Essays on Optimal Policy in Open Economies

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Dedication

This thesis is dedicated to my parents, Liliana and Aldo Dovis.

Abstract

My dissertation consists of three chapters. The common theme that unifies the chapters is the analysis of how lack of commitment and enforcement frictions shape outcomes in dynamic economies and the implications that the existence of these frictions have for policy.

In the first chapter, *"Efficient Sovereign Default,"* I show that key aspects of sovereign debt crises can be rationalized as part of the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in a production economy with informational and commitment frictions. I show that, under appropriate assumptions, the interaction between lack of commitment and private information gives rise to ex-post inefficient outcomes that exhibit many of the characteristics of sovereign debt crises in the data. Despite being ex-post inefficient, these outcomes are efficient from an ex-ante perspective because they help to provide incentives. To interpret these inefficient outcomes as debt crises, I show how the efficient allocation can be implemented as an equilibrium outcome of a sovereign debt game in which the set of securities that the sovereign government can issue is restricted to non-contingent defaultable bonds of multiple maturities. The implementation has implications for the optimal maturity composition of debt. Consistently with the data, as default is more likely, the maturity composition of debt shifts toward short term debt.

In the second chapter, *"Credit Market Frictions and Trade Liberalization,"* joint with Wyatt Brooks, we investigate whether credit market frictions reduce gains from trade liberalization. We develop a dynamic, general equilibrium trade model with heterogeneous firms and consider two specifications of credit market frictions: collateral constraints as in Evans and Jovanovic (1989) and limited enforcement as in Albuquerque and Hopenhayn (2004). Though these two specifications have similar implications for firm-level dynamics, but they have different implications for trade reform. With limited enforcement there are the same percentage gains from trade liberalization as there would be in the presence of perfect credit markets; with collateral constraints the gains

are lower. This is because the debt limits that firms face respond to profit opportunities in the first case and not in the second. Using firm-level panel data from a trade reform in Colombia, we find that the change in entry decisions in the export market after the reform is consistent with the limited enforcement specification.

In the third chapter, *"Capital Mobility and Optimal Fiscal Policy without Commitment: A Rationale for Capital Controls?,"* I study a Ramsey taxation model for a small open economy. A known result in the literature is that if the government has commitment, there is no need to introduce capital controls. In contrast, I show that when the government lacks commitment, imposing capital controls on inflows is necessary to support the efficient allocation when the economy is capital scarce. In this chapter I abstract from uncertainty. The model, augmented with shocks to productivity and the international interest rate, offers a framework for studying how the government should respond to shock to the international interest rate, and how the differential between domestic and foreign capital income taxes moves over the cycle.

Contents

List of Tables

List of Figures

Chapter 1

Efficient Sovereign Default

1.1 Introduction

In this paper, I show that the main aspects of sovereign debt crises can be rationalized as part of the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in a production economy with informational and commitment frictions. I show that the efficient allocation can be implemented as the equilibrium outcome of a sovereign debt game with non-contingent defaultable debt of multiple maturities. Defaults and *ex-post* inefficient outcomes along the equilibrium path are not a pathology. Rather, they serve the purpose of supporting the *ex-ante* efficient outcome. *Ex-post* inefficient debt crises in the model exhibit many of the characteristics of those in the data.

Specifically, the debt crises in the model are consistent with four key aspects observed in the data. First, sovereign debt crises are associated with severe output and consumption losses for the debtor country. Second, they coincide with trade disruptions. In particular, the drop in imports of intermediate goods is very large relative to the drop that occurs in episodes with recessions of similar magnitudes but without default. Third, after a default economic activity eventually recovers and these recoveries are accompanied by large trade surpluses. Moreover, eventually there is a partial repayment of the defaulted debt, after which the country regains access to international credit markets. Fourth, as the crisis approaches, interest rate spreads on sovereign debt

rise and the maturity of this debt gets shorter.[1]

I develop a model to study the optimal risk-sharing arrangement between a sovereign borrower and a large number of foreign lenders. The environment has three main ingredients. First, motivated by the data, I consider a production economy in which imported intermediate inputs are used in production. Second, the sovereign government cannot commit to repaying its debt and the only recourse available to the lenders to ensure repayment is the threat of exclusion from future borrowing, lending, and trade. Third, the sovereign borrower has some private information about the state of the domestic economy.[2]

In the baseline economy, the source of private information is the relative productivity of the domestic non-tradable sector. One interpretation of this assumption is that the government has more information about the domestic economy than the foreign lenders and has control over the released statistics and other sources of information. Rogoff (2011) suggests that the lack of explicit state contingencies in international debt contracts can be explained by the fact that sovereign borrowers have enormous discretion over the creation of statistics to be used for indexation of the contingent claims. For simplicity, I consider the extreme case where the foreign lenders have no information about the state of the domestic economy. The main insights of the paper should carry through in the less extreme case in which foreigner lenders receive some noisy signals about the state of the domestic economy. I also show that it is possible to reinterpret the baseline economy as one in which the source of private information is a *taste* shock that affects the marginal utility of consumption of the sovereign borrower, as in Atkeson and Lucas (1992). This reinterpretation is useful for establishing some of the technical results.

To derive the model's implications I proceed in two steps. First, I solve the optimal contracting problem subject to the restrictions imposed by lack of commitment and private information. This problem gives the efficient allocations for output, consumption, imports, and exports. I then implement these allocations as a *sustainable equilibrium*

[1] In Appendix A, I document the first two features of the data and discuss an extensive literature which also documents these key aspects of the data.

[2] The main results of the paper remain valid if I consider an environment in which the incentive problem arises because of moral hazard. For instance, if the sovereign government can take a costly hidden action that increases the probability of higher future realization of aggregate productivity as in Atkeson (1991). Results are available upon request.

outcome of a sovereign debt game with non-contingent defaultable debt of multiple maturities. I use this implementation to interpret some of the features of these allocations as debt crises and to derive the implications for the maturity composition of debt and for interest rate spreads.

Consider first the optimal contracting problem. At the beginning of any period, the lenders supply intermediate goods to the borrower. Then the productivity of the non-tradable sector is realized. The borrower uses the imported intermediates and domestic labor to produce a non-tradable domestic consumption good and an export good. The export good can be thought of as repayment of past debt. Since the sovereign borrower is risk averse and the foreign lenders are risk neutral, absent contracting frictions the lenders would completely insure the borrower. In particular, when the productivity of the domestic non-tradable sector is low, more resources would be devoted to non-tradables and repayments in the form of exports would be lower. Moreover, the realization of the shock would have no effect on the continuation of the allocation. Private information and the borrower's lack of commitment limit such insurance.

To understand how the insurance must be limited by the presence of private information, consider an adverse shock to productivity. If the lenders completely insured such a shock, it would be in the borrower's best interest to always claim to have low productivity and devote fewer resources to repaying the lenders, thus eliminating any possibility of insurance. Therefore, in order to make insurance payments to borrowers with currently low productivity shocks incentive compatible, there must be a cost associated with claiming to have low productivity. In a dynamic model, lenders can impose such a cost by reducing the continuation value of the borrower through lowering the borrower's future consumption levels.

In this contracting problem, the lack of commitment interacts with the incentive problem. In particular, when the continuation value of the borrower is low, the borrower is tempted to deviate from the efficient allocation by increasing current consumption by not repaying the amount prescribed and then living in autarky thereafter. To prevent such an outcome, the lenders must provide a sufficiently low amount of intermediate goods so that this kind of deviation is unprofitable.

Enforcing such an outcome is also costly for the lenders. The reason is that, by lending little when the continuation value of the borrower is low, the lender is depressing

the production level of the borrower and hence limiting the ability of the borrower to repay the pre-existing debt. Enforcing a continuation value for the borrower close to autarky is *ex-post* inefficient. That is, if the borrower and the lenders could renegotiate the terms of their agreement, committing not to do it again in the future, then both could be made better off. By increasing the borrower's value when it is close to autarky, it is possible to avoid the drop in imported intermediate inputs which depresses production and reduces the ability of the sovereign borrower to repay the lenders. The necessity of providing incentives *ex-ante* requires that these *ex-post* inefficient outcomes happen along the equilibrium path with strictly positive probability. I also show that such *ex-post* inefficiencies are recurrent.

Notice that the interaction between private information and lack of commitment is key for generating this *ex-post* inefficiency. With lack of commitment but without private information, it is optimal to backload consumption and increase the continuation value of the borrower. Thus, the outcomes are *ex-post* efficient. With private information but with commitment, the statically efficient amount of production can always be sustained; thus, there is no cost to the lenders associated with lowering the borrower's continuation value.

Next, I describe how the model can generate the features of output, consumption, imports, and exports that occur during and after debt crises. The proximate cause of a debt crisis is a sufficiently long string of low productivity shocks which lead the borrower's continuation value to decrease until it is close to the value of autarky. The lack of commitment implies that the imports of intermediates must drop to prevent a deviation by the borrower. This drop in imports reduces output, consumption, and the repayments made to the lenders. Once the economy receives a high productivity shock in the non-tradable sector, output increases, the borrower shifts some resources to the traded sector and uses these resources to run a large trade surplus to repay the foreign lenders. These repayments result in the gradual increase of the borrower's continuation value and hence consumption, production, and imported intermediate inputs used in production also increase in the future.

To interpret these outcome paths as debt crises, I then turn to implementing an efficient allocation as a sustainable equilibrium outcome of a sovereign debt game between a sovereign borrower, competitive foreign lenders, and private domestic agents. There are

several ways in which one can implement the efficient allocation. The specific elements I choose are motivated by three key facts about sovereign debt. First, in the data, the vast majority of sovereign and external debt comes in the form of non-contingent debt (see Rogoff (2011) for a discussion). Second, default episodes are infrequent. Third, defaults happen when the sovereign is highly indebted.

Motivated by these facts, I restrict the set of securities that the sovereign borrower can use to non-contingent bonds of multiple maturities. The sovereign borrower has the option to *default*, which I define as suspending the principal and coupon payments specified by the bond contracts. The borrower is excluded from credit markets until at least a partial repayment to the bond holders is made. Finally, the sovereign government can impose a tariff on the imports of intermediate goods, capturing the idea that the sovereign government cannot commit to repay foreign lenders.

Along the equilibrium path that supports an efficient allocation, defaults are recurrent but infrequent and happen when the borrower is highly indebted. In particular, they happen only when the continuation value for the sovereign is close to the value of autarky. Defaults and periods of temporary exclusion from international credit markets occur at the same time as the *ex-post* inefficient outcomes prescribed by the efficient allocation. In all the other periods, the sovereign government repays in full its debt obligations.

When there is full repayment, in the absence of contingent debt, the state contingent returns implied by the efficient allocation are replicated by exploiting the variation in the price of long-term debt after a shock. After the realization of a low productivity shock, the continuation value for the sovereign borrower decreases and the probability that there will be a default in the near future goes up. The increase in the likelihood of a future default reduces the value of the outstanding long-term debt. This reduction results in a capital gain for the borrower and provides some debt relief after an adverse shock. If the maturity composition of debt is appropriately chosen, this mechanism can exactly replicate the state-contingent returns implied by the efficient allocation.

Along the path approaching default, the maturity composition of the sovereign debt shifts toward short term debt. This shift occurs because, when the probability of future default is high, the price of the long-term debt is more sensitive to shocks. Therefore, a lower long-term debt holding is needed to replicate the debt relief that is implicit in

the efficient allocation after a low realization of productivity in the non-tradable sector. Since the overall indebtedness of the sovereign borrower is increasing along the path approaching a default, it must be that the amount of short term debt issued goes up as the probability of default increases. Thus, the maturity composition shortens as indebtedness increases, as is true in the data.

Furthermore, an equilibrium outcome path that supports an efficient allocation is consistent with the evidence on interest rate spreads in emerging economies. In particular, long term spreads are generally higher than short term spreads. During debt crises, the term spread curve inverts, as documented by Broner, Lorenzoni, and Schmukler (2010) and Arellano and Ramanarayanan (2012).

Finally, note that the policy implications of this paper differ from those that can be drawn from the existing literature. First, because *ex-post* inefficient outcomes are part of the efficient allocation, interventions by a supranational authority aimed at reducing the inefficiencies in a sovereign default episode are not beneficial from an *ex-ante* perspective. Moreover, choosing arrangements that are hard to restructure *ex-post* is consistent with *ex-ante* efficiency. One interpretation of the results in my model is that lack of coordination is desirable. In this sense, attempts by, say, international organizations to coordinate lenders during debt restructuring may lead to worse outcomes from an *ex-ante* point of view. Second, the increasing share of short term debt when a sovereign borrower accumulates external debt can be optimal when only non-contingent defaultable debt of multiple maturities is available. This feature of the data is often blamed as one of the main causes of sovereign or external debt crises. For instance, see Rodrik and Velasco (1999). In sharp contrast, my model generates an endogenous shortening of the maturity structure of the debt. My analysis suggests that interventions that penalize the issuance of short term debt might negatively affect welfare.

Related Literature This paper is related to several strands of literature. First, it is related to the quantitative incomplete market literature on sovereign default. Following Eaton and Gersovitz (1981), recent contributions include Aguiar and Gopinath (2006), Arellano (2008), Hatchondo and Martinez (2009), Benjamin and Wright (2009), Yue (2010) and Chatterjee and Eyigungor (2012a). Defaults are associated with an additional cost in the form of lower endowments in subsequent periods. Mendoza and Yue

(2012) assume that this extra cost is driven by a collapse in imports of intermediate goods, as endogenously obtained in this paper. None of these papers can address why the sovereign borrower and the lenders do not renegotiate the terms of the contracts in order to avoid the additional cost associated with default. Even models with explicit renegotiations - such as Yue (2010) and Benjamin and Wright (2009) - impose that the borrower first defaults and incurs the cost in terms of foregone output before the renegotiation can start. I contribute to this strand of the literature by endogenizing the additional default cost and providing a rationale for why it is incurred along the equilibrium path.

This paper makes a contribution to the dynamic contracting literature. Green (1987), Thomas and Worrall (1990) and Atkeson and Lucas (1992) consider environments with only private information. Kehoe and Levine (1993), Thomas and Worrall (1994), Kocherlakota (1996), Kehoe and Perri (2002, 2004), Albuquerque and Hopenhayn (2004), and Aguiar, Amador, and Gopinath (2009) consider economies with only lack of commitment. In this paper, I combine both contracting frictions and show that their interaction in a production economy can generate cycles with *ex-post* inefficient outcomes.

I build on the seminal contribution in Atkeson (1991) who considers an optimal contracting problem with both lack of commitment and an incentive problem due to the presence of moral hazard. He does not emphasize the role of *ex-post* inefficient outcomes in supporting the efficient arrangement and instead focuses on the downward sloping portion of the utility possibility frontier. Another paper that combines both frictions is Atkeson and Lucas (1995). In their model, there is no cost in terms of production associated with lower continuation values for the borrower. Hence, there are no *ex-post* inefficiencies along the efficient allocation. In parallel work, Ales, Maziero, and Yared (2012) consider both frictions in a similar environment. They allow for the principal to replace the agent; this feature of the environment rules out *ex-post* inefficiencies.

The theme that *ex-post* inefficiencies on-path are necessary to support the *ex-ante* optimal arrangement in economies with incentive problems has been explored in various contexts. For instance, see Green and Porter (1984), Phelan and Townsend (1991), and Yared (2010). In the context of firm dynamics with credit frictions, Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), and DeMarzo

and Fishman (2007) show that private information and limited liability can lead to inefficient liquidation. Hopenhayn and Werning (2008) have a similar result for an optimal contracting problem with lack of commitment and private information about the outside option of the agent. A novel feature of my paper is that there is no termination of the risk-sharing relationship. The optimal allocation has periods of temporary autarky (which are *ex-post* inefficient), but cooperation eventually restarts after the domestic economy recovers.

Finally, this paper contributes to the literature that studies the maturity composition of sovereign debt under default risk. Broner, Lorenzoni, and Schmukler (2010) and Arellano and Ramanarayanan (2012) document that the maturity composition of sovereign debt gets shorter when a default is more likely. Moreover, Rodrik and Velasco (1999) find that the ratio of short term debt to reserves is a robust predictor of an external debt crisis. The excessive reliance on short-term debt on the verge of a sovereign debt crisis is often blamed as one of the main causes of the crisis itself. Models with roll-over risk, such as Cole and Kehoe (2000), provide a rationale for such a prediction because short-term debt is more prone to roll-over risk.[3] I contribute to this literature by showing that the negative correlation between the maturity of outstanding debt and the notional value of debt (and hence the probability of a crisis) emerges as a way to support the efficient allocation. Managing the maturity composition of debt serves to replicate state contingent returns as in Kreps (1982), Angeletos (2004) and Buera and Nicolini (2004). In a related paper, Arellano and Ramanarayanan (2012) endogenize the maturity composition of debt in an Eaton and Gersovitz (1981) type of model. Consistent with my findings, the maturity composition of debt shortens when the probability of default is high. The difference between their paper and mine is that they cannot assess the efficiency of such an equilibrium outcome.

The rest of the paper is organized as follows. In section 2, I describe the baseline environment and its reinterpretation as a taste shock economy. In section 3, I define an efficient allocation. In section 4, I characterize the efficient allocation and show that the *ex-ante* efficient allocation calls for *ex-post* inefficient outcomes. In section 5, I implement the efficient allocation with non-contingent defaultable debt and active maturity management. In section 6, I present an illustrative numerical example and

[3] See Phelan (2004) for a different view.

relate the model to the evidence about sovereign default episodes. Finally, section 7 concludes and discusses potential extensions.

1.2 Environment

In this section, I lay out the baseline environment in which the source of private information is the productivity of the non-tradable sector. Under appropriate conditions, I provide a reinterpretation of this economy as a taste shock economy as in Atkeson and Lucas (1992). Because this formulation is much more tractable, I will use it in the rest of the paper.

1.2.1 Baseline Environment: Productivity Shock Economy

Time is discrete and indexed by $t = 0, 1, ...$ There are three types of agents in the economy: a large number of homogeneous domestic households, a benevolent domestic government, and a large number of foreign lenders. There are three goods: a domestic consumption good (non-traded), an export good, and an intermediate good. There are two sources of uncertainty. They are a shock to the relative productivity of the non-tradable (domestic consumption) sector and a public randomization device, ξ_t, which is distributed uniformly over the interval $[0, 1]$ and is *iid* over time.

Preferences All agents are infinitely lived. The stand-in domestic agent values a stochastic sequence of consumption of the domestic good, $\{y_t\}_{t=0}^{\infty}$, according to

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(y_t) \tag{1.1}$$

where the period utility function $U : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and strictly concave, and satisfies standard conditions and $\beta \in (0, 1)$ is the discount factor. The government is benevolent and so it shares the same preferences as the domestic households.

Foreign lenders are risk neutral and they value consumption of the export good . They discount the future with a discount factor $q \in (0, 1)$, which should be thought of as the inverse of the risk-free interest rate in international credit markets. I allow the discount factor β and q to differ, but I will restrict myself to the case where $q \geq \beta$;

that is, the domestic households discount the future at a (weakly) higher rate than the international interest rate.

Endowments and Technology Foreign lenders have a large endowment of the intermediate good. They have access to a technology that transforms one unit of the intermediate good into one unit of the export good so that the relative price between the export and the intermediate good is fixed at one.

Each domestic agent is endowed with one unit of labor in each period. There is a domestic production technology that transforms the intermediate good and labor into domestic consumption good, y, and foreign consumption good, y^*, as follows:

$$y \leq zF(m_1, \ell_1) \tag{1.2}$$

$$y^* \leq F(m_2, \ell_2) \tag{1.3}$$

where

$$m_1 + m_2 \leq m, \quad \ell_1 + \ell_2 \leq 1 \tag{1.4}$$

and m_1 and m_2 are the units of the intermediate good allocated to the production of the domestic and export good respectively, m is the total amount of intermediates used domestically, and ℓ_1 and ℓ_2 are the units of domestic labor allocated to domestic and export production respectively. The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ has constant returns to scale; it is such that $F(0,1) > 0$, so that strictly positive output can be produced in autarky; and it satisfies the Inada condition $\lim_{m\to 0} F_m(m,\ell) = +\infty \ \forall \ell > 0$, as well as other standard conditions. Let $f(m) = F(m,1)$. The relative productivity of the domestic sector, $z \in Z \equiv \{z^1, z^2, ..., z^N\}$ with $N < \infty$, is distributed according to a probability distribution μ and it is *iid* over time. Without loss generality, let $z^i < z^j$ if $i < j$. By properties of constant returns to scale technology, the technological restrictions imposed by (1.2)-(1.4) can be summarized by the following aggregate resource constraint

$$\frac{y}{z} + y^* \leq f(m) \tag{1.5}$$

along with the non-negativity conditions on y and y^*.

Timing The timing of events within the period is as follows:

1. The public randomization device $\xi_t \in [0,1]$ is realized;
2. Foreign lenders supply intermediate goods $m_t \geq 0$;
3. z_t is realized according to μ;
4. Real activity occurs: production, consumption, and exporting take place.

Let $s_t = (\xi_t, z_t)$ and $s^t = (s_0, s_1, ..., s_t)$. An allocation for this economy is a stochastic process $\mathbf{x}_z \equiv \{m(s^{t-1}, \xi_t), y(s^t), y^*(s^t)\}_{t=0}^{\infty}$. An allocation $\mathbf{x}_z$ is feasible if it satisfies (1.5).

Information Foreign lenders observe the amount of intermediate goods that the country imports, m, and the amount of exports, y^*. Moreover, they can observe the amount of resources, m_1 and ℓ_1, employed in the domestic consumption (non-tradable) sector. They cannot see the amount of output produced with the inputs because the realization of z is privately observed by the domestic government. From (1.5), they can use their information about m and y^* to infer y/z but not y and z separately. Next, I will show how under appropriate assumptions I can relabel the variables in this economy to obtain an equivalent taste shock formulation.

1.2.2 Reinterpretation: Taste Shock Economy

Suppose that the period utility function displays constant relative risk aversion:

$$U(y) = \frac{y^{1-\gamma}}{1-\gamma} \tag{1.6}$$

with $\gamma > 1$. Under this assumption, I can rewrite the baseline environment as an economy with two goods - a final and an intermediate good - where the stand-in domestic household is subject to a taste shock

$$\theta_t \in \Theta \equiv \{\theta^1, \theta^2, ..., \theta^N\} = \{(z^N)^{1-\gamma}, (z^{N-1})^{1-\gamma}, ..., (z^1)^{1-\gamma}\}$$

which is *iid* over time and is privately observed by the domestic agent. For notational convenience, let $\theta_L = \theta^1$ and $\theta_H = \theta^N$. The taste shock affects the domestic agent's marginal utility of consumption in a multiplicative fashion; a higher θ_t makes current consumption more valuable. A high taste shock corresponds to a low productivity

shock in the original baseline economy. Intuitively, after either a high taste shock or a low productivity shock in the non-tradable sector, the marginal utility of imported intermediates used for domestic consumption is high. Define

$$c = \frac{y}{z} \quad \text{and} \quad \theta = z^{1-\gamma} \tag{1.7}$$

where c is domestic consumption and θ is a taste shock. With some abuse of notation, let $s_t = (\xi_t, \theta_t)$. Under (1.6), I can write the preferences for a stand-in domestic agent over a stochastic sequence $\left\{c_t(s^t)\right\}_{t=0}^{\infty}$ as

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \theta_t U\left(c_t(s^t)\right) \tag{1.8}$$

From (1.5), the resource constraint for this economy can be written as:

$$c(s^t) + y^*(s^t) \leq f\left(m(s^{t-1}, \xi_t)\right) \tag{RC}$$

where y^* are exports as in the productivity shock formulation. An allocation for this taste shock economy is a stochastic process $\mathbf{x} \equiv \{m(s^{t-1}, \xi_t), c(s^t), y^*(s^t)\}_{t=0}^{\infty}$. The allocation is feasible if it satisfies (RC). Clearly, if $\mathbf{x}$ is feasible then

$$\mathbf{x}_z = \{m(s^{t-1}, \xi_t), c(s^t)\theta_t^{1/(1-\gamma)}, y^*(s^t)\}_{t=0}^{\infty}$$

is feasible for the baseline economy[4] and viceversa.

In the rest of the paper, I will present the results using this taste shock formulation. I assume that the primitives of the taste shock economy satisfy the following conditions:

Assumption 1.1. *$F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is C^2, strictly increasing, has constant returns to scale and is such that $F(0,1) > 0$ and $\lim_{m\to 0} F_m(m,\ell) = +\infty \ \forall \ell > 0$. Furthermore, $f'(m) > 0$ and $f''(m) < 0$. $U : \mathbb{R}_+ \to \mathbb{R}$ is C^2, strictly increasing and strictly concave, satisfies $\lim_{c\to 0} U'(c) = \infty$, $\lim_{c\to\infty} U'(c) = 0$ and is such that $\theta_L U(0) + \beta v^{**} < \theta_L U\left(f(0)\right) + \beta v_a$ where $m^* \equiv f'^{-1}(1)$ and $v^{**} \equiv \left[\theta_H U(f(m^*)) + \beta v_a\right] / \left[\theta_H(1-\beta) + \beta\right]$. Finally, $q \in [\beta, \min\{\beta/\theta_L, 1\})$.*

[4] This statement is true under the requirement that $y^* \geq 0$. In characterizing the efficient allocation for the taste shock economy, I abstract from this constraint for simplicity. This is not affecting any of the results in the paper as this constraint is potentially binding outside the region of interest.

1.3 Efficient Allocation

In this section, I first define a (constrained) efficient allocation under the two contracting frictions. Then, I show that it solves a nearly recursive problem.

1.3.1 Definition

Private information and lack of commitment by the sovereign borrower impose constraints in addition to the resource constraint (RC) which an allocation must satisfy in order to be implementable. Consider first the restriction imposed by the fact that θ is privately observed by the borrower. By the *revelation principle*, it is without loss of generality to focus on the direct revelation mechanism in which the sovereign borrower reports his type. Define the continuation utility for the sovereign borrower associated with the allocation $\mathbf{x}$ after history s^t (according to truth-telling) as:

$$v(s^t) \equiv \sum_{j=1}^{\infty} \sum_{s^{t+j}} \beta^{j-1} \Pr(s^{t+j}|s^t)\theta_{t+j} U(c(s^{t+j})) \tag{1.9}$$

An allocation $\mathbf{x}$ is incentive compatible if and only if it satisfies the following (temporary) *incentive compatibility constraint* $\forall t, s^{t-1}, \xi_t, \theta_t, \theta'$:

$$\theta_t U(c(s^{t-1}, \xi_t, \theta_t)) + \beta v(s^{t-1}, \xi_t, \theta_t) \geq \theta_t U\left(c(s^{t-1}, \xi_t, \theta')\right) + \beta v(s^{t-1}, \xi_t, \theta') \tag{IC}$$

That is, after any history, there are no gains from the borrower reporting $\theta' \neq \theta_t$.

Second, because the sovereign borrower lacks commitment, to be implementable an allocation $\mathbf{x}$ must satisfy the following *sustainability constraint* $\forall t, s^t$:

$$\theta_t U(c(s^t)) + \beta v(s^t) \geq \theta_t U(f(m(s^{t-1}, \xi_t))) + \beta v_a \tag{SUST}$$

where v_a is the value of autarky

$$v_a \equiv \frac{\sum_{\theta \in \Theta} \mu(\theta)\theta U(f(0))}{1-\beta} = \frac{\mathbb{E}\,(\theta)\, U(f(0))}{1-\beta} \tag{1.10}$$

That is, after any history, the borrower cannot gain from increasing his consumption by failing to export $y^*(s^t)$ and living in autarky forever after. As is standard in the literature, I assume that after this observable deviation, the borrower is punished with autarky. This entails two forms of punishment. First, the sovereign borrower cannot

access credit markets to obtain insurance. Second, he suffers a loss in production because he cannot use imported intermediate goods.

A feasible allocation $\mathbf{x}$ is said to be *efficient* if it maximizes the present value of net transfers to the foreign lenders, $y^* - m$, subject to (RC), (IC), (SUST), and a participation constraint for the borrower:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \theta_t U(c(s^t)) \geq v_0 \tag{PC}$$

for some feasible initial level of promised utility $v_0 \in [v_a, \bar{v}]$, with $\bar{v} \equiv \lim_{c \to \infty} \frac{\mathbb{E}(\theta)U(c)}{1-\beta}$. An efficient allocation solves:

$$J(v_0) = \max_{\mathbf{x}} \sum_{t=0}^{\infty} \sum_{s^t} q^t \Pr(s^t) \left[y^*(s^t) - m(s^{t-1}, \xi_t) \right] \tag{J}$$

subject to (RC), (PC), (IC) and (SUST). I will refer to $J : [v_a, \bar{v}] \to \mathbb{R}$ as the *Pareto frontier*.

1.3.2 Near Recursive Formulation

The problem in (J) admits a nearly recursive formulation using the borrower's promised utility, v, as a state variable. From $t \geq 1$, an efficient allocation solves the following recursive problem for $v \in [v_a, \bar{v}]$:

$$B(v) = \max_{m(\xi), c(\xi,\theta), v'(\xi,\theta)} \int_0^1 \left\{ f\left(m(\xi)\right) - m(\xi) + \sum_{\theta \in \Theta} \mu(\theta) \left[-c(\xi,\theta) + qB(v'(\xi,\theta)) \right] \right\} d\xi \tag{P}$$

subject to

$$\int_0^1 \left\{ \sum_{\theta \in \Theta} \mu(\theta) \left[\theta U(c(\xi,\theta)) + \beta v'(\xi,\theta) \right] \right\} d\xi = v \tag{1.11}$$

$$\theta U(c(\xi,\theta)) + \beta v'(\xi,\theta) \geq \theta U(c(\xi,\theta')) + \beta v'(\xi,\theta') \quad \forall \xi, \forall \theta, \theta' \tag{1.12}$$

$$\theta U(c(\xi,\theta)) + \beta v'(\xi,\theta) \geq \theta U\left(f(m(\xi))\right) + \beta v_a \quad \forall \xi, \forall \theta \tag{1.13}$$

$$v'(\xi,\theta) \geq v_a \quad \forall \xi, \forall \theta \tag{1.14}$$

where $B(v)$ is the maximal present discounted value of net transfers, $y^* - m = f(m) - c - m$, that the foreign lenders can attain subject to the constraint that the recursive allocation delivers a value of v to the sovereign borrower (the *promise keeping constraint*),

(1.11), a recursive version of the incentive compatibility constraint, (1.12), a recursive version of the sustainability constraint, (1.13), and the fact that continuation utility must be greater than the value of autarky, (1.14). The function B traces out the *utility possibility frontier*.

At $t = 0$, for all $v_0 \in [v_a, \bar{v}]$ the problem in (J) can be expressed as

$$J(v_0) = \max_{m(\xi), c(\xi,\theta), v'(\xi,\theta)} \int_0^1 \left\{ f\left(m(\xi)\right) - m(\xi) + \sum_{\theta \in \Theta} \mu(\theta) \left[-c(\xi,\theta) + qB(v'(\xi,\theta)) \right] \right\} d\xi \tag{1.15}$$

subject to (1.12), (1.13), (1.14), and the participation constraint

$$\int_0^1 \left\{ \sum_{\theta \in \Theta} \mu(\theta) \left[\theta U(c(\xi,\theta)) + \beta v'(\xi,\theta) \right] \right\} d\xi \geq v \tag{1.16}$$

The difference between J and B is that, in B, I require that the allocation delivers *exactly* the promised utility $v \in [v_a, \bar{v}]$ to the sovereign borrower - see (1.11). This is because for $t \geq 1$ the promise keeping constraint serves to maintain incentives from previous periods. In contrast, in the definition of the Pareto frontier J, the participation constraint (1.16) requires that the sovereign borrower receives *at least* v. In many applications, this asymmetry is irrelevant because the participation constraint in (J) is binding. This is not the case here because $B(v)$ has an increasing portion, as I will later show.

The constraint set in (P) is not necessarily convex because of the presence of $U \circ f(m)$, a concave function, on the right hand side of the sustainability constraint (1.13). Thus, randomization may be optimal. The programming problem in (P) can be represented as follows:

$$B(v) = \max_{\zeta \in [0,1], v_1, v_2 \in [v_a, \bar{v}]} \zeta \hat{B}(v_1) + (1-\zeta)\hat{B}(v_2) \quad \text{s.t.} \quad \zeta v_1 + (1-\zeta) v_2 = v \tag{P'}$$

where

$$\hat{B}(v) = \max_{m, c(\theta), v'(\theta)} \sum_{\theta \in \Theta} \mu\left(\theta\right) \left[f(m) - m - c(\theta) + qB(v'(\theta)) \right] \tag{$\hat{\text{P}}$}$$

subject to

$$\sum_{\theta\in\Theta} \mu(\theta)\left[\theta U(c(\theta)) + \beta v'(\theta)\right] = v \tag{1.17}$$

$$\theta U(c(\theta)) + \beta v'(\theta) \geq \theta U\left(c(\theta')\right) + \beta v'(\theta') \quad \forall \theta, \theta' \tag{1.18}$$

$$\theta U(c(\theta)) + \beta v'(\theta) \geq \theta U(f(m)) + \beta v_a \quad \forall \theta \tag{1.19}$$

$$v'(\theta) \geq v_a \quad \forall \theta \tag{1.20}$$

$\hat{B}$ is the maximal value that the lenders can attain *without* using randomization in the current period and using the convexified value for B to evaluate the continuation value. For any $v \in [v_a, \bar{v}]$, the value of $B(v)$ can be obtained from $\hat{B}$ using (P') where, without loss of generality, the randomization is between two values. It is possible to rule out randomization as part of the efficient allocation by making an additional assumption as in Aguiar, Amador, and Gopinath (2009).

Assumption 1.2. *Define* $H : [U(f(0)), U(f(m^*))] \to \mathbb{R}$ *as* $H(\underline{u}) \equiv C(\underline{u}) - f^{-1} \circ C(\underline{u})$ *with* $C = U^{-1}$. H *is concave.*

If Assumption 2 is satisfied[5] , then randomization is not optimal, $B(v) = \hat{B}(v)$ for all $v \in [v_a, \bar{v}]$ and the solution to ($\hat{\text{P}}$) is unique and continuous in the borrower's promised utility (see the appendix for a proof). When Assumption 2 does not hold, it is not guaranteed that the maximizer of ($\hat{\text{P}}$) is unique. I will assume that this is the case. In the next section, I characterize the solution to (P) using the equivalent representation given by (P')-($\hat{\text{P}}$).

1.4 Characterization: Optimality of Ex-Post Inefficiencies

In this section, I establish that under certain conditions an efficient allocation has cyclical periods with *ex-post* inefficient outcomes which resemble a sovereign default episode in the data. First, I show that the value function for the lenders (the utility possibility frontier) has an upward sloping portion. I call this the *region with ex-post inefficiencies* because both agents could do better. Second, I show that under appropriate assumptions the process for the borrower's continuation value implied by the efficient allocation

[5] Assumption 2 is satisfied if the curvature in U and f is low.

transits with strictly positive probability to the region with *ex-post* inefficiencies. Finally, I show that if $q > \beta$, there is a unique non-degenerate limiting distribution and the region with *ex-post* inefficiencies is part of its support. Thus, cycles with *ex-post* inefficient outcomes persist in the long-run.

1.4.1 Preliminaries

Before moving to the main results of this section, I first establish some preliminary results. The constraint set in $(\hat{\mathrm{P}})$ can be simplified as follows:

Lemma 1.1. *Under Assumption 1: (i) only local upward incentive compatibility constraints bind at a solution to $(\hat{P})$; (ii) if $(m, c(\theta), v'(\theta))$ is incentive compatible and sustainable for θ_H, then it satisfies the sustainability constraint for all $\theta \in \Theta$.*

Proof 1.1. *Appendix.* □

Part (i) states that the relevant incentive compatibility constraints are the ones for which the borrower of type θ_i wants to report being of type θ_{i+1} where $\theta_{i+1} > \theta_i$. This is because the efficient allocation provides more current consumption after a higher taste shock. This result is standard; see for instance Thomas and Worrall (1990). Part (ii) states that the only relevant sustainability constraint is the one for the highest taste shock type, θ_H. This is not standard. Models with lack of commitment and no incentive problem typically display the opposite binding pattern.

The next proposition establishes three properties of the efficient allocation that I will later use.

