(I^1, \ldots, I^\varrho)$ to the resolving system.

• Form an automorphic system $\mathcal{A}_\varrho$ using the resolving system solution and write the invariants in this automorphic system explicitly in terms of $(x, u^{(k)})$. Solutions of this automorphic system will satisfy the original equation $\Delta = 0$.

Example 3.19. We continue Example 3.16. A general solution to the resolving system (3.22) is easily found:

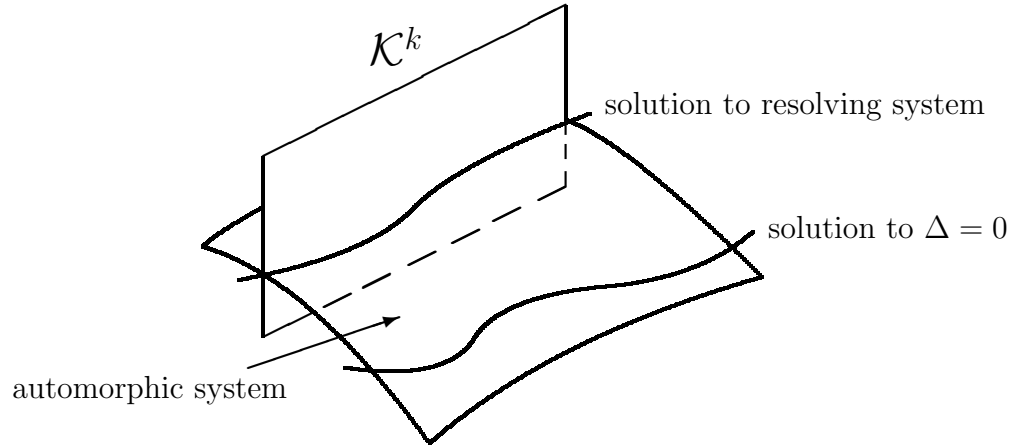$$K(H, J) = 1, \qquad L(H, J) = \frac{3}{2}J^2 + G(H), \tag{3.23}$$

Figure 3.2: The geometry of group foliation

where $G(H)$ is an arbitrary function. Substituting (3.23) and the explicit formulae (2.35) for the invariants into the automorphic system (3.13) we obtain the system of differential equations

$$\frac{uu_{xy} - u_x u_y}{u^3} = 1, \qquad \frac{u_{yy}}{u} = \frac{3}{2}\left(\frac{u_y}{u}\right)^2 + G(y). \tag{3.24}$$

This may now be a bit disappointing, because it is apparent that the method in this instance has been circular; the original equation itself appears in the final automorphic system. One might hope that the second equation of (3.24) may offer simplification, but it is in fact a consequence of the first. This unfortunate outcome will be remedied by the subject of the next section.

Example 3.20. To illustrate the algorithm for non-maximal invariant rank we consider the differential equation

$$uu_{xy} - u_x u_y = 0. \tag{3.25}$$

This equation also admits the symmetry pseudogroup (2.7). Using the same notation as Examples 3.6 and 3.9, (3.25) implies the constraint syzygy $K = 0$. The constraint syzygy implies that the independence condition (3.12) is not satisfied, and hence the resolving equations in this case are identical to the rank 1 syzygy system already computed in Example 3.9. A solution to the resolving system is easily found:

$$
\begin{aligned}
J(H) &= G(H) \\
L(H) &= G'(H) + G(H)^2,
\end{aligned}
\tag{3.26}
$$

where G is an arbitrary function. Substituting (3.26) and the explicit formulae (2.35) for the invariants into the automorphic system (3.14) we obtain the system of differential equations

$$\frac{u_y}{u} = G(y)$$

$$\frac{uu_{xy} - u_x u_y}{u^3} = 0 \tag{3.27}$$

$$\frac{u_{yy}}{u} = G'(y) + G(y)^2.$$

We won't pursue a solution of (3.27) at present. This will be done by alternative means in Example 3.33 to follow.

In Algorithm 3.18, all steps except for the last may be executed using the symbolic calculus of moving frames. It is only the last step that requires explicit knowledge of the differential invariants and, in the instance of Example 3.19, leads to a dead end in the computation. In keeping with the intent of moving frames, we propose an alternative method for reconstruction of solutions from the resolving system that is completely symbolic, and effective in certain examples — such as Example 3.19 — where the standard reconstruction method fails.

3.3 Reconstruction equations

The method of moving frames is naturally incorporated into our exposition of the group foliation method. Moving frames are not required *per se* to perform the algorithm, but they facilitate the symbolic construction of the automorphic and resolving systems using only the infinitesimal data of the pseudogroup action and the choice of a cross-section to the pseudogroup orbits. But, when the automorphic system is used to construct a solution to $\Delta = 0$ from a solution of the resolving system, as in (3.24), it becomes necessary to know the explicit formulae for the generating invariants. Also, as Example 3.19 shows, this final step of the group foliation method may result in a problem no easier to solve than the original differential equation.

To address these shortcomings, we replace the explicit automorphic system by a system of *reconstruction equations*. In essence, the reconstruction system makes use of the pseudogroup transformations to map the resolving system solution away from the cross-section, to solutions of $\Delta = 0$. More precisely: a right moving frame ρ will project the jet of an unknown solution along pseudogroup orbits onto the cross-section. This projection is identical to the intersection of the orbit of the solution with the cross-section, and hence characterized as a solution of the resolving system $\mathcal{R}_\varrho$ studied in the previous section. A left moving frame $\bar\rho$ inverts this process, mapping a resolving system solution away from the cross-section and back to solutions of $\Delta = 0$. See Figure 3.3 for the geometry of this process. We begin by returning to Example 3.2 to explain the reconstruction process by obtaining coordinate expressions for the left moving frame through explicit computation. We then introduce *recurrence relations* for the pseudogroup jets of $\mathcal{G}$, which will allow the determination of reconstruction equations in a purely symbolic manner.
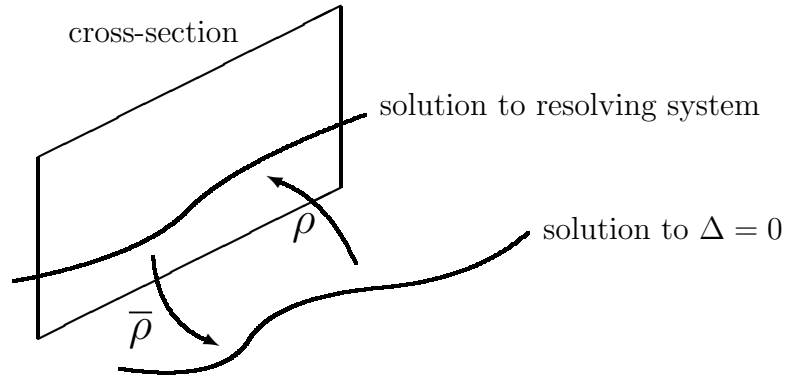

Figure 3.3: The geometry of reconstruction

Example 3.21. We return to Example 3.2 to illustrate reconstruction with as little machinery as possible. Although a moving frame was not used explicitly for the initial computations in this example, we are compelled to introduce one now. Choosing the cross-section $u = 1$ for the symmetry transformations (3.2) we deduce

$$\lambda u = 1 \qquad \Longrightarrow \qquad \lambda = \frac{1}{u}. \tag{3.28}$$

Invariantization with respect to this right moving frame produces the generating

invariants we used for foliation:

$$\iota(x) = x, \qquad \iota(t) = t, \qquad \iota(u_x) = \frac{u_x}{u} = I, \qquad \iota(u_t) = \frac{u_t}{u} = J.$$

By construction, the right moving frame projects points in jet space onto the cross-section along pseudogroup orbits. Thus, projecting the graph of a solution $u(x,t)$ to (3.1) will yield the graph of a solution to the resolving system (3.5):

$$\lambda \cdot \big(x, t, u(x,t), u_x(x,t), u_t(x,t)\big)\Big|_{\lambda = 1/u(x,t)} = \big(x, t, 1, I(x,t), J(x,t)\big).$$

Let us begin instead with a solution to the resolving system and use the inverse of the right moving frame — the left moving frame — to reverse the projection. Explicitly, the left moving frame $\overline{\lambda}$ is defined by the relation

$$\overline{\lambda}\,\lambda = 1. \tag{3.29}$$

Differentiating the defining relation (3.29) we find

$$
\begin{aligned}
D_x(\overline{\lambda}\,\lambda) = 0 \qquad &\implies \qquad D_x\overline{\lambda} = \overline{\lambda}\,(-D_x\lambda \cdot \lambda^{-1}) \\
&\implies \qquad D_x\overline{\lambda} = \overline{\lambda}\,I
\end{aligned}
\tag{3.30a}
$$

and

$$
\begin{aligned}
D_t(\overline{\lambda}\,\lambda) = 0 \qquad &\implies \qquad D_t\overline{\lambda} = \overline{\lambda}\,(-D_t\lambda \cdot \lambda^{-1}) \\
&\implies \qquad D_t\overline{\lambda} = \overline{\lambda}\,J,
\end{aligned}
\tag{3.30b}
$$

since the expression (3.28) for the right moving frame λ gives

$$-D_x\lambda \cdot \lambda^{-1} = -D_x\left(\frac{1}{u}\right)u = \frac{u_x}{u} = I \quad \text{and} \quad -D_t\lambda \cdot \lambda^{-1} = -D_t\left(\frac{1}{u}\right)u = \frac{u_t}{u} = J.$$

The equations (3.30) may be interpreted as a system of differential equations for the left moving frame $\overline{\lambda}(x,t)$; this is the *reconstruction system*.

Given a solution $\overline{\lambda}(x,t)$ to the reconstruction system, we obtain a solution to (3.1) by simply acting on the graph of the chosen resolving system solution:

$$
\begin{aligned}
\overline{\lambda} \cdot (x, t, 1, I, J) &= \big(x, t, \overline{\lambda}, \overline{\lambda}\,I, \overline{\lambda}\,J\big) \\
&= \big(x, t, u, u_x, u_t\big),
\end{aligned}
$$

yielding

$$\overline{\lambda}(x,t) = u(x,t).$$

For example, if we choose the resolving system solution $I(x,t) = x$, $J(x,t) = 1$ as before, the reconstruction equations become

$$D_x\overline{\lambda} = x\,\overline{\lambda} \qquad D_t\overline{\lambda} = \overline{\lambda},$$

resulting in same solution obtained in Example 3.2:

$$\overline{\lambda}(x,t) = u(x,t) = \mu e^{\frac{1}{2}x^2+t}, \qquad \mu > 0.$$

The general family of solutions from Example 3.2 is derived in a similar manner.

Remark 3.22. It happens by coincidence that the left moving frame $\overline{\lambda}(x,t)$ is identical to the reconstructed solution $u(x,t)$. This is a serendipitous feature of (3.2) and the choice of cross-section. No such coincidences will occur in the more involved examples to follow.

Pseudogroup jet recurrence relations

With the geometry of reconstruction clearly in mind, we now address the main purpose of the reconstruction process: to replace the final step of Algorithm 3.18 with a completely symbolic alternative to the explicit automorphic system. As such, we would like a way to arrive at reconstruction equations analogous to (3.30) of Example 3.21 without recourse to explicit formulae for either the differential invariants or the right and left moving frames. To accomplish this goal, we introduce *pseudogroup jet recurrence relations*, arising simply from exterior differentiation of the pseudogroup jets. Pull-back of these pseudogroup jet recurrence relations by the right moving frame results in an expression for the exterior derivatives of the left moving frame components in terms of "known quantities": invariant horizontal differential forms and the right moving frame pull backs of the right Maurer–Cartan forms, computed using the universal recurrence relation (2.40). Expansion of these exterior derivatives in the invariant horizontal coframe yields differential equations for the the left moving frame; after restriction to a resolving system solution, these differential equations become the *reconstruction equations*.

The pseudogroup jet recurrence relations rely on the relation between left and right Maurer–Cartan forms. Recall from Section 2.1 the right and left zero order Maurer–Cartan forms, respectively, for the full diffeomorphism pseudogroup:

$$\mu^a = dZ^a - \sum_{b=1}^m Z^a_{z^b} dz^b, \qquad \overline{\mu}^a = dz^a - \sum_{b=1}^m z^a_{Z^b} dZ^b.$$

Higher order right and left Maurer–Cartan forms $\mu^a_A = \mathbb{D}^A_Z \mu^a$ and $\overline{\mu}^a_A = \mathbb{D}^A_z \overline{\mu}^a$ are obtained via Lie differentiation with respect to, respectively,

$$\mathbb{D}_{Z^a} = \sum_{b=1}^m z^b_{Z^a} \mathbb{D}_{z^b} \qquad \text{and} \qquad \mathbb{D}_{z^a} = \sum_{b=1}^m Z^b_{z^a} \mathbb{D}_{Z^b}.$$

The Maurer–Cartan forms for a Lie pseudogroup $\mathcal{G} \subset \mathcal{D}$ are found by restricting the diffeomorphism pseudogroup Maurer–Cartan forms to the determining equations (2.5) and lifted determining equations (2.21) for $\mathcal{G}$, with the interchange $z \leftrightarrow Z$, $\mu \leftrightarrow \overline{\mu}$ for the left Maurer–Cartan forms. It is amusing to note that the difference between left and right Maurer–Cartan forms has been thus rendered a change of notation.

Using the relation (2.24) between left and right zero order Maurer–Cartan forms we find the following relations among the diffeomorphism pseudogroup jets:

$$dz^a = \sum_{b=1}^m \left(z^a_{Z^b} dZ^b - z^a_{Z^b} \mu^b \right). \tag{3.31a}$$

Similar relations among the higher order diffeomorphism pseudogroup jets z^a_A are obtained by Lie differentiation of (3.31a) with respect to $\mathbb{D}_{Z^a}$. For example, we find for the first order pseudogroup jets:

$$dz^a_{Z^c} = \sum_{b=1}^m \left(z^a_{Z^b Z^c} dZ^b - z^a_{Z^b Z^c} \mu^b - z^a_{Z^b} \mu^b_{Z^c} \right). \tag{3.31b}$$

Definition 3.23. Equations (3.31) and higher order consequences will be called *pseudogroup jet recurrence relations* for the diffeomorphism pseudogroup. For a Lie pseudogroup $\mathcal{G} \subset \mathcal{D}$, the recurrence relations for its pseudogroup jets are obtained by application of the determining system (2.5), with the interchange of $z \leftrightarrow Z$, and lifted determining system (2.21) to (3.31).

Remark 3.24. It will usually be more convenient to work with pseudogroup parameters instead of pseudogroup jets. The distinction is purely computational; we will illustrate both approaches in our running examples.

Example 3.25. Returning to the scaling action (3.2), the zero order pseudogroup jet recurrence relations are

$$dx = dX,$$
$$dt = dT, \tag{3.32}$$
$$du = \frac{u}{U}(dU - \mu^u),$$

derived by applying to (3.31a) the determining equations for (3.2),

$$x = X, \qquad t = T, \qquad u_U = \frac{u}{U}, \qquad u_X = 0, \qquad u_T = 0,$$

and identities for the Maurer–Cartan forms,

$$\mu^x = 0, \qquad \mu^t = 0, \qquad \mu_U^u = \frac{1}{U}\mu^u, \qquad \mu_X^u = 0, \qquad \mu_T^u = 0.$$

Lie differentiation of (3.32) with respect to $\mathbb{D}_X, \mathbb{D}_T, \mathbb{D}_U$ produces only one nontrivial higher order jet recurrence relation:

$$du_U = -\frac{u}{U}\mu_U^u = -\frac{u}{U^2}\mu^u. \tag{3.33}$$

We can also write the pseudogroup jet recurrence relations in terms of the parameter $\bar{\lambda}$; in this case (3.32) and (3.33) coincide:

$$d\bar{\lambda} = -\frac{\bar{\lambda}}{U}\mu^u. \tag{3.34}$$

Example 3.26. We now compute the pseudogroup jet recurrence relations for the Lie pseudogroup action (2.7). Applying to (3.31) the determining equations

$$x_Y = 0, \qquad x_U = 0, \qquad y = Y, \qquad u = \frac{U}{x_X},$$

and the lifted determining equations (2.22) for the right Maurer–Cartan forms we obtain

$$dx = x_X(dX - \mu^x)$$
$$dy = dY$$
$$du = \frac{1}{x_X}(dX - \mu^x) - \frac{U x_{XX}}{x_X}(dU + U\mu^x).$$

Lie differentiation with respect to $\mathbb{D}_X$ gives the higher order relations

$$dx_X = x_{XX}\,(dX - \mu^x) - x_X\,\mu^x_X$$

$$du_X = \frac{-x_{XX}}{x_X^2}(dX - \mu^x) - \frac{1}{x_X}\mu^x_X - \frac{U(x_{XXX}x_X - x_{XX}^2)}{x_X^2}(dU + U\mu^x) - \frac{U^2 x_{XX}}{x_X}\mu^x,$$

and so on. Writing these jet recurrence relations in terms of the pseudogroup parameters $\bar{f} = x$, $\bar{f}_X = x_X$, $\bar{f}_{XX} = x_{XX}, \ldots$ instead offers some simplification:

$$
\begin{aligned}
d\bar{f} &= \bar{f}_X\,(dX - \mu^x) \\
d\bar{f}_X &= \bar{f}_{XX}\,dX - \bar{f}_{XX}\,\mu^x - \bar{f}_X\,\mu^x_X, \\
d\bar{f}_{XX} &= \bar{f}_{XXX}\,dX - \bar{f}_{XXX}\,\mu^x - 2\bar{f}_{XX}\,\mu^x_X - \bar{f}_X\,\mu^x_{XX},
\end{aligned}
\tag{3.35}
$$

and so on. The recurrence relations involving the jets $u, u_X, u_{XX}, \ldots$ may be disregarded since they are expressible in terms of the jet parameters determined by (3.35).

The reconstruction equations

Because the right and left moving frames ρ and $\bar{\rho}$ are related by inversion, the right moving frame pull-back of the "inverse" pseudogroup jets z^a_A produces the left moving frame pull-back of the "regular" pseudogroup jets Z^a_A:

$$\bar{\rho}^*(z,\,Z^{(\infty)}) = \rho^*(Z,\,z^{(\infty)}).$$

Thus applying the right moving frame pull-back ρ^* to the pseudogroup jet recurrence relations will yield an expression for the differential $d\bar{\rho} = \big(d(\bar{\rho}^*z),\,d(\bar{\rho}^*z^a_{Z^b}),\ldots\big)$ of the left moving frame:

$$d\bar{\rho} \equiv \sum_{j=1}^{p} P_j(\bar{\rho}, H, K^{(\infty)})\,\varpi^j, \tag{3.36}$$

where $H, K^{(\infty)}$ are the collections of normalized invariants $H^i = \iota(x^i)$, $K^\alpha_J = \iota(u^\alpha_J)$, respectively, and $z = (x, u)$ as usual. The invariants H^i, K^α_J, make their appearance in (3.36) via the normalized Maurer–Cartan forms $\rho^*\mu^a_A$ and the normalized differentials ρ^*dX^i, ρ^*dU^α. Note that these quantities may all be computed symbolically via the universal recurrence relation (2.40). Giving a general expression for the functions P_j is possible but not useful for our discussion.

Definition 3.27. The system (3.36) will be called the *universal reconstruction equations* for the moving frame $\bar{\rho}$.