Proposition 1.1. *Under Assumption 1, the efficient allocation is such that:*

(i) There are distortions in production. Let m^* be the statically efficient level of intermediates, i.e. m^* such that $f'(m^*) = 1$. There exists $v^* \in (v_a, \bar{v})$ such that $m(v) = m^*$ for all $v \geq v^*$, $m(v) < m^*$ for all $v \in [v_a, v^*)$, and in particular $m(v_a) = 0$. Moreover, if Assumption 2 holds, then $m(v)$ is strictly increasing in v over $[v_a, v^*]$.

(ii) The efficient allocation is dynamic: $\forall v \in [v_a, \bar{v}]$, $c(v, \theta_H) > c(v, \theta_L)$ and $v'(v, \theta_H) < v'(v, \theta_L)$.

(iii) There is subsidization across states. Let $b(v,\theta) \equiv y^*(v,\theta) - m(v) + qB(v'(v,\theta))$ *be the lenders' value after the realization of* θ*.* $\forall v \in [v_a, \bar{v}]$ *and for all* $\theta' > \theta$*,* $b(v,\theta) \geq b(v,\theta')$*. In particular,* $b(v,\theta_L) > b(v,\theta_H)$*.*

Proof 1.2. *Appendix.* □

Part (i) states that low levels of promised utility for the borrower are associated with imported intermediates that are below the statically efficient level, m^* such that $f'(m^*) = 1$. When the continuation value for the borrower is low, imports must be low to satisfy the sustainability constraint. Whenever the sustainability constraint is binding $m < m^*$. In particular, at autarky it must be that $m(v_a) = 0$. In fact, if the foreign lenders supplied any $m > 0$, the sovereign government could unilaterally achieve a life-time utility of $U(f(m)) + \beta v_a > U(f(0)) + \beta v_a = v_a$. Thus, only $m = 0$ is consistent with the promise keeping and sustainability constraints at autarky. On the other hand, for continuation values high enough, $v \geq v^*$, the threat of autarky after an observable deviation is sufficiently harsh that the statically efficient amount of intermediate imports can be supported, $m(v) = m^*$ for all $v \geq v^*$. If Assumption 2 is satisfied, it can be shown that m is actually strictly increasing in the borrower's promised value for $v \in [v_a, v^*]$. Part (ii) states that the efficient allocation is *dynamic*, in the sense that it uses variation in the borrower's continuation utility to provide incentives, thus allowing for higher consumption after the realization of a higher taste shock. Part (iii) shows that the present value of payments received by the lenders is state-contingent; there is debt relief when the borrower has a high marginal utility of consumption. Thus, the efficient allocation provides some insurance, albeit imperfect.

1.4.2 Optimality of Ex-Post Inefficiencies

I now turn to the main result of this section: an efficient allocation calls for *ex-post* inefficient outcomes with strictly positive probability, provided that a sufficient condition is satisfied.

Region with Ex-Post Inefficiencies The next proposition establishes that the utility possibility frontier is upward sloping for borrower values that are close to autarky.

Proposition 1.2. *[Region with ex-post inefficiencies]* $\exists\ \tilde{v} \in (v_a, v^*)$ *such that* $B(v)$ *is strictly increasing over* $[v_a, \tilde{v})$ *and decreasing over* $[\tilde{v}, \bar{v}]$.

Proof 1.3. *Appendix.* □

I refer to the interval $[v_a, \tilde{v})$ as the *region with ex-post inefficiencies* because for all $v \in [v_a, \tilde{v})$ the lenders can attain a strictly higher value by providing higher utility to the borrower. This is because supporting a continuation value for the borrower that is close to the autarkic level requires that a very low level of intermediate goods is employed in production so that the sustainability constraint (1.19) is satisfied. This depresses production and, consequently, the repayments that the lenders can receive in the period. In particular, when the borrower's value is close to autarky, intermediates are close to zero (see Proposition 1 part (i)). Thus, because of the Inada condition on f, the marginal return from additional intermediates is large enough that the benefit from extra production that can be obtained by increasing the borrower's continuation value is larger than the cost to the lender of providing the additional value. Therefore, both agents can be made better off from autarky and B is upward sloping in a neighborhood of v_a. In contrast, for sufficiently high promised values, $v \geq v^*$, the statically efficient level of intermediates can be supported. For such promised values, increasing the borrower's value is costly and has no benefit for the lenders and so B is strictly decreasing for $v \geq v^*$. Therefore, because of the concavity of B, the utility possibility frontier must peak at some $\tilde{v} \in (v_a, v^*)$. Over the interval $[\tilde{v}, \bar{v}]$, which I will refer to as the *efficient region*, B is decreasing.

In the rest of the paper, I will assume that randomization may only occur in the region with *ex-post* inefficiencies:

Assumption 1.3. *An efficient allocation is such that if randomization is optimal, there exists* $v_r \in (v_a, \tilde{v})$ *such that (i) for* $v \in (v_a, v_r)$*, it is optimal to randomize between* v_a *and* v_r*, and (ii) if* $v \geq v_r$ *then there is no randomization.*

Assumption 3 states that if there is a *randomization region* (linear portion of B), this is given by the set $[v_a, v_r]$ with $v_r < \tilde{v}$. In this case, for all $v \in [v_a, v_r]$, let

$$\zeta(v) = \frac{v_r - v}{v_r - v_a} \tag{1.21}$$

be the probability that continuation utility after randomization is equal to v_a. With probability $1 - \zeta(v)$ the post-randomization continuation value is equal to v_r. This pattern for randomization is what I find in any computed example.

The Efficient Allocation Transits to the Region with Ex-Post Inefficiencies Any efficient allocation starts in the efficient region because the participation constraint, (PC), in (J) can hold as an inequality. For any borrower value, v, in the region with *ex-post* inefficiencies, (PC) does not bind and $J(v) = J(\tilde{v}) = B(\tilde{v}) > B(v)$. It is optimal for the lenders to promise at least $\tilde{v}$ to the borrower. Instead, for v in the efficient region (PC) in (J) binds and $J(v) = B(v)$. These results are illustrated in Figure 2. The question now is: Does an efficient allocation transit to the region with *ex-post* inefficiencies after some history? Or is the efficient region an ergodic set?

Provided that a *sufficient* condition is satisfied, the continuation of any efficient allocation transits to the region with *ex-post* inefficiencies after a sufficiently long (but finite) string of realizations of θ_H. The essential piece of the argument is to show that following a realization of θ_H, the continuation utility is strictly lower than the current one: $v'(v, \theta_H) < v$. This is not obvious because there is a tension between two countervailing forces. First, there is an *incentive effect* that calls for lowering $v'(v, \theta_H)$ below v. This is because lowering the continuation utility after a high taste shock helps to separate types and to provide more current consumption when the marginal utility of consumption is high. Second, there is a countervailing *commitment effect*: lowering the continuation utility tightens future sustainability constraints. As is standard in economies with only lack of commitment, there is a motive to backload payments to the sovereign borrower in order to relax future sustainability constraints and allow for lower production distortions in the future.

Two point support For simplicity, in the rest of this section, I consider the case with $N = 2$, $\Theta \equiv \{\theta_L, \theta_H\}$. Let $c_H(v) = c(v, \theta_H)$, $v'_H(v) = v'(v, \theta_H)$, $c_L(v) = c(v, \theta_L)$, and $v'_L(v) = v'(v, \theta_L)$. Let $x = (c_L, c_H, v'_L, v'_H, m)$ be the solution to ($\hat{\text{P}}$) for some v in the efficient region for which randomization is not optimal, $B(v) = \hat{B}(v)$. In order to see why the continuation value after a high taste shock must be lower than the borrower's current value and to understand the tensions in the model, consider

the following variation: for some $\varepsilon \in \mathbb{R}$ sufficiently close to zero, decrease v'_H and v'_L by ε/β and increase c_H and c_L such that both $U(c_H)$ and $U(c_L)$ increase by ε. This variation satisfies the relevant incentive compatibility constraint and the promise keeping constraint under the normalization $\mathbb{E}(\theta) = 1$. Moreover, it also relaxes the relevant sustainability constraint because it delivers a higher value after the realization of θ_H. This value increases by $(\theta_H - 1)\,\varepsilon$. Therefore, the amount of intermediates can be increased by $\varepsilon_m = \varepsilon(\theta_H - 1)/\left[\theta_H U'(f(m))f'(m)\right]$. Since x is optimal, the change in the lenders' value from this variation must be equal to zero:

$$0 = \frac{\Delta B}{\varepsilon} \approx -\left[\frac{\mu_H}{U'(c_H)} + \frac{\mu_L}{U'(c_L)}\right] - \frac{q}{\beta}\left[\mu_H B'(v'_H) + \mu_L B'(v'_L)\right] + \frac{(\theta_H - 1)\,(f'(m) - 1)}{\theta_H U'(f(m))f'(m)} \tag{1.22}$$

where the first term in square brackets is the cost of providing more consumption in the current period, the second is the benefit (or cost if v'_j is in the region with *ex-post* inefficiencies and $B'(v'_j) > 0$) of reducing continuation values, and the last term is the benefit from relaxing the current sustainability constraint. Using the fact that (see Lemma 7 in the appendix for the derivation)

$$B'(v) = -\left[\frac{\mu_H}{U'(c_H)} + \frac{\mu_L}{U'(c_L)}\right] + \frac{f'(m) - 1}{U'(f(m))f'(m)} \tag{1.23}$$

the expression in (1.22) can be rearranged as follows:

$$B'(v) = \frac{q}{\beta}\left[\mu_H B'(v'_H) + \mu_L B'(v'_L)\right] + \frac{f'(m) - 1}{\theta_H U'(f(m))f'(m)} \tag{1.24}$$

or equivalently, using the fact that $B'(v) \leq 0$ and $\beta \leq q$, as:

$$\left[B'(v_H) - B'(v)\right] \geq \mu_L\left[B'(v'_H) - B'(v'_L)\right] - \frac{\beta}{q}\frac{f'(m) - 1}{\theta_H U'(f(m))f'(m)} \tag{1.25}$$

Equation (1.25) illustrates the two forces operating in the model. The first term in square brackets on the right hand side of (1.25) stands in for the incentive effect, while the second term stands in for the commitment effect. First notice that by the concavity of B, if the right hand side of (1.25) is positive, then it must be that $v'_H(v) < v$. By Proposition 1 part (ii), $v'_L > v'_H$ and thus the first term on the right hand side of (1.25) is strictly positive. Absent any commitment problem, $f'(m) = 1$, the second term on the right hand side of (1.25) is equal to zero. Therefore the right hand side is positive

and consequently $v'_H(v) < v$. When the sustainability constraint binds, $f'(m) > 1$, the second term on the right hand side of (1.25), $-[f'(m)-1]/[\theta_H U'(f(m))f'(m)]$, is negative; it is then not obvious that the right hand side of (1.25) is positive. Thus, in this case it is not guaranteed that $v'_H(v) < v$. The next assumption guarantees that this is indeed the case.

Assumption 1.4. $\Theta = \{\theta_L, \theta_H\}$ *and either (i) the difference between* θ_L *and* θ_H *is sufficiently large or (ii)* μ_H *is sufficiently small.*

Lemma 1.2. *Under Assumptions 1, 3, and 4,* $\forall v \in [\tilde{v}, \bar{v}]$ $v'_H(v) < v$.

Proof 1.4. *Appendix.* □

Suppose that either (i) the difference between θ_H and θ_L is sufficiently large or (ii) the probability of being in state θ_H is sufficiently small. Intuitively, if $\theta_H - \theta_L$ is sufficiently large, the benefit of separating the two types is large. It is very cheap to satisfy the promise keeping constraint by providing consumption when $\theta = \theta_H$. To provide a large spread in current consumption across types in an incentive compatible way, i.e. such that $\beta[v'_L - v'_H] \geq \theta_L[U(c_H) - U(c_H)]$, it is necessary to have a large spread in continuation values, $v'_L - v'_H$. Thus, the first term in the right hand side of (1.25) is large. Moreover, if μ_H is small, the cost of tightening future sustainability constraints by reducing the continuation value after θ_H is small from an *ex-ante* perspective. Inspecting (1.25), if μ_H is low, then the first term on the right hand side is again large. Thus, if either (i) $\theta_H - \theta_L$ is sufficiently large or (ii) μ_H is sufficiently small, the benefits from lowering $v'_H(v)$ below v by relaxing the incentive compatibility constraint and the current sustainability constraint (incentive effect) are larger than the costs that arise from higher production distortions in the future after a high taste shock (commitment effect). In the appendix, I show how Lemma 2 can be extended to the general case with $N \geq 2$.

Under the assumptions in Lemma 2, for all v in the efficient region, $v'_H(v)$ lies *strictly* below the 45 degree line, as illustrated in Figure 3. Let $\Delta \equiv \min_{v \in [\tilde{v}, \bar{v}]} \{v - v'_H(v)\}$. By continuity of $v'_H(v)$, it follows that $\Delta > 0$. Thus, starting from any $v_0 \in [\tilde{v}, \bar{v})$, after a sequence of t consecutive realizations of θ_H, the borrower's continuation value is less than $v_0 - \Delta t$. Thus, after a sufficiently long string $\theta^T = (\theta_H, \theta_H, ..., \theta_H)$ with $T \leq (v_0 - \tilde{v})/\Delta$

finite, the continuation utility transits to the region with *ex-post* inefficiencies, $[v_a, \tilde{v})$. The next proposition summarizes the argument above:

Proposition 1.3. *Under Assumptions 1, 3 and 4, an ex-ante efficient allocation transits to the region with ex-post inefficiencies with strictly positive probability.*

Proof 1.5. *It follows from Lemma 2 and the discussion above.* □

Role of the Main Ingredients The interaction between lack of commitment and private information is key to having *ex-post* inefficient outcomes happening along the path. Both lack of commitment and the fact that intermediates are used in production are crucial to generating an upward sloping portion of the utility possibility frontier. However, these two features alone cannot generate *ex-post* inefficient outcomes associated with an *ex-ante* efficient allocation. Without an incentive problem, any continuation of an efficient allocation is itself efficient. Thus, an efficient allocation never transits to the region with *ex-post* inefficiencies of the utility possibility frontier. See Aguiar, Amador, and Gopinath (2009) for this result in a related environment.

Private information alone generates a downward drift of the continuation utility - see Thomas and Worrall (1990) and Atkeson and Lucas (1992) - but does not generate *ex-post* inefficiencies because with commitment there is no connection between low continuation values and production in the economy. The statically efficient amount of production can always be sustained. Low continuation values for the borrower only have distributional effects in that the lenders can appropriate larger shares of total undistorted production. Also in this case, continuations of efficient allocations are always on the Pareto frontier.

Both contracting frictions are needed to obtain *ex-post* inefficient outcomes as part of the *ex-ante* optimal arrangement (Proposition 3). Lack of commitment is crucial for having an upward sloping portion of the utility possibility frontier (Proposition 2); private information is crucial for having the efficient allocation to transit to the region with *ex-post* inefficiencies (Lemma 2).[6]

[6] Notice that if preferences are of the form $U(c, g; \theta) = U(c) + \theta G(g)$ where for instance c is private consumprion and g is public consumption and θ is private information but c is observable, then in this case the efficient region is an ergodic set. If there is one lever other than the continuation utility to use in order to provide incentives, then continuations of efficient allocation are efficient.

1.4.3 Long-Run Properties

In this section, I show that if the sovereign borrower is more impatient than the foreign lenders, $\beta < q$, then any efficient allocation converges to a unique non-degenerate stationary distribution. Moreover, under the assumptions in Lemma 2, *ex-post* inefficient outcomes are part of the long-run behavior of the economy.

For expositional simplicity, I consider again the case with a two point support.[7] As a first step to establishing the existence of a non-degenerate stationary distribution, I show that under Assumption 1 autarky is a reflecting point and not an absorbing state.

Lemma 1.3. *Under Assumption 1, at $v = v_a$ after θ_H it must be that $c_H(v_a) = f(0)$ and $v'_H(v_a) = v_a$. Instead, after θ_L it must be that $c_L(v_a) < f(0)$ and $v'_L(v_a) > v_a$.*

Proof 1.6. *Appendix.* □

Lemma 3 characterizes the efficient allocation when the borrower's value is equal to autarky. If the borrower draws θ_H, his consumption is equal to production in autarky, $f(0)$, and his continuation value is equal to autarky. When θ_L is drawn, the borrower's valuation of current consumption is low. Therefore it is efficient to deliver the value of autarky, $\theta_L U(f(0)) + \beta v_a$, by providing lower consumption in the current period, $c_L(v_a) < f(0)$, and increasing the borrower's continuation value, $v'_L(v_a) > v_a$. Thus, autarky is not absorbing.

The next lemma completes the characterization of the law of motion for promised utility implied by the efficient allocation:

Lemma 1.4. *Under Assumptions 1 and 3, (i) for all $v \in [v_a, \tilde{v})$, $v'_L(v) \geq \tilde{v}$ and $v'_L(\tilde{v}) > \tilde{v}$. (ii) If $\beta = q$ then $\forall v \in (\tilde{v}, \bar{v}]$, $v'_L(v) > v$. (iii) Instead, if $\beta < q$ then there exists $\bar{v}_q \in (\tilde{v}, \bar{v})$ such that for all $v > \bar{v}_q$, $v'_L(v) < v$.*

Proof 1.7. *Appendix.* □

Part (i) states that if v is in the region with *ex-post* inefficiencies, it transits to the efficient region the first time that θ_L is drawn, $v'_L(v) \geq \tilde{v}$ for all $v \in [v_a, \tilde{v}]$. For borrower

[7] Lemma 3 and Proposition 4 also hold for the general case with $|\Theta| \geq 2$. In particular, $c(v_a, \theta) = f(0)$ and $v'(v_a, \theta) = v_a$ for all $\theta \in \Theta \backslash \{\theta_L\}$, see the appendix.

values in the efficient region, I have to consider two cases. If $\beta = q$, part (ii) establishes that $v'_L(v) > v$ for all v. This is because lenders and the sovereign borrower discount the future at the same rate and it is optimal for incentive provision to increase continuation utility after θ_L is drawn. Finally, if the borrower is more impatient than the lenders, $\beta < q$, part (iii) states that there is a threshold, $\bar{v}_q$, after which it is optimal to have $v'_L(v) < v$. The relative impatience of the borrower eventually dominates the incentive benefits from backloading payments after θ_L.

Under Assumption 4, the laws of motion for the continuation utility, v'_L and v'_H, are shown in Figure 3 for the case $\beta < q$. They define a unique ergodic set for promised utility. The following proposition establishes this result:

Proposition 1.4. *Under Assumptions 1, 3, and 4, if $\beta < q$, then any efficient allocation converges to a unique non-degenerate stationary distribution, Ψ^*. Moreover, $[v_a, \tilde{v}] \cap Supp\Psi^* \neq \emptyset$.*

Proof 1.8. *Appendix.* □

If $\beta < q$ then, by Lemma 4, continuation utility does not grow without bound. Moreover, by Lemma 2, after a sufficiently long - and finite - string of draws of θ_H, continuation utility transits to the region with *ex-post* inefficiencies. The region with *ex-post* inefficiencies - and the value of autarky in particular - is not an absorbing state. Lemmas 3 and 4 imply that whenever θ_L is drawn, then the continuation is back in the efficient region. Thus, there is sufficient "mixing" that the existence of a unique limiting distribution is guaranteed.

The limiting distribution has perpetual cycles that transit in and out of the region with *ex-post* inefficiencies. This feature differentiates my environment from related dynamic contracting problems such as Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), and Hopenhayn and Werning (2008) which also have *ex-post* inefficiencies along the path. In all of these papers, in the long-run either the incentive problem disappears or there is an inefficient termination of the venture between the principal and the agent. In contrast, here the incentive problem does not disappear in the long-run and there is no termination of the risk-sharing relationship. The optimal allocation has periods of temporary autarky, but cooperation eventually restarts after the domestic economy recovers. This is because

the sovereign borrower is the owner of the domestic production technology that can be operated also in autarky.

When $\beta = q$, I cannot establish that a stationary distribution exists without imposing an exogenous upper bound on consumption. Preliminary results suggest that, in this case, it is still true that *ex-post* inefficiencies persist in the long-run.

Summing up, in this section I showed that *ex-post* inefficiencies are part of the *ex-ante* efficient arrangement for the economy I consider. In the next section, I provide an implementation of the efficient allocation, associating defaults with these *ex-post* inefficient outcomes.

1.5 Implementation with Non-Contingent Debt and Maturity Management

In this section, I show that any efficient allocation can be implemented as a *sustainable equilibrium* outcome of a sovereign debt game where the set of securities available to the sovereign borrower is restricted to *non-contingent defaultable bonds of multiple maturities*. A *default* is defined as an episode in which the sovereign borrower makes a lower payment than what is specified in the bond contract. Along the equilibrium outcome path that supports an efficient allocation, defaults and periods of temporary exclusion from international credit markets occur at the same time as the inefficient outcomes.

There are several ways one could implement the efficient allocation. For instance, I could assume that the sovereign government can issue securities contingent on its report about the state of the economy as in Sleet (2004) and Sleet and Yeltekin (2006). Alternatively, I could consider one period debt which is nominally non-contingent but it is understood that the sovereign will not repay the full face value of the debt after certain shocks. Partial repayments introduce *de facto* implicit state contingencies in the bond contract. This is what Grossman and Van Huyck (1988) term *excusable default.*

The specific elements that I choose are motivated by three key facts about sovereign debt. First, in the data, the vast majority of sovereign and external debt comes in the form of non-contingent debt (see Rogoff (2011) for a discussion). Second, default episodes are infrequent events. Third, defaults happen when the sovereign is highly indebted. My implementation is consistent with these three facts: only non-contingent

debt is available and there are recurrent but infrequent excusable defaults on path only when the sovereign's continuation value is low (in the region with *ex-post* inefficiencies). In all the other periods, I replicate the state contingent returns implied by the efficient allocation by exploiting the variation in the price of long-term debt, which is determined by default probabilities, after the realization of a shock. This allows me to derive implications for the optimal maturity structure of sovereign debt.

In the rest of this section, I first describe the sovereign debt game and define a sustainable equilibrium. Next, I show how I can support any efficient allocation as a sustainable equilibrium outcome. Finally, I show that the equilibrium outcome path for bond holdings and prices is qualitatively consistent with the evidence.

1.5.1 Sovereign Debt Game

Consider a game between competitive (non-strategic) foreign lenders (bond holders and exporters of the intermediate good), domestic firms, and a benevolent domestic government (the only strategic player). The sovereign government can issue two types of *non-contingent defaultable* bonds: *a one period bond*, b_S, (or foreign reserves if $b_S < 0$) and a *consol*, $b_L \geq 0$. One unit of the one period bond promises to pay one unit of the final good tomorrow in exchange for q_S units of the final good today. The consol is a perpetuity that promises to pay a coupon of one unit of the final consumption good in every period starting tomorrow in exchange for q_L units of the final good today.

The government (borrower) cannot commit to satisfy the terms of the bond contracts. It has the option to *default*: to pay less than what is contractually specified. In particular, the borrower can choose any level of repayment from a set $\mathbf{r} = \{1, r_1, r_2, ..., r_k, 0\} \subset [0, 1]$. Let the repayment decision of the borrower at time t be $\delta_t \in \mathbf{r}$. If $\delta_t = 1$, there is no default and the borrower repays in full both the one period debt and the coupon payment for the consol. If $\delta_t = r_k \in (0, 1)$, the holders of the one-period bond receive r_k units of the tradable final good, while holders of the consol receive $r_k/(1-q)$ units of the final good per unit of debt and the borrower has no further obligations.[8] I will refer to $r_k \in (0, 1)$ as the recovery rate. Finally, if $\delta_t = 0$

[8] Alternatively, I could have assumed that if r_k is chosen the holders of the consol receive a coupon payment of r_k and r_k units of the newly issued consol. This alternative specification does not alter any result.

there is no repayment in the current period. In this case, the borrower cannot access international credit markets. In the next period, he can choose a repayment policy δ_{t+1} for the notional amount of today's debt obligations. Interest payments are forgiven.

In defining an efficient allocation for the economy, I assumed that observable deviations can be punished with autarky forever after. See the definition of the sustainability constraint (SUST). Consistent with this, I assume that lenders can deny access to savings to the borrower. This assumption is common in the literature; see Atkeson (1991) and Aguiar, Amador, and Gopinath (2009), among others.

Furthermore, the government can tax the payments made by domestic firms to foreign exporters for the intermediate goods at a rate $\tau_t \in [0, 1]$. Thus, foreign exporters receive an after tax payment of $p_t(1 - \tau_t)$ per unit of intermediate good sold, where p_t is the price of the intermediates in terms of the final good. The role of this tax is to ensure that private agents choose imports that are consistent with the sustainability constraint. This is related to the necessity of capital income taxes in the implementation for the efficient allocation in economy with lack of commitment in Kehoe and Perri (2004) and Aguiar, Amador, and Gopinath (2009).

Informally, the sequence of events within the period is the following:

1. The public randomization device $\xi_t \in [0, 1]$ is realized;
2. Foreign lenders set a price for intermediate inputs p_t;
3. Domestic competitive firms choose m_t;
4. θ_t is realized and privately observed by the domestic government;
5. The government picks a policy $\pi_t = (\delta_t, \mathbf{b}_{t+1}, \tau_t)$ that consists of a repayment rule δ_t, new bond holdings, $\mathbf{b}_{t+1} = (b_{S,t+1}, b_{L,t+1})$ if $\delta_t \neq 0$ (if $\delta_t = 0$ then $\mathbf{b}_{t+1} = \mathbf{b}_t$) and a tariff on imported intermediates, τ_t ;
6. Bond prices $\mathbf{q}_t = (q_{S,t}, q_{L,t})$ are consistent with foreign lenders' optimality.

The equilibrium concept I use is an extension to an environment with private information[9] of the *sustainable equilibrium* (SE) concept developed in Chari and Kehoe

[9] See Sleet (2004) and Sleet and Yeltekin (2006) for a similar extension to a macro-policy game with private information.

(1990). Formally, for all $t \geq 0$ let $h^t = (h^{t-1}, \xi_t, p_t, m_t, \pi_t)$ be a public history up to period t and let $h^{-1} = \mathbf{b}_0 = (b_{S,0}, b_{L,0})$ be the initial outstanding debt. It is also convenient to define the following public histories when agents take action: $h_p^t = (h^{t-1}, \xi_t)$, $h_m^t = (h^{t-1}, \xi_t, p_t)$, $h_\sigma^t = (h^{t-1}, \xi_t, p_t, m_t)$, and let H_p^t, H_m^t and H_σ^t be the space of all possible such histories. The price of the intermediate good, p, the allocation rule for m, the strategy for the government, σ, and the price of bonds, $\mathbf{q}$, can be written as:

$$p = \{p_t\}_{t=0}^{\infty}, p_t : H_p^t \to \mathbb{R}_+ \tag{1.26}$$

$$m = \{m_t\}_{t=0}^{\infty}, m_t : H_m^t \to \mathbb{R}_+ \tag{1.27}$$

$$\sigma = \{\sigma_t\}_{t=0}^{\infty}, \sigma_t = (\delta_t, \mathbf{b}_{t+1}, \tau_t) : H_\sigma^t \times \Theta \to \mathbf{r} \times (\mathbb{R} \times \mathbb{R}_+) \times [0,1] \tag{1.28}$$

$$\mathbf{q} = \{q_{S,t}, q_{L,t}\}_{t=0}^{\infty}, q_{S,t}, q_{L,t} : H^t \to \mathbb{R}_+ \tag{1.29}$$

Problem of the Government Taking as given p, m, and the price schedule for bonds, $\mathbf{q}$, after any history $(h_\sigma^t, \theta) \in H_\sigma^t \times \Theta$, the strategy for the government, σ, solves the following problem:

$$W(h_\sigma^t, \theta) = \max_{c, \pi = (\delta, b'_{ST}, b'_{LT}, \tau)} \theta U(c) + \beta \mathbb{E}\left[W(h_\sigma^{t+1}, \theta_{t+1}) | h_\sigma^t, g\right] \tag{1.30}$$

subject to, if there is no default (i.e. $\delta = 1$)

$$c + (b_{S,t} + b_{L,t}) \leq y(\tau) + q_{S,t}(h_\sigma^t, \pi) b'_S + q_{L,t}(h_\sigma^t, \pi)(b'_L - b_{L,t}) \tag{1.31}$$

or, if there is partial repayment (i.e. $\delta = r_k$)

$$c + \left(b_{S,t} + \frac{b_{L,t}}{1-q}\right) r_k \leq y(\tau) + q_{S,t}(h_\sigma^t, \pi) b'_S + q_{L,t}(h_\sigma^t, \pi) b'_L \tag{1.32}$$

or, if there is default without any partial repayment (i.e. $\delta = 0$)

$$c \leq y(\tau) \quad \text{and} \quad \left(b'_S, b'_L\right) = (b_{S,t}, b_{L,t}) \tag{1.33}$$

where $y(\tau)$ is the amount of resources that are available to the sovereign borrower after production, repayments of intermediates, and the collection of the tariff revenue:

$$y(\tau) = F(m_t, 1) - p_t m_t + \tau p_t m_t \tag{1.34}$$

In (1.33), I impose the restriction that after $\delta_t = 0$ there is a temporary exclusion from international credit markets.[10]

[10] In the background, as in Amador, Aguiar, and Gopinath (2009), the stand-in domestic household supplies labor inelastically and receives lump sum transfers (or taxes if negative), LS_t, from the government. His budget constraint is $c_t = w_t + LS_t$, where $w_t = F_\ell(m_t, 1)$ is the competitive wage rate.

Bond Prices and Other Equilibrium Objects The equilibrium prices, p and $\mathbf{q}$, and the allocation rule, m, must satisfy the following conditions. The price of the imported intermediate, $p_t : H_p^t \to \mathbb{R}_+$, must be consistent with optimization by competitive foreign lenders that take the tariff level as given:

$$1 = \mathbb{E}\left[p_t(h_p^t)\left(1 - \tau_t(h_\sigma^t, \theta_t)\right)|h_p^t\right] \tag{1.35}$$

The allocation rule for the quantity of foreign intermediate goods, $m_t : H_m^t \to \mathbb{R}_+$, satisfies the optimality condition for the representative domestic competitive firm

$$F_m(m_t(h_m^t), 1) = p_t(h_p^t) \tag{1.36}$$

Finally, bond prices $q_{S,t}, q_{L,t} : H^t \to \mathbb{R}_+$ are consistent with the maximization problem of the risk-neutral foreign lenders that discount the future at a rate q given the government repayment policy. For the one period bond, if $b_{S,t+1} \geq 0$, it must be that

$$q_{S,t}(h^t) = q\mathbb{E}\left[\chi_{S,t+1}(h^{t+1})|h^t\right] \tag{1.37}$$

where $\chi_{S,t+1}$ is the *ex-post value of short-term debt*:

$$\chi_{S,t+1}(h^{t+1}) = \begin{cases} 1 & \text{if } \delta_{t+1} = 1 \\ r_k & \text{if } \delta_{t+1} = r_k \\ q\mathbb{E}\left[\chi_{S,t+2}(h^{t+2})|h^{t+1}\right] & \text{if } \delta_{t+1} = 0 \end{cases} \tag{1.38}$$

When $\delta_{t+1} = 0$, $q\mathbb{E}\left[\chi_{S,t+2}(h^{t+2})|h^{t+1}\right]$ can be interpreted as the secondary market value of defaulted debt. If instead $b_{S,t+1} < 0$, $q_{S,t}(h^t)$ can take on two values. If the government can save abroad, $q_{S,t}(h^t) = q$. Instead, if the government cannot save, I adopt the convention that $q_{S,t}(h^t) = \infty$. Finally, the price for the consol must be such that

$$q_{L,t}(h^t) = q\mathbb{E}\left[\chi_{L,t+1}(h^{t+1})|h^t\right] \tag{1.39}$$

where $\chi_{L,t+1}$ is the *ex-post value of the consol* given by

$$\chi_{L,t+1}(h^{t+1}) = \begin{cases} 1 + q_{L,t+1}(h^{t+1}) & \text{if } \delta_{t+1} = 1 \\ \frac{r_k}{1-q} & \text{if } \delta_{t+1} = r_k \\ q\mathbb{E}\left[\chi_{L,t+2}(h^{t+2})|h^{t+1}\right] & \text{if } \delta_{t+1} = 0 \end{cases} \tag{1.40}$$

(1.31)-(1.33) represent the combined budget constraints of the benevolent government and the stand-in household.

Equilibrium Definition Given a set of recovery rates $\mathbf{r}$ and initial outstanding debt $\mathbf{b}_0$, a *sustainable equilibrium* (SE) is a strategy for the government, σ, a price rule for the foreign intermediate good, p, price rules for the government bonds, q_S and q_L, and an allocation rule for the intermediate good, m, such that p, m and q_S and q_L satisfy (1.35), (1.36), (1.37) and (1.39) given σ, and $\forall(h^t_\sigma, \theta)$, σ is a solution to (1.30) taking p, m, q_S and q_L as given. The associated equilibrium outcome path is denoted by $\mathbf{y} = (\mathbf{x}, \pi, \mathbf{p})$ where $\mathbf{x} = \left\{m(s^{t-1}, \xi_t), c(s^t)\right\}_{t=0}^{\infty}$, $\pi = \left\{\delta(s^t), b_L(s^t), b_S(s^t), \tau(s^t)\right\}_{t=0}^{\infty}$ and $\mathbf{p} = \left\{p(s^{t-1}, \xi_t), \mathbf{q}(s^t)\right\}_{t=0}^{\infty}$.

1.5.2 Implementation

Suppose that $\Theta = \{\theta_L, \theta_H\}$ and $\beta < q$ so that the borrower's value is bounded in the long-run by $\bar{v}_q < \bar{v}$. In the rest of this section, I show that an efficient allocation $\mathbf{x}$ can be implemented as a sustainable equilibrium outcome of the sovereign debt game.[11] Assume that $\mathbf{x}$ satisfies Assumption 3 and the following properties:

Assumption 1.5. *(i) For all v, $v'_H(v) < v$ and in particular, there exists a $\underline{v} \in (v_a, \bar{v})$ such that $v'_H(v) = v_a$ for all $v \leq \underline{v}$. If randomization is optimal, $\underline{v} < v_r$. (ii) $v'_L(v)$ is strictly increasing and $v'_H(v)$ is strictly increasing for all $v \geq \underline{v}$.*

Part (i) of the assumption implies that starting from any v there is a strictly positive probability of reaching autarky. That is, starting from any $v \in [v_a, \bar{v})$ after a sufficiently long - but finite - string of high taste shocks, the continuation value is equal to v_a. Part (ii) requires that v'_L and v'_H are monotone increasing. Both (i) and (ii) are satisfied in my simulations (see Figure 3 for an example). Part (i) of Assumption 5 can be dispensed with, but it allows for an easier exposition of the proof for the next proposition:

Proposition 1.5. *Assume that Assumptions 1 and 3 hold, $\Theta = \{\theta_L, \theta_H\}$, and $\beta < q$. If $\mathbf{x}$ is an efficient allocation that satisfies Assumption 5, then there exist a set of recovery rates, $\mathbf{r}$, an initial debt position, $\mathbf{b}_0$, a strategy for the benevolent government, σ, prices, p and $\mathbf{q}$, and an allocation rule, m, such that: (i) $(\sigma, p, m, \mathbf{q})$ is a SE given $\mathbf{r}$ and $\mathbf{b}_0$, and (ii) $\mathbf{x}$ is the real allocation associated with the equilibrium outcome path.*

[11] If $\mathbf{y}$ is an equilibrium outcome, its associated real allocation satisfies (IC) and (SUST). Thus, given an initial level of indebtedness, $B_0 = q_{ST0}b_{ST0} + q_{LT0}b_{LT0}$, the government's *ex-ante* value of any equilibrium outcome is bounded from above by the value associated with the efficient allocation that delivers B_0 to the lenders.

The proof of the proposition consists of two main steps. First, I construct the *on-path* default rule, bond holdings, tariffs, and prices that support the efficient allocation. Second, I show that I can find out-of-path behavior that prevents deviation from the constructed plan.

Mapping Between Efficient Allocation and Equilibrium Objects on Path I now construct the candidate equilibrium outcome path that implements an efficient allocation $\mathbf{x}$. Since the efficient allocation can be represented by a time-invariant function of borrower's continuation utility and exogenous shocks, the *on-path* repayment rule, bond holdings, tariffs, and prices can also be expressed as a function of on-path continuation utility for the borrower. In particular, the repayment policy, tariff, and intermediate prices are functions of the post-randomization value:

$$\bar{\delta} : [v_a, \bar{v}] \times \Theta \to \mathbf{r} \quad \text{and} \quad \bar{\tau}, \bar{p} : [v_a, \bar{v}] \to \mathbb{R} \tag{1.41}$$

Bond holdings and prices are functions of the continuation value (for the next period):

$$\bar{q}_S, \bar{q}_L, \bar{b}_S, \bar{b}_L : [v_a, \bar{v}] \to \mathbb{R} \tag{1.42}$$

An outcome path $\mathbf{y}$ can be recovered in the natural way from (1.41), (1.42), and the law of motion for v from the efficient allocation.