Application of the universal reconstruction equations to the problem of group foliation will essentially consist of restricting (3.36) to a particular automorphic system $\mathcal{A}_\varrho$ given by a choice of resolving system solution. We first consider the case of full rank $\varrho = p$. Let $\Delta = 0$ be a $\mathcal{G}$-invariant differential equation, and suppose that a solution to a full rank resolving system $\mathcal{R}_\varrho$ is given, determining the automorphic system $\mathcal{A}_\varrho$. Let $I^1, \ldots, I^p$ be the distinguished independent invariants for the resolving system. Because of independence, (invariant horizontal projections of) the forms

$$dI^1, \ldots, dI^p$$

form a invariant horizontal coframe, which may be used in place of ϖ^i, $i = 1, \ldots, p$, in (3.36). Restricted to $\mathcal{A}_\varrho$, the universal reconstruction equations (3.36) then yield an explicit system of differential equations for the left moving frame as a function of the invariant independent variables $I^1, \ldots, I^p$ via projection onto this horizontal coframe:

$$d\bar{\rho} = \sum_{j=1}^{p} Q_j(\bar{\rho}, I^1, \ldots, I^p) \, dI^j. \tag{3.37}$$

All invariants $H, K^{(\infty)}$ are expressed as functions of the distinguished invariants via the recurrence relations. The result is a system of first order differential equations that must be satisfied by $\bar{\rho}$:

$$D_{I^j} \bar{\rho} = Q_j(\bar{\rho}, I^1, \ldots, I^p). \tag{3.38}$$

We will refer to (3.37) or (3.38) simply as *reconstruction equations* when the resolving system solution of interest is implicit.

Remark 3.28. The reconstruction system (3.37), though it appears to be an infinite system of differential equations, may be taken as finite in practice. This is because it defines an involutive exterior differential system, as may be verified directly via application of the Cartan–Kähler theorem, [**31**].

Once a solution to the reconstruction equations is found, the reconstruction process proceeds as follows. By construction, a solution to the reconstruction equations will

map the resolving system solution to the graph of a solution to the equation $\Delta = 0$. Thus, to construct a (parametrized) solution of $\Delta = 0$, we apply our reconstruction solution $\overline{\rho}$ to the graph of the resolving system solution:

$$\overline{\rho}(I^1, \ldots, I^p) \cdot \big(H(I^1, \ldots, I^p), K(I^1, \ldots, I^p)\big) = \big(x^1, \ldots, x^p, u^1(x), \ldots, u^q(x)\big),$$

where the normalized invariants $H = \big(\iota(x^1), \ldots, \iota(x^p)\big)$, $K = \big(\iota(u^1), \ldots, \iota(u^q)\big)$ are evaluated on the resolving system solution.

Remark 3.29. To simplify notation in the following examples, we will use the same notation for the pseudogroup parameters and their right moving frame pull-backs.

Example 3.30. Continuing Example 3.25, applying the pull-back by the right moving frame used in Example 3.21, the zero order pseudogroup recurrence relation (3.34) becomes

$$d\overline{\lambda} = -\overline{\lambda}\,\rho^*(\mu^u),$$

since $U = 1$ by the definition of the cross-section. Using the recurrence relations, we compute

$$\rho^*(\mu^u) = -I\,\varpi^x - J\varpi^t,$$

and hence

$$d\overline{\lambda} = \overline{\lambda}\,I\,\varpi^x + \overline{\lambda}\,J\,\varpi^t \qquad \Longrightarrow \qquad d\overline{\lambda} = \overline{\lambda}\,I\,dx + \overline{\lambda}\,J\,dt,$$

since our distinguished invariants are x, t and the invariant horizontal forms are simply the differentials: $\varpi^x = dx$ and $\varpi^t = dt$. Hence we arrive at the same reconstruction equations (3.30) found in Example 3.2, with no recourse to the explicit formulae for the invariants I, J or the moving frames $\rho, \overline{\rho}$.

Example 3.31. Continuing Example 3.19, we apply the reconstruction approach to obtain solutions to (3.20). We begin by deriving the reconstruction equations (3.37). Taking the right moving frame pull-back of the zero order pseudogroup jet recurrence relation from (3.35) yields

$$d\overline{f} = \overline{f}_X\,\varpi^x, \tag{3.39}$$

since, as found in Example 2.33,

$$\rho^*(dX) = 0 \qquad \text{and} \qquad \rho^*(\mu^x) = -\varpi^x.$$

By duality with (3.16) we find

$$\varpi^x \equiv (J^2 - L)\, dH + dJ, \qquad \varpi^y \equiv dH,$$

using the constraint syzygy $K = 1$ and writing $L = L(H, J)$ for our choice of resolving system solution from (3.23). Expressing (3.39) in this new coframe we obtain

$$d\bar{f} = (J^2 - L)\bar{f}_X\, dH + \bar{f}_X\, dJ,$$

which gives the reconstruction equations for $\bar{f}(H, J), \bar{f}_X(H, J)$:

$$D_H\, \bar{f} = (J^2 - L)\bar{f}_X \qquad D_J\bar{f} = \bar{f}_X.$$

These equations determine $\bar{f}, \bar{f}_X$, which are the only parameters needed for reconstruction. Using (3.23) the reconstruction equations may be written more explicitly as

$$D_H\, \bar{f} = \left(\frac{J^2}{2} - G(H) \right) D_J\bar{f}, \qquad D_J\bar{f} = \bar{f}_X, \tag{3.40}$$

which may be solved by the method of characteristics. Acting on the graph of the resolving system solution in the cross-section (2.29) by the left moving frame determined by the reconstruction yields a solution to the nonlinear wave equation (3.20) given parametrically, in terms of the invariants H and J:

$$x = \bar{f}(H, J), \qquad y = H, \qquad u = \frac{1}{\bar{f}_X(H, J)}.$$

Remark 3.32. The same reconstruction result obtained in Example 3.31 was derived in [**67**], obtained by other means using the machinery of symmetry reduction of exterior differential systems. Particular solutions to the reconstruction equations (3.40) for constant $G(H)$ are given in [**67**], but we will not reproduce them here as we are presently interested only in the method of solution and not the solutions themselves.

We now consider the reconstruction process for non-maximal invariant rank, $\varrho < p$. In this case, (horizontal projections of) the forms dI^1, ..., dI^ϱ cannot be used as an invariant coframe in place of the invariant forms ϖ^i. To remedy this situation, we supplement the invariant forms $dI^1, \ldots, dI^\varrho$ with $p - \varrho$ forms $\varpi^{j_1}, \ldots, \varpi^{j_{p-\varrho}}$ from

the standard invariant horizontal coframe in order to form a full invariant horizontal coframe. Thus the reconstruction equations have the modified form

$$d\overline{\rho} \equiv \sum_{j=1}^{\varrho} Q_j(\overline{\rho}, I^1, \dots, I^\varrho) \, dI^j + \sum_{i=1}^{p-\varrho} P_{j_i}(\overline{\rho}, I^1, \dots, I^\varrho) \, \varpi^{j_i}.$$

We may then use $p-\varrho$ of these equations to express the supplemental differential forms ϖ^{j_i} in terms of the differentials of $p - \varrho$ moving frame components $\overline{\rho}^{a_i} = \rho^*(z^{a_i})$, $i = 1, \dots, p - \varrho$. Solutions to these non-maximal rank reconstruction equations will then be parametrized by the invariant variables $I^1, \dots, I^\varrho$ in addition to the components $\overline{\rho}^{a_1}, \dots, \overline{\rho}^{a_{p-\varrho}}$. The addition of these $p - \varrho$ "free parameters" in the reconstruction transformations is expected; we are attempting to reconstruct the graph of a solution to $\Delta = 0$, a p-dimensional manifold, from the graph of a resolving system solution, a ϱ-dimensional manifold.

Example 3.33. We return now to Example 3.20 to illustrate reconstruction for non-maximal rank. Recall that in this example we have the single distinguished invariant H, and the resolving system solution

$$J(H) = G(H)$$
$$L(H) = G'(H) + G(H)^2.$$

We supplement the form dH with ϖ^x so that $\{dH, \varpi^x\}$ is an invariant horizontal coframe. Applying the right moving frame pull-back to the first two pseudogroup jet recurrence relations from (3.35) yields

$$d\bar{f} \equiv \bar{f}_X \, \varpi^x, \qquad d\bar{f}_X \equiv \bar{f}_{XX} \, \varpi^x - \bar{f}_X \, J \, dH. \tag{3.41}$$

The first equation of (3.41) allows us to express the invariant horizontal form ϖ^x in terms of the moving frame components:

$$\varpi^x \equiv d\bar{f}/\bar{f}_X,$$

reducing the second equation of (3.41) to

$$d\bar{f}_X \equiv \frac{\bar{f}_{XX}}{\bar{f}_X} \, d\bar{f} - \bar{f}_X \, J \, dH. \tag{3.42}$$

The component $\bar{f}$ of the moving frame may be taken as an independent variable so that

$$\bar{f}_X = \bar{f}_X(\bar{f}, H), \qquad \bar{f}_{XX} = \bar{f}_{XX}(\bar{f}, H),$$

and hence (3.42) yields differential equations for $\bar{f}_X$:

$$D_{\bar{f}} \bar{f}_X = \frac{\bar{f}_{XX}}{\bar{f}_X}, \qquad D_H \bar{f}_X = -\bar{f}_X J.$$

The first equation gives $\bar{f}_{XX}$ in terms of $\bar{f}_X$; solving the second we find

$$\bar{f}_X(\bar{f}, H) = A(\bar{f}) \, e^{-\int G(H)\,dH} = \frac{A(\bar{f})}{B(H)},$$

where $A(\bar{f}) \neq 0$, $B(H) > 0$ are arbitrary functions. Hence we find solutions to (3.25), parametrized by $\bar{f}, H$:

$$(x, y, u) = \left(\bar{f}, H, \frac{1}{\bar{f}_X} \right) = \left(\bar{f}, H, \frac{B(H)}{A(\bar{f})} \right).$$

In agreement with our explicit computation of the left moving frame in (2.31), we have $\bar{f} = x$, and conclude that $u(x, y) = B(y)/A(x)$ solves (3.25).

Relation of automorphic system with reconstruction system

To conclude the description of reconstruction, we would like to show that the two approaches to reconstruction — reconstruction equations for the moving frame and the explicit automorphic system — are equivalent.

Theorem 3.34. Suppose that a solution to the resolving system $\mathcal{R}_\varrho$ of a $\mathcal{G}$-invariant differential equation $\Delta = 0$ is fixed. The resulting reconstruction equations (3.37) are automorphic relative to $\mathcal{G}$.

Proof. Let $\bar{\rho}_1$ and $\bar{\rho}_2$ be two solutions of the reconstruction equations (3.37). Then, by construction, $\bar{\rho}_1 \cdot (H, I)$ and $\bar{\rho}_2 \cdot (H, I)$ are solutions to $\Delta = 0$ that come from the same resolving system solution. Thus their projections onto $\mathcal{K}^\infty$, and hence their signatures, overlap. By Theorem 2.47, the solutions must be locally equivalent, and hence $\bar{\rho}_1$ and $\bar{\rho}_2$ must be related by a pseudogroup transformation. $\square$

Because the reconstruction equations satisfy the automorphic property, the solutions obtained by solving the automorphic system associated to a particular solution of the resolving system coincide with those found via reconstruction. We can envision this geometrically as follows: choosing a single solution $\bar{\rho}$ to the reconstruction system yields, from a solution to the resolving system, a solution u_0 to $\Delta = 0$. Applying a pseudogroup transformation to $\bar{\rho}$ coincides with acting on the solution u_0 by the same pseudogroup transformation. This orbit of u_0 is exactly the automorphic system (3.10).

Remark 3.35. We note that, due to the automorphic property of the reconstruction equations, solutions are not unique. Acting by a transformation of $\mathcal{G}$ will produce a new reconstruction solution, and hence a new solution to $\Delta = 0$. This freedom of choice in the reconstruction solution can be seen in all of our examples.

Example 3.36. With explicit knowledge of the left moving frame, we can see directly the equivalence of the automorphic system and reconstruction equations. In this example we compare directly the automorphic system (3.24) and reconstruction equations (3.40) for our running example. Taking the exterior derivative of the invariants

$$H = y, \qquad J = \frac{u_y}{u},$$

we obtain

$$dH \equiv dy, \qquad dJ \equiv \left(\frac{uu_{xy} - u_x u_y}{u^2} \right) dx + \left(\frac{uu_{yy} - u_y^2}{u^2} \right) dy$$

so that by duality

$$D_H = D_y - \left(\frac{uu_{yy} - u_y^2}{uu_{xy} - u_x u_y} \right) D_x, \qquad D_J = \frac{u^2}{uu_{xy} - u_x u_y} D_x.$$

Substituting the values of the left moving frame, $\bar{f} = x$ and $\bar{f}_X = 1/u$, and writing out the reconstruction equations (3.40) explicitly:

$$D_H \bar{f} = \left(\frac{J^2}{2} - G(H) \right) D_J \bar{f}, \qquad D_J \bar{f} = \bar{f}_X$$

we recover exactly the automorphic system (3.24).

3.4   Further examples

Stationary boundary layer equations

This section is dedicated to several examples of group foliation in action. The first
example revisits one of the earliest appearances of a "practical implementation" of
group foliation, Ovsiannikov's solution of the stationary boundary layer equations,
[**66**]. In addition to reproducing Ovsiannikov's invariant rank 2 resolving system, we
also consider invariant rank 1 and apply the moving frames reconstruction method
introduced in Section 3.3. The second example is a nonlinear wave equation solved by
Calogero, [**12**], by other means and later considered by Lei, [**44**], from the perspective
of group foliation. The final examples considered are first order nonlinear transonic
gas equations, [**66**], and a nonlinear second order ordinary differential equation. These
examples are used to illustrate the application of group foliation to find invariant and
partially invariant solutions in the context of a Lie group action.

Example 3.37. Let $\theta(x)$ be a given function and consider the stationary boundary
layer equations

$$uu_x + vu_y + \theta(x) = u_{yy},$$
$$u_x + v_y = 0. \tag{3.43}$$

These equations admit the Lie pseudogroup action

$$X = x, \qquad Y = y + a(x), \qquad U = u, \qquad V = v + u\, a_x(x). \tag{3.44a}$$

The first order prolonged action is

$$U_X = u_x - a_x u_y, \qquad V_X = v_x - a_x v_y + a_x u_x + u a_{xx} - u_y a_x^2,$$
$$U_Y = u_y, \qquad V_Y = v_y + u_y a_x, \tag{3.44b}$$

and a cross-section to the prolonged action is

$$Y = 0, \qquad V_{X^k} = 0 \qquad k \geq 0. \tag{3.45}$$

Although these formulae will not be needed, we remark that solving the normalization
equations (3.45) results in the right moving frame

$$a = -y, \qquad a_x = -\frac{v}{u}, \qquad \dots, \tag{3.46}$$

which leads to the normalized invariants

$$\iota(x) = x, \qquad \iota(u) = u,$$

$$\iota(u_x) = u_x + \frac{v}{u}u_y, \qquad \iota(u_y) = u_y, \qquad \iota(v_y) = v_y - \frac{v}{u}u_y, \qquad \dots.$$

The infinitesimal generator of the prolonged action (3.44) is

$$\mathbf{v}^{(\infty)} = a(x)\frac{\partial}{\partial y} + ua_x\frac{\partial}{\partial v} - u_y a_x\frac{\partial}{\partial u_x} + (ua_{xx} + u_x a_x - v_y a_x)\frac{\partial}{\partial v_x} + u_y a_x\frac{\partial}{\partial v_y} + \cdots,$$

and the recurrence relations for the lifted invariants up to order 1 are

$$dX = \Omega^x, \qquad\qquad\qquad\qquad dY = \Omega^y + \mu,$$

$$dU \equiv U_X\Omega^x + U_Y\Omega^y, \qquad\qquad\qquad dV \equiv V_X\Omega^x + V_Y\Omega^y + U\mu_X,$$

$$dU_X \equiv U_{XX}\Omega^x + U_{XY}\Omega^y - U_Y\mu_x, \qquad dU_Y \equiv U_{XY}\Omega^x + U_{YY}\Omega^y,$$

$$dV_X \equiv V_{XX}\Omega^x + V_{XY}\Omega^y + U\mu_{XX} + (U_X - V_Y)\mu_X, \quad dV_Y \equiv V_{XY}\Omega^x + V_{YY}\Omega^y + U_Y\mu_X.$$

$$(3.47)$$

Let $\iota(x) = H^x$, $\iota(u_K) = I_K^u$ and $\iota(v_K) = I_K^v$. Pulling-back (3.47) by the right moving frame (3.46) we obtain

$$\mu = -\varpi^y, \qquad \mu_X \equiv -\frac{I_{01}^v}{I^u}\varpi^y, \qquad \mu_{XX} = \frac{1}{I^u}\left(\frac{I_{01}^v}{I^u}(I_{10}^u - I_{01}^v) - I_{11}^v\right)\varpi^y.$$

The recurrence relations for the non-phantom invariants become

$$dH^x = \varpi^x, \qquad dI^u \equiv I_{10}^u\varpi^x + I_{01}^u\varpi^y, \qquad dI_{01}^u \equiv I_{11}^u\varpi^x + I_{02}^u\varpi^y,$$

$$dI_{10}^u \equiv I_{20}^u\varpi^x + I_{11}^u\varpi^y + \frac{I_{01}^u I_{01}^v}{I^u}\varpi^y, \qquad dI_{01}^v \equiv I_{11}^v\varpi^x + I_{02}^v\varpi^y - \frac{I_{01}^u I_{01}^v}{I^u}\varpi^y,$$

from which we deduce the relations

$$\mathcal{D}_x I^u = I_{10}^u, \qquad \mathcal{D}_y I^u = I_{01}^u,$$

$$\mathcal{D}_x I_{10}^u = I_{20}^u, \qquad \mathcal{D}_y I_{10}^u = I_{11}^u + \frac{I_{01}^u I_{01}^v}{I^u},$$

$$\mathcal{D}_x I_{01}^u = I_{11}^u, \qquad \mathcal{D}_y I_{01}^u = I_{02}^u, \qquad\qquad (3.48)$$

$$\mathcal{D}_x I_{01}^v = I_{11}^v, \qquad \mathcal{D}_y I_{01}^v = I_{02}^v - \frac{I_{01}^u I_{01}^v}{I}.$$

From (3.48) we obtain the fundamental syzygy

$$\mathcal{D}_y I_{10}^u = \mathcal{D}_x I_{01}^u + \frac{I_{01}^u I_{01}^v}{I^u}. \qquad\qquad (3.49)$$

We take the generating set of differential invariants

$$s = x, \qquad t = u, \qquad J = I_{01}^u, \qquad K = t I_{10}^u, \qquad L = I_{10}^u + I_{01}^v.$$

Considering first solutions of maximal invariant rank, we choose the automorphic system

$$\mathcal{A}_2 : \qquad J = J(s,t), \qquad K = K(s,t), \qquad L = L(s,t).$$

Since

$$ds \wedge dt = J\varpi^x \wedge \varpi^y, \tag{3.50}$$

the variables s, t, are independent provided $J \neq 0$. By the chain rule

$$\mathcal{D}_x = (\mathcal{D}_x s)D_s + (\mathcal{D}_x t)D_t = D_s + \frac{K}{t}D_t,$$

$$\mathcal{D}_y = (\mathcal{D}_y s)D_s + (\mathcal{D}_y t)D_t = JD_t,$$

and the syzygy (3.49) yields the syzygy system

$$\mathcal{S}_2 : \qquad JD_t\left(\frac{K}{t}\right) = D_s J + \frac{K}{t}D_t J + \frac{J}{t}\left(L - \frac{K}{t}\right). \tag{3.51a}$$

The invariantization of the stationary boundary layer equations (3.43) produces the constraint syzygies

$$K + \theta(s) = JJ_t, \qquad L = 0, \tag{3.51b}$$

which together with (3.51a) form the resolving system. From (3.51b) we obtain K and L in terms of J, while (3.51a) gives the differential equation

$$J^2 J_{tt} + \theta(s)J_t = tJ_s. \tag{3.52}$$

The difficulty of solving the system (3.43) is now reduced to the solution of (3.52).