The steps to construct the candidate equilibrium outcome path $\mathbf{y}$ from an efficient allocation $\mathbf{x}$ are: (i) define the repayment policy; (ii) use the repayment policy in the optimality conditions for the foreign lenders to calculate equilibrium bond prices; (iii) choose short and long term debt to match the total value of debt (lenders' value) after a realization of θ implied by the efficient allocation, i.e. for $v \geq v_r$

$$b(v, \theta) = f(m(v)) - c(v, \theta) - m(v) + qB(v'(v, \theta)) \tag{1.43}$$

and finally (iv) use the optimality conditions for the domestic firms and the lenders to get tariffs and prices for the intermediate good.

Consider first the repayment policy which is consistent with the fact that defaults are infrequent and they happen only when the borrower is highly indebted. The borrower defaults only when his continuation value post-randomization is in $[v_a, v_r]$. If it is not

optimal to randomize, let $v_r = v_a$. For all the other borrower values, there is full repayment. That is:

$$\bar{\delta}(v,\theta) = 1 \quad \text{if } v \in (v_r, \bar{v}] \text{ for all } \theta \tag{1.44}$$

$$\bar{\delta}(v,\theta) < 1 \quad \text{if } v \in [v_a, v_r] \text{ for all } \theta \tag{1.45}$$

I will refer to $[v_a, v_r]$ as the *default region* and to $(v_r, \bar{v}]$ as the *no-default region*. If randomization is optimal, by Assumption 3, for pre-randomization value $v \in [v_a, v_r]$, the post-randomization value is equal to either v_a (with probability $\zeta(v)$) or v_r (with probability $1 - \zeta(v)$). Thus, there are four relevant outcomes for the repayment policy in the default region:

$$\bar{\delta}(v,\theta) = \begin{cases} 0 & \text{if } v = v_a \text{ and } \theta = \theta_H \\ r_{rH} & \text{if } v = v_r \text{ and } \theta = \theta_H \\ r_{aL} & \text{if } v = v_a \text{ and } \theta = \theta_L \\ r_{rL} & \text{if } v = v_r \text{ and } \theta = \theta_L \end{cases} \tag{1.46}$$

where $\bar{\delta}(v_a, \theta_H) = 0$ because from Lemma 3 it follows that, when the borrower's value is autarky, there are no capital flows: $m(v_a) = 0$ and $c(v_a, \theta_H) = f(0)$. It is worth noting that when $v = v_r$ and $\theta = \theta_H$ then there is a partial repayment today (r_{rH} is generally greater than zero) and there will be again less than full repayment the next period because $v'_H(v_r) < v_r$. I interpret this as a unique protracted default episode.[12]

The borrower is out of the default region the next period only after he draws θ_L ($v'_L(v) \geq \tilde{v} > v_r$ for all v).

Given the repayment policy, bond prices are uniquely pinned down by the lenders' optimality conditions. The price for short-term debt is given by:

$$\bar{q}_S(v) = \begin{cases} q & \text{if } v \in (v_r, \bar{v}] \\ q\bar{R}(v) & \text{if } v \in [v_a, v_r] \end{cases} \tag{1.47}$$

where $\bar{R}(v)$ is the expected recovery rate in the default region:

$$\bar{R}(v) = \zeta(v)\frac{\mu(\theta_L) r_{aL}}{1 - q\mu(\theta_H)} + [1 - \zeta(v)]\,[\mu(\theta_L) r_{rL} + \mu(\theta_H) r_{rH}] \tag{1.48}$$

[12] This is consistent with the fact that there are repeated restructurings, see Cruces and Trebesch (2012).

The price for long-term debt can be written recursively as:

$$\bar{q}_L(v) = \begin{cases} q \sum_{i=L,H} \mu(\theta_i) \left[1 + \bar{q}_L(v_i'(v))\right] & \text{if } v \in (v_r, \bar{v}] \\ \frac{q}{1-q} \bar{R}(v) & \text{if } v \in [v_a, v_r] \end{cases} \tag{1.49}$$

Outside of the default region, i.e. for all $v \in (v_r, \bar{v}]$, the price of short-term debt is equal to that of a risk-free bond. Instead, the price of long-term debt is lower than the price of a risk-free consol because there is always a positive probability that there will be a default over the relevant time horizon of the bond. The next lemma shows that $\bar{q}_{LT}$ is strictly increasing in the continuation value for the borrower.

Lemma 1.5. *Under the assumptions in Proposition 5, for a given* $\mathbf{r} = \{1, r_{rL}, r_{aL}, r_{rH}, 0\}$ *with* $r_{rL}, r_{aL}, r_{rH} \in (0,1)$, $\bar{q}_L : [v_a, \bar{v}] \to \mathbb{R}$ *is the unique fixed point of the contraction mapping defined by the right hand side of (1.49) and is strictly increasing.*

Proof 1.9. *Appendix.* □

Given the functions for bond prices $\bar{q}_S$ and $\bar{q}_L$, in the no-default region $\bar{b}_S(v)$ and $\bar{b}_L(v)$ are chosen to match the total value of debt (lenders' value) implied by the efficient allocation after θ_L and θ_H defined in (1.43):

$$b(v, \theta_L) = \bar{b}_S(v) + \bar{b}_L(v) \left[1 + \bar{q}_L(v_L'(v))\right] \tag{1.50}$$

$$b(v, \theta_H) = \bar{b}_S(v) + \bar{b}_L(v) \left[1 + \bar{q}_L(v_H'(v))\right] \tag{1.51}$$

A (unique) solution to (1.50)-(1.51) is guaranteed by the fact that $\bar{q}_L$ is strictly increasing and $v_H'(v) < v_L'(v)$, see Proposition 1 part (ii). Therefore $\bar{q}_L\left(v_H'(v)\right) < \bar{q}_L\left(v_L'(v)\right)$. Thus, outside of the default region the maturity composition of debt is uniquely pinned down. Simple algebra shows that:

$$\bar{b}_L(v) = \frac{b(v, \theta_L) - b(v, \theta_H)}{\bar{q}_L(v_L'(v)) - \bar{q}_L(v_H'(v))} \tag{1.52}$$

$$\bar{b}_S(v) = b(v, \theta_L) - \bar{b}_L(v) \left[1 + \bar{q}_L(v_L'(v))\right] \tag{1.53}$$

Notice that it is guaranteed that $\bar{b}_L(v) > 0$ because, by Proposition 1 part (iii), $b(v, \theta_L) - b(v, \theta_H) > 0$ and, as shown above, $\bar{q}_L(v_L'(v)) - \bar{q}_L(v_H'(v)) > 0$. Intuitively, given the *ex-post* variation in the price of long-term debt, $\bar{q}_L(v_L'(v)) - \bar{q}_L(v_H'(v))$, the long-term debt is chosen to replicate the amount of insurance, $b(v, \theta_L) - b(v, \theta_H)$, implied by the

efficient allocation. Instead the short-term debt holdings can be thought as being chosen to match the total value of debt.

In the default region, bond holdings are constant. For all $v \in [v_a, v_r]$, $\bar{b}_S(v) = \bar{b}_{Sr}$ and $\bar{b}_L(v) = \bar{b}_{Lr}$, and it must be that:

$$b(v_r, \theta_L) = r_{rL}\left[\bar{b}_{Sr} + \frac{\bar{b}_{Lr}}{1-q}\right] \tag{1.54}$$

$$b(v_r, \theta_H) = r_{rH}\left[\bar{b}_{Sr} + \frac{\bar{b}_{Lr}}{1-q}\right] \tag{1.55}$$

$$b(v_a, \theta_L) = r_{aL}\left[\bar{b}_{Sr} + \frac{\bar{b}_{Lr}}{1-q}\right] \tag{1.56}$$

The other possible outcome follows from (1.56) and $\delta(v_a, \theta_H) = 0$ because

$$b(v_a, \theta_H) = q\left[\mu(\theta_L)b(v_a, \theta_L) + \mu(\theta_H)b(v_a, \theta_H)\right] = \frac{q\mu(\theta_L)}{1 - q\mu(\theta_H)}b(v_a, \theta_L) \tag{1.57}$$

Thus, fixing any $r_{rL} \in (0, 1)$, it must be that

$$r_{rH} = \frac{b(v_r, \theta_H)}{b(v_r, \theta_L)}r_{rL} \text{ and } r_{aL} = \frac{b(v_a, \theta_L)}{b(v_r, \theta_L)}r_{rL} \tag{1.58}$$

If the restrictions in (1.58) are satisfied, I can choose any $\left(\bar{b}_{Sr}, \bar{b}_{Lr}\right)$ that satisfies (1.54). Consequently (1.55)-(1.57) will also be satisfied. The split between long and short term debt is indeterminate because the two are perfect substitutes if there is default for sure in the next period. I resolve this indeterminacy by assuming that $\bar{b}_{Sr}/\bar{b}_{Lr} = \lim_{v \to v_r} \bar{b}_S(v)/\bar{b}_L(v)$. The recovery rate $r_{rL} \in (0, 1)$ is a free-parameter. It can be chosen sufficiently low that $\bar{b}_S$ is strictly positive in the default region and for v close to v_r so that a non-full repayment has a natural interpretation.

Finally, I construct the on-path tariff rates and prices for the intermediate good, $\bar{\tau}, \bar{p}$, as follows:

$$f'(m(v)) = \frac{1}{1 - \bar{\tau}(v)} = \bar{p}(v) \tag{1.59}$$

Then the outcome path $\mathbf{y}$ constructed from the efficient allocation $\mathbf{x}$ using (1.44), (1.46), (1.47), (1.49), (1.52), (1.53), (1.54) and (1.59) satisfies the optimality conditions (1.35) and (1.36), and the equilibrium bond pricing equations (1.37) and (1.39). Moreover, it supports the level of consumption implied by the efficient allocation.

Deterring Deviation To complete the proof of Proposition 5, I need to verify that for any on-path history, the government does not have a strict incentive to deviate. A simple way to proceed is to consider a trigger strategy that reverts to autarky after any deviation by the government. If the government does not follow the prescription of $\bar{\delta}$, $\bar{b}$, and $\bar{\tau}$, it faces future bond prices equal to zero, cannot save, and receives no foreign intermediate goods. Zero intermediates and a price equal to zero for both short-term and long-term-debt can be supported as part of a sustainable equilibrium because if foreign lenders expect a tariff equal to 100 percent (full expropriation) and full default ($\delta = 0$) in any subsequent periods irrespective of the action chosen today by the government, then the government has no incentive to choose something different than $\tau = 1$ and $\delta = 0$, confirming the lenders' beliefs. The fact that the sovereign borrower cannot save after a deviation follows from an assumption. Therefore, the value of any deviation is equal to the value of the static deviation plus the continuation value associated with autarky. This value is bounded from above by $\theta U(f(m)) + \beta v_a$. Since the efficient allocation satisfies the sustainability constraint (SUST), the government has no strict incentive to deviate. This concludes the proof of Proposition 5.

Using reversion to autarky after any deviation to support the efficient allocation as an equilibrium outcome is *not* necessary. In particular, it is possible to find a history dependent pricing function $\mathbf{q}_t(h^t_\sigma, \cdot)$ that is continuous in new debt issuance $\mathbf{b}'$ for all (δ, τ). However, to support the efficient allocation, some degree of history dependence is needed. The efficient allocation cannot be supported by a Markovian equilibrium of the kind typically considered in the quantitative sovereign default literature. The pricing functions for bonds need to depend on history, not only on bonds issued today.

Proposition 5 can be generalized to the case with $|\Theta| = N \geq 2$, allowing for a richer maturity structure. For instance, I can use N types of the perpetuity introduced in Hatchondo and Martinez (2009) that pay a coupon that decays exponentially at rate $\alpha_n \in [0, 1]$. The one-period bond and the consol are special cases of this class of securities for α equal to 1 and 0, respectively. Provided that the return matrix satisfies a full-rank condition[13] (which is authomatically satisfied when $N = 2$), the statement

[13] For $\alpha_i \in \{\alpha_1 = 0, \alpha_2, ..., \alpha_N = 1\}$, define $\bar{q}_{\alpha_i}$ in a similar way as in (1.49):

$$\bar{q}_{\alpha_i}(v') = \begin{cases} q \sum_\theta \mu(\theta) \left[1 + (1 - \alpha_i)\bar{q}_{\alpha_i}(v'(v, \theta))\right] & \text{if } v \in (v_r, \bar{v}] \\ q \frac{\bar{R}(v)}{1-(1-\alpha_i)q} & \text{if } v \in [v_a, v_r] \end{cases}$$

in Proposition 5 generalizes to the case with $N > 2$.

The proposed implementation works for environments other than the one considered here. For instance, if $\beta < q$ and the realization of θ_t is public information[14] - as in the economy considered in Aguiar, Amador, and Gopinath (2009) - or if θ_t is persistent, the same logic can be applied.

Mechanism that replicates state-contingent returns To prove that the proposed implementation works, the crucial step was to show that it is possible to replicate the insurance provided by the efficient allocation (where by insurance I mean the fact that the NPV of net exports – total value of debt – falls after an adverse shock relative to a positive shock). When there is default, insurance is provided through partial repayments that make the non-contingent debt de facto state-contingent. The occurrence of default in equilibrium induces the prices of long-term debt to fluctuate. When there is no default, insurance is provided through capital gains/losses imposed on lenders (long-term debt holders).

When there is full repayment, the fall in the value of debt after the realization of a high taste shock is obtained by imposing a capital loss on the holders of outstanding long-term debt or, in the terminology of Chatterjee and Eyigungor (2012a, 2012b), by *diluting outstanding long-term debt.* After a high taste shock, the continuation value for the borrower decreases, the overall level of indebtedness increases, and the probability that there will be a default in the near future increases. This increase in the likelihood of a future default reduces the value of the outstanding long-term debt, resulting in a capital loss for the debt holders and a capital gain for the borrower. This capital loss on the debt holders after an adverse shock mimics the debt relief for the borrower associated with the efficient allocation[15] . Thus, debt dilution is not a negative feature

Then, if the return matrix

$$\underset{N\times N}{\bar{Q}(v)} \equiv \begin{bmatrix} 1+q_{\alpha_1}(v'(v,\theta_1)) & ... & 1+q_{\alpha_N}(v'(v,\theta_1)) \\ 1+q_{\alpha_1}(v'(v,\theta_2)) & ... & 1+q_{\alpha_N}(v'(v,\theta_2)) \\ ... & ... & ... \\ 1+q_{\alpha_1}(v'(v,\theta_N)) & ... & 1+q_{\alpha_N}(v'(v,\theta_N)) \end{bmatrix}$$

is invertible, then there exists a $\bar{b}(v) = [\bar{b}_{\alpha_1}, ..., \bar{b}_{\alpha_N}]^T$ that solves the analogue of (1.50)-(1.51) given $\bar{Q}$.

[14] In this case, defaults will not be associated with *ex-post* inefficiencies.

[15] This is consistent with the evidence in Berndt, Lustig and Yeltekin (2011) who document how the fall in the value of long-term debt provides fiscal insurance to the U.S. government.

of the equilibrium outcome path.

1.5.3 Characterization of Equilibrium Outcome Path: Debt Holdings and Spread

The bond holdings and the prices that support the efficient allocation are qualitatively consistent with two features of the data documented in Broner, Lorenzoni, and Schmukler (2010) and Arellano and Ramanarayanan (2012) for emerging markets. First, when interest rate spreads are low, long term spreads are generally higher than short term spreads. During debt crises, the gap between long and short-term spreads tends to narrow and the term spread curve flattens or even inverts. Second, during emerging market debt crises, the debt maturity shortens. An equilibrium outcome path that supports an efficient allocation shares these features of the data.

Implications for Interest Rate Spreads Define the *short-term spread* as the difference between the interest rate implied by q_S and the risk-free international interest rate: $s_S \equiv 1/q_S - 1/q$. The *long-term spread* is defined as the difference between the consol's yield to maturity[16] and the risk-free interest rate: $s_L \equiv [1 + q_L]/q_L - 1/q$. The *term premium* is the difference between the long and the short term spreads: $s_T \equiv s_L - s_S$.

Outside the default region the short term debt is risk-free, see (1.47). Thus, $q_{S,t} = q$ and $s_{S,t} = 0$. Instead, from Lemma 5 it follows that $q_{L,t} < \frac{q}{1-q}$. Therefore the short term spread is zero while the long-term spread is positive. Consequently, the term spread s_T is positive. When the borrower's continuation value is in the default region, the term spread is given by:

$$\begin{aligned} s_T(v) &= \left(1 + \frac{1}{\bar{q}_L(v)}\right) - \frac{1}{\bar{q}_S(v)} = \left(1 + \frac{1-q}{q\bar{R}(v)}\right) - \frac{1}{q\bar{R}(v)} \\ &= \frac{1 - q(1 - \bar{R}(v))}{q\bar{R}(v)} - \frac{1}{q\bar{R}(v)} < 0 \end{aligned} \tag{1.60}$$

Thus the spread for the short-term debt is higher than the long-term spread and the term structure is inverted. This behavior for the term spread is consistent with the evidence. The next proposition summarizes the argument above:

[16] That is, the implicit constant interest rate at which the discounted value of the bond's coupons equals its price. Define $q_{YM,L}$ as $q_L = \frac{q_{YM,L}}{1-q_{YM,L}}$. The consol's yield to maturity is $1/q_{YM,L} = \frac{q_L}{1+q_L}$.

Proposition 1.6. *Outside the default region s_L is higher than s_S. In the default region, the term structure is inverted: $s_L < s_S$.*

Maturity Shortens as Indebtedness Increases I now turn to the implications for the optimal maturity composition of debt. The main finding is that the maturity of outstanding debt issued by the sovereign gets shorter as its indebtedness increases. In particular, the amount of long-term debt, $b_{L,t}$, decreases while the amount of short-term debt, $b_{S,t}$, increases for all v in the efficient region. This result is illustrated in Figure 6. I cannot state a proposition for this result, but the findings are consistent in all of my numerical simulations.

To understand this result, notice that outside of the default region, the amount of long-term debt held by the borrower is determined by (1.52), reported here for convenience:

$$\bar{b}_L(v) = \frac{b(v, \theta_L) - b(v, \theta_H)}{\bar{q}_L(v'_L(v)) - \bar{q}_L(v'_H(v))}$$

The long-term debt holdings are constructed to match the debt relief implied by the optimal contract after the realization of θ_H, $b(v, \theta_L) - b(v, \theta_H)$, given the *ex-post* variation in the price of the consol, $\bar{q}_L(v'_L(v)) - \bar{q}_L(v'_H(v))$. As is shown in Figure 7, the level of cross-subsidization is approximately constant for all v over the efficient region. The *ex-post* variation in the price of the consol instead is larger the closer the borrower is to the default region. This is because as the borrower's continuation value approaches the default threshold from above, it is more likely that a realization of θ_H will push the economy into default in the near future. Hence, the long-term debt price is more sensitive to the realization of a taste shock. Therefore, a lower holding of long term debt is needed in order to replicate the same amount of insurance, i.e. the same debt relief after a high taste shock. Since the overall level of indebtedness is increasing, it must be that $\bar{b}_S$ is increasing as the borrower's continuation value approaches $\tilde{v}$ because $\bar{b}_L$ is falling at the same time. Therefore, in the efficient region, as the level of indebtedness increases, the maturity composition of debt gets shorter.

In the region with *ex-post* inefficiencies, $[v_a, \tilde{v}]$, the ratio of short-term debt to long-term debt is not always decreasing in the borrower's value under all parameterizations. This is because the *ex-post* variation in the price of long-term debt is high, but also the amount of insurance, $b(v, \theta_L) - b(v, \theta_H)$, increases a lot in this region (see Figure

7). Despite not necessarily being monotonically decreasing in this region, the maturity composition of debt is more tilted toward short-term debt than it is for continuation values associated with lower default probabilities.

To summarize, in this section I showed that an efficient allocation can be implemented with only non-contingent defaultable debt of multiple maturities. Defaults are infrequent events, are associated with *ex-post* inefficiencies, and happen on the equilibrium outcome path only when the borrower's value is minimal (or close to minimal). When there is no default, capital gains or losses on outstanding long-term debt replicate the state contingent returns implied by the efficient allocation. Moreover, the maturity of outstanding debt gets shorter as the level of indebtedness increases.

1.6 Illustrative Numerical Example

In this section, I show that an efficient equilibrium outcome path leading to a default is qualitatively consistent with the four key aspects of the data that were mentioned in the introduction. I consider the following functional forms and parameterization. Let $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma = 2$ and $F(m, \ell) = \left(\omega m^{1-1/\eta} + (1-\omega)\ell^{1-1/\eta}\right)^{\frac{\eta}{\eta-1}}$ with $\eta = 1.5$ and $\omega = .3$. The other parameters in the model are $\beta = .95$, $q = .96$, $\Theta = \{.9, 1.4\}$ and $\mu_L = .8$. The recovery rate r_{rL} is set to .6, r_{rH} and r_{aL} are set according to (1.58). This example is representative of several simulations that I perform.

Randomization is optimal under this parameterization. Consistently with Assumption 3, there is a $v_r \in (v_a, \tilde{v})$ such that B is linear over $[v_a, v_r]$. For any v in this region it is optimal to randomize between v_a and v_r. Figure 3 displays the law of motion for the borrower's continuation utility which has already been discussed. In Figure 4, I show the policy functions associated with $(\hat{\mathrm{P}})$ for intermediate imports, output, consumption, and the net transfers to the foreign lenders, $y^* - m$, as a function of the borrower's value.

The first two panels illustrate the result in Proposition 1 part (i). For low borrower values, $v \in [v_a, v^*)$, imported intermediates and output are depressed relative to the statically efficient level. The lower is the borrower's value, the higher are the distortions: imported intermediates and output are strictly increasing over the interval $[v_a, v^*)$. For borrower values sufficiently high - higher than v^*- the statically efficient level of intermediates can be sustained. The same is true for output. Then, when the

economy is in the region with *ex-post* inefficiencies or the *default region*, output and intermediate imports are very low relative to "normal" times.

In the third panel, I show the decision rules for consumption after θ_L and θ_H. After a high taste shock consumption is higher than it is after a low taste shock. Notice how $c_H(v)$ is not monotone. For continuation values sufficiently high ($v \geq v^*$), it is increasing in the borrower's value, as one would expect. In the region in which the sustainability constraint is binding ($v \leq v^*$), $c_H(v)$ may have a decreasing portion. This is because providing more consumption after a high taste shock relaxes the current sustainability constraint. This is more valuable for low borrower values.

Finally, the last panel shows the dynamics for the trade balance, $y^* - m$. The sovereign borrower experiences larger outflows after a low taste shock than it does after a high taste shock because after a high taste shock more resources are devoted to domestic consumption. In particular, when the borrower's continuation value is autarky, after θ_H there are no trade flows: $m(v_a) = 0$ and $y^*_H(v_a) = 0$ (as shown in Proposition 1 part (i) and Lemma 3 respectively). Instead, after θ_L there are positive outflows: $m(v_a) = 0$ and $y^*_L(v_a) > 0$ (as shown in Proposition 1 part (i) and Lemma 3 respectively). The set of borrower values illustrated in the picture is restricted to those that are in the support of the unique limiting distribution. The borrower experiences net outflows because it has accumulated a stock of debt. This is due to the fact that the borrower is impatient relative to the international interest rate.

Figure 5 illustrates a sample path leading to a sovereign default for output, consumption, imported intermediates, and the short-term to long-term debt ratio as well as the realizations of the taste (productivity of the non-tradable sector, $z_t = 1/\theta_t$ with $\gamma = 2$) shock. As illustrated in the first panel, the economy is hit by a sequence of high taste (low productivity) shocks that pushes the economy into the region with *ex-post* inefficiencies. In particular, at time $\tilde{t}$, the borrower's value enters the region with *ex-post* inefficiencies. At t_d, it reaches the value of autarky and there is a default. The country is stuck in autarky until the economy draws a low taste (high productivity) shock at time t_r. At t_r there is a partial repayment to the creditors and the continuation value for the sovereign borrower starts to recover.

The outcome path is qualitatively consistent with the four key facts from sovereign default episodes mentioned in the introduction. First, in the model, as in the data,

defaults are associated with output and consumption losses for the sovereign borrower (see panels 3 and 4). Second, they are associated with a drop in imports of intermediate goods (see panel 2). Third, in both the model and the data, eventual recoveries are accompanied by large trade surpluses. This can be seen from the fact that consumption after the partial repayment (at t_r) is below output. Fourth, the model generates shortening of the maturity composition of sovereign debt as default approaches (see panel 5). This mirrors the observed maturity composition of sovereign debt as the debtor country approaches default.

1.7 Final Remarks

In this paper, I show that key aspects of sovereign debt crises can be rationalized as part of the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in an economy with informational and commitment frictions. Along the outcome path that supports an efficient allocation, sovereign default episodes happen because of the need to provide incentives, despite being *ex-post* inefficient. In this economy, intervention by a supranational authority aimed at reducing the inefficiencies in a sovereign default episode is not beneficial from an *ex-ante* perspective. Moreover, the increasing share of short term debt when a sovereign country accumulates external debt is optimal when only non-contingent defaultable debt is available. My analysis suggests that interventions that penalize the issuance of short term debt might negatively affect welfare.

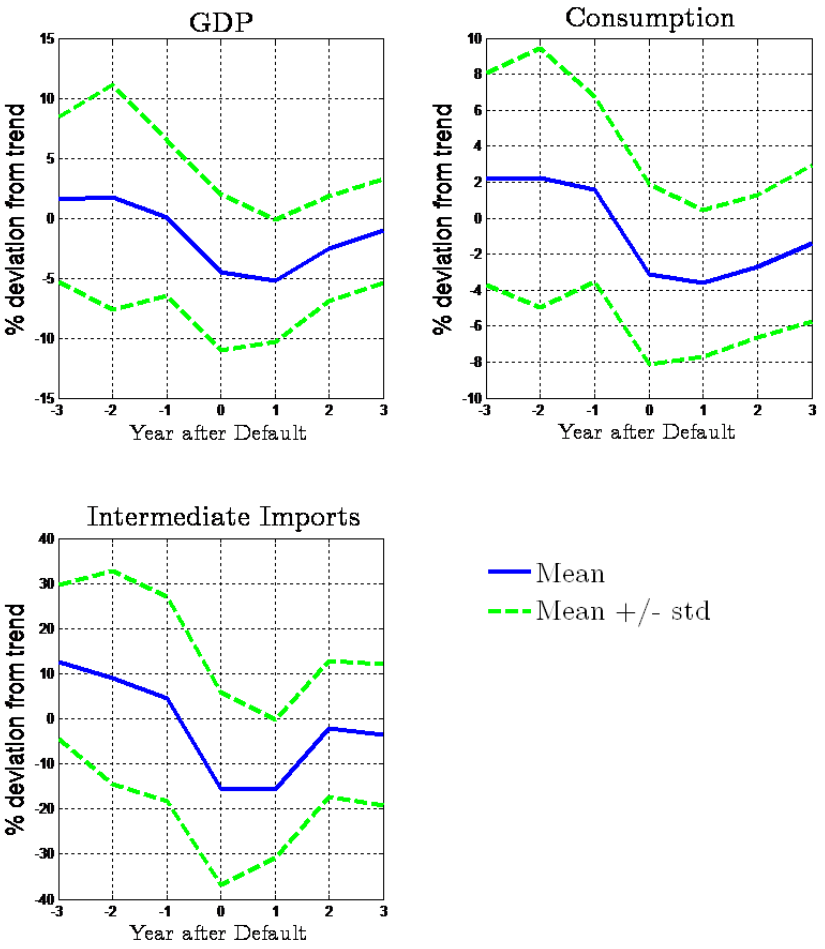
It is worth noting that the implementation I propose is applicable to environments other than the one considered here. Thus the implications for the optimal maturity composition of debt may have a more general applicability. I plan to consider possible generalizations in future research.

The simple model developed in this paper is consistent with broad patterns of the data. In future work, I plan to extend the current environment along two dimensions to be able to quantitatively evaluate the performance of the model. First, I plan to add capital accumulation, bringing the simple production economy considered here closer to a standard international business cycle model used in quantitative work. Second, in this paper I assumed that shocks were *iid* only for tractability. Introducing persistence

in the shock process is an interesting avenue for future research. In particular, it can help to account for the fact that debt restructuring is a lengthy process. Benjamin and Wright (2009) document that on average, the renegotiation process lasts 8 years. In my model, the sovereign borrower is out of the default region the first time he draws a high productivity (low taste) shock. The combination of *iid* shocks and the sufficient conditions in Assumption 4 imply that default episodes are resolved quickly. Introducing persistence in the shock process will help along this dimension.

Finally, while the efficient allocation can be implemented as a sustainable equilibrium outcome of the game that I proposed, the converse is not true. There is a *continuum* of equilibria and generically they are not efficient. Thus, despite the fact that agents are able to achieve the efficient outcome in a market setting, regulation by a supranational authority may indeed be helpful in avoiding inefficient equilibria and achieving *unique implementation*. I am planning to work on this in the future, introducing a strategic supranational authority (with and without commitment) into this framework.

Figure 1.1: Real Variables Around Sovereign Default Episodes



Data for 23 default events over the 1977- 2009 period. Same sample as in Mendoza and Yue (2012). See Data Appendix for a description of each variable.

Figure 1.2: Pareto and Utility Possibility Frontiers

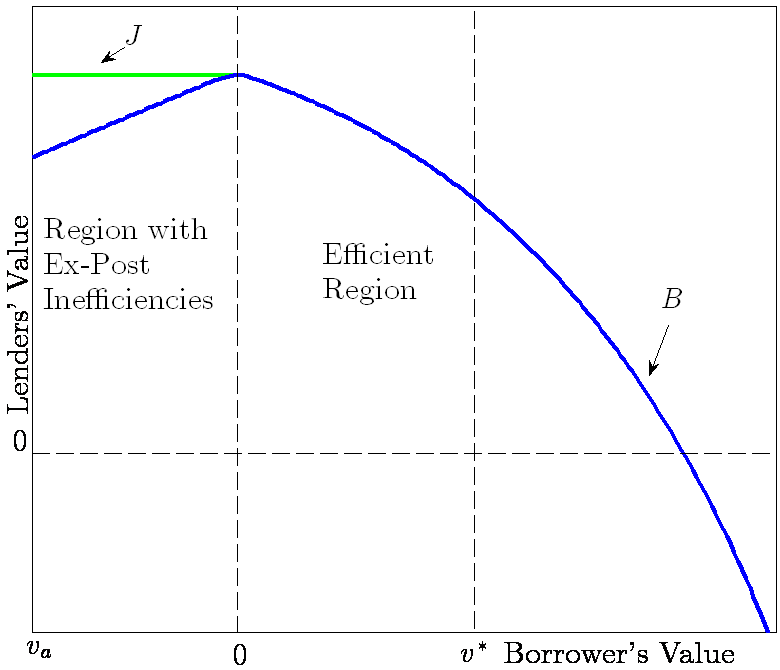


Figure 1.3: Law of Motion for Borrower's Value

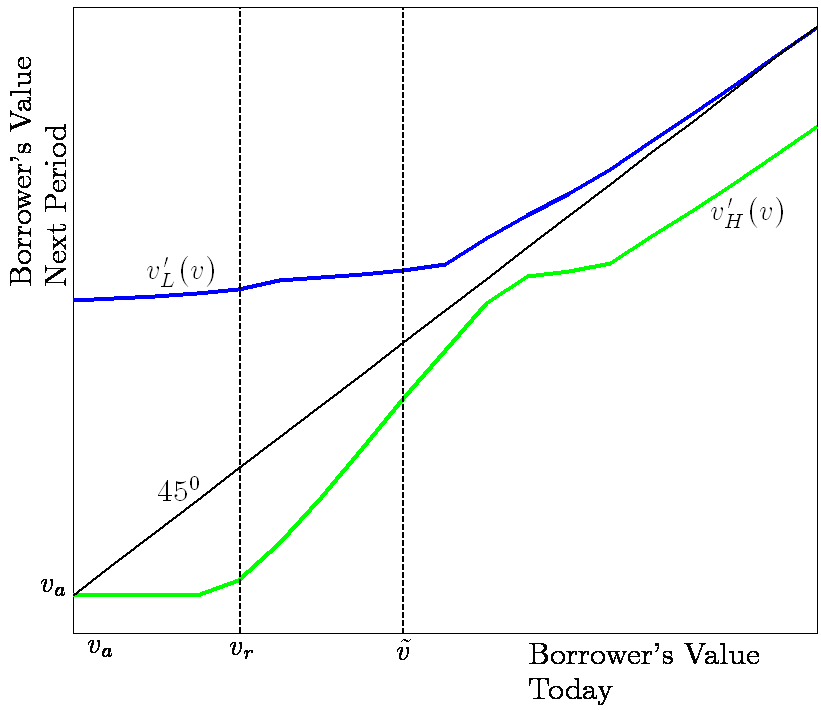


Figure 1.4: Policy Functions: $m(v), Y(v,\theta), c(v,\theta)$ and $y^*(v,\theta) - m(v)$.

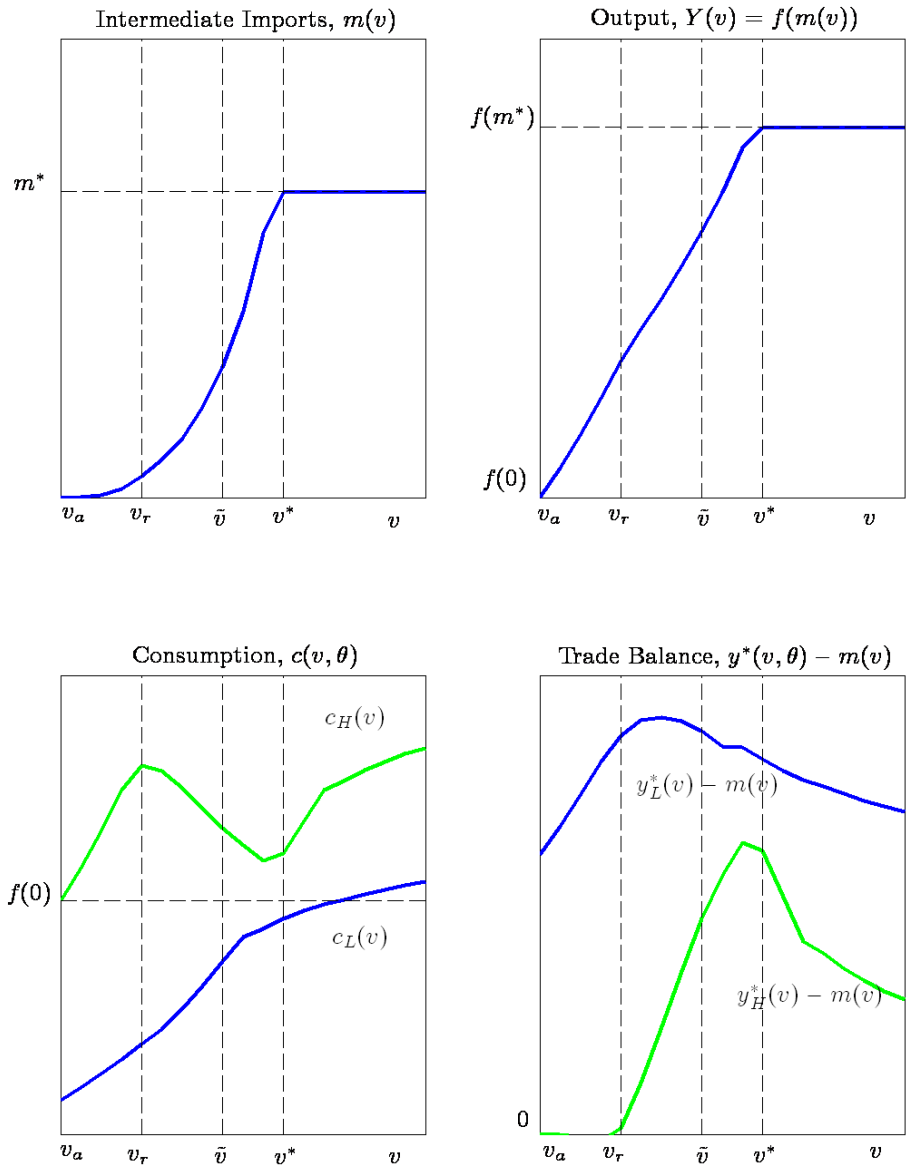


Figure 1.5: Outcome Path with Example of a Crisis

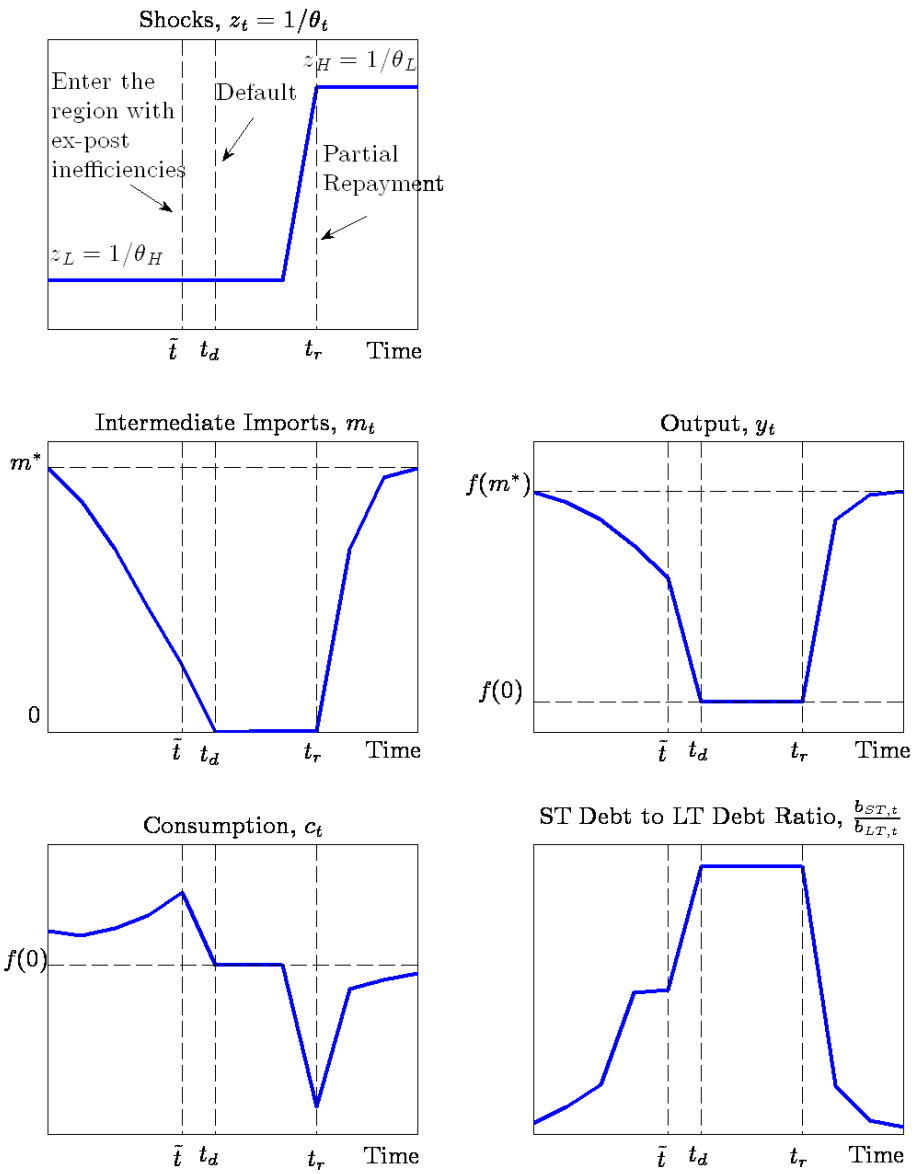


Figure 1.6: Bond Prices and Bond Holdings

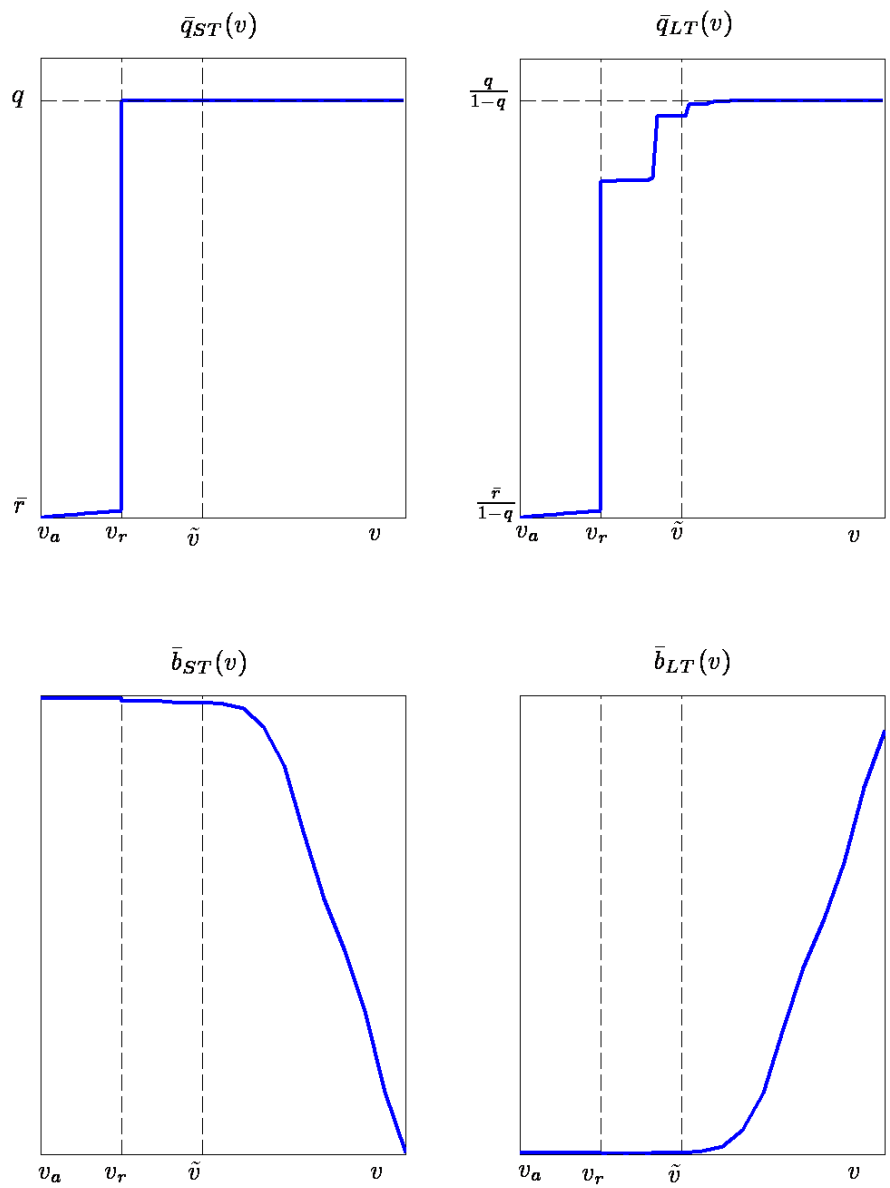
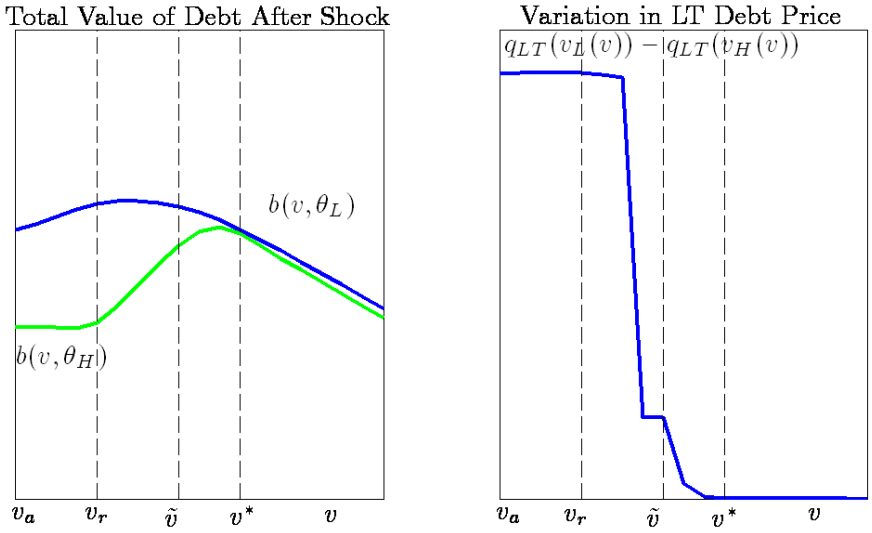


Figure 1.7: Ex-Post Variation in LT-Debt Price and Insurance



Chapter 2

Credit Market Frictions and Trade Liberalization

2.1 Introduction

Recent work has studied the role of credit constraints in economies undergoing reforms, and has concluded that financial market imperfections limit the gains from undergoing reform[1] . In this paper, we demonstrate that the way that credit constraints are modeled crucially determines their role in reform. In particular, we contrast two commonly used types of debt limits: what we refer to as *forward-looking* debt limits, following Albuquerque and Hopenhayn (2004), and *collateral constraints* or *backward-looking* debt limits. We concentrate on the role that credit frictions play in trade liberalization. Trade liberalization is an important type of reform for two reasons. First, trade liberalization is a clear example of exactly the type of reform that requires reallocation among firms[2] , as emphasized by the recent trade literature (see Melitz (2003) and Eaton and Kortum (2004)). Second, we show that trade liberalization provides a means of distinguishing between these two types of credit frictions. Using panel data from Colombia, we study the effects of a trade reform on the export behavior of firms. We show that the two specifications have opposite predictions for how young firms respond to the reform, and

[1] See, for example, Buera and Shin (2011 and 2013) and Song, Storesletten and Zilibotti (2011).

[2] Most of the episodes of reform studied in Buera and Shin (2013) include trade liberalization as a major component.

that the response observed is consistent with the forward-looking specification.

We extend a dynamic Melitz (2003) trade model to include credit market frictions in the form of debt limits. Our formulation takes both the forward-looking and backward-looking versions as special cases. With forward-looking debt limits, the amount of debt that firms can sustain is limited by the value of continuing to operate the firm (that is, the discounted stream of future income to the firm). With backward-looking debt limits (or collateral constraints), the amount that firms can borrow is at most a fraction of their capital stock. The key difference between these specifications is how credit limits are affected by the firm's future profitability. With forward-looking constraints, higher future profits allow firms to sustain more debt. With collateral constraints, future profits do not affect debt limits.

We show that both specifications of credit frictions are consistent with the empirical relationship between credit and export decisions at the firm level analyzed in a recent literature surveyed in Manova (2010). In particular, both specifications can account for the fact that access to credit affects both export participation and the amount that firms export. Moreover, these specifications have similar predictions for the life cycle path of firms. In both models, young firms are small and grow over time until they reach their optimal scale. In each, firms generally do not find it optimal to enter export markets when their capital stocks are small.

Despite these similarities, the models have different implications for gains from trade reform both at the aggregate and at the firm level. We show that the percentage increase in steady state consumption from a trade reform in the forward-looking specification is *the same* as in a corresponding model with perfect credit markets. However, with collateral constraints, the percentage gains from trade are *lower* than with perfect credit markets. We show that the important difference is on the extensive margin of adjustment. In the model with forward-looking debt limits, future exporters are able to sustain higher debt after the trade liberalization than before, even in periods *before* they enter the export market. This allows young, productive firms to start to export earlier. With collateral constraints, entering the export market requires asset accumulation. Non-exporters are less profitable after trade reform (due to increased wages) so they accumulate assets more slowly. Therefore, with collateral constraints productive, young (low net worth) firms are unable to enter export markets, while less productive,

old (high net worth) firms are able to enter. This creates perverse selection into the export market that lowers the gains from trade reform.

Since we show that these two specifications have different implications for the gains from a reform, it is important to be able to distinguish between them. We show that this can be done using panel data from before and after a trade liberalization. The two specifications disagree about whether or not young firms are able to become exporters following the reform. Using panel data from Colombia from 1981-91, which includes a trade liberalization in the mid-1980s, we show that, while all firms increase export activity following the reform, the increase is concentrated among young firms. We simulate the Colombian reforms in a small open economy version of our model and show that the forward-looking model shares this prediction. However, the backward-looking model has the opposite prediction. Because wages are higher, profits for young, non-exporters are lower and it takes firms longer to accumulate enough assets to become exporters.

We interpret this as supporting the forward-looking specification of credit constraints. This suggests that credit market frictions are not a barrier to reallocation following trade reforms. We interpret this result in two ways. First, introducing credit market frictions into trade models does make them more consistent with the empirical relationship between export behavior and access to credit, but it is reasonable to abstract from them when computing aggregate effects of trade reform. However, we stress that in general the existence of credit markets does matter for the levels of aggregate output and consumption. Second, the quality of financial markets is not an important determinant of gains from trade reform. Hence, policymakers in economies with low credit market development need not take that into account when deciding whether or not to liberalize.

Related Literature This paper is related to several literatures in international trade and macroeconomics. We build on the seminal contribution of Melitz (2003) and subsequent work, such as Alessandria and Choi (2007), who analyze the gains from trade in a model with heterogeneous monopolistic competitive firms, which emphasize the role of reallocation and selection into the export market as a driver for the gains from trade. Chaney (2005) and Manova (2008) introduce credit market frictions into a Melitz (2003)

framework. Both papers consider a static environment, and do not address how credit frictions affect the gains from trade, which is the central theme of our paper.

The model presented here is consistent with the growing empirical literature on the relationship between firm-level export behavior and access to credit (see Manova (2010) for a survey). This literature uses firm-level data from many different countries, and finds that access to credit is an important determinant of export participation (the extensive margin) and the scale of exports (the intensive margin). See Berman and Hericourt (2010), Minetti and Zhu (2011) and Gorodnichenko and Schnitzer (2013). This literature uses measures such as survey responses[3] and leverage ratios to proxy for access to credit. The models of trade and credit frictions developed in the next sections are consistent with both findings from this literature.

This paper is also related to the literature that studies how aggregate gains from a trade liberalization are affected by including institutional and technological details in trade models. Arkolakis, Costinot and Rodriguez-Clare (2011) show that all of a large class of trade models have the same implications for welfare gains from trade given *ex post* realizations of changes in trade flows. We are interested in evaluating *ex ante* how a given reduction in tariffs affects welfare with and without credit market frictions. This is similar in spirit to Atkeson and Burstein (2010), who show that modeling innovation decisions has no effect on aggregate gains from trade. Similarly, Kambourov (2009) shows that labor market frictions reduce gains from a trade liberalization.

We model credit market frictions following two specifications widely used in the macroeconomics literature. First, our forward-looking specification extends Albuquerque and Hopenhayn (2004) to a general equilibrium trade model with a discrete choice to export. See Cooley, Marimon and Quadrini (2004) for an application in a closed economy context. Second, we analyze collateral constraints following Evans and Jovanovic (1989), which has been used in many applied papers.

Finally and most importantly, our paper contributes to the literature that analyzes how credit market frictions affect reallocation in economies undergoing reform. Buera and Shin (2013) show that collateral constraints slow down the reallocation process

[3] For instance, in Minetti and Zhu (2011) they use a firm-level Italian data set that includes answers to the question, "In 2000, would the firm have liked to obtain more credit at the market interest rate?"

following a reform, because it takes time for productive but low net-worth firm to accumulate sufficient assets to start a business and operate at full scale[4] . Likewise, Song, Storesletten and Zilibotti (2011) consider a similar mechanism for the case of technological growth in China, showing that collateral constraints generate misallocation between constrained, productive private firms and unconstrained, less productive state-owned firms. These results all depend on the backward-looking nature of the financial constraints. If the debt limits have a forward-looking component, as in the specification that follows Albuquerque and Hopenhayn (2004), then productive firms can start a business and operate at a larger scale sooner after the reform or technological improvement, and they do not have to accumulate a large stock of assets to do so. Jermann and Quadrini (2007) consider a similar mechanism in the context of news shocks where they show that a signal of future productivity immediately relaxes the firms' enforcement constraints. The second contribution of our paper is to use micro-level evidence to tell these two formulations of credit market frictions apart, providing direct evidence that the forward-looking specification of the debt limits is more in line with the data.

In Sections 2, 3, and 4 we build and characterize a model of trade and consider two types of credit frictions. In Section 5 we discuss the difference in implications between these two specifications for trade reform both at the firm level and for aggregates. In Section 6 we provide a means of distinguishing between them and show that the data from a trade reform in Colombia favors the limited enforcement specification. Section 7 concludes.

2.2 Model

Time is discrete, denoted by $t = 0, 1, ...$ and there is no aggregate uncertainty. There are two symmetric countries, home and foreign, with variables for the foreign country are denoted with superscript f. Each country is populated by a unit measure of identical households, competitive final good producers, and monopolistic competitive firms each producing an intermediate differentiated product.

[4] Buera and Shin (2011) obtain similar results in an open economy environment (no intratemporal trade) considering debt limits that depend not only on the installed capital stock (collateral constraints) but also on period profits.

2.2.1 Household Problem

The stand-in household in each country inelastically supplies 1 unit of labor each period. He chooses final good consumption c_t and bond holdings b_{t+1} to maximize his lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2.1}$$

where $\beta \in (0,1)$ is the discount factor and u is increasing, differentiable and concave, subject to the sequence of budget constraints

$$c_t + q_t b_{t+1} \leq w_t + b_t + \Pi_t + T_t \qquad \forall t \geq 0 \tag{2.2}$$

expressed in terms of the final good in each country. Here w_t is the wage, q_t is the intertemporal price, Π_t is the sum of profits from the operation of firms[5] and T_t are lump-sum transfers from the government (revenue from tariffs). The problem for the stand-in household in the foreign country is symmetric.

2.2.2 Final Goods Producers

The final good in the home country is produced using the following CES aggregator:

$$y_t = \left[\omega \int_{I_t} y_{dt}(i)^{\frac{\sigma-1}{\sigma}} di + (1-\omega) \int_{I^f_{xt}} y^f_{xt}(i)^{\frac{\sigma-1}{\sigma}} di \right]^{\frac{\sigma}{\sigma-1}} \tag{2.3}$$

where I_t is the set of active domestic firms at time t, I^f_{xt} is the set of foreign firms that export at t, $y_{dt}(i)$ is the output of firm i in I_t, $y^f_{xt}(i)$ is the output of firm i in I^f_{xt}. The final good in the foreign country is produced analogously. The parameter ω indexes home bias in the production of the final good. The elasticity of substitution is $\sigma > 1$.

Final goods producers are competitive. A representative firm solves the following static problem:

$$\max_{y_t, y_{dt}, y_{xt}} y_t - \int_{I_t} p(i) y_{dt}(i) di - \int_{I^f_{xt}} (1+\tau_t) p(i) y^f_{xt}(i) di$$

[5] This assumption is standard (see Jermann and Quadrini (2012)). Suppose that there are a large number of households composed of workers and entrepreneurs that run the firms. These household members make their decisions separately but consume together.

subject to (2.3). One can then derive the inverse demand functions faced by domestic and foreign producers for the intermediated good i:

$$p_{dt}(y(i)) = \omega y_t^{\frac{1}{\sigma}} y(i)^{-\frac{1}{\sigma}} \tag{2.4}$$

$$p_{xt}^f(y(i)) = \frac{1-\omega}{1+\tau_t} y_t^{\frac{1}{\sigma}} y^f(i)^{-\frac{1}{\sigma}} \tag{2.5}$$

2.2.3 Intermediate Goods Producers

A mass of monopolistic competitive intermediate goods producers are operated by entrepreneurs in each country. In every period a mass $\delta \in (0,1)$ of entrepreneurs is born. Each operates a firm and is endowed with a new variety of the intermediate good. At birth the entrepreneur draws a type (z, ϕ), where z is the firm's productivity and $\phi \in \{0,1\}$ indicates if the firm has the ability to export or not[6] . If $\phi = 1$ the firm can pay a fixed cost f_x in any period to enter the export market the following period, while if $\phi = 0$ the firm does not have that option. For simplicity, z and ϕ are independently distributed. Productivity z is drawn from a distribution Γ, and the indicator ϕ is a Bernoulli random variable with parameter ρ. The type of the firm remains constant through time[7] . The firm can produce its differentiated variety using the following constant returns to scale technology:

$$y = zF(k,l) = zk^{\alpha}l^{1-\alpha}, \quad \alpha \in (0,1) \tag{2.6}$$

where l and k are the labor and capital employed by the firm, and y is total output produced, which the firm splits between domestic and export sales. Every period the production technology owned by the firm becomes unproductive with probability δ. To be able to export, a firm of type $\phi = 1$ must pay a sunk cost f_x in period t to be able to export in all the subsequent period conditional on surviving.

The firm has to borrow to finance its operations each period and to pay the export fixed cost f_x if it is profitable to do so. We consider a decentralization where firms have

[6] This feature of the model is useful to match the fact that there are large, productive firms that are non-exporters, and to generate reallocation after trade reform even if there are no fixed costs.

[7] Our goal is to compare the forward-looking limited enforcement model with the backward-looking collateral constraints model. Adding idiosyncratic uncertainty would require us to specify the *completeness* of debt contracts.

access to a rental market for capital. We denote the rental capital rate by r_t. Firms can save across periodsin contingent securities that pay one unit of the final good next period conditional on the firm's survival. All firms start with a_0 units of the final good, which are transferred to them by the household. Entrepreneurs are paid of dividend d_t from the operation of the firm. We are assuming that a_0 is the maximum one-time transfer that the household can make to the firm not subject to the debt limit[8] . That is, in any period it must be that

$$d_t \geq 0 \tag{2.7}$$

where d_t are the dividends distributed by the firm. Firms can issue *intra-period* debt at a zero net interest rate. We first present a general formulation, then consider two cases in the next section. The amount that can be borrowed depends on their assets at the beginning of the period:

$$b_t \leq \bar{B}_t^i(a_t; z, \phi) \tag{2.8}$$

The firm's problem can be conveniently written recursively using assets or cash on hand, a, together with its export status and type (z, ϕ) as state variables. The problem of the firm that has already paid to enter the export market can be written as choosing dividend distribution d, new assets a' to solve:

$$V_t^x(a, z, \phi) = \max_{d,a'} d + q_t(1-\delta)V_{t+1}^x(a', z, \phi) \tag{2.9}$$

subject to

$$\begin{aligned} d + (1-\delta)q_t a' &\leq \pi_t^x(a, z) \\ d &\geq 0 \end{aligned}$$

The production plan y_d, y_x, k, l and the intra-period debt b are chosen to maximize period profits $\pi_t^x(a, z)$:

$$\pi_t^x(a, z) = \max_{y_d, y_x, l, b, k} p_{dt}(y_d)y_d + p_{xt}(y_x)y_x - w_t l - r_t k - b \tag{2.10}$$

subject to

$$\begin{aligned} y_d + y_x &\leq zF(k, l) \\ k &\leq a + b \\ b' &\leq \bar{B}_t^x(a; z, \phi) \end{aligned}$$

[8] Clearly if there was not a bound on such transfers this channel would eliminate the credit friction.

For a firm that has not yet paid the fixed cost to start exporting, denoted with the superscript nx, the recursive formulation of its problem is the same, with the addition of the discrete decision to export or not:

$$V_t^{nx}(a,z,\phi) = \max_{d,a',x} d + (1-\delta)q_t \left[x\phi V_{t+1}^{x}(a',z,\phi) + (1-x)V_{t+1}^{nx}(a',z,\phi)\right] \tag{2.11}$$

subject to

$$\begin{aligned} d + (1-\delta)q_t a' + xf_x &\leq x\pi_t^x(a-f_x,z) + (1-x)\pi_t^{nx}(a,z) \\ d &\geq 0, x \in \{0,1\} \end{aligned}$$

where $\pi_t^{nx}(k,z)$ is given by the following static problem:

$$\pi_t^{nx}(a,z) = \max_{y_d,y_x,l,b,k} p_{dt}(y_d)y_d - w_t l - r_t k - b \tag{2.12}$$

subject to

$$\begin{aligned} y_d &\leq zF(k,l) \\ k &\leq a+b \\ b' &\leq \bar{B}_t^{nx}(a;z,\phi) \end{aligned}$$

We will denote the policy functions of the firms associated with the above problems as $\{d_t^{nx}, a_t'^{nx}, k_t^{nx}, b_t^{nx}, y_{dt}^{nx}, y_{xt}^{nx}, l_t^{nx}, x_t\}_{t=0}^{\infty}$ and $\{d_t^{x}, a_t'^{x}, k_t^{x}, b_t^{x}, y_{dt}^{x}, y_{xt}^{x}, l_t^{x}\}_{t=0}^{\infty}$ for non-exporters and exporters respectively.

2.2.4 Equilibrium

To define an equilibrium for the economy we need to keep track of the evolution of the measure of operating firms over (a,z,ϕ) and export status. Denote by λ_t^{nx} and λ_t^{x} the measure of non-exporting and exporting firms at the beginning of the period over (a,z,ϕ) respectively after the entry of new firms, and let $\lambda_t = (\lambda_t^{nx}, \lambda_t^{x})$. The evolution over time for λ_t^{nx} and λ_t^{x} is given by: $\forall\ (A,Z,\Phi) \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}_+) \times P(\{0,1\})$:

$$\begin{aligned} \lambda_{t+1}^{x}(A,Z,\Phi) = (1-\delta)\int 1\left\{a_t'^{x}(a,z,\phi) \in A,\ z \in Z, \phi \in \Phi\right\} d\lambda_t^{x} + \\ + (1-\delta)\int 1\left\{x_t(a,z,\phi) = 1, a_t'^{nx}(a;z,\phi) \in A,\ z \in Z, \phi \in \Phi\right\} d\lambda_t^{nx} \end{aligned} \tag{2.13}$$

$$\lambda_{t+1}^{nx}(A,Z,\Phi) = (1-\delta)\int 1\left\{x_t(a,z,\phi)=0, a'^{nx}(a,z,\phi)\in A,\ z\in Z, \phi\in\Phi\right\} d\lambda_t^{nx} \tag{2.14}$$

$$+\delta\rho\int_Z 1\{\ a_0\in A, z\in Z,\ 1\in\Phi\}\, d\Gamma$$

$$+\delta(1-\rho)\int_Z 1\{\ a_0\in A, z\in Z,\ 0\in\Phi\}\, d\Gamma$$

Given debt limits $\{\bar{B}_t\}_{t=0}^{\infty}$, an initial distribution of firms $\lambda_0 = \lambda_0^f$, capital stock $K_0 = K_0^f$, bonds holdings $b_0 = b_0^f$, and a sequence of tariff $\{\tau_t, \tau_t^f\}_{t=0}^{\infty}$ such that $\tau_t = \tau_t^f$, a *symmetric equilibrium* consists of household's allocations $\{c_t, b_{t+1}\}_{t=0}^{\infty}$, prices $\{p_t, w_t, r_t, q_t\}_{t=0}^{\infty}$, inverse demand functions $\{p_{xt}, p_{dt}\}_{t=0}^{\infty}$, aggregate capital $\{K_t\}_{t=0}^{\infty}$, firms decision rules $\left\{\left\{d_t^i, b_t'^i, k_t'^i, y_{dt}^i, y_{xt}^i, l_t^i\right\}_{i\in\{nx,x\}}, x_t\right\}_{t=0}^{\infty}$, and measure of firms $\{\lambda_t\}_{t=0}^{\infty}$ such that:

1. The households' allocations solve the problem (2.1) subject to (2.2) where the aggregate dividend distribution is given by

$$\Pi_t = \sum_{i\in\{nx,x\}}\int d_t^i(a,z,\phi)d\lambda_t^i \tag{2.15}$$

and the lump-sum transfers are given by

$$T_t = \tau_t\left\{\sum_{i\in\{nx,x\}}\int p_{xt}\left[y_{xt}^{if}(a,z,\phi)\right] y_{xt}^{if}(a,z,\phi)d\lambda^{if}\right\} \tag{2.16}$$

2. The firms' decision rules are optimal for (2.11) and (2.9);

3. The inverse demand functions are given by (2.4) and (2.5);

4. The capital rate is $r_t = 1/q_t - (1-\delta_k)$ so that competive intermediaries earn zero profits;

5. The markets for final good, labor and bonds clear. Market clearing in the final good market requires that

$$y_t = c_t + K_{t+1} - (1-\delta_k)K_t + y_{ft} \tag{2.17}$$

Market clearing in the rental capital market requires that

$$K_t = \sum_{i\in\{nx,x\}}\int k_t^i(a,z,\phi)d\lambda_t^i \tag{2.18}$$

and y_{ft} is the total investment in export fixed cost in period t:

$$y_{ft} = f_x \int x_t(a, z, \phi) d\lambda_t^{nx} \tag{2.19}$$

The labor market feasibility is given by

$$1 = \sum_{i \in \{nx, x\}} \int l_t^i(a, z, \phi) d\lambda_t^x \tag{2.20}$$

For the bond market to clear, it must be that

$$b_t + b_t^f + A_t + A_t^f = K_t + K_t^f \tag{2.21}$$

where A_t is the aggregate amount of assets held by firms:

$$A_{t+1} = (1 - \delta) \sum_{i \in \{nx, x\}} \int a_t'^i(a, z, \phi) d\lambda_t^i + \delta \int_Z a_t'^{nx}(a_0, z, \phi) d\Gamma \tag{2.22}$$

6. The measures of firms evolve according to (2.13) and (2.14).

In such an economy, prices will be equal in both countries. In our analysis we will focus on a *symmetric stationary equilibrium* for the economy.

2.3 Credit Market Frictions

Throughout the paper, we will contrast two popularly used specifications of debt limits $\bar{B}$ in (2.8) that are widely used in the literature. We refer to the first as the *forward-looking* specification, which follows Albuquerque and Hopenhayn (2004), and to the second as the *backward-looking* specification, following Evans and Jovanovic (1989) among others.

2.3.1 Forward-Looking Specification

In our first specification for (2.8), we derive debt limits faced by the firm that arise from the inability of firms to commit to repay their debt obligations. Credit contracts are not enforceable in the sense that every period the entrepreneur can choose to default on their outstanding debt.[9] . After default, the entrepreneur can divert a proportion

[9] This is consistent with a formulation in which households are composed of workers and entrepreneurs. If credit markets are anonymous, entrepreneurs do not internalize the loss to the household from default.

θ of the funds advanced for the next period's capital stock for personal benefits that are consumed immediately. Also with probability $1-\xi$, the entrepreneur loses its production technology. If the technology survives the default, the entrepreneur is able to continue to operate the firm without the assets or debt previously accumulated.[10] The corresponding debt limit $\bar{B}^i$for $i = x, nx$ is implicitely defined by:

$$V_t^i(a) = \theta\left[\bar{B}_t^i(a; z, \phi) + a\right] + \xi v_0(z, \phi) \tag{2.23}$$

where $v_0(z, \phi) = V_{t+1}^{nx}(0, z, \phi)$. This corresponds to the debt limit being "*not too tight*" in the terminology of Alvarez and Jermann (2000). The parameters θ and ξ index to the quality of financial markets[11] . If $\theta = 0$, then entrepreneurs have nothing to gain from default and credit constraints never bind. In this formulation, firms are able to borrow even if they have zero assets. Therefore we set $a_0 = 0$.

The key feature of this specification is that debt limits depend on the future profitability of the firm. That is, the higher the present value of the firm, $V^i(a)$, the more debt it can sustain.

2.3.2 Backward-Looking Specification

The backward-looking specification is a collateral constraint, with a debt limit (2.8) for $i = x, nx$ given by:[12]

$$\bar{B}_t^i(a; z, \phi) = \frac{1-\theta}{\theta} a \tag{2.24}$$

for some $\theta \in [0, 1]$. That is, a firm can borrow only up to a multiple $(1-\theta)/\theta$ of its assets. A common interpretation for this formulation is that entrepreneur cannot commit to repay his intra-period debt but the only punishment for doing so is the loss of a fraction $1-\theta$ of the capital stock. In particular, default does not result in the destruction of the firm's technology nor in exclusion from credit markets.[13] In this case, new entrepreneurs must be endowed with some assets in order to begin operation, $a_0 > 0$. Again, θ parameterizes the quality of financial markets, where higher values of θ imply lower financial market quality.

[10] Within a stationary equilibrium, this is equivalent to a period of exclusion from financial markets.

[11] As in Jermann-Quadrini (2012), we do not restrict $\theta \in [0, 1]$. This can be interpreted as there being some probability that, following default, the entrepreneur cannot be punished.

[12] This is equivalent to require that $b \leq \theta k$.

[13] This can be thought of as a version of the limited enforcement model in which there is no dynamic punishment for default. See Rampini and Viswanathan (2010).

The backward-looking debt limits depend only on the amount of profits that the firm has reinvested in the past, a, and not on future profitability. This aspect contrasts with the forward looking case. This difference is crucial for the two specifications to differ in their implications for the response of the economy to a trade reform.

2.4 Characterization of Equilibrium

Before analyzing the effect of a trade reform, we characterize the symmetric stationary equilibrium for the economy. We show that both specifications of credit market frictions are able to account for the relationship between export behavior and access to credit documented in the empirical literature: (i) the probability that a firm is an exporter is decreasing with measures of firm-level financial constraints, and (ii) firms' sales and exports grow over time and are decreasing in the credit constraints it faces.

In a stationary equilibrium, all prices and aggregate quantities are constant over time. Therefore, we will drop the dependence on time in this section. First we consider a relaxed problem where the borrowing constraint is dropped. It is easy to see that the production decisions are independent of the firm's debt level, and solve the following static problem:

$$\pi^*(z,\phi) = \max_{l,k,y_d,y_x,x} \omega y^{1/\sigma} y_d^{1-1/\sigma} + x\phi \frac{1-\omega}{1+\tau} y^{1/\sigma} y_x^{1-1/\sigma} - wl - rk - x\phi(1-q(1-\delta))f_x \tag{2.25}$$

subject to

$$y_d + xy_x \leq zF(k,l) \tag{2.26}$$

Given prices w, q, tariff τ and aggregate final output y, denote the solutions to this problem $\{l^*(z,\phi), k^*(z,\phi), y_d^*(z,\phi), y_x^*(z,\phi), x^*(z,\phi)\}$. These would be the firms' decision rules in a standard Melitz (2003) model. We say that a firm reaches its optimal scale whenever $k = k^*(z,\phi)$.

The following proposition fully characterizes the evolution of a firm over time. The proof is relegated to the online appendix.[14]

Proposition 2.1. *When debt limits are given by (2.23) or (2.24) then:*

[14] This characterization extends Albuquerque and Hopenhayn (2004) to an environment with a discrete choice of increasing the number of markets in which the firm operates.

(i) Firms issue no dividends until they reach their optimal scale[15] *;*

(ii) $\exists$ *cut-off productivity level* z_x *s.t. the firm will eventually export iff* $\phi = 1$ *and* $z \geq z_x$ *;*

(iii) $\forall z \geq z_x \; \exists \; \hat{a}(z, 1)$ *s.t. firms export iff* $\phi = 1$ *and* $a \geq \hat{a}(z, 1)$*;*

(iv) If $z' > z \geq z_x$ *and* $T(z)$ *is the age when a firm starts exporting, then* $T(z') \leq T(z)$*.*

Part (i) states that dividend distributions are back-loaded. Because the firm discounts at the equilibrium interest rate, the value of increasing their assets is always greater than the value of distributing dividends whenever the debt limit is binding. Therefore, the firm distributes no dividends so that it can increase its assets as quickly as possible. This allows the firm's capital stock to grow over time until it reaches its unconstrained scale $k^*(z, \phi)$. After reaching optimal scale the dividend policy of the firm is arbitrary, so long as they maintain enough assets to sustain their optimal scale of capital[16] .

Part (ii) states that only more productive firms will export, as in Melitz (2003), but now with the qualification that they will *eventually* export. When a firm is constrained to operate at an inefficiently low scale they may find it profitable to wait several periods to enter the export market. Part (iii) states that the firm's export status depends on both productivity and assets. For each productivity type z, there is an asset cut-off $\hat{a}(z, 1)$ such that it is profitable to start to export only if a firm has assets above that threshold. Firms with low assets are borrowing constrained and their capital stock is too low to make it profitable to pay the fixed cost to export.