Suppose that a solution $J(s,t)$ has been found. We now proceed with the reconstruction step. The pseudogroup jet recurrence relations (3.31) give

$$d\bar{a} = \bar{a}_X dX - \mu.$$

Using the same notation for pseudogroup parameters and their pull-backs and pulling back the pseudogroup jet recurrence relations by the right moving frame (3.46) we obtain

$$d\bar{a} = \bar{a}_X \varpi^y + \bar{a}_X ds.$$

Restricting to the automorphic system specified by our resolving system solution $J(s,t)$ we obtain the reconstruction equations

$$d\bar{a} \equiv \left(\bar{a}_X - \frac{K}{tJ}\right)ds + \frac{dt}{J} \qquad \Longleftrightarrow \qquad D_t\bar{a} = \frac{1}{J}, \qquad \bar{a}_X = D_s\bar{a} + \frac{K}{tJ}. \qquad (3.53)$$

The solution to (3.53) is

$$\bar{a}(s,t) = \int \frac{dt}{J} + f(s), \qquad \bar{a}_X(s,t) = -\int \frac{J_s}{J^2}dt - \left(\frac{JJ_t - \theta(s)}{tJ}\right) + f'(s).$$

Hence, the parametrized solution to the boundary layer equations (3.43) is given by

$$(x,y,u,v) = \bar{\rho}\cdot(s,0,t,0) = \left(s, \int \frac{dt}{J} + f(s), t, t\left(\frac{\theta(s)}{J} - J_t - \int \frac{J_s}{J^2}dt + f'(s)\right)\right), \qquad (3.54)$$

where $J(s,t)$ is any solution to (3.52). As expected, the freedom in the choice of $f(s)$ in the parametric solution (3.54) is a reflection of the automorphic property of the reconstruction equations (3.53).

We now search for solutions of invariant rank $\varrho = 1$. According to the independence condition (3.50), rank $\varrho = 1$ occurs when $J = 0$. We choose the rank 1 automorphic system

$$\mathcal{A}_1 : \qquad t = t(s), \qquad J = J(s) = 0, \qquad K = K(s), \qquad L = L(s).$$

By the chain rule,

$$\mathcal{D}_x = (\mathcal{D}_x s)D_s = D_s, \qquad \mathcal{D}_y = (\mathcal{D}_y s)D_s = 0,$$

and the syzygy (3.49) is thus trivially satisfied. On the other hand, the recurrence relations (3.48) yield the restriction syzygy

$$D_s t = \frac{K}{t}. \qquad (3.55a)$$

When $J = 0$ the constraint syzygies (3.51b) reduce to

$$K + \theta(s) = 0, \qquad L = 0. \qquad (3.55b)$$

Equations (3.55) form the rank 1 resolving system. Their solution may be reduced to that of the ordinary differential equation

$$t \cdot D_s t = -\theta(s). \qquad (3.56)$$

To reconstruct a solution of (3.43) from a solution of the resolving system we consider the pseudogroup jet recurrence relations

$$d\bar{a} = -\mu + \bar{a}_X dX, \qquad d\bar{a}_X = -\mu_X + \bar{a}_{XX} dX.$$

After pull-back by the right moving frame and restriction to the automorphic system given by a solution of (3.55) we obtain

$$d\bar{a} = \varpi^y + \bar{a}_X ds, \qquad d\bar{a}_X = \left(\frac{L}{t} - \frac{K}{t^2}\right)\varpi^y + \bar{a}_{XX} ds. \tag{3.57}$$

Using the first equation to express the supplemental form ϖ^y in terms of the pseudogroup parameters we find

$$\varpi^y = d\bar{a} - \bar{a}_X ds, \tag{3.58}$$

from which we choose $\bar{a}$ and s as variables through which to express the other pseudogroup parameters $\bar{a}_X, \bar{a}_{XX}, \dots$. Substituting (3.58) in the second equation of (3.57) we obtain the reconstruction equations

$$D_{\bar{a}} \bar{a}_X = \frac{\theta(s)}{t^2(s)}, \qquad \bar{a}_{XX} = D_s \bar{a}_X + \frac{\bar{a}_X \, \theta(s)}{t^2(s)}.$$

Solving the first equation we conclude that

$$\bar{a}_X(\bar{a}, s) = \frac{\bar{a} \, \theta(s)}{t(s)^2} + f(s),$$

where $f(s)$ is an arbitrary differentiable function. Hence, rank 1 solutions to (3.43) are given by

$$(x, y, u, v) = \overline{\rho} \cdot (s, 0, t, 0) = \left(s, \bar{a}, t(s), \frac{\bar{a} \, \theta(s)}{t^2(s)} + f(s)\right),$$

where $t(s)$ is any solution of the ordinary differential equation (3.56), and $f(s)$ is an arbitrary analytic function.

Calogero's nonlinear wave equation

Example 3.38. In this example, we apply the group foliation method to the nonlinear Calogero wave equation, [**12**],

$$u_{xt} + u u_{xx} = F(u_x). \tag{3.59}$$

The equation (3.59) admits the infinite-dimensional symmetry group

$$X = x + a(t), \qquad T = t, \qquad U = u + a'(t), \tag{3.60}$$

where $a(t)$ is an arbitrary differentiable function of t. The Lie pseudogroup action (3.60) is generated by the vector fields

$$\mathbf{v} = a(t)\frac{\partial}{\partial x} + a'(t)\frac{\partial}{\partial u},$$

whose prolongation is

$$\mathbf{v}^{(\infty)} = a(t)\frac{\partial}{\partial x} + a_t\frac{\partial}{\partial u} + (a_{tt} - u_x\,a_t)\frac{\partial}{\partial u_t} - u_{xx}\,a_t\frac{\partial}{\partial u_{xt}} + (a_{ttt} - u_x\,a_{tt} - 2u_{xt}\,a_t)\frac{\partial}{\partial u_{tt}} + \cdots.$$

The recurrence relations (2.39) for the lifted invariants are

$$\begin{aligned}
dX &= \Omega^x + \mu, \qquad dT = \Omega^t, \\
dU &\equiv U_X\,\Omega^x + U_T\,\Omega^t + \mu_T, \\
dU_X &\equiv U_{XX}\,\Omega^x + U_{XT}\,\Omega^t, \\
dU_T &\equiv U_{XT}\,\Omega^x + U_{TT}\,\Omega^t + \mu_{TT} - U_X\,\mu_T, \\
dU_{XX} &\equiv U_{XXX}\,\Omega^x + U_{XXT}\,\Omega^t, \\
dU_{XT} &\equiv U_{XXT}\,\Omega^x + U_{XTT}\,\Omega^t - U_{XX}\mu_T, \\
dU_{TT} &\equiv U_{XTT}\,\Omega^x + U_{TTT}\,\Omega^t + \mu_{TTT} - U_X\,\mu_{TT} - 2U_{XT}\,\mu_T,
\end{aligned} \tag{3.61}$$

and so on. A cross-section to the pseudogroup orbits is given by

$$X = 0, \qquad U_{T^k} = 0, \qquad k \geq 0,$$

which leads to the normalized Maurer–Cartan forms

$$\mu = -\varpi^x, \qquad \mu_T = -I_{10}\,\varpi^x, \qquad \mu_{TT} = -(I_{11} + I_{10}^2)\varpi^x, \qquad \dots. \tag{3.62}$$

Substituting (3.62) into (3.61) we obtain, up to order 2, the recurrence relations

$$\begin{aligned}
\mathcal{D}_x I_{10} &= I_{20}, & \mathcal{D}_t I_{10} &= I_{11}, \\
\mathcal{D}_x I_{20} &= I_{30}, & \mathcal{D}_t I_{20} &= I_{21}, \\
\mathcal{D}_x I_{11} &= I_{21} + I_{10}I_{20}, & \mathcal{D}_t I_{11} &= I_{12}.
\end{aligned} \tag{3.63}$$

Eliminating I_{21} from (3.63) we find the fundamental syzygy

$$\mathcal{S}_2: \qquad \mathcal{D}_x I_{11} = \mathcal{D}_t I_{20} + I_{10} I_{20}. \qquad (3.64)$$

A generating set for the algebra of differential invariants is given by

$$t, \qquad s = I_{10}, \qquad K = I_{11}, \qquad L = I_{20}.$$

The independence condition for t and s is

$$ds \wedge dt \equiv L \, \varpi^x \wedge \varpi^y,$$

and thus we require $L \neq 0$ and search for rank 2 solutions of (3.59). The rank 2 automorphic system is

$$\mathcal{A}_2: \qquad K = K(s,t), \qquad L = L(s,t).$$

By the chain rule

$$\mathcal{D}_x = (\mathcal{D}_x t) \, D_t + (\mathcal{D}_x s) \, D_s = L \, D_s,$$
$$\mathcal{D}_t = (\mathcal{D}_t t) \, D_t + (\mathcal{D}_t s) \, D_s = D_t + K \, D_s,$$

and thus the syzygy (3.64) gives the rank 2 syzygy system

$$L(K_s - s) = L_t + K L_s. \qquad (3.65a)$$

Additionally, invariantization of the differential equation (3.59) gives the constraint syzygy

$$K = F(s). \qquad (3.65b)$$

The equations (3.65) form the rank 2 resolving system for (3.59). Substituting (3.65b) into (3.65a) we obtain the first order differential equation

$$L_t + F L_s = L(F_s - s) \qquad (3.66)$$

for the invariant L. Assuming $F(s) \neq 0$, the solution to (3.66) is

$$L(s,t) = F(s) \, A\left(t - \int \frac{ds}{F(s)} \right) \exp\left[-\int \frac{s}{F(s)} ds \right],$$

where A is an arbitrary differentiable function.

We now proceed with reconstruction. The pseudogroup jet recurrence relations yield, after pull-back by the right moving frame,

$$d\bar{a} = \varpi^x + \bar{a}_T \, dt.$$

Restricting to the automorphic system given by a solution $L(s,t)$ of (3.66) we arrive at the reconstruction equations

$$d\bar{a} = \frac{1}{L}ds + \left(\bar{a}_T - \frac{K}{L}\right)dt, \qquad \Longleftrightarrow \qquad D_s\bar{a} = \frac{1}{L}, \qquad \bar{a}_T = D_t\bar{a} + \frac{K}{L}.$$

Hence,

$$\bar{a}(s,t) = \int \frac{ds}{L} + a(t) \qquad \text{and} \qquad \bar{a}_T = -\int \frac{L_t}{L^2}ds + \frac{F(s)}{L} + a'(t),$$

where $a(t)$ is an arbitrary function. Solution to (3.59) of invariant rank 2 are then given by

$$(x,t,u) = \bar{\rho} \cdot (0,t,0) = \left(\int \frac{ds}{L} + a(t), t, -\int \frac{L_t}{L^2}ds + \frac{F(s)}{L} + a'(t)\right). \qquad (3.67)$$

We now assume that $L = 0$, and search for solutions of rank 1. The rank 1 automorphic system is now given by

$$\mathcal{A}_1 : \qquad s = s(t), \qquad K = K(t), \qquad L = L(t) = 0,$$

and by the chain rule,

$$\mathcal{D}_x = 0, \qquad \mathcal{D}_t = D_t.$$

The recurrence relations (3.63) yield the restriction syzygy

$$D_t s = K,$$

while the constraint syzygy (3.65b) still holds. These equations form the rank 1 resolving system. The function $s(t)$ must then be a solution of the ordinary differential equation

$$D_t s = F(s). \qquad (3.68)$$

Now for reconstruction. After pull-back by the right moving frame the pseudogroup jet recurrence relations become

$$d\bar{a} = \varpi^x + \bar{a}_T \, dt,$$

$$d\bar{a}_X = s \, \varpi^x + \bar{a}_{TT} \, dt,$$

which yields

$$\varpi^x = d\bar{a} - \bar{a}_T \, dt.$$

Using the variables $\bar{a}$ and t to parametrize pseudogroup jets $\bar{a}_T$, $\bar{a}_{TT}$, $\ldots$ we arrive at the reconstruction equations

$$D_{\bar{a}}(\bar{a}_T) = s, \qquad \bar{a}_{TT} = D_t(\bar{a}_T) + s \, \bar{a}_T.$$

Hence

$$\bar{a}_T(\bar{a}, t) = \bar{a} \cdot s(t) + a(t),$$

where $s(t)$ is a solution of (3.68) and $a(t)$ is an arbitrary differentiable function. We obtain the solutions of rank 1 given by

$$(x, t, u) = \overline{\rho} \cdot (0, t, 0) = (\bar{a}, t, \bar{a} \cdot s(t) + f(t)).$$

Remark 3.39. The full rank solution (3.67) also appears in [**44**]. It can be seen by comparison with this author's computations that the moving frame approach yields the solution in a completely systematic manner and does not require explicit formulae for the invariants s, K and L.

Invariant and partially invariant solutions via group foliation

We illustrate in the next two examples the group foliation method for finite dimensional Lie groups. In particular we show how the group foliation method subsumes existing algorithms for obtaining invariant, [**52**], and partially invariant solutions [**64, 66**]. In the context of finite dimensional Lie groups, the dimension of the automorphic system is bounded between p and $p + r$, where r is the dimension of the Lie group. Let $p + \delta$ be the dimension of the automorphic system, $0 \leq \delta \leq r$.

The number δ is called the *defect* of the solution generating the automorphic system. Invariant solutions have defect $\delta = 0$, while partially invariant solutions satisfy $0 < \delta < r$. By limiting our search to resolving systems of rank $p + \delta - r$, we discover invariant and partially invariant solutions of rank δ.

Example 3.40. We consider the equations

$$u_y - v_x = 0, \qquad u\,u_x + v_y = 0 \tag{3.69}$$

of a transonic gas flow, [**66**]. We first obtain an invariant solution of (3.69) by foliating with respect to the symmetry group of dilations

$$X = \lambda\,x, \qquad Y = \lambda\,y, \qquad U = u, \qquad V = v,$$

and searching for defect $\delta = 0$ solutions, which must have invariant rank 1. Choosing the cross-section

$$\mathcal{K} = \{y = 1\},$$

the recurrence relations imply that

$$dH = \varpi^x - H\,\varpi^y, \tag{3.70}$$

and

$$I_{i+1,j} = \mathcal{D}_x I_{i,j}, \qquad I_{i,j+1} = \mathcal{D}_y I_{i,j} + (i+j)I_{i,j},$$
$$J_{i+1,j} = \mathcal{D}_x J_{i,j}, \qquad J_{i,j+1} = \mathcal{D}_y J_{i,j} + (i+j)J_{i,j},$$

where

$$H = \iota(x), \qquad I_{i,j} = \iota(u_{x^i y^j}), \qquad J_{i,j} = \iota(v_{x^i y^j}).$$

A generating set for the algebra of differential invariants is given by

$$H, \quad I, \quad J.$$

Modulo the commutator syzygies arising from the commutator relation

$$[\mathcal{D}_y, \mathcal{D}_x] = \mathcal{D}_x,$$

there are no fundamental syzygies. Now, searching for invariant rank 1 solutions, we choose the automorphic system given by

$$\mathcal{A}_1: \qquad I = I(H), \qquad J = J(H).$$

By the chain rule

$$\mathcal{D}_x = (\mathcal{D}_x H)D_H = D_H, \qquad \mathcal{D}_y = (\mathcal{D}_y H)D_H = -H\,D_H. \tag{3.71}$$

Examining the recurrence relations subject to (3.71) we find no restriction syzygies. Thus invariantization of (3.69) yields the resolving system

$$H\,D_H I + D_H J = 0, \qquad I\,D_H I - H\,D_H J = 0.$$

This is an ordinary differential equation whose non-constant solutions are easily found:

$$I(H) = -H^2, \qquad J(H) = \frac{2}{3}H^3 + C,$$

where C is an arbitrary constant. Implementing the reconstruction step, we obtain the reconstruction equation

$$\varpi^y = \frac{d\overline{\lambda}}{\overline{\lambda}}. \tag{3.72}$$

Viewing $\overline{\lambda}$ as an independent variable, the invariant solution is given by

$$(x, y, u, v) = \overline{\lambda} \cdot (H, 1, I, J) = (t\,\overline{\lambda}, \overline{\lambda}, -H^2, \frac{2}{3}H^3 + C).$$

Example 3.41. Consider again the transonic gas equations (3.69). To find partially invariant solutions of defect $\delta = 1$, we foliate with respect to

$$X = \lambda\,x, \qquad Y = \lambda\,y, \qquad U = u, \qquad V = v + \epsilon. \tag{3.73}$$

A cross-section is given by

$$\mathcal{K} = \{y = 1,\ v = 0\}$$

and a basis of invariants is given by

$$H = \iota(x), \qquad I = \iota(u), \qquad J = \iota(v_x), \qquad K = \iota(v_y).$$

This time around there is one fundamental syzygy

$$\mathcal{D}_y J = \mathcal{D}_x K - J. \tag{3.74}$$

Restricting to the rank 1 automorphic system

$$I = I(H), \qquad J = J(H), \qquad K = K(H),$$

the corresponding resolving system is given by

$$J + H\, D_H I = 0, \qquad K + I\, D_H I = 0, \qquad D_H\big((H^2 + I)D_H I\big) = 0, \qquad (3.75)$$

where the first two equations come from the invariantization of (3.69) and the third equation is a consequence of the syzygy (3.74). Hence, provided $I(H)$ is a solution of

$$(H^2 + I)D_H I = C,$$

where C is a constant, the invariants J and K are completely determined by (3.75). Implementing the reconstruction step, the first of two reconstruction equations is given by (3.72). From (3.70), which still holds, we conclude that

$$\varpi^x = dH + H\frac{d\overline{\lambda}}{\overline{\lambda}}.$$

Hence, integrating the second reconstruction equation

$$d\overline{\epsilon} = J\,\varpi^x + K\,\varpi^y = -H\, D_H I\, dt - C\,\frac{d\overline{\lambda}}{\overline{\lambda}}$$

we obtain

$$\overline{\epsilon} = -\ln\overline{\lambda} - \int H D_H I\, dH.$$

This produces the partially invariant solution

$$(x, y, u, v) = (\overline{\lambda}, \overline{\epsilon}) \cdot (H, 1, I, 0) = \Big(H\,\overline{\lambda}, \overline{\lambda}, I(H), -\ln\overline{\lambda} - \int H\, D_H I\, dH\Big).$$

Our final example illustrates the use of the group foliation method to reduce the order of an ordinary differential equation. Foliating a second order ordinary differential equation with respect to a two dimensional Lie group, we obtain a resolving system of order zero, i.e. an algebraic equation.