Finally, part (iv) states that more productive firms enter export markets younger. This is true for two reasons. First, the value of being an exporter is increasing in the productivity of the firm. The minimal amount of assets necessary to justify the fixed cost to be an exporter, $\hat{a}(z, 1)$, is decreasing in z. Second, more productive firms accumulate

[15] *This statement is minorly qualified. The period before the firm reaches its optimal scale it is only required to distribute a low enough level of dividends that it will still be able to operate at full scale in the next period. In that period, zero dividends is optimal, but not uniquely optimal.*

[16] Two extreme cases would be the following: 1) after reaching optimal scale, all firms maintain just enough cash on hand to sustain their optimal capital stock and always distribute the rest of its profit in dividends, or 2) all firms retain all of their earnings by saving in risk-free debt. Either of these dividend policies, or any intermediate case, is consistent with the same allocation within a stationary equilibrium. In the first case, the household receives the dividends directly from the firms, while in the second they receive them indirectly through increased borrowing.

assets more quickly because they earn higher profits. Moreover, in the forward looking specification (2.23), more productive firms are able to borrow more because the value of the firm (left hand side of (2.23)) is increasing in z: For a given value of assets, default is less attractive the higher is the productivity of the firm.

The typical life-cycle path predicted by the model is as follows. After the initial productivity draw there is no uncertainty (except for exogenous exit) and firms are fully characterized by their productivity and their age. The amount of capital that a firm can sustain is initially low, then it increases over time as firms use period profits to accumulate assets (no dividend distributions). Likewise, labor usage and domestic sales (which are the static solutions to (2.12) above) are also initially low and grow over time with the capital stock. More productive firms eventually find it optimal to pay the fixed cost to enter the export market because they are able to sustain a larger capital stock, which increases the value of being an exporter. Then labor, domestic sales and export sales for a given capital stock are the solution to (2.10). Again, export sales remain at suboptimal levels as long as the firm's capital stock is constrained below its optimal scale. In finite time, the firm is able to sustain its optimal capital stock, and labor, domestic sales and export sales are constant forever after that.

Thus credit market frictions in the form of debt limits (2.23) or (2.24) affect firm level export decisions along the extensive and intensive margin: Firms whose debt limits are binding are both less likely to be exporters and export at smaller scale. This is consistent with the findings of the empirical literature on the relationship between export behavior and access to credit discussed before.[17] Despite having similar implications for firm-level dynamics in a stationary equilibrium, the two specifications of credit market frictions have different implications for the aggregate effects of a trade reform, as discussed in the next section.

2.5 Effects of Trade Liberalization

In this section, we evaluate if credit market imperfections reduce the gains from a bilateral tariff reduction. We show that with forward-looking debt limits the gains from trade

[17] Notice that firms with binding debt limits have higher leverage ratios and would identify themselves as constrained in survey responses.

are *not* affected by the quality of financial markets, while with backward-looking debt limits the gains from trade are lower than in an economy with perfect credit markets. The key mechanism that we will highlight throughout is how the debt limits that firms face respond to trade reform. With the backward-looking constraint, the amount firms are able to borrow depends only on their history of capital accumulation, and does not directly respond to the reform. However, with the forward-looking constraint, the fact that exporting firms are more profitable makes default less attractive and increases their debt limits. We begin by showing this quantitatively, then illustrating the mechanism with special cases.

2.5.1 Quantitative Exercise

To evaluate the effects of a trade liberalization in general equilibrium, we calibrate both specifications of the model and analyze the response to a bilateral, unforeseen reduction of tariffs.

Calibration

To calibrate our model we make use of the Colombian *Annual Survey of Manufactures* (ASM), which is described in detail in Roberts and Tybout (1997). This dataset covers all manufacturing plants with ten or more employees and provides data on items including sales, exports, input usage (employees, capital and energy), age, and subsidies at the plant level. Plants are classified by 3 digit SIC industry. There are 66,921 plant-year observations after removing observations that are inconsistently coded. A trade liberalization occured in 1985-86 in Colombia, so we calibrate our model to the 1981-84 period. A detailed description of the reform can be found in the next section.

Table 1 lists the parameter values used. The parameters α (the Cobb-Douglas parameter), β (the discount factor), σ (the elasticity of substitution) and δ_k (capital depreciation) are set to standard values. The survival probability is set to match the average age of operating firms in the data set during the pre-liberalization period. We assume that the ex-ante productivity distribution Γ is distributed log-normal$(0,s)$.

We calibrate the model with forward-looking and backward-looking constraints separately, with parameter values given in columns (a) and (b) in Table 1. We have six parameters to calibrate in each model: f_x, ω, s, ρ, ξ, and θ under the forward-looking

specification and f_x, ω, ρ, a_0, and θ in the backward-looking specification. They are set jointly to match six moments from Colombia in the years 1981-1984. These moments are: 1) the fraction of firms that export, 2) exports as a fraction of GDP, 3) the average difference in labor usage between exporters and non-exporters, 4) the average annual growth rate in labor usage before age 10, and 5) the fraction of firms that export before age 10. Age 10 was chosen because that is the first age for which the average growth rate of firms is 0%. The values of these moments in the model and data are given in Table 2. For comparison, we do a third calibration for the model without credit constraints shown in column (c) in Table 1. Here we only have four parameters to calibrate (f_x, ω, s,and ρ) and we match the first four listed moments.

Results

We consider the effects on the model economy of an unforeseen, bilateral reduction in tariffs from 50% to 13% (these are the Colombian manufacturing tariff rates before and after the reform from Attanasio et al (2004)). The results are reported in Table 3. We compare the steady state effects of a bilateral tariff reduction in the calibrated version of the model under the forward-looking and backward-looking specifications, each compared to an economy with perfect credit markets. As reported in Table 3, we can see that the gains from trade in the forward-looking specification are very similar to those in the model with perfect credit markets with the same parameters (changes in consumption of 7.1% versus 7.2%). However, the gains are lower under the backward-looking specification compared to the perfect credit markets benchmark by more than a full percentage point (6.4% versus 7.5%).

We also compare the effects predicted by our calibrated model with credit market frictions to a calibrated model with perfect credit market (calibration (c)). Again, Table 3 shows us that the gains from trade with the forward-looking specification are similar to those with perfect credit markets, while the gains under the backward-looking specification are lower.

Sensitivity

We test the sensitivity of these findings to variation in parameter values. We vary the fixed cost f_x, the elasticity of substitution σ and the quality of financial markets θ.

The first panel of Figure 1 shows the results of varying the fixed cost to export from one third to three times its calibrated value in both specifications of the model. In this figure, we plot the percentage gains from the trade reform in the perfect credit markets model (horizontal axis) and the model with each specification of debt limits (vertical axis). We see that high values of the fixed cost does increase the gap between percentage gains under perfect credit market and financial frictions under each specification. However, the difference with the forward-looking constraint is at most 0.7 percentage points (7.7% versus 8.4%) when the fixed cost is three times its calibrated value. This effect is more dramatic for the backward-looking specification, where the difference is as large as 2 percentage points (6.8% versus 8.8%). The example in the next subsection will demonstrate the importance of the fixed cost in determining this difference.

The second panel of Figure 1 shows how the gains from trade change as the elasticity of substitution σ is varied. Here it ranges from 1.5 to 10. As σ is varied, the gains from trade for the same change in tariffs also varies widely. However, we can see that the same basic story holds: under the forward-looking specification the gains from trade are similar to those with perfect credit markets, while with the backward-looking specification they are lower. It is noteworthy that when σ is low (between 1.5 and 3) gains from trade are actually somewhat higher with the forward-looking constraint than with perfect credit markets (by at most 1 percentage point when $\sigma = 2.5$), so that the forward-looking and backward-looking specifications have opposite predictions about how the gains from trade relate to the perfect credit markets case. This amplification disappears when the $\sigma = 2.5$ economy is calibrated to the moments described above.

Lastly, Figure 2 compares the gains predicted by the model as θ is varied. In both panels, the solid lines are pre-reform and post-reform levels of final output as θ is varied. The dotted line is the percentage increase in output from the perfect credit markets model applied to the pre-reform level in each specification. Therefore, if the model with debt constraints and the perfect credit markets model have exactly the same percentage increase in output, the dotted line and upper solid line would coincide. We see that this is the case for the forward-looking specification, but not for the backward-looking specification. In the backward-looking case, we can see that the gap between the two widens as θ is increased (worse credit markets).

From these results we can then conclude that under the backward-looking specification gains are limited by the presence of financial frictions. However, under the forward-looking specification, the difference in gains from trade is not significant. Next we illustrate the mechanism for why this is the case by looking at a special case.

2.5.2 Forward-Looking Case: Analytical Result

To understand why the gains from trade are the same with credit constraints of the form (2.23) as with no credit constraint, we first consider a special case by setting $f_x = 0$. All firms with $\phi = 1$ are exporters both before and after the trade liberalization, while all firms with $\phi = 0$ are not. Then because the set of exporting firms is not affected by trade reform, the only margin of adjustment is the intensive margin. With perfect credit markets, trade liberalization causes factors of production to be reallocated from non-exporters to exporters. In principle, financial frictions could be a barrier to that reallocation. The following proposition shows *analytically* that this is not true with the forward-looking specification of borrowing constraints.

Proposition 2.2. *Under the forward-looking specification with $f_x = 0$, for any change in tariffs the steady state percentage changes in aggregate output and wages are independent of θ and ξ. Furthermore, firm-by-firm the percentage change in capital usage is independent of θ and ξ.*

A formal proof of Proposition 2 is provided in the appendix, but here we sketch our approach. When tariffs are reduced exporting firms are more profitable, so for any debt level and capital stock, the value of not defaulting has increased. Therefore, exporting firms can sustain higher debt levels than before the liberalization allowing the firm to operate at a greater scale. The opposite is true for non-exporters who, because wages have increased, are less profitable after the tariff reduction than before.

In particular, the relaxation of the borrowing constraints for exporters is such that all exporting firms increase their scale by exactly the same proportion for every age and every productivity level as in the model with perfect credit markets. This is because the optimal scale of production and the present value of firms' dividends increase by exactly the same proportion. Consider a reform in which tariffs have been reduced from τ to τ'. Let $\mathbf{s} = (y, w, \tau)$ be the aggregate state before the reform and conjecture that

the aggregate post reform is given by $\mathbf{s}' = (\Delta_y y, \Delta_w w, \tau')$ where Δ_y and Δ_w are the changes in output and wages in the economy with perfect credit markets. Letting Δ_k be the change in the capital stock of exporting firms in the economy with perfect credit markets after the reform, we have that:

$$\Delta_k \pi^j(k; \mathbf{s}) = \pi^j(\Delta_k k; \mathbf{s}') \text{ and } \Delta_k V^x(a; z, \mathbf{s}) = V^x(\Delta_k a; z, \mathbf{s}')$$

This implies that if the path $\{a_t, k_t, b_t, d_t\}_{t=0}^{\infty}$ is feasible and optimal for a firm of type (z, ϕ) with aggregate state $\mathbf{s}$, then $\{\Delta_k a_t, \Delta_k k_t, \Delta_k b_t, \Delta_k d_t\}_{t=0}^{\infty}$ is feasible and optimal for the firm given the aggregate state $s' = (\Delta_y y, \Delta_w w, \tau')$. Debt limits (2.23) are satisfied with this new path because, noting that $v_0(z, \mathbf{s}) = V^x(0; z, \mathbf{s})$:

$$\begin{aligned} V^x(a; z, \mathbf{s}) \quad &= \quad \theta \left[\bar{B}^x(a; z, \mathbf{s}) + a\right] + \xi v_0(z, \mathbf{s}) \\ &\Longrightarrow \quad V^x(\Delta_k a; z, \mathbf{s}') = \theta \left[\bar{B}^x(\Delta_k a; z, \mathbf{s}') + \Delta_k a\right] + \xi v_0(z, \mathbf{s}') \end{aligned}$$

We can then verify that markets clear at the new allocation and therefore

$$\mathbf{s}' = (\Delta_y y, \Delta_w w, \tau')$$

is the aggregate state after the tariff reduction. Hence, though credit frictions do create an inefficient allocation of inputs across firms, they do not limit the reallocation of inputs and the percentage change in output following a trade reform.

This result extends to other types of reform that affect the indirect profit function of the firm multiplicatively, such as taxes on revenues or inputs. Moreover, it also holds for other specifications of the right hand side of (2.23). For instance, the same result goes through if, instead of the capital stock, the entrepreneur was able to abscond with working capital, period revenues, period profits, or any linear combination thereof.

We can see why if $f_x > 0$ we are not able to prove the analogue of Proposition 2. The presence of the fixed cost breaks down the value function's homogeneity property that is used in the proof. Despite not holding exactly, the numerical results clearly indicate that the difference in the percentage change in consumption, output, and exports that follows a bilateral tariff reduction between an economy with perfect credit markets and one with debt limits of the form (2.23) is very small. Furthermore, as the sensitivity analysis shows, the larger are fixed costs, the further from this result we get. However, we would need a counterfactually high fixed cost to make the difference significant (that is, one that implies a very small number of exporting firms).

2.5.3 Extensive Margin in the Backward-Looking Case

Under the backward-looking specification, the inability of young firms to borrow sufficiently to enter the export market is the key factor that lowers gains from trade. We illustrate this directly with an example in which the fixed cost to export is zero and the model is calibrated under both specifications of the borrowing constraint[18] . The results are reported in Table 4. As we can see, with no extensive margin of trade the difference between the perfect credit markets model and both specifications of the borrowing constraint are small. That the percentage gains with the forward-looking specification and perfect credit markets are exactly the same is analytically true by Proposition 2.

The reason that the extensive margin is so important in the backward-looking specification is as follows. All firms start as non-exporters and must accumulate sufficient assets to be able to become exporters. Since trade reform makes non-exporters less profitable (because wages have increased), they accumulate assets more slowly after the reform than before. Under the backward-looking specification, this directly implies that they cannot borrow as much after the reform compared to before. Notice that this is not the true in the forward-looking case. There, the fact that the firm will be an exporter in the future allows it to borrow more from the beginning of its life. Therefore, whether or not young firms are able to become exporters is the key factor that determines how financial frictions affect gains fromt trade.

We can then conclude that we have two possible answers to our original question: do financial frictions limit gains from trade reform? In the backward-looking model gains are less than in a perfect credit markets benchmark. However, in the limited enforcement model, the difference in gains is negligible. It is then important to distingiush between these two forms of debt limits. As discussed in Section 4, these models are difficult to distinguish using firm level data from a stationary environment because they have very similar implications for firms dynamics. In the next section we will argue that a trade reform provides a means of distinguishing them.

[18] The calibration strategy is the same as before, except that we have one fewer parameter to calibrate (f_x), and two fewer moments (the size difference between exporters and non-exporters, and the percentage of young firms that export). Since the model is overidentified, we arbitrarily set ξ in the forward-looking specification and a_0 in the backward-looking specification to the values from the baseline calibration.

2.6 Distinguishing Between Credit Constraints

In this section, we provide a means of distinguishing between the forward-looking and backward-looking specifications of the debt limts using data from a trade reform. We will show that the experience of Colombia in the 1980s provides evidence in favor of the forward-looking specification.

2.6.1 Colombian Reform

We use data from the Colombian *Annual Survey of Manufacturers*, as detailed in the previous section. The major benefit of using this data is that there was a major period of reform in the middle of sample. Through the early 1980s, Colombia had increasingly high tariff rates and quotas (see Roberts (1996)). This trend reversed in 1985, when Colombia agreed to a Trade Policy and Export Diversification Loan from the World Bank. Tariffs were substantially reduced and trade subsequently increased (see Fernandes (2007)). Figure 3 shows large increases in exports at both the aggregate and firm level. Also, changes in exchange rate policy led to a major real exchange rate depreciation (see Figure 4). Though not equivalent to a trade liberalization, a large real exchange rate devaluation has, for our purposes, the same effect of a reduction in tariff from the foreign country: an increase in the value of being an exporter compared to being a non-exporter.

2.6.2 Difference in Implications

As the previous section demonstrates, the important difference between the two specifications of debt limits is whether or not credit constraints restrict the ability of firms to become exporters following trade reform. In the backward-looking case, firms are only able to export once they have accumulated sufficient assets. Since the profitability of young, non-exporting firms is decreased after the reform, it takes longer to accumulate assets and, therefore, credit constraints diminish the extensive margin of exporting. This predicts that the incidence of export activity across firms will be shifted away from young firms (who are more credit constrained) and toward older firms (who are less credit constrained). Alternatively, under the forward-looking specification firms

that will eventually export are able to borrow more from the beginning of their lifetimes, which allows them to become exporters. Furthermore, since the profitability of exporting has increased, firms may choose to become exporters earlier in their lives.

In the case of Colombia, we illustrate the export activity of firms by age in Figures 5 and 6. Figure 5 shows the fraction of firms that export conditional on age before and after the reform. As we can see, there is a very large increase in export activity among the youngest firms. In Figure 6, we create a cumulative distribution function of exporters aged 1 to 20 before and after the reform. This shows that export activity increased by the most among young firms, independent of overall changes in export activity. Given the argument above and in the previous section, this provides support for the forward-looking case relative to the backward-looking case. We formalize this with simulations in the next subsection.

2.6.3 Small Open Economy Simulation

To precisely compare the predictions of both models with the outcome of the reform in Colombia, we now simulate the effects of a trade reform and real exchange rate depreciation in line with those experienced in Colombia. To do this, we reformulate the model as a small open economy for two reasons. First, during this period Colombia was very different from its trading partners and accounted for a small share of world trade, so that we think of the small open economy case as being more realistic than an economy with two identical countries. Second, the reforms that Colombia underwent were highly asymmetric in nature: reductions in import tariffs and real exchange rate devaluation. Considering a small open economy allows us to impose a real exchange rate decline by exogenously changing the price level in the rest of the world.

In the small open economy model[19] intermediate goods are exported abroad, and the domestic country buys a single foreign intermediate good that has price P. The country is small in the sense that P is exogenous. We analyze the effect of the following reform: a simultaneous reduction of import tariffs from 50% to 13% (the average manufacturing tariff rates before and after reform from Attanasio et al. (2004)), and an increase in P. Since the domestic price level is numeraire, an increase in P is equivalent to a real exchange rate depreciation. In our baseline exercise, we match the observed

[19] A formal description of the small open economy framework is in the appendix.

44% depreciation.

We calibrate the small open economy model to match the same moments as in the two country case. Additionally, we have the initial price of the imported good P, which we choose to match imports as a fraction of GDP. The calibrated values and targets are in Tables 5 and 6. Then we can compare the incidence of exporting across firms of different ages to what was observed in the data. The first panel of Figure 7 shows the results for the forward-looking case, and the second panel for the backward-looking case. This shows the pattern described above. In the forward-looking case, the increase in export activity is concentrated among the young, while in the backward-looking case it is concentrated among older firms. The two panels of Figure 8 show why this is true. Here, the horizontal axis is firm productivity, and the vertical axis is the age when a firm first becomes an exporter. In the forward-looking model, firms of every productivity level become exporters earlier in their lives after the reform than before. In the backward-looking case, just the opposite is true. Though more firms are exporters after the reform than before, young firms actually take longer to become exporters after the reform than before. This is completely at odds with the change in the pattern of export status observed among young firms in the Colombian reform.

The large real exchange rate depreciation generates a counterfactually large increase in exports as a fraction of GDP. As an alternative exercise, we consider the case where the increase in P is calibrated to match the change in exports as a fraction of GDP observed in Colombia before and after the reform. The results are given in both panels of Figure 9. As in the baseline case, the forward-looking case agrees with the predictions from the data. Furthermore, in this case, the shift in the composition of export activity is quantitatively similar to that observed in the data. However, again the backward-looking case has the opposite prediction.

2.7 Conclusion

This paper shows that a trade reform provides a means of distinguishing between two widely used specifications of borrowing constraints: the forward-looking specification and the backward-looking or collateral constraint specification.We first show that these

two models have importantly different predictions for the gains from undergoing a reform. Our analysis provides evidence in favor of the forward-looking specification. Although the model and data are related to a trade reform, we believe these results are more widely applicable.

We intrepret our results as demonstrating that models with fixed collateral constraints may be misleading when analyzing economies undergoing reform or structural change. While a collateral constraint may be a good approximation to an underlying financial market imperfection in a stationary economy, it fails to address the endogenous response of financial markets in economies undergoing change. In many contexts this effect may be important. Apart from this paper, the results in work by Buera and Shin (2013) and Song et al. (2011) could be significantly altered if borrowing constraints were forward-looking in the sense discussed in this paper: reform affects the value of projects for borrowers, allowing them to sustain more debt. This allows for a rapid reallocation of resources at the reform date, even absent any type of financial reform.

In future work, we would like to consider the case where borrowers are privately informed about their types. We have assumed here that lenders set debt limits according to the publically known type of the borrower. If the profitability of exporting was known only to the borrower, they may be able to misrepresent it to lenders, be able to borrow more than they otherwise would, then default on their debts. Debt contracts that take this possibility into account may respond very differently to reform than those considered here.

In this paper we abstracted from idiosyncratic productivity shocks to isolate the difference between these two financial environments. This abstraction is potentially important[20] , but introducing them would require one to specify whether or not debt is contingent on realizations of individual productivity. Typically, models with collateral constraints have only non-contingent debt[21] , while models with limited enforcement usually have contingent debt. In future work we plan to explore this distinction.

[20] Midrigan and Xu (2013) show that volatility in firm-level productivity can have large effects on the ability of firms to overcome their collateral constraints through self-financing, though they provide evidence that productivity is very persistent in the micro data.

[21] Rampini and Viswanathan (2010) is a notable exception.

Table 2.1: Parameters Value:
(a) Calibration for the economy with forward-looking debt limits
(b) Calibration for the economy with backward-looking debt limits
(c) Calibration for the economy with perfect credit markets

Parameter	Symbol	(a)	(b)	(c)
Discount Factor	β	0.96	0.96	0.96
Cobb-Douglass Parameter	α	0.3	0.3	0.3
Capital Depreciation	δ_k	0.05	0.05	0.05
Elasticity of Substitution	σ	5	5	5
High Tariffs	τ_H	0.50	0.50	0.50
Low Tariffs	τ_L	0.13	0.13	0.13
Survival Probability	δ	0.07	0.07	0.07
Std of Productivity	s	0.39	0.41	0.40
Export fix cost	f_x	0.14	0.15	0.19
Home Bias	ω	0.50	0.50	0.50
% Firms that can Export	ρ	0.54	0.57	0.52
Enforcement Parameter	θ	1.71	0.37	-
Probability of Starting New Firm	ξ	0.96	-	-
Initial Assets	a_0	-	0.005	-

Table 2.2: Target Statistics: Data and Model

Target	Data	Model		
		(a)	(b)	(c)
Exports/GDP	11%	11%	11%	11%
% Firms Exporters	12%	12%	12%	12%
Average Exporter Size Difference	3.9	3.9	3.9	3.9
St. Deviation of Log(Employees)	1.1	1.1	1.1	1.1
Annual Firm Growth, 1 to 10 years	5%	5%	5%	-
% Firms Exporting, 1 to 10 years	8%	8%	8%	-

Data: IMF IFS and Colombian Survey of Manufacturers using years 1981-1984.

Table 2.3: Steady State Comparison

	% Change in			
	c	y	w	$\Delta(x/y)$
Forward-looking (a)	7.1	7.5	5.8	7.4
Perfect credit markets (a)	7.2	7.7	6	7.1
Difference	-2%	-2%	-3%	
Backward-looking (b)	6.4	6.8	5.6	6.8
Perfect credit markets (b)	7.5	8.0	6.2	7.1
Difference	*-16%*	*-15%*	*-10%*	
Perfect credit markets (c)	7.3	7.8	5.9	7.5

$\Delta(x/y)$ is the change in export as a fraction of output

Table 2.4: Steady State Comparison, $f_x = 0$ case

	% Change in			
	c	y	w	$\Delta(x/y)$
Forward-looking (a)	5.4	5.7	4.4	4.4
Perfect credit markets (a)	5.4	5.7	4.4	4.4
Difference	*0%*	*0%*	*0%*	
Backward-looking (b)	5.3	5.5	4.2	4.1
Perfect credit markets (b)	5.5	5.8	4.4	4.3
Difference	*-4%*	*-4%*	*-3%*	
Perfect credit markets (c)	5.4	5.7	4.3	4.3

$\Delta(x/y)$ is the change in export as a fraction of output

Table 2.5: Small Open Economy, Target Statistics: Moments and Data

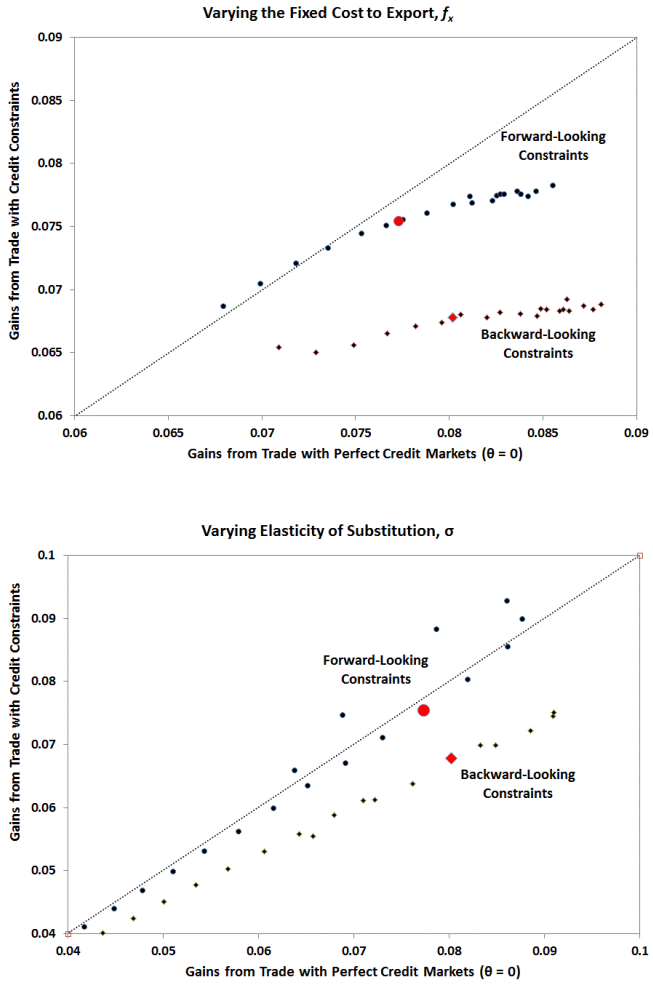
Target	Data	Model		
		LE (a)	CC (b)	PM (c)
Exports/GDP	11%	11%	11%	11%
% Firms Exporters	12%	12%	12%	12%
Imports/GDP	14%	14%	14%	14%
Average Exporter Size Difference	3.9	3.9	3.9	3.9
St. Deviation of Log(Employees)	1.1	1.1	1.1	1.1
Annual Firm Growth, 1 to 10 years	5%	5%	5%	-
% Firms Exporting, 1 to 10 years	8%	8%	8%	-

Data: IMF IFS and Colombian Survey of Manufacturers using years 1981-1984.

Table 2.6: Small Open Economy Parameter Values
(a) Calibration for the economy with forward-looking debt limits
(b) Calibration for the economy with backward-looking debt limits

Parameter	Symbol	(a)	(b)
Discount Factor	β	0.96	0.96
Cobb-Douglass Parameter	α	0.3	0.3
Capital Depreciation	δ_k	0.05	0.05
Elasticity of Substitution	σ	5	5
Import Tariffs, Pre-Reform	τ_I	0.50	0.50
Import Tariffs, Post-Reform	τ_I'	0.13	0.13
Export Tariffs	τ_X	0.05	0.05
Survival Probability	δ	0.07	0.07
Foreign Price, Pre-Reform	P	0.90	0.91
Foreign Price, Post-Reform (i)	P'	1.58	1.60
Foreign Price, Post-Reform (ii)	P'	1.03	1.04
Std of Productivity	s	0.35	0.39
Export fix cost	f_x	1.84	1.42
Home Bias	ω	0.39	0.36
% Firms that can Export	ρ	0.53	0.49
Enforcement Parameter	θ	0.73	0.20
Probability of Starting New Firm	ξ	0.96	-
Initial Assets	a_0	-	0.03

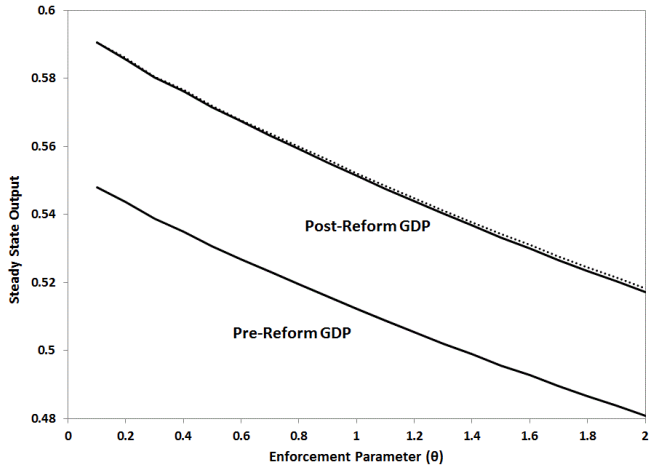
Figure 2.1: Gains from trade varying fixed costs and elasticity of substitution



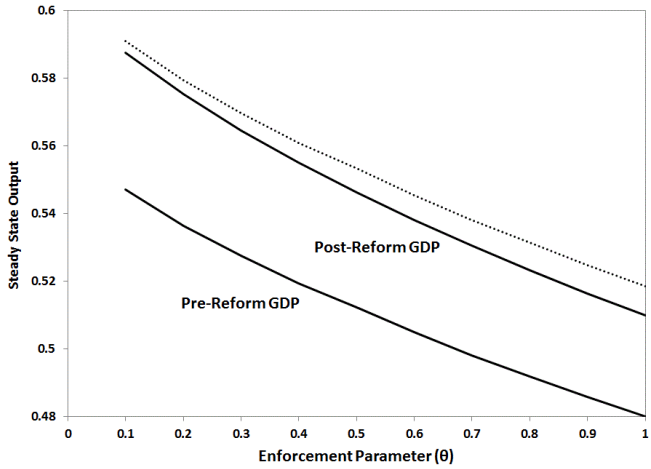
Percentage increase in steady state consumption from the decrease in tariffs in both specifications of borrowing constraints compared to the model with no credit constraints. Gains in the calibrated economies are emphasized.

Figure 2.2: Changes in Steady State Output, varying credit market quality

Forward-Looking Debt Limits:

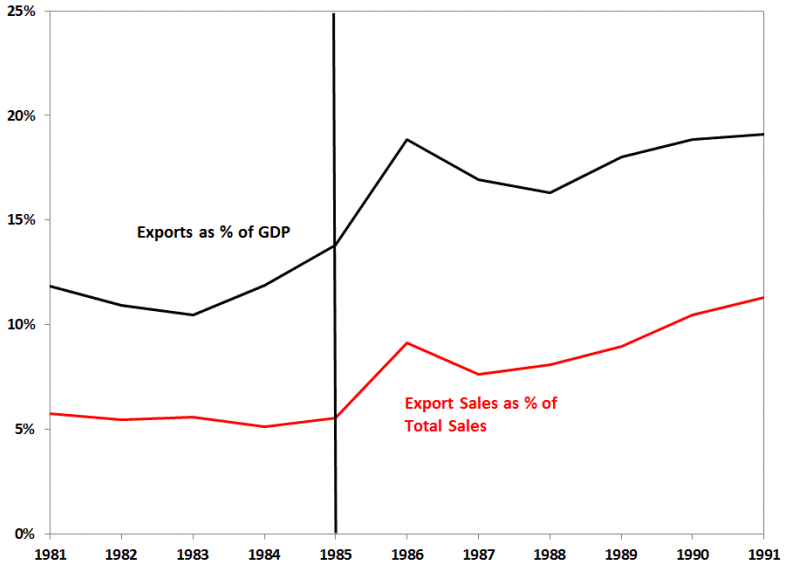


Backward-Looking Debt Limits



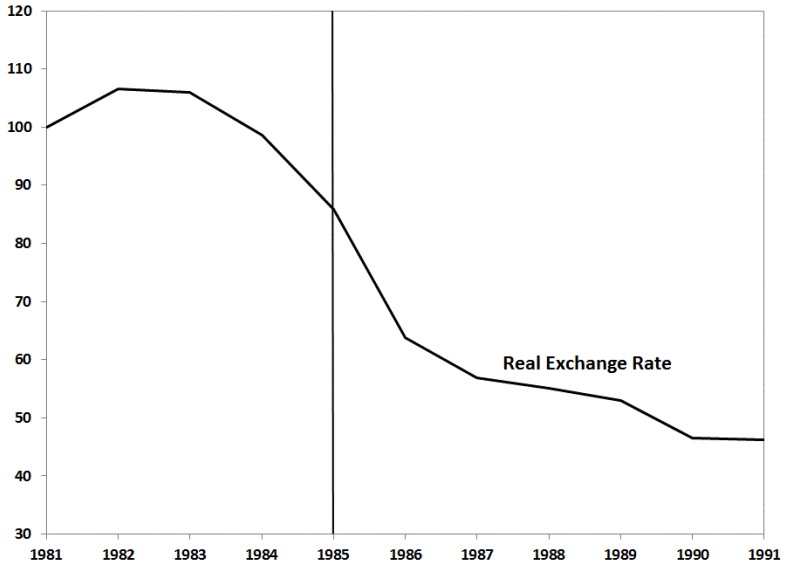
The dotted line is the pre-reform level of output increased by the same percentage as in the model with perfect credit markets.