Example 3.42. Consider the nonlinear second order ordinary differential equation

$$x^2 u_{xx} = (x\, u_x - u)^2, \qquad x > 0. \qquad (3.76)$$

The equation (3.76) is invariant under the two dimensional Lie group action

$$X = \lambda\, x, \qquad U = u + \epsilon\, x, \qquad \lambda > 0 \quad \text{and} \quad \epsilon \in \mathbb{R}.$$

The order zero lifted recurrence relations for this action are

$$dX = \Omega^x + X\,\mu_X^x, \qquad dU \equiv U_X\,\Omega^x + X\,\mu_X^u.$$

Choosing the cross-section $\mathcal{K} = \{x = 1,\, u = 0\}$, these recurrence relations yield the normalized Maurer–Cartan forms

$$\mu^x = -\varpi^x, \qquad \mu_X^u \equiv -\iota(u_x)\,\varpi^x.$$

A generating set of invariants is given by

$$z = \iota(u_x), \qquad v = \iota(u_{xx}).$$

Now, consider the rank 1 automorphic system

$$\mathcal{A}_1: \qquad v = v(z).$$

Since

$$dz \equiv v\,\varpi^x \tag{3.77}$$

we require $v \neq 0$. There are no syzygies, and invariantization of (3.76) yields the resolving system

$$v(z) = z^2.$$

We now proceed with reconstruction. The pseudogroup jet recurrence relations yield, after pull-back by the right moving frame,

$$d\bar{\lambda} = \bar{\lambda}\,\varpi^x, \qquad d\bar{\epsilon} = (\bar{\epsilon} + z)\,\varpi^x.$$

Using (3.77) we find $\varpi^x = dz/v$, and the reconstruction equations become

$$D_z\bar{\lambda} = \frac{\bar{\lambda}}{z^2}, \qquad D_z\bar{\epsilon} = \frac{\bar{\epsilon} + z}{z^2}. \tag{3.78}$$

Solving (3.78) we obtain

$$\bar{\lambda}(z) = Ae^{-1/z}, \qquad \bar{\epsilon}(z) = e^{-1/z}\left[\int \frac{e^{1/z}}{z}dz + B\right], \tag{3.79}$$

where A and B are two constants. By construction, the parametric curve given by

$$(x, u) = \bar{\rho}(z) \cdot (1, 0) = \left(\bar{\lambda}(z), \bar{\epsilon}(z)\right)$$

is then a solution of (3.76). To recover the solution in the form $u(x)$ it suffices to express the parameter z as a function of x using the first equation in (3.79):

$$u(x) = -x \left[\int \frac{dx}{x^2(\ln x + A)} + B \right].$$

3.5   Historical overview

We conclude this chapter with an overview of the historical development of group foliation. The story begins in 1895 with Sophus Lie, who suggested the basic strategy of the algorithm using the Lie pseudogroup (2.7), along with other similar examples. We recall Lie's treatment of (2.7) in detail in the first section. Shortly after Lie's suggestion, Ernest Vessiot gave a more thorough account of the method, intended to work generally for any differential equation admitting a symmetry pseudogroup. For this reason, the origin of the method is often attributed to Vessiot. We recall Vessiot's treatment of Lie's example (2.7) and his description of the general method in the second section. To the best of our knowledge, further investigation of group foliation did not appear for a half century after Vessiot. H. H. Johnson devoted a single article in 1962 to a reconsideration of Vessiot's approach in an attempt to make it fully rigorous. Judging from the number of times it has been cited, Johnson's article seems to have since passed largely unnoticed. We summarize the approach of this article in the final section and relate it to the constructions of this chapter. Finally we briefly survey more recent publications and research directions.

Lie's suggestive examples

Sophus Lie learned about Galois theory for algebraic equations as early as 1862 from lectures of Ludvig Sylow. Influenced by Galois's ideas and collaboration with Klein beginning in 1870, Lie set out to develop a version of Galois theory for differential equations based on his new theory of continuous groups, [32]. In analogy with the action of a Galois group, Lie considered continuous groups permuting the solutions of a differential equation and discovered methods by which these groups could be

used to integrate the differential equation. In one of his last papers, *Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung*, [**46**], Lie proved that group invariant solutions to a system of differential equations may be found via the solution of a reduced system of differential equations involving fewer independent variables. This method of symmetry reduction of differential equations has since shown itself tremendously useful for obtaining exact solutions to nonlinear differential equations, [**8**, **52**, **66**].

In the final chapter of this same paper Lie introduced another method for the integration of differential equations which has received much less attention than symmetry reduction. This is the method of *group foliation* (also called group splitting or group stratification in some references). Lie called his idea, in rough translation, a "general theory of integration for all partial differential equations admitting a given group". Because it is the first historical appearance of group foliation, we outline here in Lie's original notation one of his key examples, from Sections 83–86 of [**46**]. This example is identical to our running example, (2.7). Lie also includes other similar examples in the same paper.

Example 3.43. Lie considered the pseudogroup of infinitesimal transformations of x, y, z given by

$$\xi(x)\frac{\partial}{\partial x} + \xi'(x)z\frac{\partial}{\partial z} \tag{3.80}$$

and computed a complete set of differential invariants up to second order, denoted

$$\mu = y \quad \nu = \frac{q}{z} \quad u = \frac{zs - pq}{z^3} \quad v = \frac{t}{z}, \tag{3.81}$$

where p, q, r, s, t are the jet coordinates $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$, respectively. He asserts that any invariant differential equation of second order must then have the form

$$\Omega(\mu, \nu, u, v) = 0.$$

Lie observes, conceptually, that the differential invariants μ, ν may be used as independent variables to describe a geometric surface, i.e. that they may be considered as independent variables. Computing the invariant derivatives

$$u_\mu = \frac{\partial u}{\partial \mu} \quad u_\nu = \frac{\partial u}{\partial \nu} \quad v_\mu = \frac{\partial v}{\partial \mu} \quad v_\nu = \frac{\partial v}{\partial \nu}$$

he finds eight differential invariants

$$\mu, \nu, u, v, u_\mu, u_\nu, v_\mu, v_\nu,$$

of order less than four. Arguing that there are only seven independent differential invariants of order less than four he arrives at the syzygy

$$W(\mu, \nu, u, v, u_\mu, u_\nu, v_\mu, v_\nu) = 0.$$

The method of integration then proceeds as follows. Use the equation $\Omega = 0$ to find an explicit expression for v: $v = V(\mu, \nu, u)$, which may then be used to eliminate v, v_μ, v_ν from the expression $W = 0$. The result is a differential equation

$$\Pi(\mu, \nu, u, u_\mu, u_\nu) = 0,$$

whose solution $u = U(\mu, \nu)$ should produce a solution to the original equation by means of substituting the original differential invariant expressions (3.81)

$$u = U(\mu, \nu) \quad v = V(\mu, \nu) \quad \longleftrightarrow \quad \frac{zs - pq}{z^3} = U(y, \frac{q}{z}) \quad \frac{t}{z} = V(y, \frac{q}{z}) \qquad (3.82)$$

and solving the resulting differential equations for the desired $z(x, y)$. The equations $\Omega = 0$ and $W = 0$ make up the *resolving system* and the final expression (3.82) forms the *automorphic system.*

Vessiot's general method

Less than a decade later, in his 1904 paper *Sur l'intégration des systèmes différentiels qui admettent des groupes continus de transformations*, [**78**], Vessiot gave a more extensive account of the method indicated by Lie's example. Vessiot's focus was twofold: he first investigated more deeply the notion of an automorphic system, showing that the integration of an automorphic system can always be reduced to the integration of automorphic systems for simple pseudogroups. Second, he formalized Lie's solution technique, and exposed it in a way that could be seen to apply to any differential equation admitting a pseudogroup of symmetries. Paraphrasing [**78**]: "Our method produces, in the general case, the reduction of the integration of a

system of differential equations into two successive problems: 1) integration of the resolving system and 2) integration of an automorphic system." We recount here Vessiot's version of Example 3.43, [**78**, Chapitre II, §I.], then outline his description of the general method, [**78**, Chapitre II, §III.].

Example 3.44. Vessiot considered again the Lie pseudogroup given by the infinitesimal generator (3.80), transforming the variables x, y, z. Introducing an untransformed independent variable u and using u, y as invariant independent variables he arrives at the differential invariants

$$z\frac{\partial x}{\partial u}, \qquad z\frac{\partial x}{\partial y}, \qquad \frac{D(x,z)}{D(u,y)},$$

where

$$\frac{D(x,z)}{D(u,y)} = \frac{1}{z}\frac{\partial z}{\partial u} - \frac{1}{z}\frac{\partial z}{\partial y}\frac{\frac{\partial x}{\partial u}}{\frac{\partial x}{\partial y}}.$$

Vessiot posits a surface of the form

$$x = f(u,y), \qquad z = g(u,y)$$

and observes that any surface related to it by a pseudogroup transformation must be a solution of the *automorphic system*

$$z\frac{\partial x}{\partial u} = \alpha(u,y), \qquad z\frac{\partial x}{\partial y} = \beta(u,y), \qquad \frac{D(x,z)}{D(u,y)} = \gamma(u,y), \qquad (3.83)$$

where α, β, γ must satisfy the integrability condition

$$\frac{\partial \alpha}{\partial y} - \frac{\partial \beta}{\partial u} = \gamma.$$

He then deduces from the automorphic system the equations

$$\begin{cases} \dfrac{\partial x}{\partial u} = \dfrac{\alpha}{z}, & \dfrac{\partial x}{\partial y} = \dfrac{\beta}{z}, \\[2mm] \dfrac{\partial z}{\partial u} = p\dfrac{\alpha}{z}, & \dfrac{\partial z}{\partial y} = p\dfrac{\beta}{z} + q \end{cases}$$

and accompanying integrability condition

$$q\alpha - \gamma z = 0,$$

where $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$, keeping with Lie's notational convention. Supposing that γ and α are not identically zero (a trivial case), he arrives finally at the condition

$$\frac{q}{z} = \frac{\gamma}{\alpha}. \tag{3.84}$$

In this way, the differential invariant $\dfrac{q}{z}$ is expressed in terms of the unknown functions α, β, γ from the automorphic system (3.83). Vessiot then explains that all differential invariants may be expressed in terms of y, α, β and the derivatives $\dfrac{\partial \alpha}{\partial u}, \dfrac{\partial \alpha}{\partial y}, \dfrac{\partial \beta}{\partial u}, \dfrac{\partial \beta}{\partial y}, \dots$ through successive differentiation of the relation (3.84) with respect to the invariant total differential operators

$$\frac{1}{z}\frac{d}{dx} \quad \text{and} \quad \frac{d}{dy}.$$

Supposing that these expressions take the form

$$J_k(x, y, z, p, q, r, \dots) = H_k(y, \alpha, \beta, \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial y}, \dots), \qquad k = 1, \dots, \mu,$$

and an invariant system of differential equations

$$F_h(J_1, \dots, J_\mu) = 0, \qquad h = 1, \dots, \rho, \tag{3.85}$$

is given, we must then have the relations

$$F_h(H_1, \dots, H_\mu) = 0, \qquad h = 1, \dots, \rho, \tag{3.86}$$

among $y, \alpha, \beta, \dfrac{\partial \alpha}{\partial u}, \dfrac{\partial \alpha}{\partial y}, \dots$. Vessiot calls (3.86) the *resolving system*, which is the origin of the terminology. The integration of a given invariant system of differential equations (3.85) then reduces to, first, the integration of the resolving system (3.86), resulting in known functions α, β, γ; and second, the solution of the automorphic system (3.83).

Vessiot's general method proceeds in a similar fashion. Consider independent variables $x_1, \dots, x_m$ and dependent variables $z_1, \dots, z_q$, together with a system (A) of differential equations in these variables invariant under a Lie pseudogroup (G). A solution M to the system (A) is an m dimensional manifold in $m + q$ dimensional

space, so Vessiot introduces the coordinates $y_1, \ldots, y_n$ for this $n = m + q$ dimensional space and the untransformed variables $t_1, \ldots, t_m$ to parametrize the solution as

$$y_k = f_k(t_1, \ldots, t_m), \qquad k = 1, \ldots, n.$$

Considering a family of manifolds, all given by transformations of this chosen solution M, he observes that this family is given by an automorphic system

$$U_s(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1 \ldots \beta_m)}, \ldots) = \theta_s(t_1, \ldots, t_m), \qquad s = 1, \ldots, p, \tag{3.87}$$

where U_s, $s = 1, \ldots, p$, is a complete set of differential invariants for (G) and

$$y_k^{(\beta_1 \ldots \beta_m)} = \frac{\partial^{\beta_1 + \cdots + \beta_m} y_k}{\partial t_1^{\beta_1} \cdots \partial t_m^{\beta_m}}.$$

The functions θ_s are considered as unknown, but must satisfy the integrability conditions of (3.87); let

$$\Psi_j\left(\theta_1, \ldots, \theta_p, \ldots, \frac{\partial \theta_s}{\partial t_k}, \ldots\right) = 0, \qquad j = 1, \ldots, \rho, \tag{3.88}$$

be these integrability conditions. To (3.88) it is necessary to join information given by the original equations (A). This is done as follows: observe that the invariant differential equations (A) may be written as a zero locus of differential invariants. Let one such differential invariant be

$$J(x_1, \ldots, x_m, z_1, \ldots, z_q, \ldots, z^{(a_1 \ldots a_m)}, \ldots),$$

where

$$z_i^{(\alpha_1 \ldots \alpha_m)} = \frac{\partial^{\alpha_1 + \cdots + \alpha_m} z_i}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}.$$

Evaluating this differential invariant on the solution M we find

$$J(x_1, \ldots, x_m, z_1, \ldots, z_q, \ldots, z^{(a_1 \ldots a_m)}, \ldots) = \eta(t_1, \ldots, t_m),$$

for some function $\eta(t_1, \ldots, t_m)$. Because the differential invariants U_s form a complete system, J may be written as a function H of the U_s and their t_i derivatives. As such, Vessiot arrives at the identity

$$J(x_1, \ldots, x_m, z_1, \ldots, z_q, \ldots, z^{(a_1 \ldots a_m)}, \ldots) = H\left(\theta_1, \ldots, \theta_p, \ldots, \frac{\partial^{\gamma_1 + \cdots + \gamma_m} \theta_s}{\partial t_1^{\gamma_1} \cdots \partial t_m^{\gamma_m}}, \ldots\right).$$

If differential invariants $J_1, \ldots, J_\mu$ figure in the expression for (A), a system of relations (B) between functions $H_1, \ldots, H_\mu$ of the θ_s and their derivatives results. The equations (B), together with the integrability conditions (3.88), constitute the *resolving system* for (A). The integration of the system (A) is then split into the problem of the integration of the *resolving system* (B), (3.88), followed by the integration of the automorphic system (3.87).

Remark 3.45. Although Vessiot's introduction of untransformed independent variables simplifies the process of generating invariants through invariant differentiation, it introduces significant complications in the practical implementation of the group foliation method. Indeed, suppose that we begin with a single dependent variable u and two independent variables x, y. Introducing new independent variables s, t, the partial derivatives are expressed in terms of the new variables as

$$\frac{\partial}{\partial x} = \frac{y_t}{x_s y_t - y_s x_t} \frac{\partial}{\partial s} - \frac{y_s}{x_s y_t - y_s x_t} \frac{\partial}{\partial t} \text{ and } \frac{\partial}{\partial y} = -\frac{x_t}{x_s y_t - y_s x_t} \frac{\partial}{\partial s} + \frac{x_s}{x_s y_t - y_s x_t} \frac{\partial}{\partial t}.$$

Rewriting a simple differential expression such as u_{xy} in terms of these new variables is an elementary calculation with a surprisingly complex result:

$$u_{xy} = \left[\frac{x_s}{\Delta} \left(\frac{y_t}{\Delta} \right)_t - \frac{x_t}{\Delta} \left(\frac{y_t}{\Delta} \right)_s \right] u_s + \left[\frac{x_t}{\Delta} \left(\frac{y_s}{\Delta} \right)_s - \frac{x_s}{\Delta} \left(\frac{y_t}{\Delta} \right)_t \right] u_t$$
$$+ \frac{x_t y_s + x_s y_t}{\Delta^2} u_{ts} - \frac{x_t y_t}{\Delta^2} u_{ss} - \frac{x_s y_s}{\Delta^2} u_{tt},$$

where $\Delta = x_s y_t - y_s x_t$. Because these new variables may be viewed as an arbitrary reparametrization, extra syzygies may be introduced arbitrarily to simplify expressions such as the above. These syzygies may be viewed as restrictions on the reparametrization. See [**48**] for examples of solving ordinary differential equations using this viewpoint.

Johnson's rigorous approach

Over a half century passed until the problem of group foliation again appeared in research literature. Spurred on by the work of Kuranishi on exterior differential systems, H. H. Johnson returned to the problem in 1962, [**33**]. He does not provide examples of the efficacy of the method for solving differential equations, but instead

concerns himself with rigorous discussion of the method using Ehressman's language of jets and tools of exterior differential systems. In particular, he articulates the hypotheses under which a system of differential equations invariant under a pseudogroup $\mathcal{G}$ possesses Vessiot's claimed decomposition into resolving system (called *resolvant* by Johnson) and automorphic system. Johnson also notices that Lie and Vessiot implicitly restrict their considerations to *absolutely invariant* equations and *weakly automorphic* systems and addresses these assumptions. We now outline Johnson's ideas and make some comparisons with the approach to group foliation in this thesis.

Because Johnson works with general pseudogroups rather than Lie pseudogroups, he is careful to isolate the conditions on a pseudgroup which make group foliation possible. The central condition is the notion of strong completeness.

Definition 3.46 (Johnson's Definition 3.1)**.** A pseudogroup $\mathcal{G}$ is called *strongly complete* of order ℓ if there is a complete set of differential invariants

$$I^1, \ldots, I^\ell \tag{3.89}$$

such that any transformation leaving (3.89) invariant belongs to $\mathcal{G}$.

To relate the notions of a strongly complete pseudogroup and a Lie pseudogroup, we present the following proposition.

Proposition 3.47. Let $\mathcal{G}$ be the symmetry pseudogroup of $\Delta(x, u^{(n)}) = 0$. If $\mathcal{G}^{(n)}$ acts semi-regularly on an open subset $\mathcal{W}^n \subset \mathrm{J}^n$ containing the solution space of $\Delta = 0$, then $\mathcal{G}$ is strongly complete with respect to any complete set of invariants $I^1, \ldots, I^\ell$ of order n.

Proof. Let $I^1, \ldots, I^\ell$ be a complete set of $\mathcal{G}$-invariants on J^n. First observe that since $\mathcal{G}$ acts semi-regularly on its solution space, the equation $\Delta = 0$ may be expressed in terms of the invariants $I^1, \ldots, I^\ell$, [**52**, Proposition 2.56]. Thus

$$\Delta(x, u^{(n)}) = \widetilde{\Delta}(I^1, \ldots, I^\ell).$$

Define the pseudogroup $\overline{\mathcal{G}}$ to be the largest pseudogroup admitting this complete set of invariants:

$$\overline{\mathcal{G}} = \{\varphi \in \mathcal{D} \,|\, (\mathrm{j}_n\varphi)^* I^i = I^i, \quad i = 1, \ldots, \ell\}.$$

Clearly $\mathcal{G} \subset \overline{\mathcal{G}}$. On the other hand, for any $\varphi \in \overline{\mathcal{G}}$ we have

$$(\mathrm{j}_n \varphi)^* \Delta = (\mathrm{j}_n \varphi)^* \widetilde{\Delta} = \widetilde{\Delta},$$

and hence $\overline{\mathcal{G}} \subset \mathcal{G}$. $\qquad\square$

   Thus if a Lie pseudogroup considered arises as the collection of symmetries of a differential equation, it is automatically strongly complete. Additionally, it is not difficult to prove that if a Lie pseudogroup is strongly complete, a generating set of invariants will determine the Lie pseudogroup transformations. Thus, if we consider only strongly complete Lie pseudogroups, we may use a generating set of invariants for Johnson's set (3.89). Simple examples of Lie pseudogroups which are not strongly complete exist; for example the pseudogroups

$$X = x, \qquad U = f(u) \qquad \text{and} \qquad X = x, \qquad U = f(x, u),$$

both possess only the single invariant x but are clearly distinct.

   Assume that a strongly complete pseudogroup $\mathcal{G}$ is given which acts only on the dependent variables $u^1, \ldots, u^q$ and leaves the independent variables $x^1, \ldots, x^p$ invariant. As Vessiot explains, if the independent variables are not invariant, one can simply introduce untransformed parameters and use chain rule computations to rewrite the system under consideration. For this reason, Johnson assumes $p \leq q$ and restricts considerations only to jets of maximal rank. To establish an object which will encode the resolving system, Johnson defines the *resolvant* of $\Delta = 0$. Before recalling this definition, we review the idea of the prolongation of a system of differential equations.