Figure 2.3: Evidence of Liberalization: Colombia 1981-1991 (Data)



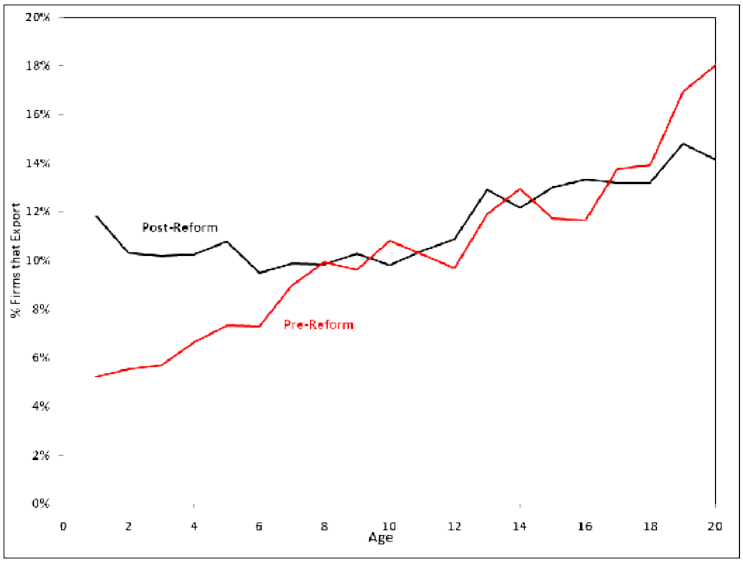
Sources: IMF IFS, and Colombian Survey of Manufacturers

Figure 2.4: Real Exchange Rates (Data)



Source: IMF IFS

Figure 2.5: Trade Reform: Extensive Margin for Young Firms (Data)

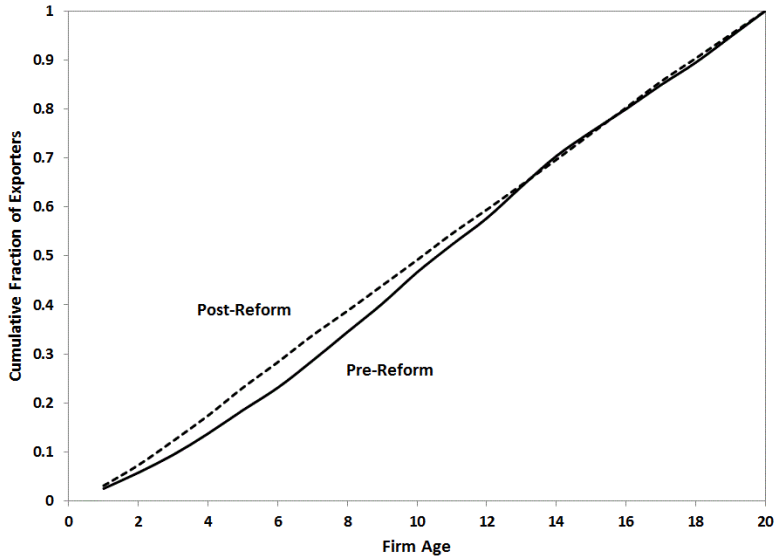


Source: Colombian Survey of Manufacturers, 1981-1991

Pre-Reform: 1981-1984

Post-Reform: 1986-1991

Figure 2.6: Cumulative Distribution Function of Age for Exporters (Data)



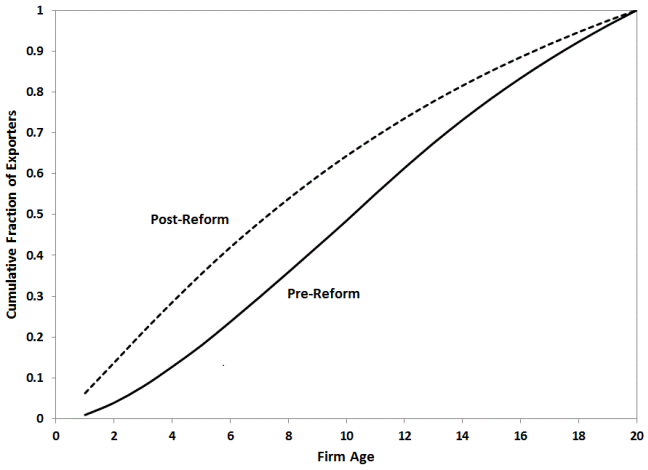
Source: Colombian Survey of Manufacturers, 1981-1991

Pre-Reform: 1981-1984

Post-Reform: 1986-1991

Figure 2.7: Cumulative Distribution Function of Age for Exporters, SOE Model, case (i)

Forward-Looking Debt Limits



Backward-Looking Debt Limits

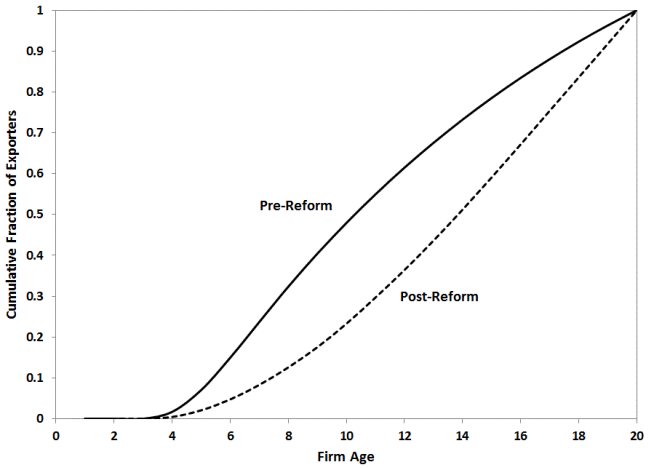
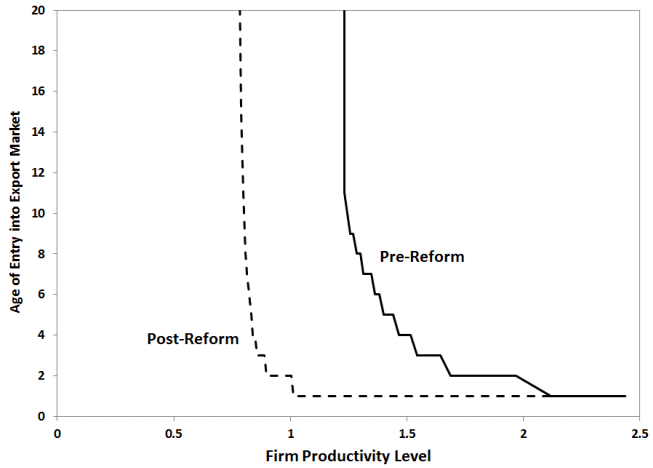


Figure 2.8: Age of Entering the Export Market by Productivity

Forward-Looking Debt Limits



Forward-Looking Debt Limits

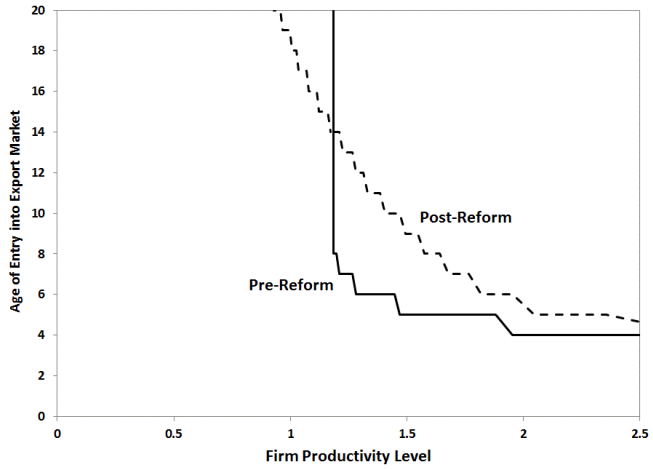
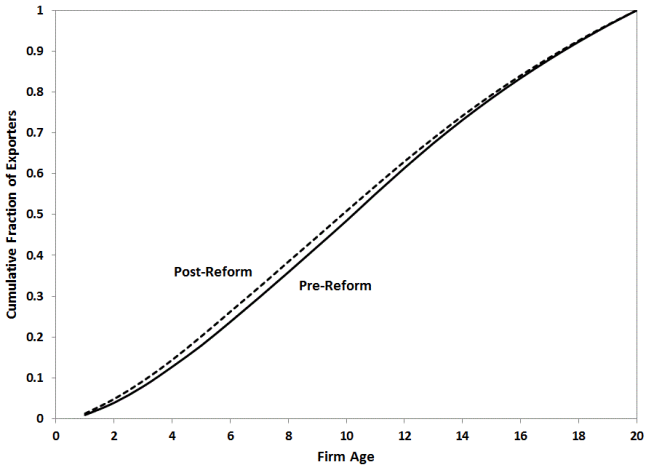
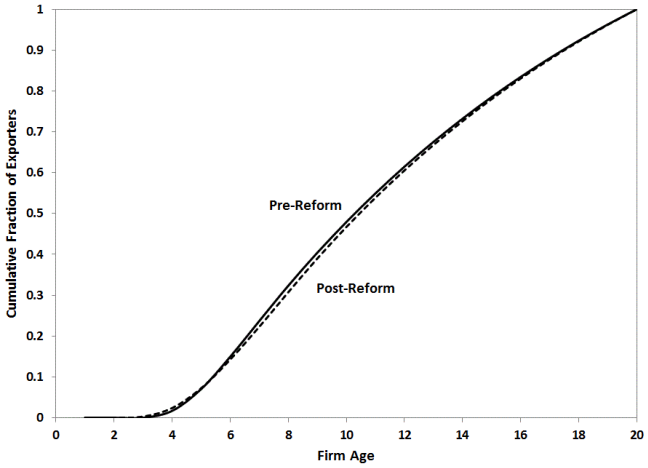


Figure 2.9: Cumulative Distribution Function of Age for Exporters, SOE Model, case (ii)

Forward-Looking Debt Limits



Backward-Looking Debt Limits



Chapter 3

Capital Mobility and Optimal Fiscal Policy without Commitment: A Rationale for Capital Controls?

3.1 Introduction

In the last two decades, there has been a rise in private capital flows. In particular, FDI to low and middle income economies increased substantially since the 1990s[1] . Despite this rise in international capital flows, Aizenman, Pinto and Radziwill (2007) document that approximately 90 percent of the stock of capital in developing countries is self-financed and that countries with higher self-financing ratios grew faster in the 1990s than those with lower ratios[2] . In this paper, I provide a mechanism through which policies aimed at increasing self-financing ratios - capital controls on inflows - are welfare and growth improving. In particular, I investigate whether or not it is optimal for a benevolent government to influence private foreign asset positions by taxing capital income of domestic residents and foreigners at different rates. That is,

[1] See for instance Broner et al. (2011), Table 8.

[2] This fact is related to the "Capital Allocation Puzzle" in Gourinchas and Jeanne (2011).

should a benevolent government impose capital controls?

To answer these questions, I study the best equilibrium outcome path of a Ramsey taxation model for a small open economy (SOE) when the government cannot commit. Under commitment (see Correia (1996) and Atkeson et al. (1999)), the optimal Ramsey policy for a SOE prescribes no capital controls; under standard assumptions, taxes on capital income are zero in every period for both the domestic and the foreign residents. In contrast, if the government cannot commit to future policies, I show that it is optimal to impose capital controls on inflows. The optimal policy prescribes that *capital income of foreigners is taxed more heavily than that of domestic households* on the transition to the steady state. Moreover, this wedge between the domestic and foreign after-tax-return on capital invested in the country is decreasing over time. In particular, if the international interest rate is equal to the inverse of the discount factor of the domestic agents, then capital controls are vanishing over time and the steady state tax rate on capital income is zero for both foreign and domestic residents as in the Ramsey policy. If the international interest rate is lower than the inverse of the discount factor, then the optimal policy dictates that the wedge between domestic and foreign tax rates, although decreasing, persists also in the steady state.

The optimality of taxing capital income of foreigners at a higher rate may look counter-intuitive, especially for a country with a low capital stock that might greatly benefit from receiving capital inflows from foreign investors to smooth consumption over time and to increase the capital stock in the country. One would naively suggest that it may be optimal to provide some sort of tax advantage to induce foreigners to invest. This argument is not correct if the government lacks commitment. Intuitively, if a large fraction of the capital stock is owned by foreigners, the domestic government finds it difficult to resists pressure from domestic agents who will want an increase in tax on capital income in exchange for lower taxes on labor and more government expenditure. Anticipating this, foreigners are not willing to invest today. On the other hand, the government is able to sustain a lower level of taxes on capital income - and hence a higher capital stock - if a larger share of domestic capital is owned by domestic agents.

Back-loading leisure and consumption - both private and public - is optimal, as typical in an economy with one-sided lack of commitment. In fact, back-loading consumption is not only trading off current utility with future utility, but it is also helping

to relax the future commitment problem. Since the government does not directly control private consumption, it has to subsidize household savings in order to induce the domestic household to save more (or to borrow less) than what they would if they face the international interest rate. This implies that the government taxes the capital income of foreign residents at a higher rate. Equivalently, the government could tax foreign debt held by domestic residents at a higher rate. Differential tax rates are necessary because private domestic households do not internalize the fact that by postponing consumption, they are relaxing the commitment problem for the government. This allows the economy to sustain a higher level of capital, and hence, higher wages and welfare.

I then study the limiting behavior of the optimal outcome path for the SOE under two assumptions about the international interest rate. If the international interest rate is equal to the rate of time preference of households then there are no capital controls in the long-run; the domestic household and the domestic government accumulate sufficient assets such that the continuation of a solution with commitment can be sustained. Capital income taxes for domestic and foreigners converge to zero. Instead, if the international interest rate is lower than the rate of time preference, then the economy converges to a steady state with higher capital income taxes for foreign residents than for their domestic counterparts.

To assess the potential welfare gains of adopting the optimal policy, I compare the best equilibrium outcome with the one in which I impose the restriction that the tax rate on capital income must be equal for both domestic and foreign agents. This restriction eliminates the possibility that the government might manipulate the saving decision of the domestic household; there is no wedge between the international interest rate and the after tax rate of return faced by domestic households. I show, in numerical simulations, that the gains from manipulating domestic savings depend crucially on the expected TFP growth of the country. The gains are limited (on the order of a 0.2 % increase in life-time consumption) if there is no growth in TFP, but can be substantial if the country is expecting high TFP growth in the future. Thus, the normative recommendations of this paper are more relevant for fast growing economies.

Finally, the results derived for a SOE generalize to a two-country general equilibrium environment. In particular, I show that capital controls are a feature of a best equilibrium outcome also under coordination between asymmetric countries. Capital

controls are not the result of a "prisoner's dilemma" argument. Thus, banning capital controls through some form of international agreement is not welfare improving.

Related Literature This paper is related to the vast literature on one-sided lack of commitment. Several contributions emphasize the optimality of *back-loading* in such economies because it relaxes future commitment problems. In particular, with respect to government lack of commitment in an open economy context, the closest works to this paper are Aguiar, Amador and Gopinath (2009) and Aguiar and Amador (2011b). Both papers, following Thomas and Worrall (1994), study the best equilibrium outcome of a game between a benevolent domestic government and competitive foreign investors. Abstracting from different incentives between the government and the households in the country, competitive private domestic agents in the country are not explicitly modeled. The main contribution of my paper is to explicitly model the behavior of the private domestic sector. This allows me address the central question of the paper: How should capital income earned by foreigners and domestic residents be taxed? In fact, as noted in Gourinchas and Jeanne (2011), these papers "*rely, implicitly or explicitly, on the assumption that the government can control the volume of net capital flows. This is not true in the frictionless neoclassical model, because the accumulation of reserves by the government should be offset one-for-one by higher capital inflows. This must be prevented by friction, either natural (low financial development) or policy-induced (capital controls)*"[3] . In this paper, I characterize the optimal capital controls.

This paper is also closely related to Aguiar and Amador (2011a). They consider the same policy game as I do in this paper, but they focus on a different aspect of the optimal policy. Their main result is that labor income taxes are front-loaded when the domestic stand-in household is impatient relative to the international interest rate. In particular, there are no taxes on labor and positive taxes on capital in the steady state, the opposite of what happens in a closed economy Ramsey problem with commitment. They do not characterize the path of capital income taxes for domestic and foreign investors on the transition to the steady state of the economy.

This paper is also related to the Ramsey taxation literature, see Chari and Kehoe (1999). In particular, the optimal Ramsey policy for a SOE when the government can

[3] See also Jeanne (2011)

commit has been studied by Correia (1996) and Atkeson et al. (1999). This paper also relates to the literature that studies optimal policy when the government cannot commit in a closed economy, as in Chari and Kehoe (1990), Phelan and Stacchetti (2001), Fernandez-Villaverde and Tsyvinski (2002), Reis (2008) and Benhabib and Rustichini (1997). Quadrini (2005) and Klein et al (2005) consider a two-country open economy environment, but restrict themselves to a Markov-Perfect equilibrium and assume that the government must balance its budget period by period.

Other papers examine reasons why it might be optimal to impose capital controls. The recent literature on "macro-prudential" policies emphasizes the beneficial role of capital controls in economies with credit market frictions and pecuniary externalities, see for instance Caballero and Lorenzoni (2007), Jeanne and Korineck (2010), Mendoza and Bianchi (2011) and the related recent IMF policy work by Ostry et al (2010). Closely related are Jeske (2006) and Wright (2006), who show how subsidies to capital inflows can be welfare improving in an open economy environment with decentralized credit markets and limited enforcement of debt contracts because the subsidies correct for a pecuniary externality. In this paper, I abstract from the existence of a pecuniary externality. Finally, Costinot, Lorenzoni and Werning (2011) study optimal capital controls motivated by terms of trade manipulation. In their environment, capital controls are inefficient and there are gains from coordination. In a two-country extension of my environment, I show that this result no longer holds; capital controls arise also under coordination.

The rest of the paper is organized as follows. In section 2, I describe the environment and I define a sustainable equilibrium for the policy game. In section 3, I characterize the best equilibrium outcome and prove the main results of the paper. In section 4, I compare the best equilibrium outcome with one in which the government cannot use capital controls for economies under various time profiles for TFP. In section 5, I generalize the results for a general equilibrium 2 country model. Section 6 concludes.

3.2 Environment

Consider a SOE version of the economy considered in Chari and Kehoe (1999) and Phelan and Stacchetti (2001). Time is discrete and denoted by $t = 0, 1, ...$ There are three

agents in the economy: a continuum of homogeneous domestic households, a domestic government, and a risk-neutral foreign lender. The preference of each domestic household over sequences of consumption, hours worked, and public good are represented by a time-separable utility function:

$$\sum_{t=0}^{\infty} \beta^t \left[U(c_t, n_t) + G(g_t)\right] \tag{3.1}$$

where $\beta \in (0,1)$ is the discount factor, c_t is consumption, n_t are hours worked, and g_t is the public good.

Assumption 3.1. *$U(c,n) : \mathbb{R}_+^2 \to \mathbb{R}$ is strictly increasing in c and decreasing in n, strictly concave, C^1, and satisfies the following Inada conditions: $\lim_{c\to 0} U_c(c,n) = +\infty$ $\forall n$, $\lim_{n\to 0} U_n(c,n) = 0$ $\forall c$, and $\lim_{n\to \bar{N}} U_n(c,n) = -\infty$ $\forall c$. $G(g) : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, C^1, and $\lim_{g\to 0} G'(g) = +\infty$.*

In each period the unique final good y_t can be produced domestically using a constant return to scale technology:

$$y_t = F(K,N) = K^{\alpha} N^{1-\alpha} + (1-\delta_k)K \tag{3.2}$$

where K and N are aggregate capital and hours worked respectively, $\alpha \in (0,1)$, and $\delta_k \in (0,1]$ is the capital depreciation rate.

Each domestic households is initially endowed with k_0 unit of capital installed in the domestic technology. Foreigners have k_0^* unit of capital installed in the domestic technology. We will denote the total capital installed in the domestic technology as $K = k + k^*$. Physical capital fully depreciates within the period.

The government is benevolent. In each period, it can make lump-sum transfers T_t to the households, but it cannot impose lump-sum taxes. To finance government expenditures, g_t, it must levy linear taxes on labor and capital income. Thus, a policy for the government is $\pi_t = (\tau_t^n, \tau_t^k, \tau_t^a, \tau_t^*, T_t, \delta_t) \in \Pi \equiv \mathbb{R}^5 \times [0,1]$, where:

1. $\tau_t^n \in [\underline{\tau}, 1]$ is the labor income tax.

2. $\tau_t^k \in [\underline{\tau}, 1]$ is the tax on capital income earned in the country for domestic households

3. $\tau^a \in [\underline{\tau}, 1]$ is the tax on capital income earned abroad for domestic households

4. $\tau^* \in [\underline{\tau}, 1]$ is the tax on capital income earned in the country for foreign investors

5. $T_t \geq 0$ are lump-sum trasfers to the domestic households

6. $\delta \in [0, 1]$ is the fraction of debt the government is repaying (partial default).

where $\underline{\tau} < 0$ is some lower bound on taxes that will be chosen sufficiently low so that it won't bind. I fix the upper bound of the tax rate to be equal to one in order to be able to characterize the worst equilibrium of the game analytically. The results derived in section 3 can be generalized to the case where taxes are bounded by some $\bar{\tau} < 1$ as in Phelan and Stacchetti (2001).

The government budget constraint in period t is:

$$\delta_t b_t + g_t + T_t = \tau_t^n w_t n_t + \tau_t^k R_t k_t + \tau_t^a R^* a_t + \tau_t^* R_t k_t^* + q_{t+1} b_{t+1} \tag{3.3}$$

The government does not posses a *commitment technology*. Taxes and the default decision are chosen sequentially, *after* capital has been installed and capital therefore is inelastically supplied for the period. This is the source of the time inconsistency problem.

The timing protocol of the game is as follows:

1. Enter the period with (k_t, k_t^*, a_t, b_t)

2. The government sets $\pi_t = (\tau_t^n, \tau_t^k, \tau_t^a, \tau_t^*, T_t, \delta_t) \in \Pi$

3. Each households chooses $(c_t, n_t, k_{t+1}, s_{t+1})$ that satisfy the budget constraint

$$c_{t+1} + k_{t+1} + a_{t+1} \leq (1 - \tau_t^k) R_t k_t + (1 - \tau_t^a) R^* a_t + (1 - \tau_t^n) w_t n_t + T_t \tag{3.4}$$

   and foreigners choose $k_{t+1}^* \geq 0$.

4. Finally the government collects tax revenues and finances g_t with tax collection according to (3.3).

Before formally defining a sustainable equilibrium for the policy game, it is useful to define a competitive equilibrium for the economy

3.2.1 Competitive Equilibria

Definition 3.1. *Given* (k_0, k_0^*, a_0, b_0) *and a sequence of international interest rates* $\{R_t^*\}_{t=0}^{\infty}$*, a competitive equilibrium for the SOE is a sequence of policies* $\{g_t, \pi_t\}_{t=0}^{\infty}$*, an allocation for the stand-in domestic household* $\{c_t, n_t, k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$*, an allocation for the foreign investors* $\{k_t^*\}_{t=0}^{\infty}$*, and prices* $\{w_t, R_t, q_{t+1}\}_{t=0}^{\infty}$ *such that:*

1. *Given prices and policies, the stand-in household maximizes his utility (3.1) subject to the sequence of budget constraints (3.4)*

2. *Optimality conditions for the foreign investors:*

$$\begin{aligned} R_{t+1}^* &\geq (1-\tau_{t+1}^*)R_t, \quad "="if k_{t+1}^* > 0 \quad \forall t \geq 0 \\ q_{t+1} &= \frac{\delta_{t+1}}{R_{t+1}^*} \quad \forall t \geq 0 \end{aligned}$$

   are satisfied

3. $w_t = F_n(K_t, n_t), \quad R_t = F_k(K_t, n_t) \quad \forall t \geq 0$

4. *The government budget constraint (3.3) holds* $\forall t \geq 0$.

If an allocation x is implementable with the "FDI" equilibrium defined above, then $\tilde{x} = \{c_t, n_t, g_t, K_t = k_t + k_t^*, B_t = k_t^* - a_t\}_{t=0}^{\infty}$ can be decentralized with an alternative equilibrium notion in which all the investment in the country is done by domestic households that are allowed to *borrow*, and the government tax instruments are: tax on labor, $\tilde{\tau}_t^n$, and capital income (all earned by domestic residents), $\tilde{\tau}_t^k$, and tax on foreign borrowing and lending, $\tilde{\tau}_t^B$. The household budget constraint in this case can then be written as

$$c_t + K_{t+1} - B_{t+1} \leq (1-\tilde{\tau}_t^k)R_t K_t + (1-\tilde{\tau}_t^k)w_t n_t - (1-\tilde{\tau}_{t+1}^B)R_t^* B_t$$

There are capital controls if $\tilde{\tau}_{t+1}^B \neq 0$[4] . This decentralization may offer a clearer interpretation of capital controls. However, if one assumes, as it is natural, that private agents also cannot commit to repay their debt obligations and the only punishment they face after default is the exclusion from credit markets in the future, then I can

[4] It is easy to check that $\tilde{\tau}_{t+1}^B \neq 0 \iff \tau_{t+1}^* \neq \tau_{t+1}^k$. In particular there are capital controls on inflows $\tilde{\tau}_{t+1}^B < 0 \iff \tau_{t+1}^* > \tau_{t+1}^k$.

show that no private debt can be supported in equilibrium. That is, the debt limit would be $B_{t+1} \leq 0$, as in my "FDI" decentralization, where I impose that $a_t \geq 0$. So, if private agents cannot commit, the only form of *private* foreign liability that can be supported in a deterministic equilibrium is FDI. I will later show that the domestic government can instead support a positive amount of foreign debt because a government default can trigger a self-fulfilling expectation of high capital income taxes. Therefore, defaulting is more costly for the government than for an individual atomistic agent. This observation can rationalize the fact that most of the international capital flows to emerging economies are either FDI or are intermediated through the government.

3.2.2 Sustainable Equilibria

I now formally define a symmetric[5] subgame perfect equilibrium for the policy game. I will use the concept of a sustainable equilibrium (SE) developed in Chari and Kehoe (1990) in order to accomodate the fact that in the game I consider there is one strategic player (the government), and a continuum of non strategic players (the domestic households and the foreign investors). Let $h^t = (\pi_0, \pi_1, ..., \pi_t) \in H^t$ denote a public history of events up to time t, and let $h^{-1} = \emptyset$. The government strategy, $\sigma = \{\sigma_t\}_{t=0}^{\infty}$, the domestic household's strategy, $f = \{f_t\}_{t=0}^{\infty}$, and the foreigners' strategy, $f^* = \{f_t^*\}$, are measurable functions of public histories[6] . That is:

$$\sigma_t : H^{t-1} \to \Pi, \qquad f_t : H^t \to \mathbb{R}^4, \qquad f_t^* : H^t \to \mathbb{R}$$

with $\sigma_t = \pi_t$, $f_t = (c_t, n_t, k_{t+1}, a_{t+1})$, and $f_t^* = k_{t+1}^*$. With some abuse of notation, I will use the same letter to refer to strategies and outcomes. For any strategy, define the on path utility for the domestic household after history h^{t-1} as follows:

$$W(\sigma, f, f^*|h^{t-1}) = \sum_{r=0}^{\infty} \beta^r [U(c_{t+r}(h^{t+r}), n_{t+r}(h^{t+r})) + G(g_{t+r}(h^{t+r}))]$$

where $h^{t+r} \succeq h^{t-1}$ is induced by σ. Finally, define $p = (w, R, q)$ as follows:

$$\begin{aligned} w_t(h^t; \sigma, f, f^*) &= F_n(K_t(h^{t-1}), n_t(h^t)), \quad R_t(h^t; \sigma, f, f^*) = F_k(K_t(h^{t-1}), n_t(h^t)) \\ q_t(h^t; \sigma, f, f^*) &= \frac{\delta_{t+1}(h^t)}{R_{t+1}^*} \end{aligned}$$

[5] All domestic households follows the same strategy. The same is true for the foreign investors.

[6] Restricting households and foreign investors to condition their actions on the public history is without loss of generality, see Phelan and Stacchetti (2001).

Definition 3.2. *A sustainable equilibrium (SE) is a triple* (σ, f, f^*) *such that* $\forall\ (t, h^{t-1})$ *and associated* $(k_t(h^{t-1}), k_t^*(h^{t-1}), a_t(h^{t-1}), b_t(h^{t-1}))$*:*

1. $\forall \tilde{\sigma} \quad W(\sigma, f, f^*|h^{t-1}) \geq W(\tilde{\sigma}, f, f^*|h^{t-1})$
2. $\{f_{t+r}(h^{t+r})\}_{r=0}^{\infty}, \{f_{t+r}^*(h^{t+r})\}_{r=0}^{\infty}, \{\pi_{t+r}(h^{t+r-1})\}_{r=0}^{\infty}, \{p_{t+r}(h^{t+r})\}_{r=0}^{\infty}$*, where* $h^{t+r} \succeq h^{t-1}$ *is induced by* σ*, constitutes a competitive equilibrium.*

In a SE, the requirement of optimality for the strategic player, the government, is the standard game theoretic one. Domestic households and foreign investors are non-strategic: they take current policies, prices and the evolution of future histories as unaffected by their actions. Optimality for these players is captured by the requirement that equilibrium strategies induce allocations and prices that constitute a competitive equilibrium. Moreover, this notion of equilibrium requires that the government, the domestic households and the foreign investors behave optimally after every history, not just the ones induced by the equilibrium strategy σ. This perfection requirement captures the lack of a commitment technology on the side of the government.

In the rest of the paper, I will characterize the equilibrium outcome of the *best* SE of the policy game, that is, the one that attains the highest utility for the stand-in domestic household. I then contrast it with the solution of a Ramsey problem.

3.3 Characterization of the Best Equilibrium Outcome

Following the seminal work of Abreau (1988) and Chari and Kehoe (1990), I can characterize the *best* SE outcome by solving a simple programming problem, using the fact that any equilibrium outcome path can be supported with a trigger strategy that calls for the worst SE (in terms of utility) after any deviation.

3.3.1 Set of Equilibrium Outcomes

First, I can characterize the set of allocations that can be implemented as a competitive equilibrium as follows:

Lemma 3.1. *Given* τ_0^k*,* τ_0^a*,* τ_0^**,* δ_0*, and* (k_0, k_0^*, a_0, b_0)*, an* interior *allocation* $x \equiv$

$\{c_t, n_t, k_{t+1}, k^*_{t+1}, a_{t+1}, g_t\}_{t=0}^{\infty}$ *is part of a competitive equilibrium for the SOE iff* x *satisfies*

$$\sum_{t=0}^{\infty} \beta^t \left[U_{ct}c_t + U_{nt}n_t\right] = A_0 \tag{3.5}$$

where $A_0 = U_{c0}\left[(1-\tau_0^k)F_{k0}k_0 + (1-\tau_o^a)R^*a_0\right]$, $\forall\ t \geq 1$

$$\sum_{s=0}^{\infty} \beta^s \left[U_{ct+s}c_{t+s} + U_{nt+s}n_{t+s}\right] \geq 0 \tag{3.6}$$

$$\sum_{t=0}^{\infty} Q_t \left[g_t + c_t + k_{t+1} + R^*k_t^* - F(K_t, n_t)\right] - R^*a_0 + \tau_0^* F_k(K_0, n_0)k_0^* = -b_0\delta_0 \tag{3.7}$$

where $Q_0 = 1$ *and* $\forall t \geq 1$ $Q_t \equiv \prod_{s=1}^{t}(1/R_s^*)$.

Proof 3.1. *Appendix.* □

Denoting $\underline{V}(k, k^*, a, b)$ as the value of the *worst* SE starting from (k, k^*, a, b), I can characterize the set of equilibrium outcome paths as follows.

Lemma 3.2. *Given* (k_0, k_0^*, a_0, b_0), *an* interior *allocation* $x \equiv \{c_t, n_t, k_{t+1}, k^*_{t+1}, a_{t+1}, g_t\}_{t=0}^{\infty}$ *is part of the outcome path of a SE iff it satisfies (3.5), (3.6), (3.7) with* $\delta_0 = 0$ *if* $b_0 > 0$ *and* $\delta_0 = 1$ *if* $b_0 \leq 0$, *and* $\tau_0^k = \tau_0^* = \tau_0^a = 1$, *and* $\forall t \geq 1$, *it satisfies the "sustainability" constraint:*

$$\sum_{s=0}^{\infty} \beta^s [U(c_{t+s}, n_{t+s}) + G(g_{t+s})] \geq \underline{V}(k_t, k_t^*, a_t, b_t) \tag{3.8}$$

Proof 3.2. *Standard.* □

Given this characterization of the set of allocations that can be supported as a SE, it follows that the best equilibrium outcome path solves the following programming problem:

$$\overline{V}(k_0, k_0^*, a_0, b_0) = \max_{\{c_t, n_t, k_{t+1}, a_{t+1}, k^*_{t+1}, g_t\}} \sum_{t=0}^{\infty} \beta^t [U(c_t, n_t) + G(g_t)] \tag{P}$$

subject to (3.5), (3.6), (3.7) and (3.8) where $\delta_0 = 0$ when $b_0 > 0$ and $\delta_0 = 1$ when $b_0 \leq 0$, and $\tau_0^k = \tau_0^* = \tau_0^a = 1$. To characterize (P), one has to characterize the value of the *worst* SE, $\underline{V}$. One way to do this is to use the technique developed in Abreu et

al (1990) and Phelan and Stacchetti (2001) to numerically characterize the set of SE of the policy game. Given the unitary upper bound on capital income taxes, τ_t^k, τ_t^a and τ_t^*, I can characterize the worst SE of the policy game analytically. The predictions of the model do not change if I exogenously specify a function $\underline{V}(k, k^*, a, b) : \mathbb{R}^4 \to \mathbb{R}$ to characterize the punishment for a government that deviates from the plan, as in Benhabib and Rustichini (1997).

3.3.2 Worst SE of the Policy Game

Allowing the government to tax 100 percent of capital income[7] , $\tau_t^k, \tau_t^a, \tau_t^* \leq 1$, the worst SE of the policy game is quite simple. On path (i) in any period, the government is taxing capital income of any sources to the maximum extent; (ii) no new investment is undertaken by domestic households and foreign investors, so the capital stock in the country is depreciating at a rate δ_k, $K_t = (1-\delta_k)^t K_0$; (iii) the government is defaulting on its foreign debt whenever $b \geq 0$ and it is not allowed to borrow abroad, but it can save abroad to smooth public and private consumption over time; and (iv) labor income taxes and lump-sum transfers solve a simple static Ramsey problem.

Formally, let $(\underline{\sigma}, \underline{f}, \underline{f^*})$ be the strategy that attains the worst SE. The strategy for the domestic household is defined as follows: $\forall\ t \geq 0$, $\forall\ h^t \in H^t$

$$\begin{aligned} \underline{k}_{t+1}(h^t) &= (1-\delta)\underline{k}_t(h^{t-1}) \\ \underline{a}_{t+1}(h^t) &= 0 \end{aligned}$$

and $\underline{c}(h^{t-1}, \pi_t)$ and $\underline{n}(h^{t-1}, \pi_t)$ are such that they solve the following two equations:

$$\begin{aligned} v'\left(\underline{n}(h^{t-1}, \pi_t)\right) &= u'\left(\underline{c}(h^{t-1}, \pi_t)\right)(1-\tau^n(\pi_t))\, F_n\left(\underline{n}(h^{t-1}, \pi_t), K(h^{t-1})\right) \\ \underline{c}(h^{t-1}, \pi_t) &= (1-\tau^n(\pi_t))\, F_n\left(\underline{n}(h^{t-1}, \pi_t), K(h^{t-1})\right)\underline{n}(h^{t-1}, \pi_t) + T(\pi_t) \end{aligned}$$

The investment strategy for the foreign investor is simply given by $\underline{k}^*_{t+1}(h^t) = (1-\delta)\underline{k}^*_t(h^{t-1})\ \forall\ t \geq 0$, $\forall\ h^t \in^t H^t$.

[7] For the characterization of the worst SE it does not matter if the government imposes taxes on capital income or wealth, as long as I allow the maximal rate to be 100%.

The strategy for the government is given by:

$$\begin{aligned}
\underline{\tau}^k(h^{t-1}) &= \underline{\tau}^*(h^{t-1}) = \underline{\tau}^a(h^{t-1}) = 1 \quad \forall\, t \geq 0, \forall\, h^t \in H^t \\
\underline{\delta}(h^{t-1}) &= \begin{cases} 1 & \text{if } h^{t-1} \text{ is associated with } b(h^{t-1}) \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, t \geq 0, \forall\, h^t \in H^t
\end{aligned}$$

Finally, $\forall t \geq 0$, $\forall h^{t-1}$ and associated $(k_t(h^{t-1}), k_t^*(h^{t-1}), a_t(h^{t-1}), b_t(h^{t-1}))$, let $\underline{\tau^n}(h^{t-1})$ and $\underline{T}(h^{t-1})$ be implied by the solution to the following *static* Ramsey problem:

$$\omega_t(K, \text{fa}) = \max_{c,n,g} U(c,n) + G(g) \tag{3.9}$$

subject to

$$\begin{aligned}
u'(c)c - v'(n)n &\geq 0 \\
c + g &\leq A_t F(K,n) + R_t^* \text{fa} + \underline{b}'(h^{t-1})
\end{aligned}$$

given that $\underline{b}(h^{t-1})$ solves

$$\Omega(\underline{K}(h^{t-1}), \text{fa}(h^{t-1})) = \max_{b' \leq 0} \omega_t(\underline{K}(h^{t-1}), R^* \text{fa}(h^{t-1}) + b') + \beta\Omega((1-\delta)\underline{K}(h^{t-1}), -b') \tag{3.10}$$

where $fa \equiv a + \max\{0, -b\}$.

Lemma 3.3. *$\forall$ (k, k^*, a, b), $(\underline{\sigma}, \underline{f}, \underline{f^*})$ is the worst SE of the policy game, and its value is given by*

$$\underline{V}(k, k^*, a, b) = \Omega(k + k^*, a + \max\{-b, 0\})$$

Proof 3.3. *Appendix.* □

Then, given $\underline{V}(k, k^*, a, b) = \Omega(k + k, a + \max\{0, -b\})$, I can characterize the best equilibrium outcome by solving (P).

3.3.3 Solution with Commitment

Before I move to characterizing the best equilibrium outcome, I first consider the solution to the problem when the government has a commitment technology. The optimal plan under commitment, the Ramsey plan, is the solution to a relaxed version of the probelm (P), where we can drop the sustainability constraints (3.6) and (3.8). The next proposition characterizes the solution to the Ramsey problem for the SOE.