Definition 3.48. Let $\Delta(x, u^{(n)}) = 0$ be an n-th order system of differential equations. Define the $(n + k)$-th order system

$$\Delta^{(k)}(x, u^{(n+k)}) = 0$$

to be the collection of $\binom{p+k-1}{k} \cdot l$ equations

$$D_J \Delta_\nu(x, u^{(n+k)}) = 0, \qquad \nu = 1, \ldots, l, \quad \#J \leq k.$$

This is called the *k-th prolongation* of Δ_ν.

Definition 3.49 (Johnson's Definition 4.2). Let $\mathcal{G}$ be a strongly complete pseudogroup and let $I^1, \ldots, I^\ell$ be differential invariants of $\mathcal{G}$ of order n or less. For $k \geq 0$, define $\sigma^k : \mathrm{J}^n(X, U) \to \mathrm{J}^k(X, \mathbb{R}^\ell)$ by

$$\sigma^k(x, u^{(n)}) = (I^1(x, u^{(n)}), \ldots, I^\ell(x, u^{(n)})), \ldots, D_{K_j} I^j(x, u^{(n)}), \ldots),$$

where $\#K_j \leq k$, $j = 1, \ldots \ell$. Let $\Delta(x, u^{(n)}) = 0$ be a system of differential equations. The image

$$R^k = \sigma^k(\{\Delta^{(k)}(x, u^{(n+k)}) = 0\}) \subset \mathrm{J}^k(N, \mathbb{R}^\ell),$$

will be called the *order k resolvant of Δ with respect to $I^1, \ldots, I^\ell$*.

Remark 3.50. The concept of a resolvant is somewhat analogous to that of a signature. The resolvant encodes syzygies among the invariants when restricted to $\Delta = 0$.

A function $r : X \to \mathbb{R}^\ell$ whose prolonged graph lies in the resolvant R^k plays the role of the resolving system solution. If one obtains a function $u_0 : X \to U$ such that

$$r(x) = \sigma(x, u_0^{(n)}),$$

the function u_0 will be a solution to $\Delta = 0$. To find such a u_0, for $k = 0, 1, 2, \ldots$, Johnson defines the system

$$\Lambda^k : \quad \Delta^{(k)}(x, u^{(n)}) = 0 \qquad D_{K_j}(I^j(x, u^{(n)}) - r^j(x)) = 0, \quad \#K_j \leq k, \quad j = 1, \ldots, \ell,$$

and shows that any solution $u_0 : X \to U$ of Λ^k, $k = 0, 1, 2, \ldots$, will suffice.

Proposition 3.51 (Johnson's Proposition 4.2). Let $\mathcal{G}$ be a strongly complete pseudogroup and $I^1, \ldots, I^\ell$ a complete set of invariants for $\mathcal{G}$. Let $\Delta(x, u^{(n)}) = 0$ be a system of differential equations and let R^k be the resolvant with respect to $I^1, \ldots, I^\ell$. Suppose that $r : X \to \mathbb{R}^\ell$ is a function whose prolonged graph lies in the resolvant and define the system Λ^k as above. Then there is a $u_0 : X \to U$ such that $r(x) = \sigma(x, u_0^{(n)})$.

The proof of this proposition consists of an application of the Cartan–Kuranishi theorem. With the function u_0 in hand, it is possible to define an automorphic system from which other solutions to Δ will be found. Because of the special form of this automorphic system, Johnson calls it *weakly automorphic*.

Definition 3.52 (Johnson's Definition 3.2)**.** A system of differential equations is called *weakly $\mathcal{G}$-automorphic* if there are differential invariants $I^1, \ldots I^\ell$ of $\mathcal{G}$, order n or less, such that the solution space of the system may be written as

$$\{(x, u(x)) \,:\, I^i(x, u^{(n)}) = I^i(x, u_0^{(n)}), \quad i = 1, \ldots, \ell\}$$

for some fixed solution u_0.

Remark 3.53. There seems to be no distinct benefit to distinguishing the notion of weakly automorphic and automorphic apart from making the nature of the equations resulting from group foliation more explicit. Also, not all automorphic systems are weakly automorphic. For example, solutions to $u_x = 0$ are all constant, and this system is clearly automorphic with respect to the pseudogroup of transformations of the dependent variable $U = \lambda u + a$. Meanwhile, invariants of this pseudogroup all have order larger than 1, so the automorphic system cannot be written in the form of Definition 3.52.

For coherence, it is important to know that a weakly automorphic system does in fact satisfy the automorphic property. Johnson proves the following theorem. This theorem also appears in Ueno, [**76**].

Proposition 3.54 (Johnson's Theorem 3.1)**.** Let $\mathcal{G}$ be a strongly complete pseudogroup and let $I^1, \ldots I^\ell$ be a complete set of differential invariants of order n. Let $u_0 : X \to U$ have maximal rank and define the system of differential equations

$$A(u_0): \qquad I^i(x, u^{(n)}) - I^i(x, u_0^{(n)}) = 0, \qquad i = 1, \ldots, \ell.$$

Then $A(u_0)$ is a $\mathcal{G}$-automorphic system admitting u_0 as a solution.

We can then summarize rigorously the group foliation algorithm in the following theorem. Amusingly, this "version" of group foliation does not seem to lend itself well to application.

Theorem 3.55 (Johnson's Theorem 4.1)**.** Suppose $\Delta(x, u^{(n)}) = 0$ is a system of partial differential equations invariant under a strongly complete pseudogroup $\mathcal{G}$. Let $I^1, \ldots, I^\ell$ be a complete set of invariants for $\mathcal{G}$.

- Let $r\colon X \to \mathbb{R}^\ell$ be a solution to the resolvant R^k of Δ for $k \geq 0$, and take $u_0\colon X \to U$ to be a function satisfying $r(x) = \sigma(x, u_0^{(n)})$, guaranteed by Proposition 3.51. Then any solution u to the weakly automorphic system $A(u_0)$ is a solution to Δ.

- Suppose that $u_0\colon X \to U$ is a solution to Δ. Then, trivially, $r(x) = \sigma(x, u_0^{(n)})$ is a solution to the resolvant of Δ, and u_0 is a solution to the automorphic system $A(u_0)$.

We conclude with a discussion of Johnson's observation that Lie and Vessiot considered only *absolutely invariant* differential equations.

Definition 3.56. A system of differential equations is called *absolutely invariant* under $\mathcal{G}$ if it may be written as a level set of a finite number of differential invariants of $\mathcal{G}$.

A simple example of a $\mathcal{G}$-invariant equation which is not absolutely invariant is the equation $u_x = 0$, which admits the symmetry group

$$X = x, \qquad U = \lambda\, u + a, \qquad \lambda > 0.$$

but cannot be written as a zero locus of differential invariants since, as observed earlier, all differential invariants have order greater than 1. Exceptions such as the above are singular cases in a sense that is well understood and explained through the theory of Lie determinants, [**53**]. The Lie determinant describes the singular subset of jet space on which $\mathcal{G}$ does not act regularly. Away from the singular set described by the Lie determinant, every invariant differential equation is absolutely invariant.

It is interesting to note that, although the result of Theorem 3.55 is not restricted to absolutely invariant systems, the computational method described in this thesis *is* limited to this context. It would be an interesting project to extend the computational methods of this chapter to singular equations; as indicated by preliminary computation this seems possible through the use of *partial moving frames*, [**80**].

Recent application and development of group foliation

The group foliation algorithm seems not to have been used classically to solve differential equations of interest beyond illustrative examples. It was not until L. Ovsiannikov formulated the method in modern language, [66], that substantial applications began to appear. Starting in the 1950's, Ovsiannikov and other Soviet mathematicians studied applications of symmetry methods to differential equations, eventually leading to the publication in 1982 of an English translation of a textbook on symmetry methods, [66]. In this text is presented a version of foliation very close to Lie's original method, which does not assume invariance of the independent variables. Included is the full rank group foliation of the stationary boundary layer equations, (3.43), considered much earlier by Ovsiannikov, [65]. Group foliation is also applied to the non-stationary boundary layer equations in another Russian language publication, [77]. English language publications on group foliation began to appear in the 1980's. Equations studied include the Lamé equations, [15], Calogero equation, [44], Monge–Ampère and "heavenly" equations, [49, 50], Euler equations, [24], Navier–Stokes and gas dynamics equations, [25], non-linear diffusion and reaction-diffusion equations, [1, 68], and Flierl-Petviashvili equation, [18].

The method has found recent reformulation using the language of exterior differential systems, [3, 67]. This new geometrical perspective indicates many potential applications of the group foliation method beyond the construction of exact solutions of differential equations, including the study of nonlocal symmetries, conservation laws, Bäcklund transformations, recursion operators and deformations of nonlinear equations, [4, 39, 40]. It is our hope that the perspective of equivariant moving frames advocated in this thesis may provide some insight into these geometric applications. As evidence for this possibility, we offer in Chapter 4 two applications particularly amenable to the moving frames approach: inductive construction of Bäcklund transformations (based on existing techniques in exterior differential systems) and reconstruction equations for invariant flows.

Chapter 4

Further applications

In this chapter we outline two further applications of the reduction and reconstruction process utilized in the group foliation method. These applications should not be viewed as fully developed ideas, but as suggestions for lines of future investigation.

The first application is to the construction of Bäcklund transformations. Bäcklund transformations are systems of differential equations that serve as a "bridge" from the solutions of one differential equation to another. Recent work has established a systematic process for constructing Bäcklund transformations using symmetry reduction of exterior differential systems, [**3**, **6**]. Approaching these constructions from the point of view of group foliation, the machinery of moving frames lends itself naturally to the method and facilitates the computations involved. In particular, the newly developed theory of inductive moving frames, [**36**, **59**, **81**], may be incorporated into the constructions, as we illustrate in Example 4.5.

The second application is to the theory of invariant submanifold flows. A $\mathcal{G}$-invariant submanifold flow is a motion of a submanifold governed by a $\mathcal{G}$-invariant evolutionary partial differential equation. Using moving frames, techniques have been developed for determining the evolution of the differential invariants and differential invariant signature of a submanifold subject to an invariant flow, [**58**]. Because the resolving system of group foliation is essentially a differential invariant signature, we

find that it is possible to apply the reconstruction process to invariant submanifold flows, splitting the flow into the evolution of a differential invariant signature and the evolution of a set of reconstruction parameters derived from the framework of Chapter 3. We illustrate this splitting using Euclidean invariant flows in Examples 4.22 and 4.23.

4.1 Inductive moving frames

Let $\mathcal{H}$ be a Lie pseudogroup, and let $\mathcal{G} \subset \mathcal{H}$ be a Lie subpseudogroup. The object of the inductive theory of moving frames is to use the moving frame $\rho_{\mathcal{G}}$ for the subpseudogroup $\mathcal{G}$ to simplify construction of the moving frame $\rho_{\mathcal{H}}$ for the larger pseudogroup $\mathcal{H}$. As an application, inductive moving frames may be used to write the invariants of $\mathcal{H}$ as functions of the invariants of $\mathcal{G}$.

To distinguish the target coordinates of $\mathcal{H}$ we use bars:

$$Z = g \cdot z, \qquad g \in \mathcal{G}, \qquad \overline{Z} = h \cdot z, \qquad h \in \mathcal{H}.$$

The inductive construction relies on the interpretation of the target coordinates Z of $\mathcal{G}$ as source coordinates for $\mathcal{H}$,

$$\overline{Z} = h \cdot z = (h \cdot g^{-1}) \cdot Z = \overline{h} \cdot Z, \qquad \overline{h} \in \mathcal{H}. \tag{4.1}$$

Assume that the normalization equations used to find $\rho_{\mathcal{H}}$ and $\rho_{\mathcal{G}}$ are given by cross-sections $\mathcal{K}_{\mathcal{H}}$ and $\mathcal{K}_{\mathcal{G}}$, respectively,

$$\mathcal{K}_{\mathcal{H}}: \quad \overline{c}^1 = \overline{Z}^1, \qquad \ldots \qquad \overline{c}^s = \overline{Z}^s \quad \text{and} \quad \mathcal{K}_{\mathcal{G}}: \quad c^1 = Z^1, \qquad \ldots \qquad c^r = Z^r.$$

We obtain *intermediate normalization equations* for $\mathcal{H}$ by restricting to source points which lie in the cross-section $\mathcal{K}_{\mathcal{G}}$:

$$\overline{c}^1 = \overline{h} \cdot c^1, \quad \ldots \quad \overline{c}^r = \overline{h} \cdot c^r, \qquad \overline{c}^{r+1} = \overline{h} \cdot \iota_{\mathcal{G}}(z^{r+1}), \quad \ldots \quad \overline{c}^s = \overline{h} \cdot \iota_{\mathcal{G}}(z^s), \tag{4.2}$$

where $\iota_{\mathcal{G}}(z^{r+1}) = \rho_{\mathcal{G}}(z) \cdot z^{r+1}, \ldots, \iota_{\mathcal{G}}(z^s) = \rho_{\mathcal{G}}(z) \cdot z^s$ are the normalized invariants of $\mathcal{G}$. Solving (4.2) for the transformation $\overline{h}$ yields an *intermediate moving frame*

$\rho_{\mathcal{G}}^{\mathcal{H}} : \mathcal{K}_{\mathcal{G}} \to \mathcal{H}$ satisfying

$$\rho_{\mathcal{H}}(z) = \rho_{\mathcal{G}}^{\mathcal{H}}\big(\rho_{\mathcal{G}}(z) \cdot z\big) \cdot \rho_{\mathcal{G}}(z). \tag{4.3}$$

It follows from (4.3) that the intermediate moving frame $\rho_{\mathcal{G}}^{\mathcal{H}}$ provides expressions for invariants of $\mathcal{H}$ in terms of those of $\mathcal{G}$ without requiring knowledge of $\rho_{\mathcal{H}}$ itself:

$$\begin{aligned}
\iota_{\mathcal{H}}(z^i) &= \rho_{\mathcal{H}}(z) \cdot z^i \\
&= \rho_{\mathcal{G}}^{\mathcal{H}}\big(\rho_{\mathcal{G}}(z) \cdot z\big) \cdot \rho_{\mathcal{G}}(z) \cdot z^i \\
&= \rho_{\mathcal{G}}^{\mathcal{H}}\big(\iota_{\mathcal{G}}(z^{r+1}), \ldots, \iota_{\mathcal{G}}(z^s)\big) \cdot \iota_{\mathcal{G}}(z^i).
\end{aligned} \tag{4.4}$$

To serve the discussion of the construction of Bäcklund transformations to follow, we now give two examples of the inductive construction of differential invariants. We will focus on the actions on (x, y, v, w) of the finite dimensional groups $\mathbb{R} \ltimes \mathbb{R}^2$ and $SL(2)$,

$$\begin{aligned}
\overline{\mathcal{H}} : \quad & \overline{X} = x, \quad \overline{Y} = y, \quad \overline{V} = \lambda v + a, \quad \overline{W} = \lambda w + b, \\
\widetilde{\mathcal{H}} : \quad & \widetilde{X} = x, \quad \widetilde{Y} = y, \quad \widetilde{V} = \frac{av + b}{cv + d}, \quad \widetilde{W} = \frac{aw - b}{-cw + d},
\end{aligned} \tag{4.5}$$

and their common subgroup $\mathbb{R} \ltimes \mathbb{R}$,

$$\mathcal{G} : \quad X = x, \quad Y = y, \quad V = \lambda v + a, \quad W = \lambda w - a. \tag{4.6}$$

To simplify our expressions in the following examples we introduce the notation

$$\iota_{\mathcal{G}}(v) = I, \qquad \iota_{\mathcal{G}}(w) = J,$$

$$\iota_{\mathcal{G}}(v_x) = I_{10}, \qquad \iota_{\mathcal{G}}(v_y) = I_{01}, \qquad \iota_{\mathcal{G}}(w_x) = J_{10}, \qquad \iota_{\mathcal{G}}(w_y) = J_{01}, \qquad \ldots,$$

for the normalized invariants of $\mathcal{G}$ and use bars and tildes for the normalized invariants of $\overline{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$:

$$\iota_{\overline{\mathcal{H}}}(v) = \overline{I}, \qquad \iota_{\overline{\mathcal{H}}}(w) = \overline{J}, \qquad \iota_{\widetilde{\mathcal{H}}}(v) = \widetilde{I}, \qquad \iota_{\widetilde{\mathcal{H}}}(w) = \widetilde{J}, \qquad \ldots.$$

Example 4.1. In this example we compute the intermediate moving frame $\rho_{\mathcal{G}}^{\overline{\mathcal{H}}}$ and write the invariants of $\overline{\mathcal{H}}$ in terms of those of $\mathcal{G}$. The adapted action (4.1) of $\overline{\mathcal{H}}$ on the target variables of $\mathcal{G}$ becomes (after prolongation)

$$\begin{aligned}
\overline{V} &= \lambda V + a, \quad \overline{W} = \lambda W + b, \\
\overline{V}_{\overline{X}} = \lambda V_X, \quad \overline{V}_{\overline{Y}} &= \lambda V_Y, \quad \overline{W}_{\overline{X}} = \lambda W_X, \quad \overline{W}_{\overline{Y}} = \lambda W_Y, \qquad \ldots.
\end{aligned} \tag{4.7}$$

We fix the cross-sections

$$\mathcal{K}_{\mathcal{G}}: \quad \{v = 0, w = 1\} \quad \text{and} \quad \mathcal{K}_{\overline{\mathcal{H}}}: \quad \{v = 0, w = 1, v_x = 1\}.$$

Applying the normalizations $V = 0, W = 1$ for $\mathcal{G}$ and $\overline{V} = 0, \overline{W} = 1, \overline{V}_{\overline{X}} = 1$ for $\overline{\mathcal{H}}$ to (4.7) we find

$$a = 0, \quad \lambda + b = 0, \quad \lambda V_X = 1,$$

which may be solved for the intermediate moving frame

$$\rho_{\mathcal{G}}^{\overline{\mathcal{H}}}: \quad a = 0, \quad b = -\frac{1}{V_X}, \quad \lambda = \frac{1}{V_X}.$$

Restricted to the cross-section $\mathcal{K}_{\mathcal{G}}$ the target variables V_X, V_Y become I_{10}, I_{01} and hence we may write inductively, using (4.4),

$$\overline{I}_{01} = \frac{I_{01}}{I_{10}}, \quad \overline{J}_{10} = \frac{J_{10}}{I_{10}}, \quad \overline{J}_{01} = \frac{J_{01}}{I_{10}}.$$

Example 4.2. We now compute the intermediate moving frame $\rho_{\mathcal{G}}^{\widetilde{\mathcal{H}}}$ and write the invariants of $\widetilde{\mathcal{H}}$ in terms of those of $\mathcal{G}$. The adapted action (4.1) of $\widetilde{\mathcal{H}}$ on the target variables of $\mathcal{G}$ becomes (after prolongation)

$$\widetilde{V} = \frac{aV + b}{cV + d}, \quad \widetilde{W} = \frac{aW - b}{-cW + d}, \quad \widetilde{V}_{\widetilde{X}} = \frac{V_X}{(cV + d)^2},$$
$$\widetilde{V}_{\widetilde{Y}} = \frac{V_Y}{(cV + d)^2}, \quad \widetilde{W}_{\widetilde{X}} = \frac{W_X}{(-cW + d)^2}, \quad \widetilde{W}_{\widetilde{Y}} = \frac{W_Y}{(-cW + d)^2}, \quad \dots \tag{4.8}$$

Choosing the cross-section

$$\mathcal{K}_{\widetilde{\mathcal{H}}}: \quad \{v = 0, w = 1, v_x = 1\}$$

and applying the normalizations $V = 0, W = 1$ for $\mathcal{G}$ and $\widetilde{V} = 0, \widetilde{W} = 1, \widetilde{V}_{\widetilde{X}} = 1$ for $\widetilde{\mathcal{H}}$ to (4.8) we find

$$b = 0, \quad \frac{a}{-c + d} = 1, \quad \frac{V_X}{d^2} = 1.$$

Solving these equations produces the intermediate moving frame

$$\rho_{\mathcal{G}}^{\widetilde{\mathcal{H}}}: \quad a = \frac{1}{\sqrt{V_X}}, \quad b = 0, \quad c = \frac{V_X - 1}{\sqrt{V_X}}, \quad d = \sqrt{V_X}.$$

Hence we may write inductively

$$\widetilde{I}_{01} = \frac{I_{01}}{I_{10}}, \quad \widetilde{J}_{10} = J_{10}I_{10}, \quad \widetilde{J}_{01} = J_{01}I_{10}.$$

4.2   Construction of Bäcklund transformations

A Bäcklund transformation is a relation between the solutions of two differential equations. Bäcklund transformations have many interesting connections with differential geometry, integrable equations and exterior differential systems, [**31**,**69**], but in their simplest incarnation they provide a means for solving differential equations by linking the solution of a difficult equation with those of a simpler equation. A famous example of such a link (and the example we will use to illustrate our constructions) is the Bäcklund transformation between the Liouville equation and the linear wave equation.

Example 4.3. Consider the system of equations

$$
\begin{aligned}
u_x^1 - u_x^2 &= \sqrt{2}\exp\left(\frac{u^1 + u^2}{2}\right) \\
u_y^1 + u_y^2 &= -\sqrt{2}\exp\left(\frac{u^1 - u^2}{2}\right).
\end{aligned}
\tag{4.9}
$$

Differentiating the first equation by y and the second by x we arrive at the integrability conditions

$$
u_{xy}^1 = \exp(u^1) \qquad \text{and} \qquad u_{xy}^2 = 0.
\tag{4.10}
$$

The first equation of (4.10) is called Liouville's equation while the second is the linear wave equation in characteristic coordinates. The system (4.9) relating the two equations is a Bäcklund transformation. These observations provide a method for solving Liouville's equation: insert any solution $u^2(x, y)$ of the wave equation into the Bäcklund transformation (4.9) and solve the result for $u^1(x, y)$. The result must be a solution to Liouville's equation by construction.

The usefulness of a Bäcklund transformation is evident from this example, but the question should immediately arise: how was (4.9) constructed? We will now describe an algorithm for the construction of Bäcklund transformations based on the group foliation method. This algorithm is directly related to the method for constructing integrable extensions of exterior differential systems discovered by Anderson and Fels, [**4**–**6**]. We emphasize that our approach is not yet fully developed or rigorous.

Let $\Delta = 0$ be a system of equations admitting Lie pseudogroups $\mathcal{H}, \mathcal{G}$ with $\mathcal{G} \subset \mathcal{H}$. Consider the diagram of resolving systems:

$$
\begin{array}{ccc}
\Delta = 0 & & \\
& \searrow^{\rho_{\mathcal{G}}} & \\
\rho_{\mathcal{H}} \Big\downarrow & & \mathcal{R}_{\mathcal{G}} \\
& \swarrow_{\rho_{\mathcal{G}}^{\mathcal{H}}} & \\
\mathcal{R}_{\mathcal{H}}. & &
\end{array}
$$

The moving frame maps indicate projections at the level of solutions: given a solution to $\mathcal{R}_{\mathcal{G}}$, one may project to a solution of $\mathcal{R}_{\mathcal{H}}$ by applying the intermediate moving frame $\rho_{\mathcal{G}}^{\mathcal{H}}$, and the diagram of projections is commutative by virtue of (4.3). Conversely, a solution to $\mathcal{R}_{\mathcal{H}}$ "lifts" to a family of solutions of $\mathcal{R}_{\mathcal{G}}$ as follows: choosing a solution to $\mathcal{R}_{\mathcal{H}}$, we find a family of solutions to $\Delta = 0$ via the reconstruction process. This family then projects via $\rho_{\mathcal{G}}$ to a family of solutions of $\mathcal{R}_{\mathcal{G}}$. At the level of equations this lift may be accomplished by inductively writing the generating invariants of $\mathcal{H}$ in terms of the generating invariants of $\mathcal{G}$.

Example 4.4. Take the system

$$v_y = 0, \qquad w_x = 0, \tag{4.11}$$

as our starting point $\Delta = 0$. It is simple to verify that these equations possess the symmetry group $\overline{\mathcal{H}}$ given by (4.5) and subgroup $\mathcal{G}$ given by (4.6). We first compute the resolving systems $\mathcal{R}_{\mathcal{G}}$, $\mathcal{R}_{\overline{\mathcal{H}}}$ for (4.11), then examine how a solution to $\mathcal{R}_{\overline{\mathcal{H}}}$ lifts to a family of solutions of $\mathcal{R}_{\mathcal{G}}$. Throughout the example we retain the notation from Section 4.1.

Begin with the computation of the resolving system $\mathcal{R}_{\mathcal{G}}$. The cross section $\mathcal{K}_{\mathcal{G}} = \{v = 0, w = 1\}$ gives the recurrence relations

$$
\begin{aligned}
\mathcal{D}_x I_{10} &= I_{20} - (I_{10} + J_{10})I_{10}, & \mathcal{D}_y I_{10} &= I_{11} - (I_{01} + J_{01})I_{10}, \\
\mathcal{D}_x I_{01} &= I_{11} - (I_{10} + J_{10})I_{01}, & \mathcal{D}_y I_{01} &= I_{02} - (I_{01} + J_{01})I_{01}, \\
\mathcal{D}_x J_{10} &= J_{20} - (I_{10} + J_{10})J_{10}, & \mathcal{D}_y J_{10} &= J_{11} - (I_{01} + J_{01})J_{10}, \\
\mathcal{D}_x J_{01} &= J_{11} - (I_{10} + J_{10})J_{01}, & \mathcal{D}_y J_{01} &= J_{02} - (I_{01} + J_{01})J_{01}.
\end{aligned}
$$

The terms I_{11} and J_{11} appear twice in the above equations, yielding the syzygies

$$\mathcal{D}_x I_{01} + I_{01} J_{10} = \mathcal{D}_y I_{10} + I_{10} J_{01} \tag{4.12}$$

and

$$\mathcal{D}_x J_{01} + I_{10} J_{01} = \mathcal{D}_y J_{10} + I_{01} J_{10}. \tag{4.13}$$

The constraint syzygy given by (4.11) is

$$I_{01} = 0, \qquad J_{10} = 0. \tag{4.14}$$

Taking as invariant independent and dependent variables, respectively, x, y and $K = I_{10}, L = J_{01}$, the combined syzygies (4.12), (4.13), (4.14), result in the resolving equations

$$\mathcal{R}_{\mathcal{G}}: \qquad L_x + KL = 0, \qquad K_y + KL = 0. \tag{4.15}$$

Next we find the resolving system $\mathcal{R}_{\overline{\mathcal{H}}}$. Choosing the cross section $\mathcal{K}_{\overline{\mathcal{H}}} = \{v = 0, w = 1, v_x = 1\}$ yields the recurrence relations

$$\begin{aligned}
\mathcal{D}_x \overline{I}_{01} &= \overline{I}_{11} - \overline{I}_{20}\overline{I}_{01}, & \mathcal{D}_y \overline{I}_{01} &= \overline{I}_{02} - \overline{I}_{11}\overline{I}_{01}, \\
\mathcal{D}_x \overline{J}_{10} &= \overline{J}_{20} - \overline{I}_{20}\overline{J}_{10}, & \mathcal{D}_y \overline{J}_{10} &= \overline{J}_{11} - \overline{I}_{11}\overline{J}_{10}, \\
\mathcal{D}_x \overline{J}_{01} &= \overline{J}_{11} - \overline{I}_{20}\overline{J}_{01}, & \mathcal{D}_y \overline{J}_{01} &= \overline{J}_{02} - \overline{I}_{11}\overline{J}_{01}, \\
\mathcal{D}_x \overline{I}_{20} &= \overline{I}_{30} - \overline{I}_{20}^2, & \mathcal{D}_y \overline{I}_{20} &= \overline{I}_{21} - \overline{I}_{11}\overline{I}_{20}, \\
\mathcal{D}_x \overline{J}_{02} &= \overline{J}_{12} - \overline{I}_{20}\overline{J}_{02}, & \mathcal{D}_y \overline{J}_{02} &= \overline{J}_{03} - \overline{I}_{11}\overline{J}_{02}.
\end{aligned}$$

The differential invariants $x, y, \overline{J}_{01}, \overline{J}_{10}, \overline{I}_{01}$ form a generating set. Based on the constraint syzygies

$$\overline{I}_{01} = 0 \qquad \overline{J}_{10} = 0,$$

we choose $M = \overline{J}_{01}$ as the invariant dependent variable. Applying $\mathcal{D}_x$ to

$$\mathcal{D}_y \overline{J}_{01} = \overline{J}_{02} - \overline{I}_{11}\overline{J}_{01},$$

and using the equations

$$\mathcal{D}_x \overline{J}_{02} = \overline{J}_{12} - \overline{I}_{20}\overline{J}_{02}, \quad \mathcal{D}_x \overline{J}_{01} = \overline{J}_{11} - \overline{I}_{20}\overline{J}_{01}$$

together with the constraint syzygies yields the resolving system

$$\mathcal{R}_{\overline{\mathcal{H}}}: \qquad MM_{xy} = M_x M_y. \tag{4.16}$$

From Example 4.1 we have

$$M = \frac{L}{K}, \tag{4.17}$$

and thus any solution $M(x, y)$ of (4.16) lifts to a family of solutions $L(x, y), K(x, y)$ of (4.15) satisfying (4.17). This lift is accomplished by direct substitution of the relation (4.17) into either of the equations (4.15). For example, choosing $M = 1$, we obtain the family

$$K(x, y) = L(x, y) = \frac{-1}{x + y + C}.$$

Suppose now that $\Delta = 0$ is a system of differential equations admitting two Lie pseudogroups $\overline{\mathcal{H}}$ and $\widetilde{\mathcal{H}}$, along with a shared Lie subpseudogroup $\mathcal{G}$. Consider the following diagram, consisting of resolving systems of $\Delta = 0$:
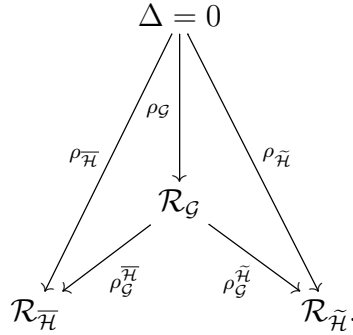
Suppose that a solution to the resolving system $\mathcal{R}_{\overline{\mathcal{H}}}$ is given. As just described, this solution lifts to a family of solutions to $\mathcal{R}_{\mathcal{G}}$, which then projects via $\rho_{\mathcal{G}}^{\widetilde{\mathcal{H}}}$ to solutions of $\mathcal{R}_{\widetilde{\mathcal{H}}}$. Similarly, solutions of $\mathcal{R}_{\widetilde{\mathcal{H}}}$ lift and project via $\rho_{\mathcal{G}}^{\overline{\mathcal{H}}}$ to solutions of $\mathcal{R}_{\overline{\mathcal{H}}}$. This allows us to interpret $\mathcal{R}_{\mathcal{G}}$ as a Bäcklund transformation relating the resolving systems $\mathcal{R}_{\overline{\mathcal{H}}}$ and $\mathcal{R}_{\widetilde{\mathcal{H}}}$.

Example 4.5. We return again to the system $\Delta = 0$ and groups $\mathcal{G}, \overline{\mathcal{H}}$ considered in Example 4.4. The equations (4.11) also admit the symmetry group $\widetilde{\mathcal{H}}$ given by (4.5). Since the resolving systems $\mathcal{R}_{\mathcal{G}}$ and $\mathcal{R}_{\overline{\mathcal{H}}}$ were found in Example 4.4, we compute the resolving system $\mathcal{R}_{\widetilde{\mathcal{H}}}$, then examine how the inductive moving frame yields $\mathcal{R}_{\mathcal{G}}$ as a Bäcklund transformation between $\mathcal{R}_{\overline{\mathcal{H}}}$ and $\mathcal{R}_{\widetilde{\mathcal{H}}}$.

Choosing the cross section $\mathcal{K}_{\widetilde{\mathcal{H}}} = \{v = 0,\, w = 1,\, v_x = 1\}$ results in the recurrence relations

$$\mathcal{D}_x \widetilde{I}_{01} = \widetilde{I}_{11} - \widetilde{I}_{20}\widetilde{I}_{01}, \qquad\qquad \mathcal{D}_y \widetilde{I}_{01} = \widetilde{I}_{02} - \widetilde{I}_{11}\widetilde{I}_{01},$$
$$\mathcal{D}_x \widetilde{J}_{10} = \widetilde{J}_{20} - 2\widetilde{J}_{10}^2 - 2\widetilde{J}_{10} + \widetilde{J}_{10}\widetilde{I}_{20}, \qquad \mathcal{D}_y \widetilde{J}_{10} = \widetilde{J}_{11} - 2\widetilde{J}_{10}\widetilde{J}_{01} - 2\widetilde{J}_{10}\widetilde{I}_{01} + \widetilde{J}_{10}\widetilde{I}_{11},$$
$$\mathcal{D}_x \widetilde{J}_{01} = \widetilde{J}_{11} - 2\widetilde{J}_{10}\widetilde{J}_{01} - 2\widetilde{J}_{01} + \widetilde{I}_{20}\widetilde{J}_{01}, \quad \mathcal{D}_y \widetilde{J}_{01} = \widetilde{J}_{02} - 2\widetilde{J}_{01}^2 - 2\widetilde{I}_{01}\widetilde{J}_{01} + \widetilde{I}_{11}\widetilde{J}_{01}.$$

and

$$\mathcal{D}_x \widetilde{I}_{20} = \widetilde{I}_{30} - \widetilde{I}_{20}^2 + 2\widetilde{J}_{10} + 2 - 2\widetilde{I}_{20},$$
$$\mathcal{D}_y \widetilde{I}_{20} = \widetilde{I}_{21} - \widetilde{I}_{11}\widetilde{I}_{20} + 2\widetilde{J}_{01} + 2\widetilde{I}_{01} - 2\widetilde{I}_{11},$$
$$\mathcal{D}_x \widetilde{J}_{11} = \widetilde{J}_{21} - \widetilde{I}_{20}\widetilde{J}_{11} - 2(\widetilde{J}_{10} + 1 - \widetilde{I}_{20})(\widetilde{J}_{10}\widetilde{J}_{01} + \widetilde{J}_{11}),$$
$$\mathcal{D}_y \widetilde{J}_{11} = \widetilde{J}_{12} - \widetilde{I}_{11}\widetilde{J}_{11} - 2(\widetilde{J}_{01} + \widetilde{I}_{01} - \widetilde{I}_{11})(\widetilde{J}_{10}\widetilde{J}_{01} + \widetilde{J}_{11}),$$
$$\mathcal{D}_x \widetilde{J}_{02} = \widetilde{J}_{12} - \widetilde{I}_{20}\widetilde{J}_{02} - 2(\widetilde{J}_{10} + 1 - \widetilde{I}_{20})(\widetilde{J}_{01}^2 + \widetilde{J}_{02}),$$
$$\mathcal{D}_y \widetilde{J}_{02} = \widetilde{J}_{03} - \widetilde{I}_{11}\widetilde{J}_{02} - 2(\widetilde{J}_{01} + \widetilde{I}_{01} - \widetilde{I}_{11})(\widetilde{J}_{01}^2 + \widetilde{J}_{02}).$$

The differential invariants $x, y, \widetilde{J}_{01}, \widetilde{J}_{10}, \widetilde{I}_{01}$ form a generating set. Based on the constraint syzygies

$$\widetilde{I}_{01} = 0, \qquad \widetilde{J}_{10} = 0$$

we choose $N = \widetilde{J}_{01}$ as our invariant dependent variable. Using the constraint syzygies it can be see from the recurrence relations that

$$\mathcal{D}_x \widetilde{J}_{01} = -2\widetilde{J}_{01} + \widetilde{I}_{20}\widetilde{J}_{01} \qquad \Longrightarrow \qquad \frac{N_x}{N} + 2 = \widetilde{I}_{20}$$

and

$$\mathcal{D}_y \widetilde{J}_{01} = \widetilde{J}_{02} - 2\widetilde{J}_{01}^2 \qquad \Longrightarrow \qquad N_y + 2N^2 = \widetilde{J}_{02}.$$

Applying $\mathcal{D}_x$ to the latter equation and using the syzygy (subject to constraint syzygies)

$$\mathcal{D}_x \widetilde{J}_{02} = -\widetilde{I}_{20}\widetilde{J}_{02} - 2(1 - \widetilde{I}_{20})(\widetilde{J}_{01}^2 + \widetilde{J}_{02}),$$

we arrive at the resolving equations

$$\mathcal{R}_{\widetilde{\mathcal{H}}}: \qquad NN_{xy} = N_x N_y + 2N^3.$$

All of the preparatory calculations are now complete. To find the Bäcklund transformation between $\mathcal{R}_{\overline{\mathcal{H}}}$ and $\mathcal{R}_{\widetilde{\mathcal{H}}}$ we write the resolving system $\mathcal{R}_{\mathcal{G}}$ in terms of the invariant variables M, N chosen for the group foliation. From Examples 4.1 and 4.2 we have

$$M = \frac{L}{K}, \qquad N = KL,$$

and thus

$$L = \sqrt{MN}, \qquad K = \sqrt{\frac{N}{M}}. \tag{4.18}$$

Substituting (4.18) into (4.15) we find

$$\sqrt{\frac{M}{N}}\left(\frac{MN_y - NM_y}{M^2}\right) + 2N = 0, \qquad \frac{1}{\sqrt{MN}}(MN_x + N_x) + 2N = 0, \tag{4.19}$$

a Bäcklund transformation relating solutions of

$$MM_{xy} = M_x M_y \qquad \text{and} \qquad NN_{xy} = N_x N_y + 2N^3.$$

To see how this Bäcklund transformation relates to Example 4.3, we make the change of variables

$$u^1 = -\log M, \qquad u^2 = \log 2N.$$

With this change of variables, (4.19) becomes the familiar Bäcklund transformation

$$u_x^1 - u_x^2 = \sqrt{2}\exp\left(\frac{u^1 + u^2}{2}\right), \qquad u_y^1 + u_y^2 = -\sqrt{2}\exp\left(\frac{u^2 - u^1}{2}\right),$$

while the resolving systems $\mathcal{R}_{\overline{\mathcal{H}}}$ and $\mathcal{R}_{\widetilde{\mathcal{H}}}$ become, respectively,

$$u_{xy}^1 = \exp(u^1) \qquad \text{and} \qquad u_{xy}^2 = 0.$$

4.3 Invariant flows and signature evolution

Before introducing the idea of reconstruction for invariant submanifold flows, we briefly review the basic theory of the invariant variational bicomplex and its connection with the evolution of differential invariant signatures under invariant flows.

For a more detailed introduction to the invariant variational bicomplex and its non-invariant progenitor, we refer the reader to [**2**, **37**, **38**]. The approach to submanifold flows reviewed here follows [**58**]. Our presentation varies slightly from these references in that we adopt the formalism of moving frames for Lie pseudogroups to be consistent with the exposition of Chapter 2.