Proposition 3.1. *[Ramsey Solution] The Ramsey plan and associated tax rates are such that*

1. $\tau_t^* = 0 \ \forall t \geq 2$
2. *If* $\beta R_t^* = 1$ *or* $U(c,n) = \frac{c^{1-\eta}}{1-\eta} - v(n)$ *for some* $\eta \geq 0$ *then* $\tau_t^k = 0 \ \forall t \geq 2$
3. *If* $\beta R_t^* = 1 \ \forall t$ *then* $c_t = c$, $n_t = n$, $g_t = g$ *and* $K_t = K$ *for all* $t \geq 1$.
4. *If* $\lim_{t\to\infty} \beta R_t^* < 1$ *then* $\{c_t^R\} \downarrow 0$, $\{g_t^R\} \downarrow 0$, $\{n_t^R\} \to \bar{N}$, $\{K_t^R\} \to K^R > 0$

Proof 3.4. *See Atkeson et al (1999) and Correia (1996).* □

The first part of the proposition states that if the government has commitment, then it is not optimal to tax capital of foreign investors. That is, the marginal product of capital installed in the country is equal to the international interest rate

$$F_{kt} = R_t^* \quad \forall t \geq 2 \tag{3.11}$$

The second part of the proposition says that if $\beta R^* = 1$ or $u(c)$ is CRRA then it is also optimal to not tax capital income of domestic residents for all $t \geq 2$. Hence, in this case, it is not optimal to impose capital controls. The case with $u(c)$ being CRRA is analogous to the closed economy set-up in Chari and Kehoe (1999). The case with $\beta R^* = 1$ strengthen the seminal Chamley (1986) - Judd (1987) result about the optimality of zero capital income taxes in steady state for a closed economy under general preferences. In fact, if $\beta R_t = 1$ for $t \geq 1$, then the economy reaches its steady state in period $t = 1$ and consuption, leisure, government consumption and capital stock are all constant for all $t \geq 1$. See Atkeson et al (1999) for a complete discussion of these results. Finally, the last part of the proposition states that if $\lim_{t\to\infty} \beta R_t^* < 1$ then consumption - both private and public - and leisure are converging to zero. Because the domestic household is relatively impatient with respect to the international interest rate, it is optimal to front-load consumption and leisure. The economy then is running a current account deficit that is initially large and later it repays its debt.

Before constrasting the solution with commitment to the one without, it is usefull to understand under which conditions (if any) the Ramsey plan can be sustained as an

equilibrium outcome when the government lacks commitment, that is, when $\forall t \geq 1$

$$V_t^R \equiv \sum_{s=0}^{\infty} \beta^s \left[U(c_{t+s}^R, n_{t+s}^R) + G(g_{t+s}^R)\right] \geq \underline{V}(k_t^R, k_t^{*R}, a_t^R, b_t^R)$$

where a superscript R denotes the Ramsey plan.

Proposition 3.2. *[Sustainability of the Ramsey Plan] (i) Suppose first that $\beta R_t^* = 1$ $\forall t \geq 1$. Then, $\forall \beta \in (0,1)$ $\exists$ $\bar{A} > 0$ sufficiently large such that $\forall\, (a_0 - b_0) \geq \bar{A}$ the Ramsey plan is sustainable. (ii) If, instead, $\beta R_t^* < 1$, $\forall (k, k^*, a, b)$ the solution of the problem with commitment is not sustainable.*

Proof 3.5. *Appendix.* □

Consider first the case with $\beta R_t^* = 1$ $\forall t \geq 1$. From part 3 of Proposition 1, it follows that one only has to check that

$$V_1^R \equiv \frac{U(c^R, n^R) + G(g^R)}{1-\beta} \geq \underline{V}(k^R, k^{*R}, a^R, b^R)$$

If the government and/or the domestic household has a sufficiently large assets position then the solution to the problem with commitment is sustainable. If the government asset position is sufficiently large (i.e. if $-b_0$ is sufficiently large), then it can finance the desidered amount of public expenditure using almost entirely public savings, without having to rely on distortionary labor income taxes. In this case, the incentive to renege on the promise to not tax capital income is weak because the benefits from substituting distortionary labor tax with a non-distortionary capital levy are small. The exact same argument works if household wealth at $t = 0$ is sufficiently large. In fact, at time zero the government can impose a full capital levy and save its proceeds. Thus, part (i) of Proposition 1 emphasizes the importance of government savings as a determinant of the ability of the government to maintain its promises: the higher are the assets of the government (the lower is its debt), the lower is its incentive to renege on its promises not to tax capital income.

Instead, if $\beta R_t^* < 1$, the continuation of a Ramsey plan is *not* sustainable, no matter what the initial asset positions of the domestic households and the governments are. As shown in part 4 of Proposition 1, if $\beta R_t^* < 1$ leisure and consumption (public and private) are converging to zero as time goes to infinity, while the capital stock is converging to

a positive constant. Thus, the continuation utility for the stand-in households in the country is converging to its lower bound

$$\lim_{t\to\infty} V_t^R = \frac{u(0) - v(\bar{N}) + G(0)}{1-\beta} = 0$$

Because the domestic capital stock is owned entirely by foreigners, the goverment will then find it optimal to impose a capital levy and default on its debt, achieving a higher continuation utility, $\Omega(K, FA < 0) = \Omega(K, 0) > 0$.

3.3.4 Optimality of Capital Controls

I now characterize the best sustainable outcome path under the assumption that preferences are separable with respect to consumption and leisure and they are characterized by constant elasticity of substitution:

Assumption 3.2. *$U(c,n) : \mathbb{R}_+^2 \to \mathbb{R}$ is given by $U(c,n) = \frac{c^{1-\eta}}{1-\eta} - \chi\frac{n^{1+\gamma}}{1+\gamma}$ with $\gamma > 0$ and $\eta \in (0,1)$.*

The main result of the paper is that it is optimal to more heavily tax the capital income of foreigners whenever the sustainability constraint is binding.

Proposition 3.3. *[Capital Controls] Under A2, the best sustainable outcome path is such that $\forall t \geq 2$*

1. *$\tau_t^* \geq 0$, strictly if the sustainability constraint binds at t*
2. *$\tau_t^k \leq \tau_t^*$, strictly if the sustainability constraint binds at t*

Proof 3.6. *Appendix.* □

A formal proof of the statement that deals with the non-convexity of the constraint set is relagated to the appendix. To get some intuition, let λ, $Q_t\phi_t$,$\beta^t\mu_t$ be the Lagrangian multipliers associated with the implementability constraint (3.5), the consolidated budget constraint (3.7), and the sustainability constraint (3.8), respectively. A necessary condition for an optimal allocation is:

$$Q_t\phi_t\left[F_{kt+1} - R_{t+1}^*\right] = \beta^{t+1}\mu_{t+1}\Omega_{k,t+1} \qquad \forall t \geq 1 \tag{3.12}$$

Marginally increasing the stock of capital owned by foreigners in the country increases the resources available to the country as a whole by $F_{kt+1} - R^*_{t+1}$, i.e. the output produced minus the compensation (net of taxes) to foreign investors. This increases the objective function in (P) by $Q_t\phi_t\left[F_{kt+1} - R^*_{t+1}\right]$. When the government cannot commit, there is an extra cost for increasing the stock of capital. This is captured by the term on the right hand side of (3.12). Increasing the stock of capital tightens the sustainability constraint. This introduces a wedge between the international interest rate and the marginal product of capital in the SOE that is not present in the case with commitment. Combining this optimality condition with the necessary first order condition for foreign investors which must hold in any interior competitive equilibrium,

$$R^*_{t+1} = (1 - \tau^*_{t+1})F_{kt+1} \tag{3.13}$$

I can obtain the following expression for τ^k_{t+1}:

$$\tau^*_{t+1} = \frac{\beta^{t+1}\mu_{t+1}}{Q_t\phi_t}\frac{\Omega_{k,t+1}}{F_{k,t+1}} \in [0,1) \tag{3.14}$$

Thus, the foreign investors capital income is taxed at a strictly positive rate whenever the sustainability constraint is binding, i.e. when $\mu_{t+1} > 0$.

To understand why it is optimal to tax domestic residents at a lower rate than foreign investors, consider the following necessary condition for an interior optimal allocation:

$$u'(c_t) = \beta R^*_{t+1}u'(c_{t+1}) + \frac{\mu_{t+1}}{1 + (1-\eta)\lambda + M_t}R^*_{t+1}u'(c_{t+1}) \tag{3.15}$$

where $M_t \equiv \sum_{s=0}^{t}\mu_s$ is the cumulative Lagrangian multiplier on the sustainability constraints (3.8). Condition (3.15) equates the marginal cost of postponing consumption from t to $t+1$, $u'(c_t)$, with its marginal benefits. Postponing consumption not only increases tomorrow's utility by $\beta R^*_{t+1}u'(c_{t+1})$, but it also helps to relax the sustainability constraint at $t+1$. The latter effect is captured by the second term on the right hand side of (3.15). Combining (3.15) with the Euler equation for the domestic household which must hold in any interior competive equilibrium

$$u'(c_t) = \beta u'(c_{t+1})(1 - \tau^k_{t+1})F_{kt+1} \tag{3.16}$$

and (3.13), I obtain the following expression for the optimal τ^k_{t+1} $\forall t \geq 1$:

$$\tau^k_{t+1} = \tau^*_{t+1} - (1 - \tau^*_{t+1})\frac{\mu_{t+1}}{1 + (1-\eta)\lambda + M_t} \leq \tau^*_{t+1} \tag{3.17}$$

Thus, because $\tau^*_{t+1} \leq 1$, it follows from (3.17) that whenever the sustainability constraint is binding, $\tau^k_{t+1} < \tau^*_{t+1}$. That is, the capital income of domestic residents is taxed at a lower rate than that of foreigners or, equivalently, there are capital controls on inflows.

The intuition for the optimality of capital controls is that when the fraction of installed capital owned by foreigners, k^*/K, is high, the government finds it more difficult to resist the temptation to impose a big capital levy. In fact, if the ratio k^*/K is high, imposing a capital levy allows the government to raise funds via a non distortionary tax, and to transfer resources from foreign investors to domestic households. As k^*/K decreases, the second effet is no longer present. Hence, it is easier for the government to credibly commit to low capital income taxes[8] .

The optimality of capital controls is a robust feature of the environment. Above, I characterized the best SE of the policy game, using the worst SE to punish a deviation by the government. If instead I followed Benhabib and Rustichini (1997) and exogenously specified a continuation value $\Omega(K, fa) : \mathbb{R}^2_+ \to \mathbb{R}$ for a government that chooses to deviate from the equilibrium path of plays, I can show the result still holds; the conclusions of Proposition 3 hold true for all Ω that are strictly increasing in K, (weakly) increasing in fa, and differentiable in K[9] . Therefore, my results are not specific to using the value of the worst SE as a punishment for a deviation. In particular, optimality of capital controls holds if Ω is such that $\forall K, fa, fa'$ $\Omega(K, fa) = \Omega(K, fa')$,as in Aguiar and Amador (2011a, 2011b).

An interesting feature of the model is that, despite the fact that the government can save after a deviation (default), a strictly positive level of debt can be sustained in equilibrium, contrary to the finding in Bulow and Rogoff (1989). In fact, in my formulation, the prospect of a default is worse than in Bulow-Rogoff (1989) in that a deviation by the government not only triggers exclusion from borrowing, but also a drop in investment and therefore in output. However, despite the possibility of sustaining positive government debt, it is efficient to run down public debt. In closely related papers, Aguiar and Amador (2011a, 2011b) emphasize the role of public foreign asset accumulation in relaxing the sustainability constraint for the government. The same

[8] See Gourinchas and Jeanne (2005) for a static model with a similar mechanism.

[9] The differentiability of Ω is just a technical requirement and it is not necessary, as shown in the proof of Proposition 3 in the appendix. In fact, the value of the worst SE is not differentiable at $(K, 0)$ $\forall K \geq 0$.

conclusion carries over to my environment with one caveat. As argued in Proposition 2 part 1, a higher government assets position - or lower public debt - leads to loosening of the sustainability constraints. In fact, the government has little to gain from raising funds through a non-distortionary taxsince it can finance g_t mainly using its positive assets position and relying very little on distortionary labor income taxation. However, I can show that the government is accumulating assets only if the domestic capital stock is owned entirely by the domestic households ($k/K = 1$). This is the content of Proposition 4.

Proposition 3.4. *The best sustainable outcome path is such that* $\forall\ t \geq 1$*, whenever the sustainability constraint is binding, either (i)* $k_t = K_t$ *and* $b_t \in \mathbb{R}$*,* $a_t \geq 0$*, or (ii)* $k_t < K_t$ *and* $b_t \geq 0$*,* $a_t = 0$*.*

3.3.5 Convergence to a Steady State

I now characterize the long-run behavior of the best equilibrium outcome. Are capital controls temporary or permanent? I show that if the international interest rate, R_t^*, converges to the inverse of the discount factor β, that is $\lim_{t\to\infty} \beta R_t^* = 1$, then the economy is converging to a steady state with no taxes on capital income and no capital controls. Instead, if the international interest rate is lower than the discount factor, $\lim_{t\to\infty} \beta R_t^* < 1$, then the economy is converging to a steady state with positive taxes on foreign investment and capital controls on inflows. Consider first the case with $\lim_{t\to\infty} \beta R_t^* = 1$.

Proposition 3.5. *[Steady State with* $\beta R_\infty^* = 1$*] Under A2, if* $\lim_{t\to\infty} \beta R_t^* = 1$ *then the economy converges to an interior steady state such that: (i) the sustainability constraint is not binding and (ii):* $\tau^k, \tau^{k*}, \tau^a \to 0$ *and* $\tau^n \to \tau_\infty^n > 0$*.*

Proof 3.7. *Appendix.* □

From (3.15) and the fact that $\lim_{t\to\infty} \beta R_t^* = 1$, for t sufficiently large, it follows that the consumption profile for the domestic household is increasing whenever the sustainability constraint is binding ($\mu_{t+1} > 0$) and is constant otherwise. A similar argument can be made for public consumption and leisure. Hence, $\left\{c_{t+s}, g_{t+s}, \bar{N} - n_{t+s}\right\}_{t=0}^{\infty}$ and, therefore, the continuation utility is weakly increasing. It will be strictly increasing if

the sustainability constraint is binding. The gist of the proof is to show that, in the limit, the sustainability constraint cannot bind ($\mu_t \to 0$). The logic of the proof is by contradiction. Suppose that $\lim_{t=0} \mu_t > 0$, then it must be that $c_t, g_t \to \infty$ and $n_t, K_t \to 0$. I can then show that for t large enough there is a continuation plan (with constant consumption, hours worked, and capital) which attains higher utility, is resource feasible, and is sustainable. Then the original plan cannot be optimal. Thus, when $\beta R^* = 1$, the optimal outcome path is such that consumption (public and private) and leisure are back-loaded in order to relax the sustainability constraint until it is no longer binding. When the sustainability constraint no longer binds, the optimal policy with commitment can be implemented. Therefore, hours worked and private and public consumption are constant and there are no capital income taxes for both domestic households and and foreign investors. Thus, capital controls are not a feature of the optimal policy in the long-run.

A typical best sustainable outcome path for the case with $\beta R_t^* = 1 \ \forall t \geq 1$ is displayed in Figure 1, together with the best outcome under commitment. First, notice that initially capital income of the domestic agents is taxed at a lower rate than the capital income of foreign investors. Both taxes are strictly positive and approach zero as time elapses, as stated in Proposition 5. Second, because it is optimal to back-load utility in order to relax future sustainability constraints, private and public consumption are increasing over time and converge to their steady state level from below. Hours worked are also increasing over time. This is not guaranteed, as there are two opposing forces: (i) on one hand, to relax future sustainability constraints, the government wants to backload leisure $(1 - n_t)$, and (ii) on the other hand, it is less efficient to work at early dates since capital is increasing over time.

Next, I consider the case with $\lim_{t\to\infty} \beta R_t^* < 1$. I show that in this case capital controls are a feature of the optimal policy also in the long-run.

Proposition 3.6. *[Steady State with $\beta R_\infty^* < 1$] Under A2, if $\lim_{t\to\infty} \beta R_t^* < 1$ then the economy converges to an interior steady state $(c_\infty, n_\infty, g_\infty, K_\infty) \in \mathbb{R}^4_{++}$ such that (i) the sustainability constraint is binding, and (ii) $\tau_t^* \to \tau_\infty^* > 0$, $\tau_t^k \to \tau_\infty^k < \tau_\infty^*$, and $\tau^n \to \tau_\infty^n = 0$.*

Proof 3.8. *Appendix.* □

To understand the logic behind Proposition 6, notice that because $\lim_{t\to\infty} \beta R_t^* < 1$, for T sufficiently large, $\beta R_T^* < 1$. Thus, the domestic residents are impatient relative to the international interest rate. From equation (3.15) one can see that there are two opposing forces: (i) relative impatience induces the domestic household to *front-load* consumption, and (ii) binding sustainability constraints calls for *back-loading* consumption. The same is true for government consumption and for leisure. As t goes to infinity, two things can happen: either the economy is converging to an interior steady state $(c_\infty, n_\infty, g_\infty, K_\infty) \in \mathbb{R}^4_{++}$ or it is converging to an "immiseration" steady state $(c_\infty, n_\infty, g_\infty, K_\infty) = (0, 0, 0, 0)$. The idea of the proof is to show that no optimal path can converge to the immiseration steady state. I can find a sustainable plan starting from the "immiseration" steady state that attains higher utility by having positive foreign capital invested in the country and using the increased output to increase government consumption. The government can commit not to tax this capital at expropriatory rate because the marginal return on capital in unbounded for K greater than but sufficiently close to zero. The economy then must converge to an interior steady state. For consumption to converge to some positive constant (instead of converging to zero), from (3.15), it must be that $\mu_t \to \mu_\infty > 0$. This and (3.14) imply that $\tau_t^* \to \tau_\infty^* > 0$. Moreover, combining the necessary household Euler equation, (3.16), with the one for the foreign investors, (3.13), it follows that

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta R^* \frac{1-\tau_{t+1}^k}{1-\tau_{t+1}^*} \to 1 \Rightarrow \lim_{t\to\infty} \tau_t^k = \tau_\infty^k < \tau_\infty^*$$

Thus it is optimal to have capital controls also in the long run. Finally, as shown in parallel work by Aguiar and Amador (2011b), labor income taxes converge to zero if $\beta R^* < 1$. See their paper for an exhaustive discussion of the result about labor income taxes.

In summary, the optimal policy with commitment calls for no capital controls; there are no taxes on capital income for both domestic and foreign investors. Instead, when the government cannot commit, capital controls on inflows are optimal on the transition to the steady state. In the long run, capital controls are a feature of the best equilibrium outcome only if the domestic agents are impatient relative to the international interest rate ($\beta R^* < 1$). When $\beta R^* = 1$, the government and the private domestic agents

accumulate sufficient assets that the government can credibly commit not to tax capital income[10] . In that case, both the marginal product of capital and the intertemporal marginal rate of substitution of the domestic households are equated to the international interest rate.

3.4 Comparison to an Environment without Capital Controls

To assess the potential welfare gains of adopting the optimal policy, I compare the best equilibrium outcome with the one in which the government cannot impose capital controls, i.e. where $\tau_t^k = \tau_t^*$. This restriction eliminates the possibility for the government to manipulate the saving decision of the domestic households. In any equilibrium outcome, it must be that

$$u'(c_t) = \beta R_{t+1}^* u'(c_{t+1}) \tag{3.18}$$

That is, there is no wedge between the international interest rate and the intertemporal marginal rate of substitution for the domestic household.

Assuming that $\beta R_t^* = 1 \ \forall t \geq 1$, the optimal outcome path for the economy without capital controls is shown in Figure 2, compared to the best outcome path for the economy analyzed in the prevous sections. To allow for a clean comparison, I assume that the government can impose a very high capital levy on the domestic household ($\tau_0^k = 1$), so that the right hand side of (3.5) is always equal to zero, making the first order conditions with respect to time zero variables symmetric to the ones for $t \geq 1$. When $\tau_t^k = \tau_t^* \ \forall t \geq 1$, consumption is constant over time because of (3.18). Leisure is back-loaded; this implies that domestic households are working more in periods with low levels of capital than in periods when the capital stock is high. Moreover, it also implies that if the economy is experiencing positive TFP growth, households work less when productivity is high. This is because, with a constant consumption profile, the government has a stronger incentive to back-load labor and public consuption to increase future continuation utility and to relax future sustainability constraints. To do

[10] As show in Proposition 2, if $\beta R^* = 1$ and the economy has a sufficiently large foreign asset position, the transition to the steady state can last one period only. It is slower for lower initial foreign asset positions.

so, the labor income tax will be increasing over time, while it is decreasing over time if the government can impose capital controls. As shown in Figure 2, the economy is converging to a steady state with no capital income taxes for both domestic households and foreigners. So, the sustainability constraint is not binding in the limit. The difference with the case where the government is allowed to impose capital controls is that the government must rely too heavily on back-loading government consumption and leisure when it cannot manipulate the consumption profile of the domestic households. This generates welfare losses.

An interesting finding is that the gains from being able to tax capital income of domestic and foreign residents at different rates crucially depend on the profile of TFP over time. Modify the production function (3.2) to be such that $y_t = Z_t F(K_t, n_t)$ for some deterministic sequence for TFP, $\{Z_t\}_{t=0}^{\infty}$, such that $Z_t \to Z_\infty$ as $t \to \infty$. With this specification, all the results derived in the previous sections go through. In my numerical examples, the gains - measured in terms of CEV in consumption - are modest when $\{Z_t\}_{t=0}^{\infty}$ is constant over time, on the order of 0.2% increase in lifetime consumption. However, they can be substantial if $\{Z_t\}_{t=0}^{\infty}$ is growing over time. Moreover, the gains are increasing in the growth rate of TFP. This suggests that the normative recommendations of this paper are more relevant for fast growing economies.

3.5 General Equilibrium: 2-Country Case

The results derived in the previous sections generalize to a general equilibrium environment where the international interest rate is endogenous and there is strategic interaction amongst governments. In particular, capital controls arise under coordination among heterogeneous countries and they are not the result of a "prisoner's dilemma" sort of argument. Any outcome in the set of best sustainable equilibrium outcome of the game between the two governments and the households in the two countries cannot be improved in a Pareto sense by a planner subject to the constraint that the outcome (i) has to be a competitive equilibrium for the economy, and (ii) has to satisfy the sustainability constraints for each country. Thus, "coordination" between governments cannot improve upon a best sustainable outcome of the game between the two.

Moreover, a ban on capital controls will generate welfare losses. It amounts to restricting the set of instruments available to the government without generating any benefits from "coordination". These results are interesting in relation to the work in Costinot, Lorenzoni and Werning (2011) which studies the optimality of capital controls motivated by manipulation of the terms of trade for a two-country pure exchange economy. The policy implications of their model for a growing economy closely resembles those from my model, despite the fact that the mechanisms are completely different. However, in their economy, capital controls are inefficient so it would be optimal for an international organization like the IMF to ban capital controls since imposing capital controls is individually optimal but there are gains from coordination. In the case considered here, with lack of commitment and a production economy, it is no longer efficient to ban capital controls.

3.6 Conclusion

In this paper, I provide a mechanism through which policies aimed at increasing the self-financing ratio - controls on capital inflows - are welfare improving. It can account for the fact that countries with higher self-financing ratios are growing faster during the 1990s, as is documented in Aizenman et al. (2007). When the government doesn't have commitment, it is optimal to tax capital income of domestic residents at a lower rate to stimulate domestic capital accumulation, which helps to relax future commitment problems. Capital controls persist in the long-run only if the domestic households are impatient relative to the international interest rate. I further show that the policy recomendations are more relevant for countries that face higher TFP growth.

One important question that is left for future research is: Can the mechanism in this paper account for the capital allocation puzzle? Gourinchas and Jeanne (2011) documented that the direction of capital flows to developing countries is at odds with the predictions of the standard neoclassical growth model. Capital does not flow to countries that grow and invest more; the opposite is true. Moreover, the capital allocation puzzle is driven mainly by public flows. There is a strong positive correlation between TFP growth and accumulation of foreign assets by the government, as documented in Aguiar and Amador (2011b) and Gourinchas and Jeanne (2011). As Gourinchas and Jeanne

(2011) suggest: "*the solution to the allocation puzzle should be looked for at the nexus between growth, saving and reserve accumulation. Why do countries that grow more also accumulate more reserves, and why is this reserve accumulation not offset by capital inflows to the private sectors*?". The basic insights of my paper can be generalized to a model where the government cannot commit to costly reforms that are optimal ex-ante but not ex-post.[11] Modifying the environment in such a way will generate an endogenous link between TFP growth, capital controls, and accumulation of foreign assets by the government.

Furthermore, in this paper I abstracted from uncertainty. This model, augmented with shocks to productivity and the international interest rate, offers a framework for studying how the government should respond to shock to the international interest rate, and how the differential between domestic and foreign capital income taxes moves over the cycle.

[11] Gourinchas and Jeanne (2005) explore something similar in a static environment.

Figure 3.1: Example of outcome path with and without commitment on the government side

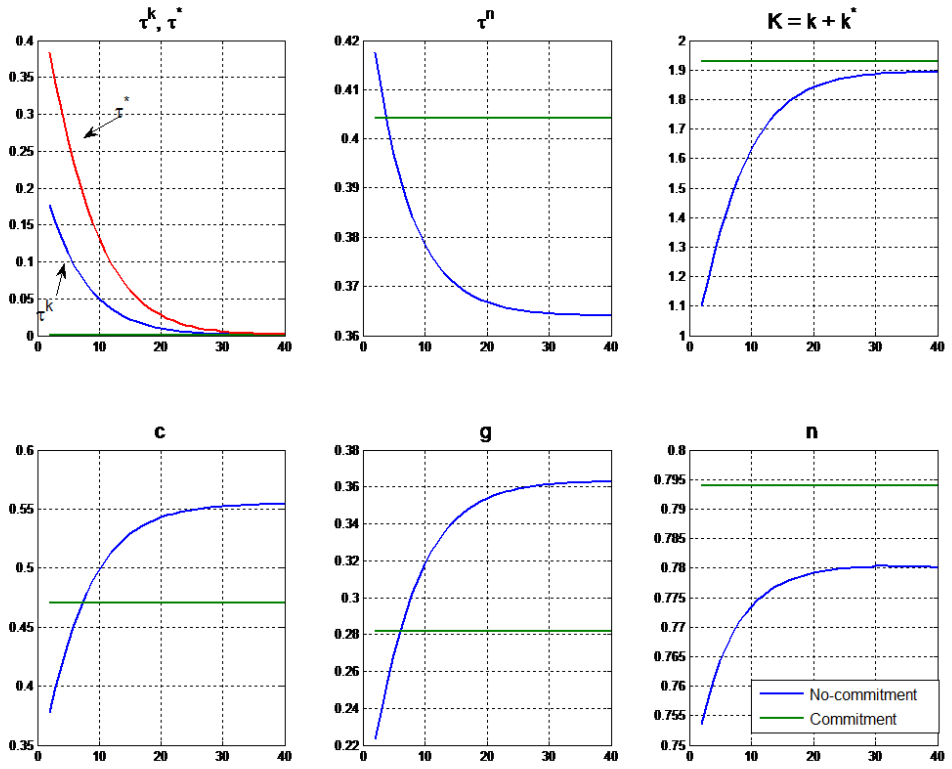
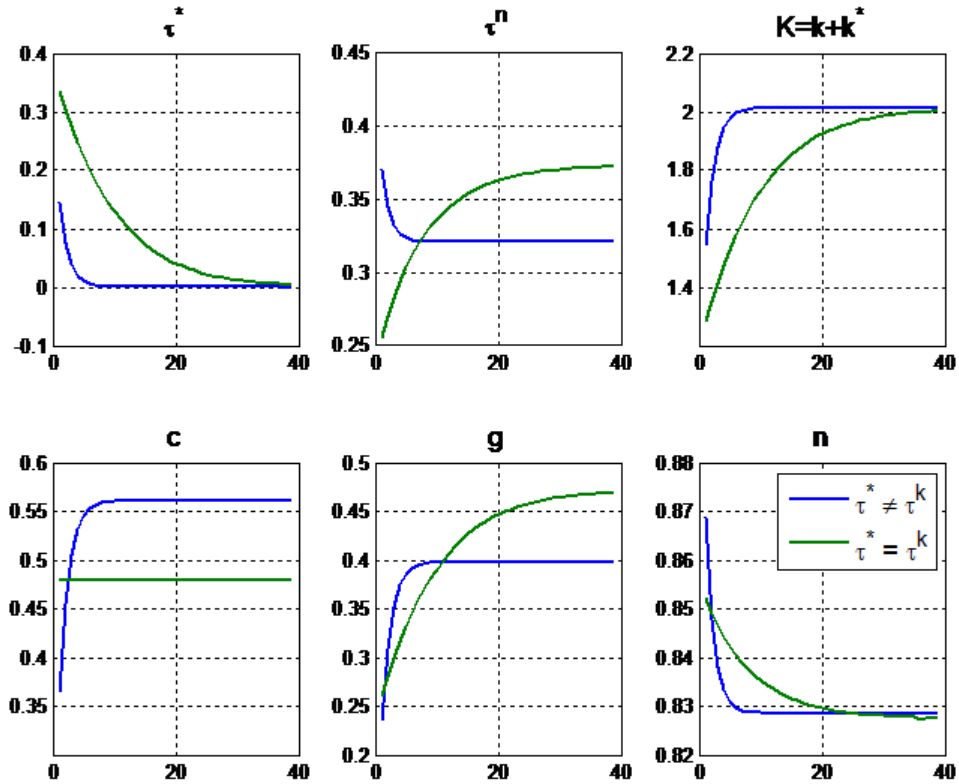


Figure 3.2: Example of outcome path without commitment with and without capital controls



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Appendix A

Appendix to Chapter 1

A.1 Data and Facts

In this Appendix, I document the behavior of GDP, consumption, and imports of intermediate goods around sovereign default episodes. Moreover, I discuss an extensive literature which also documents these and other key aspects of the data.

Data Description I consider the same 23 default events as Mendoza and Yue (2012): Argentina (1982, 2002), Chile (1983), Croatia (1992), Dominican Republic (1993), Ecuador (1999), Indonesia (1998), Mexico (1982), Moldova (2002), Nigeria (1983, 1986), Pakistan (1998), Peru (1983), Philippines (1983), Russia (1998), South Africa (1985, 1993), Thailand (1998), Ukraine (1998), Uruguay (1990), Uruguay (2003), Venezuela (1995), Venezuela (1998).

Annual data for GDP and consumption are gathered from the World Development Indicators (WDI). These are measured in real US dollars. As in Mendoza and Yue (2012), imported intermediates are the sum of categories for intermediate goods based on the Broad Economic Category (BEC) classification. The categories for intermediate goods are: (111) Food and beverages, primary, mainly for industry, (121) Food and beverages, processed, mainly for industry, (21) Industrial supplies not elsewhere specified, primary, (22) Industrial and lubricants, processed, (other than motor spirit), (42) Parts and accessories of capital goods (except transport equipment), (53) Part and accessories of transport equipment. For years 1962 through 2000, data is available from Feenstra

et al. (2005) but is classified using the Standard International Trade Classification, revision 4 (SITC4). I use UN concordances to map SITC4 into BEC codes. For years 1976 through 2010, data is available through the World Bank's World Integrated Trade Solution (WITS) database, which has information from the UN's Comtrade database. This database provides the series for the above BEC codes when available. When I have data from both sources, I use the WITS data, which does not rely on the concordances. For years in which both sources provide data, I have cross referenced the values. Although the levels are not exactly the same, deviations from the trend (my variable of interest) are very similar across the two sources. I deflate the intermediate import data using the US producer price index (PPI) from the Bureau of Economic Analysis (BEA). Each annual series is logged and HP-filtered with a smoothing parameter of 100.

Sovereign debt crises are associated with severe output and consumption losses for the debtor country The first two panels of Figure 1 illustrate the dynamics for GDP and consumption around a sovereign default episode. On average, in the 23 default episodes considered, output is 4.5% and 5.2% below trend in the year of a default and the year after, respectively. On average, consumption is 3.1% and 3.6% below trend in the year of a default and the year after, respectively. The same is true if I instead consider the median. This confirms the findings in Mendoza and Yue (2012). This pattern has been documented in several studies. See the references in the survey by Panizza, Sturzenegger, and Zettelmeyer (2009). Tomz and Wright (2007) is a notable exception. They find only a weak association between default episodes and output being below trend.

Sovereign debt crises are associated with trade disruptions A large literature documents that sovereign default episodes are accompanied by large drops in trade. For instance, see Rose (2005) and the references in the survey by Panizza, Sturzenegger, and Zettelmeyer (2009). Also, Borensztein and Panizza (2008) document that a default has a negative impact on trade credit. Arteta and Hale (2008) show that foreign credit to the private sector collapses in the aftermath of a default. Fuentes and Saravia (2006) show that defaults lead to a fall in FDI flows into the country.

As noted in Mendoza and Yue (2012), the drop in imports of intermediate goods

is very large: it drops on average from 4.4% above trend the year before a default to about 15.5% below trend the year of a default and the year after that. See the third panel in Figure 1. This drop is larger than in recessions of similar magnitudes. To document this fact, I regress imported intermediates at time t on a constant, GDP at time t, and dummy variables that take value of one if there is a default in the country at time t, $t-1$, $t-2$ and $t-3$ from 1962 to 2010 for the 18 countries in the sample for which I have data on intermediate imports. The result for this simple regression are reported in the table below. The drop in intermediate inputs in the year of and the year following a sovereign default is more than 10 percent larger than what one would expect from a drop in output of the same magnitude, absent default. This drop can have a non-trivial impact on the economy. Gopinath and Neiman (2012) present a model calibrated to replicate the crisis in Argentina in 2002 and show that the decline in imports of intermediate goods can account for up to a 5 percentage point decline in the welfare relevant measure of productivity.

Table A.1: Effect of Sovereign Default on Intermediate Imports
OLS Regression: Intermediate Imports at time t

Variable	Coefficient Estimate	Standard Error
Constant	0.007	0.960
GDP at t	1.810	0.145
Default at t	-0.119	0.044
Default at $t-1$	-0.108	0.044
Default at $t-2$	-0.040	0.044
Default at $t-3$	-0.005	0.043

R^2=0.225; Number of observations = 714
Intermediate imports and GDP are logged and HP-filtered.

Recoveries are accompanied by trade surpluses Periods following a sovereign default are associated with sustained trade surpluses as the economy recovers, see Mendoza and Yue (2012), among others. Moreover, typically as the economy recovers from the recession associated with the default episode, there is a partial repayment of the defaulted debt, after which the country regains access to international credit markets. Benjamin and Wright (2009) document that settlements tend to occur when output has

returned to trend.

Maturity of debt shortens when a default is more likely, as measured by interest rate spreads Broner, Lorenzoni, and Schmukler (2010) use data from 11 emerging economies from 1990 to 2009 to document that during emerging market debt crises (when spreads are high), the maturity of debt issued shortens. Moreover, when the spreads are low, long term spreads are generally higher than short term spreads. During debt crises, the gap between long and short-term spreads tends to narrow and the term spread curve flattens or even inverts. Arellano and Ramanarayanan (2012) confirm these findings.

A.2 Proofs

Preliminaries To characterize the efficient allocation, I use the equivalent formulation for (P) given by (P')-($\hat{\text{P}}$). Denote the decision rules associated with ($\hat{\text{P}}$) as $m(v) : [v_a, \bar{v}] \to \mathbb{R}$ and $c(v,\theta), v'(v,\theta) : [v_a, \bar{v}] \times \Theta \to \mathbb{R}$. Moreover, define $\omega(v,\theta) \equiv \theta U(c(v,\theta)) + \beta v'(v,\theta)$. Let $V_{nr} \subset [v_a, \bar{v}]$ be the (non-empty) set of promised utility values for which randomization in the current period is *not* optimal, i.e. $B(v) = \hat{B}(v)$. Let $V_r \subset [v_a, \bar{v}]$ be the *randomization region*. That is, the (possibly empty) set of promised utility values for which it is optimal to randomize, $B(v) > \hat{B}(v)$. Without loss of generality, the randomization is between two values in V_{nr} and it solves (P').

The next lemma establishes that B is concave and that the region over which $\hat{B}$ is strictly concave.