The invariant variational "bicomplex"

Local coordinates $(x, u) = (x^1, \dots, x^p, u^1, \dots, u^q)$ on M give a natural way of splitting differential one-forms on $\mathrm{J}^\infty(M, p)$ into *horizontal forms*, spanned by $dx^1, \dots, dx^p$, and *vertical* or *contact* forms, spanned by the basic contact forms

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha \, dx^i \qquad \alpha = 1, \dots, q, \qquad \#J \geq 0.$$

This splitting induces a bigrading on all differential forms, $\boldsymbol{\Omega}^*(\mathrm{J}^\infty)$; roughly speaking a form Ω of horizontal degree r and vertical degree s consists of a (finite) sum of wedge products of r horizontal one-forms and s vertical one-forms, i.e. a sum of forms

$$P(x, u^{(\infty)}) \, dx^{i_1} \wedge \cdots dx^{i_r} \wedge \theta_{J_1}^{\alpha_1} \wedge \cdots \wedge \theta_{J_s}^{\alpha_s}.$$

The collection of forms of horizontal degree r and vertical degree s will be denoted $\boldsymbol{\Omega}^{r,s}$.

The projection $\pi_{r,s} : \boldsymbol{\Omega}^* \to \boldsymbol{\Omega}^{r,s}$ of any differential form onto the r, s bigrade can be defined in an obvious manner. The exterior derivative interacts with the bigrading as follows:

$$d : \boldsymbol{\Omega}^{r,s} \to \boldsymbol{\Omega}^{r+1,s} \oplus \boldsymbol{\Omega}^{r,s+1}.$$

Hence we may define a splitting $d = d_H + d_V$ of the exterior derivative via

$$d_H = \pi_{r+1,s} \circ d : \boldsymbol{\Omega}^{r,s} \to \boldsymbol{\Omega}^{r+1,s} \qquad d_V = \pi_{r,s+1} \circ d : \boldsymbol{\Omega}^{r,s} \to \boldsymbol{\Omega}^{r,s+1}.$$

The resulting double complex of differential forms is known as the variational bicom-

plex:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & & & \vdots & & \vdots \\
\big\uparrow d_V & & \big\uparrow d_V & & & & \big\uparrow d_V & & \big\uparrow d_V \\
\mathbf{\Omega}^{0,1} & \xrightarrow{d_H} & \mathbf{\Omega}^{1,1} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \mathbf{\Omega}^{p-1,1} & \xrightarrow{d_H} & \mathbf{\Omega}^{p,1} \\
\big\uparrow d_V & & \big\uparrow d_V & & & & \big\uparrow d_V & & \big\uparrow d_V \\
\mathbf{\Omega}^{0,0} & \xrightarrow{d_H} & \mathbf{\Omega}^{1,0} & \xrightarrow{d_H} & \cdots & \xrightarrow{d_H} & \mathbf{\Omega}^{p-1,0} & \xrightarrow{d_H} & \mathbf{\Omega}^{p,0}.
\end{array}
$$

The invariant variational bicomplex is obtained from the ordinary variational bicomplex through invariantization of the ordinary bigrading. The title invariant variational bicomplex is a misnomer; its columns are not complexes, and there is an anomalous third differential operator appearing in the construction. As such, it is sometimes referred to as a *quasi-tricomplex*.

Assume that a Lie pseudogroup $\mathcal{G}$ acting on M is given, and suppose that $\mathcal{G}$ admits a moving frame. We begin by using the moving frame to invariantize the basic horizontal and contact forms

$$
\varpi^i = \iota(dx^i) \qquad \vartheta^\alpha_J = \iota(\theta^\alpha_J) \qquad i = 1, \ldots, p, \quad \alpha = 1, \ldots, q, \quad \#J \geq 0. \tag{4.20}
$$

These invariant differential forms constitute a coframe on J^∞. Thus any one-form can be uniquely decomposed into a linear combination (with arbitrary function coefficients) of invariant horizontal and invariant contact one-forms, called the *invariant horizontal* and *invariant vertical* components, respectively. This invariant coframe (4.20) may be used to bigrade the space of differential forms on J^∞:

$$
\mathbf{\Omega}^* = \bigoplus_{r,s} \widetilde{\mathbf{\Omega}}^{r,s}, \tag{4.21}
$$

where $\widetilde{\mathbf{\Omega}}^{r,s}$ is the space of forms of invariant horizontal degree r and invariant vertical degree s. Again, we may define the projection $\widetilde{\pi}_{r,s} : \mathbf{\Omega} \to \widetilde{\mathbf{\Omega}}^{r,s}$ in an obvious manner.

Remark 4.6. As will be seen in our main example, the invariant horizontal forms are not actually horizontal forms, but may contain contact components. The invariant vertical forms are indeed vertical in the sense of the ordinary bigrading. Thus this bigrading is in fact different from the ordinary bigrading in general.

We now observe how the exterior derivative splits with respect to the bigrading (4.21). Beginning with the observation that, for $\Omega \in \boldsymbol{\Omega}^{r,s}$, $d\Omega \in \boldsymbol{\Omega}^{r+1,s} \oplus \boldsymbol{\Omega}^{r,s+1}$ and $\mathbf{v}^{(\infty)}(\Omega) \in \boldsymbol{\Omega}^{r,s} \oplus \boldsymbol{\Omega}^{r-1,s+1}$, it follows from the recurrence formula (2.40) that $d\iota(\Omega) \in \widetilde{\boldsymbol{\Omega}}^{r+1,s} \oplus \widetilde{\boldsymbol{\Omega}}^{r,s+1} \oplus \widetilde{\boldsymbol{\Omega}}^{r-1,s+2}$. In fact, since arbitrary forms in $\widetilde{\boldsymbol{\Omega}}^{r,s}$ are a sums of invariant forms in $\widetilde{\boldsymbol{\Omega}}^{r,s}$ with (possibly non-invariant) function coefficients, the same decomposition applies for non-invariant forms. Thus there is a general splitting $d = d_{\mathcal{H}} + d_{\mathcal{V}} + d_{\mathcal{W}}$, where

$$d_{\mathcal{H}} : \widetilde{\boldsymbol{\Omega}}^{r,s} \to \widetilde{\boldsymbol{\Omega}}^{r+1,s}, \qquad d_{\mathcal{V}} : \widetilde{\boldsymbol{\Omega}}^{r,s} \to \widetilde{\boldsymbol{\Omega}}^{r,s+1}, \qquad d_{\mathcal{W}} : \widetilde{\boldsymbol{\Omega}}^{r,s} \to \widetilde{\boldsymbol{\Omega}}^{r-1,s+2}.$$

This gives the invariant bigrading the structure of a *quasi-tricomplex*:

$$d_{\mathcal{H}}^2 = 0, \quad d_{\mathcal{W}}^2 = 0,$$

$$d_{\mathcal{H}}d_{\mathcal{V}} + d_{\mathcal{V}}d_{\mathcal{H}} = 0, \quad d_{\mathcal{V}}d_{\mathcal{W}} + d_{\mathcal{W}}d_{\mathcal{V}} = 0, \quad d_{\mathcal{V}}^2 + d_{\mathcal{H}}d_{\mathcal{W}} + d_{\mathcal{W}}d_{\mathcal{H}} = 0.$$

Pictorially represented,

Computation in the invariant variational bicomplex is done via the recurrence relation (2.40). We will use the finite dimensional Lie pseudogroup of Euclidean transformations as our running example.

Example 4.7. Consider the finite dimensional Lie pseudogroup consisting of Euclidean transformations of planar curves,

$$X = x \cos \phi - u \sin \phi + a \qquad U = x \sin \phi + u \cos \phi + b.$$

The infinitesimal generator $\mathbf{v} = \xi(x, u)\dfrac{\partial}{\partial x} + \varphi(x, u)\dfrac{\partial}{\partial u}$ satisfies the infinitesimal determining equations

$$\xi_x = \varphi_u = 0, \qquad \xi_u = -\varphi_x.$$

Hence the prolonged infinitesimal generator has the general form

$$\mathbf{v}^{(\infty)} = \xi(u)\frac{\partial}{\partial x} + \varphi(x)\frac{\partial}{\partial u}$$
$$- \xi_u\big(1 + u_x^2\big)\frac{\partial}{\partial u_x} - 2\xi_u u_x u_{xx}\frac{\partial}{\partial u_{xx}} - \xi_u\big(3u_{xx}^2 + 4u_x u_{xx}\big)\frac{\partial}{\partial u_{xxx}} + \cdots .$$

Implementing the recurrence relations (2.40) for the normalized invariants

$$H = \iota(x), \qquad I = \iota(u), \qquad I_x = \iota(u_x), \qquad \iota(u_{xx}) = I_{xx}, \qquad \ldots$$

we find

$$dH = \varpi + \nu^x$$
$$dI = I_x\varpi + \vartheta + \nu^u$$
$$dI_x = I_{xx}\varpi + \vartheta_x - (1 + I_x^2)\nu_U^x \tag{4.22}$$
$$dI_{xx} = I_{xxx}\varpi + \vartheta_{xx} - 3I_x I_{xx}\nu_U^x$$
$$dI_{xxx} = I_{xxxx}\varpi + \vartheta_{xxx} - (3I_{xx}^2 - 4I_x I_{xx})\nu_U^x$$

and so on, where $\nu^x = \iota(\xi)$, $\nu^u = \iota(\varphi)$, $\nu_U^x = \iota(\xi_u)$. From the cross-section normalizations $H = I = I_x = 0$ these recurrence relations yield the expressions for the pulled-back Maurer–Cartan forms:

$$\nu^x = -\varpi, \qquad \nu^u = -\vartheta, \qquad \nu_U^x = I_{xx}\varpi + \vartheta_x.$$

Using these equations in (4.22) we obtain

$$dI_{xx} = I_{xxx}\,\varpi\vartheta_{xx},$$
$$dI_{xxx} = \big(I_{xxxx} - 3I_{xx}^3\big)\,\varpi - 3I_{xx}^2\,\vartheta_x + \vartheta_{xxx},$$

et cetera. Projecting onto the invariant horizontal and vertical bigrading yields

$$\begin{aligned}
d_{\mathcal{H}} I_{xx} &= I_{xxx}\,\varpi & d_{\mathcal{V}} I_{xx} &= \vartheta_{xx}, \\
d_{\mathcal{H}} I_{xxx} &= \big(I_{xxxx} - 3I_{xx}^3\big)\,\varpi & d_{\mathcal{V}} I_{xxx} &= -3I_{xx}^2\,\vartheta_x + \vartheta_{xxx}.
\end{aligned} \tag{4.23}$$

Similarly we may apply the recurrence relations to the forms $\varpi, \vartheta, \vartheta_x, \ldots$ to obtain

$$d_{\mathcal{H}}\varpi = 0 \qquad\qquad d_{\mathcal{V}}\varpi = I_x\varpi \wedge \vartheta \qquad\qquad d_{\mathcal{W}}\varpi = -\vartheta \wedge \vartheta_x$$

$$d_{\mathcal{H}}\vartheta = \varpi \wedge \vartheta_x \qquad\qquad\qquad\qquad\qquad d_{\mathcal{V}}\vartheta = 0$$

$$d_{\mathcal{H}}\vartheta_x = \varpi \wedge \left(\vartheta_{xx} - I_{xx}^2\,\vartheta\right) \qquad\qquad\qquad d_{\mathcal{V}}\vartheta_x = -I_{xx}\,\vartheta_x \wedge \vartheta$$

$$d_{\mathcal{H}}\vartheta_{xx} = \varpi \wedge \left(\vartheta_{xxx} - 3I_{xx}^2\,\vartheta_x - I_{xx}I_{xxx}\,\vartheta\right) \qquad d_{\mathcal{V}}\vartheta_{xx} = -I_{xxx}\,\vartheta_x \wedge \vartheta.$$

and so on.

One particular computation emerging from the invariant variational bicomplex will form the crux of the discussion of differential invariant signature evolution to follow. This is the invariant linearization of a differential invariant.

Definition 4.8. Let K be a differential invariant. Then $d_{\mathcal{V}}K$ is an invariant contact form which may be written as

$$d_{\mathcal{V}}K = \mathcal{A}_K(\vartheta),$$

where $\mathcal{A}_K$ is an invariant differential operator called the *invariant linearization* of K.

To compute the invariant linearization, we require the following lemma.

Lemma 4.9. Let ϑ be an invariant contact form. Then

$$d_{\mathcal{H}}\vartheta = \varpi \wedge \mathcal{D}\,\vartheta.$$

Example 4.10. As we have already seen in Example 4.7,

$$d_{\mathcal{H}}\vartheta = \varpi \wedge \vartheta_x,$$

$$d_{\mathcal{H}}\vartheta_x = \varpi \wedge \left(\vartheta_{xx} - I_{xx}^2\,\vartheta\right),$$

$$d_{\mathcal{H}}\vartheta_{xx} = \varpi \wedge \left(\vartheta_{xxx} - 3I_{xx}^2\,\vartheta_x - I_{xx}I_{xxx}\,\vartheta\right),$$

and so on. By repeated application of Lemma 4.9 we find

$$\vartheta_x = \mathcal{D}\,\vartheta,$$

$$\vartheta_{xx} = \left(\mathcal{D}^2 + I_{xx}^2\right)\vartheta \qquad\qquad\qquad (4.24)$$

$$\vartheta_{xxx} = \left(\mathcal{D}^3 + 4I_{xx}^2\mathcal{D} + 3\,I_{xx}\,I_{xxx}\right)\vartheta.$$

We may then use these identities to find invariant linearizations. From (4.23) and (4.24) we find

$$d_{\mathcal{V}} I_{xx} = \vartheta_{xx} = \left(\mathcal{D}^2 + I_{xx}^2\right)\vartheta,$$

$$d_{\mathcal{V}} I_{xxx} = -3I_{xx}^2\,\vartheta_x + \vartheta_{xxx} = \left(\mathcal{D}^3 + I_{xx}^2\mathcal{D} + 3\,I_{xx}\,I_{xxx}\right)\vartheta,$$

and hence

$$\mathcal{A}_{I_{xx}} = \mathcal{D}^2 + I_{xx}^2, \qquad \mathcal{A}_{I_{xxx}} = \mathcal{D}^3 + I_{xx}^2\mathcal{D} + 3\,I_{xx}\,I_{xxx}.$$

In the computations to follow, we will use the curvature κ and its arclength derivatives instead of the normalized invariants:

$$\kappa = I_{xx}, \qquad \kappa_s = \mathcal{D}\kappa = I_{xxx}, \qquad \kappa_{ss} = \mathcal{D}\kappa_s = I_{xxxx} - 3I_{xx}^3, \qquad \ldots\,.$$

For these invariants we find the invariant linearizations

$$\mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2, \qquad \mathcal{A}_{\kappa_s} = \mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s,$$

$$\mathcal{A}_{\kappa_{ss}} = \mathcal{D}^4 + \kappa^2\,\mathcal{D}^2 + 5\,\kappa\,\kappa_s\,\mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2,$$

and so on.

Invariant submanifold flows

Let $\mathbf{V}$ be a vector field on M defined along a submanifold S, i.e. a section of the bundle $TM \to S$. An *invariant submanifold flow* is a $\mathcal{G}$-invariant partial differential equation

$$\frac{\partial S}{\partial t} = \mathbf{V}\big|_{S(t)} \tag{4.25}$$

giving the motion of a p-dimensional submanifold. A solution of this equation will be a family of submanifolds $S(t)$ preserved under transformations of $\mathcal{G}$.

Suppose that $\mathcal{G}$ admits a moving frame. Let

$$\varpi^1, \ldots, \varpi^p, \qquad \vartheta^1, \ldots, \vartheta^q$$

be the invariant coframe on M obtained through invariantization of the basic horizontal and zero order contact forms, as in (4.20). Given a submanifold $S \subset M$ (in

the domain of the moving frame), evaluating the coefficients of this coframe on the jet $j_\infty S$ will produce, point-wise, a basis for the tangent spaces of M along S. Let

$$\mathbf{t}_1, \ldots, \mathbf{t}_p, \qquad \mathbf{n}_1, \ldots, \mathbf{n}_q \qquad (4.26)$$

be the vectors dual to this basis, i.e.

$$\langle \mathbf{t}_i \,;\, \varpi^j \rangle = \delta_i^j, \qquad \langle \mathbf{t}_i \,;\, \vartheta^\alpha \rangle = 0, \qquad \langle \mathbf{n}_\alpha \,;\, \varpi^i \rangle = 0, \qquad \langle \mathbf{n}_\alpha \,;\, \vartheta^\beta \rangle = \delta_\alpha^\beta. \qquad (4.27)$$

Example 4.11. For our running example of Euclidean transformations of curves we have

$$\varpi = \frac{dx + u_x dx}{\sqrt{1 + u_x^2}}, \qquad \vartheta = \frac{du - u_x dx}{\sqrt{1 + u_x^2}},$$

and hence, applying the relations (4.27) we find

$$\mathbf{t} = \frac{1}{\sqrt{1 + u_x^2}} \left(\frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} \right), \qquad \mathbf{n} = \frac{1}{\sqrt{1 + u_x^2}} \left(-u_x \frac{\partial}{\partial x} + \frac{\partial}{\partial u} \right),$$

the usual unit tangent and unit normal to a curve.

For any section $\mathbf{V}$ of $TM \to S$, we have the splitting according to (4.26) into tangential and normal components

$$\mathbf{V} = \mathbf{V}_T + \mathbf{V}_N = \sum_{i=1}^p I^i \, \mathbf{t}_i + \sum_{\alpha=1}^q J^\alpha \, \mathbf{n}_\alpha.$$

The vector field $\mathbf{V}$ will determine an invariant flow, as in (4.25), if and only if the coefficients I^i, J^α, $i = 1, \ldots, p$, $\alpha = 1, \ldots, q$ are differential invariants. For simplicity, we will from now on restrict considerations to the geometry of planar curves, i.e. $p = q = 1$. Thus the vector field $\mathbf{V}$ specifying an invariant curve flow may be written

$$\mathbf{V} = I \, \mathbf{t} + J \, \mathbf{n},$$

where I, J are differential invariants.

The tangential component $\mathbf{V}_T$ of the flow serves only to change the parametrization of S. Hence, if we are interested only in the images $S(t)$ of the flow we may consider only normal flows, i.e. $\mathbf{V} = \mathbf{V}_N$, without loss of generality.

Example 4.12. Some well known Euclidean invariant flows include

- $\mathbf{V} = \mathbf{n}$, the grassfire flow,

- $\mathbf{V} = \kappa\,\mathbf{n}$, the curve shortening flow,

- $\mathbf{V} = \kappa_s\,\mathbf{n}$, the mKdV flow.

Discussions and applications of these flows appear in many references, [**9**, **23**, **26**].

In contrast to most normal flows, curve flows that preserve arclength parametrization will be called *intrinsic*. We now recall two formulae valid for intrinsic flows.

Lemma 4.13. If $\mathbf{V}$ determines an intrinsic flow and $\mathcal{A}$ is any invariant differential operator,

$$\mathbf{V} \lrcorner \mathcal{A}(\vartheta) = \mathcal{A}(\mathbf{V} \lrcorner \vartheta)$$

for any invariant contact form ϑ.

Theorem 4.14. If the invariant curve flow given by $\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$ is intrinsic and K is any differential invariant, then the evolution of K concurrent with the curve flow is given by

$$\frac{\partial K}{\partial t} = \mathbf{V}(K) = \mathcal{A}_K(J) + I\,\mathcal{D}K. \tag{4.28}$$

Proof. The proof consists of a computation in conjunction with Lemma 4.13:

$$\mathbf{V}(K) = \mathbf{V} \lrcorner dK = \mathbf{V} \lrcorner \big(\mathcal{A}_K(\vartheta) + \mathcal{D}(K)\,\varpi\big)$$
$$= \mathcal{A}_K(J) + I\,\mathcal{D}K.$$

□

Given any flow, it is possible to create an extrinsically identical intrinsic flow though the addition of a (possibly non-local) tangential component. Thus the formula (4.28) for intrinsic flows may be naturally adapted to normal flows.

Corollary 4.15. Given an invariant normal curve flow $\mathbf{V} = J\,\mathbf{n}$ and a differential invariant K, the evolution of K concurrent with the curve flow is given by

$$\frac{\partial K}{\partial t} = \mathbf{V}(K) = \mathcal{A}_K(J). \tag{4.29}$$

Proof. Adding a (possibly nonlocal) tangential component $\mathbf{V}_T$, we may assume that the flow given by $\mathbf{V}$ is intrinsic. Applying Theorem 4.14 and removing the terms originating from the tangential component yields the desired formula. $\square$

Example 4.16. For a Euclidean invariant normal flow $\mathbf{V} = J\,\mathbf{n}$, formula (4.29) gives the evolution of the invariants κ, κ_s, κ_{ss}, ... :

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = \left(\mathcal{D}^2 + \kappa^2\right)J,$$

$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = \left(\mathcal{D}^3 + \kappa^2\,\mathcal{D} + 3\,\kappa\,\kappa_s\right)J,$$

$$\frac{\partial \kappa_{ss}}{\partial t} = \mathcal{A}_{\kappa_{ss}}(J) = \left(\mathcal{D}^4 + \kappa^2\,\mathcal{D}^2 + 5\,\kappa\,\kappa_s\,\mathcal{D} + 4\,\kappa\,\kappa_{ss} + 3\,\kappa_s^2\right)J,$$

where we have used the expressions from Example 4.10. These formulae may be easily specialized to any of the curve flows given in Example 4.12.

Since the normal flow given by $\mathbf{V}$ is not intrinsic, the formulae of Example 4.16 are not partial differential equations, but rather an infinite coupled system of ordinary differential equations for the variables $\kappa, \kappa_s, \kappa_{ss}, \ldots$. This presents obvious difficulty. To address this difficulty, it is possible to use the differential invariant signature to "close off" this system at finite order.

Definition 4.17. The *differential invariant signature* of a planar curve C is the subset of $\mathbb{R}^2$ parametrized by the invariants (κ, κ_s).

This definition is a variant of Definition 2.46: if the relation between the invariants κ and κ_s is known, relations among all other invariants can be determined through differentiation. For example, if we assume that the signature may be written as a graph,

$$\kappa_s = \Phi(\kappa),$$

then it follows from invariant differentiation that

$$\kappa_{ss} = \Phi_\kappa(\kappa)\kappa_s = \Phi_\kappa(\kappa)\Phi(\kappa),$$

$$\kappa_{sss} = \Phi(\kappa)^2\Phi_{\kappa\kappa} + \Phi(\kappa)\Phi_\kappa(\kappa)^2,$$

and so on.

Suppose now that an invariant curve flow is given. The family of curves $S(t)$ given by the flow produces in turn a family of signatures, which we write as

$$\kappa_s = \Phi(\kappa, t).$$

A small computation in conjunction with (4.29) produces the evolution of this family of differential invariant signatures.

Theorem 4.18. If $\mathbf{V} = J\,\mathbf{n}$ is an invariant curve flow, the evolution of the differential invariant signature concurrent with curve evolution is given by

$$\frac{\partial \Phi}{\partial t} = \mathcal{A}_{\kappa_s}(J) - \Phi_\kappa\,\mathcal{A}_\kappa(J). \tag{4.30}$$

Example 4.19. Let us consider the three flows of Example 4.12 in turn.

- $\mathbf{V} = \mathbf{n}$: $J = 1$ and equations (4.30) become

$$\frac{\partial \Phi}{\partial t} = 3\kappa\,\Phi - \kappa^2\,\Phi_\kappa.$$

- $\mathbf{V} = \kappa\,\mathbf{n}$: $J = \kappa$ and equations (4.30) become, using $\kappa_s = \Phi(\kappa, t)$,

$$\frac{\partial \Phi}{\partial t} = \Phi^2\,\Phi_{\kappa\kappa} - \kappa^3\,\Phi_\kappa + 4\kappa^2\,\Phi.$$

- $\mathbf{V} = \kappa_s\,\mathbf{n}$: $J = \kappa_s$ and equations (4.30) become, using $\kappa_{ss} = \Phi(\kappa, t)\Phi_\kappa(\kappa, t)$,

$$\frac{\partial \Phi}{\partial t} = \Phi^3\,\Phi_{\kappa\kappa\kappa} + 3\Phi^2\,\Phi_\kappa\,\Phi_{\kappa\kappa} + 3\kappa\,\Phi^2.$$

4.4   Reconstruction equations for invariant flows

The differential invariant signature may be interpreted as the projection of a curve onto the cross-section via the right moving frame. Hence, the left moving frame may be used to reconstruct a submanifold from its signature with the same reconstruction process utilized for group foliation, as seen geometrically in Figure 4.1. The question then arises: as the signature evolves concurrent with the submanifold flow, what is corresponding evolution of the reconstruction parameters? The evolution of the reconstruction parameters is given by the Lie derivative with respect to the vector field $\mathbf{V}$ defining the flow, and the reconstruction equations derived in Chapter 3 provide the information needed to evaluate this Lie derivative explicitly, as we now explain.
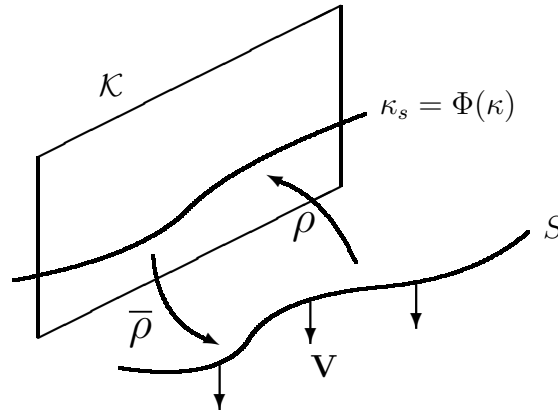

Figure 4.1: Reconstruction of invariant flows

As in the previous section, we restrict considerations to planar curve flows. All results obtained may be generalized to any submanifold flow in a straightforward manner. Let $\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$ be the generator of an invariant curve flow, and suppose that the family of signatures under this curve evolution is given by $\kappa_s = \Phi(\kappa, t)$. The invariants κ, κ_s form a generating set of invariants, and hence the signature defines an automorphic system for each fixed value of t. Pulling back the pseudogroup jet recurrence relations (3.31) by the right moving frame and restricting to this automorphic

system yields the reconstruction system

$$d\overline{\rho} = P(\overline{\rho}, \kappa)\, \varpi + \sum_{j \geq 0} Q^j(\overline{\rho}, \kappa)\, \vartheta_j \tag{4.31}$$

for the left moving frame pull-backs of the pseudogroup jets. It will be useful in this discussion to retain the portion of this equality containing the invariant contact forms ϑ_j.

Example 4.20. We compute the reconstruction equations for our running example of planar Euclidean curve flows. Begin with the pseudogroup jet recurrence relations (3.31) for our transformation:

$$dx = -u_U\, \mu^x + u_X \mu^u - u_U\, dX + u_X\, dU$$
$$du = -(u_X\, \mu^x + u_U \mu^u + u_X\, dX + u_U\, dU)$$
$$du_X = -u_U \mu^u_X,$$

and so on. As usual, it will be more convenient to work with the group parameters instead of the jets, so we write these jets in terms of the (inverse) parameters $\bar{a}, \bar{b}, \bar{\phi}$:

$$dx = \cos\bar{\phi}\, dX - \sin\bar{\phi}\, dU - (U\cos\bar{\phi} + X\sin\bar{\phi})\, d\bar{\phi} + d\bar{a}$$
$$du = \sin\bar{\phi}\, dX + \cos\bar{\phi}\, dU + (X\cos\bar{\phi} - U\sin\bar{\phi})\, d\bar{\phi} + d\bar{\phi}$$
$$du_X = \cos\bar{\phi}\, d\bar{\phi}.$$

Pulling back by the right moving frame yields the reconstruction equations

$$d\bar{a} = \cos\bar{\phi}\, \varpi - \sin\bar{\phi}\, \vartheta$$
$$d\bar{b} = \sin\bar{\phi}\, \varpi + \cos\bar{\phi}\, \vartheta \tag{4.32}$$
$$d\bar{\phi} = \kappa\varpi + \vartheta_x.$$

Suppose now that an invariant curve flow is given by $\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$. For a given curve evolution with corresponding family of signatures $\kappa_s = \Phi(\kappa, t)$ there will be a corresponding family of reconstruction parameters $\bar{\rho}(t)$. At each time t, $\bar{\rho}(t)$ will be a solution to the reconstruction equations (4.31), where P, Q now vary in time since the signature $\Phi(\kappa, t)$ is changing. This family of reconstruction parameters will evolve according to the Lie derivative

$$\frac{\partial\bar{\rho}}{\partial t} = \mathbf{V}(\bar{\rho}).$$

To find this Lie derivative we use the reconstruction equations (4.31):

$$
\begin{aligned}
\mathbf{V}(\bar{\rho}) &= \mathbf{V} \lrcorner \, d\bar{\rho} \\
&= \mathbf{V} \lrcorner \left(P(\bar{\rho}, \kappa)\,\varpi + \sum_{\#J \geq 0} Q^j(\bar{\rho}, \kappa)\,\vartheta_j \right) \\
&= P(\bar{\rho}, \kappa)\,I + \sum_{j \geq 0} Q^j(\bar{\rho}, \kappa)\left(\mathbf{V} \lrcorner \,\vartheta_j\right) \\
&= P(\bar{\rho}, \kappa)\,I + \sum_{j \geq 0} Q^j(\bar{\rho}, \kappa)\left(\mathbf{V} \lrcorner \,\mathcal{A}_j(\vartheta)\right),
\end{aligned}
$$

where Lemma 4.9 was used to write the higher order invariant contact forms as Lie derivatives $\vartheta_j = \mathcal{A}_j(\vartheta)$ of the basic zero order contact form ϑ. Assuming further that $\mathbf{V}$ generates an intrinsic flow, we may apply Lemma 4.13 to find

$$
\begin{aligned}
\mathbf{V}(\bar{\rho}) &= P(\bar{\rho}, \kappa)\,I + \sum_{j \geq 0} Q^j(\bar{\rho}, \kappa)\left(\mathbf{V} \lrcorner \,\mathcal{A}_j(\vartheta)\right) \\
&= P(\bar{\rho}, \kappa)\,I + \sum_{j \geq 0} Q^j(\bar{\rho}, \kappa)\,\mathcal{A}_j(J).
\end{aligned}
\tag{4.33}
$$

We thus arrive at the following reconstruction parameter evolution for normal flows.

Theorem 4.21. Suppose $\mathbf{V} = J\,\mathbf{n}$ gives an invariant curve flow and $C(t)$ is the corresponding family of curves. Then the evolution of the reconstruction parameters for $C(t)$ is given by the partial differential equation

$$
\frac{\partial \bar{\rho}}{\partial t} = \sum_{j \geq 0} Q^j(\bar{\rho}, \kappa)\,\mathcal{A}_j(J).
\tag{4.34}
$$

Proof. Following the same strategy as Corollary 4.15, we first add a (possibly nonlocal) tangential term so that $\mathbf{V}$ is an intrinsic flow. Applying (4.33) and deleting the terms originating from reparametrization yields the desired formula. $\qquad\square$

Example 4.22. Returning again to Euclidean curves, assuming that we have a normal flow and combining formulae (4.34) and (4.32) we obtain

$$
\begin{aligned}
\frac{\partial \bar{a}}{\partial t} &= -J \sin \bar{\phi} \\
\frac{\partial \bar{b}}{\partial t} &= J \cos \bar{\phi} \\
\frac{\partial \bar{\phi}}{\partial t} &= \mathcal{D}\,J.
\end{aligned}
\tag{4.35}
$$

Let us consider the three flows of Example 4.12 in turn.

- $\mathbf{V} = \mathbf{n}$: $J = 1$ and equations (4.35) become

$$\frac{\partial \bar{a}}{\partial t} = -\sin \bar{\phi}, \qquad \frac{\partial \bar{b}}{\partial t} = \cos \bar{\phi}, \qquad \frac{\partial \bar{\phi}}{\partial t} = 0.$$

- $\mathbf{V} = \kappa \mathbf{n}$: $J = \kappa$ and equations (4.35) become, using $\kappa_s = \Phi(\kappa, t)$,

$$\frac{\partial \bar{a}}{\partial t} = -\kappa \sin \bar{\phi}, \qquad \frac{\partial \bar{b}}{\partial t} = \kappa \cos \bar{\phi}, \qquad \frac{\partial \bar{\phi}}{\partial t} = \Phi(\kappa, t).$$

- $\mathbf{V} = \kappa_s \mathbf{n}$: $J = \kappa_s$ and equations (4.35) become, using $\kappa_{ss} = \Phi(\kappa, t)\Phi_\kappa(\kappa, t)$,

$$\frac{\partial \bar{a}}{\partial t} = -\Phi(\kappa, t) \sin \bar{\phi}, \qquad \frac{\partial \bar{b}}{\partial t} = \Phi(\kappa, t) \cos \bar{\phi}, \qquad \frac{\partial \bar{\phi}}{\partial t} = \Phi(\kappa, t)\Phi_\kappa(\kappa, t).$$

Coupled with the signature flows computed in Example 4.19, these reconstruction equations split in the invariant curve evolution into a coupled system of evolutionary partial differential equations which may be solved or studied as an alternative to the flow itself.

Example 4.23. Given an intrinsic Euclidean flow $\mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$, formula (4.33) yields the reconstruction evolution

$$\bar{a}_t = I \cos \bar{\phi} - J \sin \bar{\phi}$$
$$\bar{b}_t = I \sin \bar{\phi} + J \cos \bar{\phi}$$
$$\bar{\phi}_t = I\kappa + \mathcal{D}J.$$

Consider the intrinsic flow $\mathbf{V} = \frac{1}{2}\kappa^2\,\mathbf{t} + \kappa_s\,\mathbf{n}$, extrinsically identical to the mKdV normal flow consider previously. The resulting evolution of κ is given by the modified Korteweg–deVries equation

$$\kappa_t = \kappa_{sss} + \frac{3}{2}\kappa^2 \kappa_s,$$

which provides the origin of the name of the flow. The concurrent reconstruction evolution is then given by

$$\frac{\partial \bar{a}}{\partial t} = \frac{1}{2}\kappa^2 \cos \bar{\phi} - \kappa_s \sin \bar{\phi}$$
$$\frac{\partial \bar{b}}{\partial t} = \frac{1}{2}\kappa^2 \sin \bar{\phi} + \kappa_s \cos \bar{\phi}$$
$$\frac{\partial \bar{\phi}}{\partial t} = \frac{1}{2}\kappa^3 + \kappa_{ss}.$$

Chapter 5

Future research

The key innovation that drives ideas and computations in this thesis is the theory of equivariant moving frames. Using this framework, we have attempted to provide a unified and computationally clear approach to group foliation and associated processes of symmetry reduction. The relative newness and broad applicability of equivariant moving frame theory brings fresh insight to old algorithms and as such, many unexplored directions present themselves. We list here several possibilities for further research based on the group foliation algorithm and applications described in Chapter 4.

One of the most obvious applications of group foliation is to the solution of differential equations. There are many physically interesting equations that may be particularly amenable to our version of group foliation because of the complexity of their symmetry pseudogroups. Three particularly interesting examples are: the Davey–Stewartson equations, [13],

$$i\psi_t + \psi_{xx} + \epsilon\psi_{yy} - \delta|\psi|^2 - \psi w = 0,$$
$$w_{xx} - \epsilon w_{yy} - \alpha(|\psi|^2)_{yy} = 0,$$

where $\epsilon, \delta = \pm 1$ and $\alpha > 0$; the Infeld–Rowlands equation, [20],

$$u_t + 2u_x u_{xx} + u_{xxxx} + u_{xy} = 0,$$

and the potential Kadomstev–Petviashvili equation, [**17**],

$$\left(u_t + \frac{3}{4}u_x^2 + \frac{1}{4}u_{xxx}\right)_x \pm \frac{3}{4}u_{yy} = 0.$$

Analysis of the differential invariant algebras of the first two examples is presented in [**79**] and would serve as an excellent starting point for this project.

As Vessiot observed in [**78**], the true difficulty in utilizing the group foliation algorithm for solving differential equations lies in process of solving the resolving system. Thus, any strategy by which the solution of the resolving system can be simplified may be helpful for practical implementation of the method. Johnson outlines such a strategy, [**33**], observing that after symmetry reduction by a normal Lie subpseudogroup $\mathcal{G} \subset \mathcal{H}$, the resolving system $\mathcal{R}_{\mathcal{G}}$ admits the quotient symmetry pseudogroup $\mathcal{H}/\mathcal{G}$. By then foliating $\mathcal{R}_{\mathcal{G}}$ by $\mathcal{H}/\mathcal{G}$, we arrive at an iterative approach to the search for resolving system solutions, reducing the problem to the solution of a resolving system for a simple Lie pseudogroup, followed by a sequence of reconstruction steps. The practicality of this method has yet to be explored.

Many constructions in this thesis, including the core group foliation algorithm, can be adapted to finite difference equations through the use of joint moving frames and joint invariants, [**55**, **56**]. This adaptation is the subject of a work in progress, [**74**]. It would be an interesting project to investigate the possibility of discrete group foliation as a numerical method for solving differential equations. It may also be worthwhile to pursue the construction of Bäcklund transformations for finite difference or differential-difference equations and compare these results with similar notions from discrete differential geometry and integrable systems, [**7**, **16**].

In the case of group foliation by finite dimensional Lie groups, non-maximal rank resolving systems correspond to what are called partially invariant solutions, [**66**]. The question of when a partially invariant solution is *irreducible*, i.e. not obtainable as an invariant or partially invariant solution for a subgroup, has been studied by Ondich, [**64**]. This allows for an extension of the classification of group invariant solutions, [**52**], to partially invariant solutions. It may be interesting to investigate the extension of such a classification and the notion of irreducibility in the context of group foliation.

The systematic construction of Bäcklund transformations using symmetry reduction of exterior differential systems introduced by Anderson and Fels applies only to Lie groups. The group foliation/inductive moving frames approach outlined in Section 4.2 does not have this limitation. It would be worthwhile to pursue the possibility of constructing new Bäcklund transformations by realizing systems of interest as resolving systems for infinite-dimensional Lie pseudogroups; these ideas are also most likely closely related to the reduction methods for infinite dimensional Lie pseudogroups introduced by Pohjanpelto, [**67**]. The investigation of non-maximal rank resolving systems could also produce interesting examples. Finally, as indicated by Example 4.4, the group foliation algorithm in conjunction with inductive moving frames may provide a means for constructing coverings of differential equations, [**29**, **39**].

Finally, invariant submanifold flows find applications in a diversity of fields such as control theory, [**51**], elasticity theory, [**43**], and computer vision, [**14**, **71**], and it is possible that the idea of invariant flow reconstruction presented in Chapter 4 may provide insight in some of these areas. Theoretical application of invariant flow reconstruction is also worth exploring. For example, Mansfield and van der Kamp, [**47**], have studied the question of when the integrability (in the sense of possessing infinitely many symmetries) of a differential invariant signature flow "lifts" to integrability of the flow itself. We suspect that their results could be reinterpreted within our framework.

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