Lemma A.1. *Under Assumption 1, B is concave. If Assumption 2 holds then B is strictly concave, $B = \hat{B}$, and $V_{nr} = [v_a, v_r]$. If Assumption 2 does not hold then I can only establish that $\hat{B}$ is strictly concave over $[v^*, \bar{v}]$.*

Proof A.1. *Concavity of B follows from randomization. Rewrite ($\hat{P}$) using a change of variable: instead of $(m, c(\theta), v'(\theta))$, consider choosing $(\underline{u}, u(\theta), v'(\theta))$ where $\underline{u} = U(f(m))$ and $u = U(c)$. With this change of variable, ($\hat{P}$) can be written as*

$$\hat{B}(v) = \max_{\underline{u}, u(\theta), v'(\theta)} H(\underline{u}) + \sum_{\theta \in \Theta} \mu(\theta) \left[-C(u(\theta)) + qB(v'(\theta)) \right] \tag{$\hat{\text{P}}$'}$$

subject to

$$\begin{aligned}
\sum_{\theta\in\Theta}\mu(\theta)\left[\theta u(\theta)+\beta v'(\theta)\right] &= v \\
\theta u(\theta)+\beta v'(\theta) &\geq \theta u(\theta')+\beta v'(\theta') \quad \forall\theta,\theta' \\
\theta u(\theta)+\beta v'(\theta) &\geq \theta\underline{u}+\beta v_a \quad \forall\theta \\
v'(\theta) &\geq v_a \quad \forall\theta
\end{aligned}$$

where $C:[U(0),U(\infty)]\to\mathbb{R}$ *is* $C=U^{-1}$ *and* $H(\underline{u})\equiv f\circ\kappa(\underline{u})-\kappa(\underline{u})=C(\underline{u})-\kappa(\underline{u})$ *with* $\kappa:[U(f(0)),U(f(m^*))]\to[0,m^*]$ *is* $\kappa=f^{-1}\circ C$ *so that* $\underline{u}=U\left(f\left(\kappa(\underline{u})\right)\right)$*. The constraint set is linear in the choice variables,* $(\underline{u},u(\theta),v'(\theta))$*. Under Assumption 2,* H *is concave. Therefore by standard arguments* $\hat{B}$ *is strictly concave. Then clearly* $B(v)=\hat{B}(v)$ *for all* $v\in[v_a,\bar{v}]$*. Thus* $V_r=\emptyset$ *and* $V_{nr}=[v_a,\bar{v}]$*.*

In general, H *is not globally concave because both* C *and* κ *are convex. Hence, randomization over* v *can provide a higher value. I now claim that* $\hat{B}$ *is strictly concave over* $[v^*,\bar{v}]$*. Suppose for contradiction that is not. Then there exist* $v_1,v_2\in[v^*,\bar{v}]$ *and* $\alpha\in(0,1)$ *such that* $\hat{B}(\alpha v_1+(1-\alpha)v_2)\leq\alpha\hat{B}(v_1)+(1-\alpha)\hat{B}(v_2)$*. This cannot be the case. In fact, let* $x(v)=(\underline{u}(v),u(\theta,v),v'(\theta,v))$*. Because* $x=\alpha x(v_1)+(1-\alpha)x(v_2)$ *is attainable for* $\alpha v_1+(1-\alpha)v_2$ *and it attains a higher value than* $\alpha\hat{B}(v_1)+(1-\alpha)\hat{B}(v_2)$*. This is true by strict concavity of* C*, weak concavity of* B*, and the fact that* $\underline{u}(v_1)=\underline{u}(v_2)=U\left(f(m^*)\right)$ *by part (i) of Proposition 1. Then* $\hat{B}$ *is strictly concave over* $[v^*,\bar{v}]$*.*□

The next Lemma establishes that B is differentiable.

Lemma A.2. *Under Assumption 1,* $B:[v_a,\bar{v}]\to\mathbb{R}$ *is differentiable.*

Proof A.2. *To see that* B *is differentiable, notice that for* $v\in V_r$*,* B *is linear and, therefore, is differentiable. For* $v\in V_{nr}$*, differentiability can be established by applying the Benveniste and Scheinkman theorem, see Theorem 4.10 in Stokey, Lucas, and Prescott (SLP henceforth). For any* $v_0\in V_{nr}\cap(v_a,\bar{v})$*, let* $x=(m,u(\theta),v'(\theta))$ *be the solution that attains* $B(v_0)=\hat{B}(v_0)$*. First notice that* $m>0$ *and* $u(\theta)>U(0)$*. The next Lemma establishes this result.*

Lemma A.3. *Under Assumption 1, for all* $v\in(v_a,\bar{v})$*, if* $(m,u(\theta),v'(\theta))$ *is the solution to (*$\hat{P}$*) then* $m>0$ *and* $u(\theta)>U(0)$*.*

Proof A.3. *[Sketch] Consider m first. Suppose for contradiction that $m = 0$. By Lemma 1 part (ii), the relevant sustainability constraint is for type θ_H. There are two cases. If $\omega(\theta_H) > \theta_H U(f(0)) + \beta v_a$, then it is possible to increase m without violating the sustainability constraint and increasing the lenders' value, thus arriving at a contradiction. If instead $\omega(\theta_H) = \theta_H U(f(0)) + \beta v_a$, there are two cases. Consider for simplicity the case with $N = 2$. If $c(\theta_H) = f(0)$ then it follows that*

$$\omega(\theta_L) > \theta_L U\left(f(0)\right) + \beta v_a = \theta_L U\left(c(\theta_H)\right) + \beta v_a$$

then the relevant incentive compatibility constraint is slack. It is possible to increase consumption after θ_H and decrease it after θ_L. This increases $\omega(\theta_H)$ above $\theta_H U(f(0)) + \beta v_a$ and allows m to be greater than zero without violating incentive compatibility. This variation increases the lenders' value because of the Inada condition on f. If $c(\theta_H) < f(0)$ it must be that $v'(\theta_H) > v_a$. Then it is possible to decrease $v'(\theta_H)$, increase $c(\theta_H)$, and decrease either $c(\theta_L)$ or $v'(\theta_L)$ in a way that is consistent with the incentive compatibility and the sustainability constraints. This variation increases $\omega(\theta_H)$ above $\theta_H U(f(0)) + \beta v_a$, allowing m to be greater than zero. Moreover, it increases the lenders' value because of the Inada condition on f.

Now, consider $u(\theta)$. Notice that if $u(\theta) = U(0)$ for some θ, then it must be that $u(\theta_L) = U(0)$. Given the assumption that $\theta_L U(0) + \beta v^{**} < \theta_L U(f(0)) + \beta v_a$, if $u(\theta_L) = U(0)$ then it must be that $v'(\theta_L) \geq v^{**} > \tilde{v}$ (where $\tilde{v}$ is defined in Proposition 2; notice that this proposition does not rely on the differentiability of B). Thus, $v'(\theta_L)$ is on the downward sloping portion of B. Then consider decreasing $v'(\theta_L)$ and increasing $u(\theta_L)$, leaving utility unchanged. This is incentive compatible as the incentive compatibility constraint for type $\theta > \theta_L$ claiming to be θ_L is slack at the optimal solution. This has a positive effect on the objective because of the Inada condition on U. $\square$

Now, consider a neighborhood of v_0, $D(v_0, \varepsilon) = (v_0 - \varepsilon, v_0 + \varepsilon)$ for some small $\varepsilon > 0$. Given the interiority of m and $u(\theta)$, define $\hat{x}(v) = (\hat{m}, \hat{u}(v, \theta), \hat{v}'(v, \theta))$ for any $v \in D(v_0, \varepsilon)$ as follows:

$$\hat{m}(v) = m + \frac{v - v_0}{U'\left(f(m)\right) f'(m)}, \quad \hat{u}(v, \theta) = u(\theta) + v - v_0, \quad \hat{v}'(v, \theta) = v'(\theta)$$

so that, by construction, $\hat{x}(v)$ is feasible in $(\hat{P})$ for all $v \in D(v_0, \varepsilon)$ for $\varepsilon > 0$ sufficiently small. The fact that $\hat{x}(v)$ satisfies promise keeping and incentive compatibility is obvious.

For the sustainability constraint, notice that:

$$\begin{aligned}\theta U\left(f\left(\hat{m}(v)\right)\right)+\beta v_a &= \theta U\left(f(m)\right)+\int_0^{\frac{v-v_0}{U'(f(m))f'(m)}}\theta U'(f(m+x))f'(m+x)dx+\beta v_a\\ &\leq \theta U\left(f(m)\right)+\beta v_a+\frac{\theta U'\left(f(m)\right)f'(m)}{U'\left(f(m)\right)f'(m)}\left(v-v_0\right)\\ &= \theta U\left(f(m)\right)+\beta v_a+\theta\left(v-v_0\right)=\theta\hat{u}(\theta)+\beta\hat{v}'(\theta)\end{aligned}$$

where I use the fact that $U'(f(x))f'(x)$ is decreasing in x. Then $\hat{x}(v)$ is feasible in ($\hat{P}$) for all $v\in D(v_0,\varepsilon)$ for $\varepsilon>0$ sufficiently small. Then, define $\underline{B}:D(v_0,\varepsilon)\to\mathbb{R}$ as

$$\underline{B}(v)=f\left(\hat{m}(v)\right)-\hat{m}(v)+\sum_{\theta\in\Theta}\mu(\theta)\left[-C(\hat{u}(\theta))+qB(v'(\theta))\right]$$

$\underline{B}(v)$ is concave and differentiable in v, for all $v\in D(v_0,\varepsilon)$, $\underline{B}(v)\leq\hat{B}(v)\leq B(v)$ because $\hat{x}$ is feasible at v and $\underline{B}(v_0)=\hat{B}(v_0)=B(v_0)$. Thus, the Benveniste and Scheinkman theorem applies: B is differentiable at v_0 and

$$\begin{aligned}B'(v_0) &= \underline{B}'(v_0) \qquad\qquad (A.1)\\ &= \frac{f'\left(m\right)-1}{U'\left(f(m)\right)f'(m)}-\sum_{\theta\in\Theta}\mu(\theta)C'\left(u(\theta)\right)=\frac{f'\left(m\right)-1}{U'\left(f(m)\right)f'(m)}-\sum_{\theta\in\Theta}\frac{\mu(\theta)}{U'\left(c(\theta)\right)}\end{aligned}$$

This concludes the proof. □

The next Lemma establishes the continuity of the policy functions in ($\hat{\text{P}}$).

Lemma A.4. *Under Assumption 1, the set of maximizers $m,v'(\theta),c(\theta):[v_a,\bar{v}]\rightrightarrows\mathbb{R}$ is a compact-valued upper hemicontinuous (UHC) correspondence. If, in addition, Assumption 2 holds, then the correspondence is single valued and $m,v'(\theta),c(\theta):[v_a,\bar{v}]\to\mathbb{R}$ are continuous in v.*

Proof A.4. *The first part follows from the Theorem of the Maximum. For the second part, under Assumption 2 the objective function in ($\hat{P}$) is strictly concave and, therefore, it admits a unique solution. This and the first part of the lemma imply the continuity of the decision rule.* □

When for some $v\in[v_a,\bar{v}]$ the solution in ($\hat{\text{P}}$) is not unique, I consider an efficient allocation obtained from a selection from the UHC correspondence $m,v'(\theta),c(\theta):[v_a,\bar{v}]\rightrightarrows\mathbb{R}$. I assume that the selection is continuous.

Proof of Lemma 1 For part (i) see Lemma 4 part (i) in Thomas and Worrall (1990). For part (ii), consider choosing $(\underline{u}, u(\theta), v'(\theta))$ instead of $(m, c(\theta), v'(\theta))$, where $u(\theta) = U(c(\theta))$. Let $(m, u(\theta), v(\theta))$ be incentive compatible and such that $v(\theta) \geq v_a$ for all θ. Furthermore, let it be such that it satisfies the sustainability constraint for θ_H:

$$\omega(\theta_H) \geq \theta_H U(f(m)) + \beta v_a$$

where $\omega(v, \theta) \equiv \theta u(\theta) + \beta v(\theta)$. Consider two cases. First, if $U(f(m)) \geq u(\theta_H)$ then for all $\theta \in \Theta$ it follows that

$$\begin{aligned}
\omega(\theta) &\geq \omega(\theta_H) - (\theta_H - \theta)\, u(\theta_H) \\
&\geq \theta_H U(f(m)) + \beta v_a - (\theta_H - \theta)\, u(\theta_H) \\
&= \theta U(f(m)) + \beta v_a + (\theta_H - \theta)\left[U(f(m)) - u(\theta_H)\right] \\
&\geq \theta U(f(m)) + \beta v_a
\end{aligned}$$

where in the first line I use the fact that $(m, u(\theta), v(\theta))$ is incentive compatible; in the second line, the sustainability at θ_H; in the third line I add and subtract $\theta U(f(m))$; and finally in the fourth line, I use the fact that $U(f(m)) \geq u(\theta_H)$. Then the sustainability constraint holds for all $\theta \in \Theta$. Now suppose that $U(f(m)) < u(\theta_H)$. In this case, the sustainability constraint is slack at θ_H, $\omega(\theta_H) > \theta_H U(f(m)) + \beta v_a$. Suppose for contradiction that there exists $\theta \in \Theta$ such that

$$\omega(\theta) \leq \theta U(f(m)) + \beta v_a$$

Then, notice that

$$\begin{aligned}
\omega(\theta_H) &\leq \omega(\theta) + (\theta_H - \theta) u(\theta_H) \\
&\leq [\theta U(f(m)) + \beta v_a] + (\theta_H - \theta) u(\theta_H) = \theta_H u(\theta_H) + \beta v_a + \theta\left[U(f(m)) - u(\theta_H)\right] \\
&< \theta_H u(\theta_H) + \beta v_a \leq \omega(\theta_H)
\end{aligned}$$

where the first line follows from incentive compatibility; in the second line, I use the contradiction hypothesis; in the third line, I used the fact that, by assumption, $U(f(m)) < u(\theta_H)$; and finally that $v(\theta_H) \geq v_a$. This results in a contradiction and the result follows.

Proof of Proposition 1 Part (i). First, let $v = v_a$. Combining the the sustainability constraint and the promise keeping constraint, it follows that

$$\begin{aligned} v_a &= v = \sum_{\theta\in\Theta} \mu(\theta)\left[\theta U(c(\theta)) + \beta v'(\theta)\right] \geq \sum_{\theta\in\Theta} \mu(\theta)\left[\theta U(f(m)) + \beta v_a\right] \\ &> \sum_{\theta\in\Theta} \mu(\theta)\left[\theta U(f(0)) + \beta v_a\right] = v_a \quad \text{if } m > 0 \end{aligned}$$

Then it must be that $m(v_a) = 0$. For $v \in (v_a, \bar{v}]$, at an interior solution, the optimality condition for m can be written as

$$f'(m) - 1 = \lambda_{sust}\theta_H U'(f(m))\, f'(m) \geq 0$$

where $\lambda_{sust} \geq 0$ is the Lagrange multiplier on the sustainability constraint (for θ_H, the relevant one by Lemma 1). Hence $m \leq m^*$ and $m = m^*$ iff $\lambda_{sust} = 0$. First, I argue that if $\lambda_{sust}(v_1), \lambda_{sust}(v_2) > 0$ for $v_1 < v_2$ then it must be that $m(v_1) < m(v_2)$. To do so, I need to use Assumption 2. That is, H is concave. This implies that λ_{sust} is monotone decreasing in m. Suppose for contradiction that $\lambda_{sust}(v_1), \lambda_{sust}(v_2) > 0$ and $m(v_1) \geq m(v_2)$. Then it must be that:

$$\omega_H(v_1) \geq \omega_H(v_2), \quad \lambda_{sust}(v_1) \leq \lambda_{sust}(v_2), \quad \lambda_{pkc}(v_1) < \lambda_{pkc}(v_2) \tag{A.2}$$

where $\lambda_{pkc}(v)$ is the Lagrange multiplier associated with the promise keeping constraint. By the envelope condition, this is equal to

$$\lambda_{pkc}(v) = -B'(v)$$

and it is strictly increasing in v by strict concavity of B under Assumption 2. Consider the case $N = 2$ for simplicity. An interior optimum must also satisfy the following necessary conditions:

$$\frac{1}{\theta_H U'(c_H(v))} = \lambda_{pkc}(v) + \frac{\lambda_{sust}(v)}{\mu_H} - \frac{\lambda_{ic}(v)}{\mu_H}\frac{\theta_L}{\theta_H} = \Lambda(v) - \frac{\lambda_{ic}(v)}{\mu_H}\frac{\theta_L}{\theta_H} \tag{A.3}$$

$$-\frac{q}{\beta}B'(v'_H(v)) = \lambda_{pkc}(v) + \frac{\lambda_{sust}(v)}{\mu_H} - \frac{\lambda_{ic}(v)}{\mu_H} = \Lambda(v) - \frac{\lambda_{ic}(v)}{\mu_H} \tag{A.4}$$

where $\lambda_{ic}(v)$ is the Lagrange multiplier on the relevant incentive compatibility constraint (type θ_L reporting θ_H, see again Lemma 1) and $\Lambda(v) \equiv \lambda_{pkc}(v) + \lambda_{sust}(v)/\mu_H$. By (A.2) it follows that $\Lambda(v_2) > \Lambda(v_1)$. Now consider two cases. First, if

$$\Lambda(v_1) - \Lambda(v_2) \geq \frac{\theta_L}{\theta_H}\left[\frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_H}\right] \tag{A.5}$$

then it follows that $c_H(v_1) \geq c_H(v_2)$. Moreover, $\theta_L/\theta_H \in (0,1)$ and the term in square brackets is negative. It follows that

$$\Lambda(v_1) - \Lambda(v_2) \geq \frac{\theta_L}{\theta_H}\left[\frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_H}\right] > \frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_H}$$

Hence $v'_H(v_1) > v'_H(v_2)$. Notice that at an optimal solution, the relevant incentive compatibility constraint must hold with equality (see the discussion of part (ii) below for a proof). Therefore, if the relevant incentive compatibility constraint binds for v_1, then the incentive compatibility constraint must not bind at the solution for v_2. In fact:

$$\begin{aligned} \omega_L(v_2) &> \omega_L(v_1) = \theta_L U\left(c_H(v_1)\right) + \beta v'_H(v_1) \\ &> \theta_L U\left(c_H(v_2)\right) + \beta v'_H(v_2) \end{aligned}$$

where the first inequality follows from the fact that $\omega_H(v_1) > \omega_H(v_2)$ and $v_2 > v_1$; the second follows from a binding incentive compatibility constraint for v_1; and the last follows from $c_H(v_1) \geq c_H(v_2)$ and $_H(v_1) \geq \omega_H(v_2)$. This is a contradiction. Consider now the case in which (A.2) does not hold. This implies, together with (A.2) and (A.3), that $c_H(v_1) < c_H(v_2)$. Notice that a binding incentive compatibility constraint implies that

$$\omega_L(v) = \omega_H(v) - (\theta_H - \theta_L)U\left(c_H(v)\right)$$

Using this equality in the promise keeping constraint, I can write:

$$\begin{aligned} v &= \mu_H\omega_H(v) + \mu_L\omega_L(v) = \omega_H(v) - \mu_L(\theta_H - \theta_L)U\left(c_H(v)\right) \\ &\iff \omega_H(v) = v + \mu_L(\theta_H - \theta_L)U\left(c_H(v)\right) \end{aligned}$$

Hence, the fact that $c_H(v_1) < c_H(v_2)$ implies that $\omega_H(v_1) < \omega_H(v_2)$, a contradiction. For $N > 2$ an induction argument extends this logic to the general case.

I now turn to showing that there exists a v^* such that for all $v \geq v^*$ it must be that $m(v) = m^*$. Consider a relaxed version of ($\hat{\text{P}}$) in which the sustainability constraint is dropped. In this relaxed problem, it can be shown that $\omega_H(v) \geq \theta_H(1-\beta)v + \beta v$. Hence, if $v \geq v^{**} \equiv \left[\theta_H U(f(m^*)) + \beta v_a\right] / \left[\theta_H(1-\beta) + \beta\right]$, the solution of this relaxed problem is a solution to the original problem. Thus, $m(v) = m^*$ for all $v \geq v^{**}$. If in addition Assumption 2 holds, combining this with the fact that $m(v)$ is strictly increasing (and continuous) when λ_{sust} is binding, it follows that there must exist some $v^* \in (v_a, v^{**})$ for which $m(v) < m^*$ for all $v \in [v_a, v^*)$ and $m(v) = m^*$ for all $v \geq v^*$.

Part (ii). Consider $N = 2$ to simplify notation. First notice that the relevant incentive compatibility constraint must bind at an optimal solution. In fact, suppose for contradiction that it is slack. Then the optimality conditions imply that $v'_H \geq v'_L$ and $c_H > c_L$ or, equivalently, that $u_L > u_H$ using the change of variables in ($\hat{\text{P}}$'). Clearly this is not incentive compatible. Suppose for contradiction that $u_L \geq u_H$. For the relevant incentive compatibility constraint to be binding, it must be that $v'_H \geq v'_L$. By Lemma 3 it follows that $v'_L > v_a$. Hence the solution is interior. Thus, I can combine the first order necessary conditions with respect to v'_H and v'_L, (A.9) and (A.10) below, to get

$$\frac{\lambda_{ic}}{\mu_L} \leq \frac{\lambda_{sust} - \lambda_{ic}}{\mu_H} < \frac{\lambda_{sust} - \lambda_{ic}\frac{\theta_L}{\theta_H}}{\mu_H} \tag{A.6}$$

where λ_{ic} and λ_{sust} are the Lagrange multiplier on the incentive compatibility constraint and the sustainability constraint respectively, and the last strict inequality follows from the fact that $\theta_L/\theta_H \in (0, 1)$. Combining (A.6) with the first order conditions for u_H and u_L, (A.7) and (A.8) below, implies that

$$\frac{C'(u_H)}{\theta_H} - \frac{C'(u_L)}{\theta_L} = \frac{\lambda_{sust} - \lambda_{ic}\frac{\theta_L}{\theta_H}}{\mu_H} - \frac{\lambda_{ic}}{\mu_L} > 0 \Rightarrow u_H > u_L$$

which is a contradiction. Hence, for all v it must be that $u_H(v) > u_L(v) \iff c_H(v) > c_L(v)$. Consequently, incentive compatibility requires that $v'_L(v) > v'_H(v)$ for all v, as wanted.

Part (iii). For the cross-subsidization part, as in Thomas and Worrall (1990) Lemma 4 part (ii), suppose for contradiction that $b(v, \theta_L) < b(v, \theta_H)$ for some v. Then, consider offering the pooling allocation: $\hat{c}(v, \theta_L) = \hat{c}(v, \theta_H) = c(v, \theta_H)$ and $\hat{v}'(v, \theta_L) = \hat{v}'(v, \theta_H) = v'(v, \theta_H)$. Because the incentive compatibility constraint is binding at the optimal allocation, it follows that

$$\hat{\omega}(v, \theta_L) = \theta_L U(\hat{c}(v, \theta_L)) + \beta v'(v, \theta_L) = \theta_L u(c(v, \theta_H)) + \beta v'(v, \theta_H) = \omega(v, \theta_L)$$

Hence, the promise keeping constraint is satisfied at the proposed solution. Incentive compatibility and sustainability are also trivially satisfied. Therefore, the proposed alternative pooling solution is feasible for v and is such that

$$\mu(\theta_L)\hat{b}(v, \theta_L) + \mu(\theta_H)\hat{b}(v, \theta_H) = b(v, \theta_H) > \hat{B}(v) = \mu(\theta_L)b(v, \theta_L) + \mu(\theta_H)b(v, \theta_H)$$

This is a contradiction. So, it must be that $b(v, \theta_L) \geq b(v, \theta_H)$. Suppose now that $b(v, \theta_L) = b(v, \theta_H)$. Then it must be that the pooling allocation is a solution to ($\hat{\text{P}}$). By part (ii) the allocation is dynamic, $c_H(v) > c_L(v)$ and $v'_L(v) > v'_H(v)$, hence the pooling allocation cannot be a solution.

Proof of Proposition 2 Suppose for contradiction that B is (weakly) decreasing over $[v_a, \bar{v}]$ and so $v_a \in \arg\max_{v \in V} B(v)$. I am now going to show that a level of indebtedness strictly higher than $B(v_a)$ can be supported by delivering $v > v_a$, contradicting the fact that B is decreasing over its entire domain. Denote by x_a the allocation that attains $B(v_a)$. Consider the following variation for some $\varepsilon > 0$ sufficiently small:

$$m = \varepsilon > 0, \quad c(\theta) = c_a(\theta) + \varepsilon_c(\theta), \quad v'(\theta) = v'_a(\theta) \quad \forall \theta$$

where $\varepsilon_c(\theta) > 0$ is such that for all θ

$$U\left(c_a(\theta) - \varepsilon_c(\theta)\right) - U\left(c_a(\theta)\right) = \varepsilon_u \equiv U'\left(f(\varepsilon)\right) f'(\varepsilon)\varepsilon$$

Then, by construction, the proposed variation satisfies the incentive compatibility and the sustainability constraints. This variation attains a value for the borrower equal to $v_a + \varepsilon_u > v_a$. Thus, I am left to show that it increases the lenders' value too. The change in the lenders' value can be written as

$$\frac{\Delta B}{\varepsilon} \approx -\sum_{\theta \in \Theta} \mu(\theta) \left(\frac{U'\left(f(\varepsilon)\right)}{U'\left(c_a(\theta)\right)} \right) f'(\varepsilon) + \left[f'(\varepsilon) - 1\right] = f'(\varepsilon)\left[1 - \phi\right] - 1$$

where

$$\phi \equiv \sum_{\theta \in \Theta} \mu(\theta) \left(\frac{U'\left(f(\varepsilon)\right)}{U'\left(c_a(\theta)\right)} \right) < 1$$

because from Lemma 3, it follows that $c_a(\theta) \leq f(0) < f(\varepsilon)$ and, in particular, $c_a(\theta_L) < f(0)$. Thus, $\varepsilon > 0$ can be chosen to be sufficiently small that, by the Inada condition on f, $\Delta B/\varepsilon > 0$. Therefore it must be that $B(v_a + \varepsilon_u) \geq B(v_a) + \Delta B \varepsilon > B(v_a)$. Hence, B is not strictly decreasing, a contradiction. B is increasing in a neighborhood of v_a. Moreover, B is strictly decreasing over $[v^*, \bar{v}]$. In fact, $\lambda_{sust}(v) = 0$ for $v \geq v^*$ and therefore $m(v) = m^*$. Then, from (A.1) it follows that for all $v \geq v^*$

$$B'(v) = -\sum_{\theta \in \Theta} \mu(\theta) C'\left(u(\theta, v)\right) = -\sum_{\theta \in \Theta} \frac{\mu(\theta)}{U'\left(c(\theta, v)\right)} < 0$$

Thus, B is strictly decreasing over $[v^*, \bar{v}]$. The continuity and concavity of B imply that there exists $\tilde{v} \in (v_a\ , v^*)$ such that B is increasing for all $v \in [v_a, \tilde{v})$ and B is strictly decreasing over $[\tilde{v}, \bar{v}]$.

Proof of Lemma 2 Consider any v in the efficient region for which there is no randomization, $v \in [\tilde{v}, \bar{v}] \cap V_{nr}$. Letting λ_{ic}, λ_{sust}, and λ_{pkc} be the Lagrange multipliers on the incentive compatibility, sustainability, and promise keeping constraints, an interior solution must satisfy the following first order necessary conditions (fonc):

$$c_L \quad : \quad -\lambda_{pkc} = -\frac{1}{\theta_L U'(c_L)} + \frac{\lambda_{ic}}{\mu_L} \tag{A.7}$$

$$c_H \quad : \quad -\lambda_{pkc} = -\frac{1}{\theta_H U'(c_H)} - \frac{\lambda_{ic}}{\mu_H}\frac{\theta_L}{\theta_H} + \frac{\lambda_{sust}}{\mu_H} \tag{A.8}$$

$$v'_L \quad : \quad -\lambda_{pkc} = \frac{q}{\beta}B'(v'_L) + \frac{\lambda_{ic}}{\mu_L} \tag{A.9}$$

$$v'_H \quad : \quad -\lambda_{pkc} = \frac{q}{\beta}B'(v'_H) - \frac{\lambda_{ic} - \lambda_{sust}}{\mu_H} \tag{A.10}$$

and the envelope condition $B'(v) = -\lambda_{pkc}$. Combining the envelope condition with the fonc for v'_H, I can write:

$$B'(v) \leq \frac{\beta}{q}B'(v) = B'(v'_H) - \frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_H}\right)$$

where I used the fact that $B'(v) \leq 0$ for all $v \in [\tilde{v}, \bar{v}]$ and that, by Assumption 1, $\beta/q < 1$. If $\lambda_{ic} > \lambda_{sust}$, I can rewrite the above inequality as

$$B'(v) \leq \frac{\beta}{q}B'(v) = B'(v'_H) - \frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_H}\right) < B'(v'_H)$$

By concavity of B, it follows that $v'_H(v) < v\ \forall v \in [\tilde{v}, \bar{v}]$. Hence, it is sufficient to show that $\lambda_{ic} > \lambda_{sust}$.

Consider first the case in which there is partial insurance, i.e. $\theta_H U'(c_H) \geq \theta_L U'(c_L)$. Combine the foncs with respect to c_L and c_H to get

$$0 \geq \frac{1}{\theta_H U'(c_H)} - \frac{1}{\theta_L U'(c_L)} = \frac{1}{\mu(\theta_H)}\left[\lambda_{sust} - \lambda_{ic}\frac{\theta_L}{\theta_H}\right] - \frac{1}{\mu(\theta_L)}\lambda_{ic}$$

Rearranging terms, I obtain

$$\lambda_{sust} \leq \lambda_{ic}\left(\frac{\mu(\theta_H)}{\mu(\theta_L)} + \frac{\theta_L}{\theta_H}\right) = \lambda_{ic}\left(\frac{\mathbb{E}(\theta)}{\mu_L\theta_H}\right) \leq \lambda_{ic}$$

where in the last step I use the assumption that $\mu_L\theta_H \geq \mathbb{E}(\theta)$.

Consider now the case with $\theta_H U'(c_H) < \theta_L U'(c_L)$.[1] I contend that the following conditions cannot be jointly satisfied at a solution: (i) $v \in [\tilde{v}, v^*)$, (ii) $\theta_H U'(c_H) < \theta_L U'(c_L)$ and (iii) $v'_H(v) \geq v$. Suppose for contradiction that (i)-(iii) hold. From (i) it follows that $B(v) \leq 0$; thus, it must be that

$$\lambda_{sust}\theta_H = \frac{f'(m)-1}{U'(f(m))f'(m)} \leq \mu_L C'(u_L) + \mu_H C'(u_H) = \mathbb{E}\left(C'(u(\theta))\right) \tag{A.11}$$

From (ii) it follows that

$$\frac{C'(u_H)}{\theta_H} = \left[\mu_L C'(u_H)\frac{\theta_L}{\theta_H} + \mu_H C'(u_H)\right] > \mu_L C'(u_L) + \mu_H C'(u_H) \tag{A.12}$$

Furthermore, notice that the incentive compatibility constraint and the promise keeping constraint imply that

$$\begin{aligned} v &= \mu_H\left[\theta_H u_H + \beta v'_H\right] + \mu_L\left[\theta_L u_L + \beta v'_L\right] = \mu_H\left[\theta_H u_H + \beta v'_H\right] + \mu_L\left[\theta_L u_H + \beta v'_H\right] \\ &= \mathbb{E}(\theta)\, u_H(v) + \beta v'_H(v) \Rightarrow v'_H(v) = \frac{v - \mathbb{E}(\theta)\, u_H(v)}{\beta} \end{aligned}$$

Combining this with (iii) implies (using the normalization $\mathbb{E}(\theta) = 1$) that

$$u_H \leq v(1-\beta) \tag{A.13}$$

Thus, combining (A.11),(A.12) and (A.13), and using the fact that $C = U^{-1}$ is convex, I obtain:

$$\frac{C'(v(1-\beta))}{\theta_H} \geq \frac{C'(u_H)}{\theta_H} > \mathbb{E}\left(C'(u(\theta))\right) \geq \frac{f'(m)-1}{U'(f(m))f'(m)} = \lambda_{sust}\theta_H \tag{A.14}$$

Further notice that it must be that $m \leq \underline{m}$, defined as

$$\theta_H v(1-\beta) + \beta v = \theta_H U(f(\underline{m})) + \beta v_a \tag{A.15}$$

Equivalently, using a change of variable $\underline{u} = U(f(\underline{m}))$, I can write

$$\underline{u} = v(1-\beta) + \frac{\beta(v - v_a)}{\theta_H} \tag{A.16}$$

[1] This case never arises in any of my numerical simulation.

Then, assuming that H' is decreasing (which is always true under Assumption 2), it must be that

$$\lambda_{sust}\theta_H = \frac{f'(m)-1}{U'(f(m))f'(m)} \geq \frac{f'(\underline{m})-1}{U'(f(\underline{m}))f'(\underline{m})} = H'\left(v(1-\beta)+\frac{\beta(v-v_a)}{\theta_H}\right) \quad \text{(A.17)}$$

Then, I have a contradiction of (A.14) if the following condition is satisfied for all v in the relevant region:

$$H'\left(v(1-\beta)+\frac{\beta(v-v_a)}{\theta_H}\right) \geq \frac{C'\left(v(1-\beta)\right)}{\theta_H} \quad \text{(A.18)}$$

There exists a $\hat{v} \in (v_a, v^*)$ such that (A.18) holds for all $v \in [v_a, \hat{v}]$. Moreover, the larger is θ_H, the closer is $\hat{v}$ to v^*. If θ_H is sufficiently large then it follows that (A.18) holds for all v in the relevant region, obtaining a contradiction. Therefore it must be that $v'_H(v) < v$ for all $v \in [\tilde{v}, \bar{v}] \cap V_{nr}$ as desired.

Sufficient Conditions for Proposition 3 for $|\Theta| > 2$ Consider the general case with $|\Theta| = N \geq 2$. Letting λ_{sust} and λ_{pkc} be the Lagrange multipliers on the sustainability and promise keeping constraints and λ_n be the Lagrange multiplier associated with the incentive constraint for type θ_n (reporting θ_{n+1}), the foncs for an interior optimum can be written as:

$$\begin{aligned}
c_1 \quad &: \quad \frac{1}{\theta_1 U'(c_1)} = \lambda_{pkc} + \frac{1}{\mu(\theta_1)}\lambda_1 \\
c_n \quad &: \quad \frac{1}{\theta_n U'(c_n)} = \lambda_{pkc} + \frac{1}{\mu(\theta_n)}\left[\lambda_n - \lambda_{n-1}\frac{\theta_{n-1}}{\theta_n}\right], \qquad n = 2, ..., N-1 \\
c_N \quad &: \quad \frac{1}{\theta_N U'(c_N)} = \lambda_{pkc} + \frac{1}{\mu(\theta_N)}\left[\lambda_{sust} - \lambda_{N-1}\frac{\theta_{N-1}}{\theta_N}\right] \\
v'_1 \quad &: \quad -\frac{q}{\beta}B'(v'_1) = \lambda_{pkc} + \frac{1}{\mu(\theta_1)}\lambda_1 \\
v'_n \quad &: \quad -\frac{q}{\beta}B'(v'_n) = \lambda_{pkc} + \frac{1}{\mu(\theta_n)}\left[\lambda_n - \lambda_{n-1}\right], \qquad n = 2, ..., N-1 \\
v'_N \quad &: \quad -\frac{q}{\beta}B'(v'_N) = \lambda_{pkc} + \frac{1}{\mu(\theta_N)}\left[\lambda_{sust} - \lambda_{N-1}\right]
\end{aligned}$$

To show that $v'_N(v) < v$, it suffices to prove that $\lambda_{sust} - \lambda_{N-1} < 0$. In the relevant case with $\theta_{n+1}U'(c_{n+1}) \geq \theta_n U'(c_n)$, combining the foncs with respect to c_N and c_{N-1},

I obtain:

$$\begin{aligned} 0 &\geq \frac{1}{\theta_N U'(c_N)} - \frac{1}{\theta_{N-1} U'(c_{N-1})} \\ &= \frac{1}{\mu(\theta_N)}\left[\lambda_{sust} - \lambda_{N-1}\frac{\theta_{N-1}}{\theta_N}\right] - \frac{1}{\mu(\theta_{N-1})}\left[\lambda_{N-1} - \lambda_{N-2}\frac{\theta_{N-2}}{\theta_{N-1}}\right] \end{aligned}$$

which can be rearranged as

$$\begin{aligned} \lambda_{sust} &\leq \frac{\mu(\theta_N)}{\mu(\theta_{N-1})}\left[\lambda_{N-1} - \lambda_{N-2}\frac{\theta_{N-2}}{\theta_{N-1}}\right] + \lambda_{N-1}\frac{\theta_{N-1}}{\theta_N} \\ &= \lambda_{N-1}\left(\frac{\mu(\theta_N)}{\mu(\theta_{N-1})} + \frac{\theta_{N-1}}{\theta_N}\right) - \lambda_{N-2}\frac{\mu(\theta_N)}{\mu(\theta_{N-1})}\frac{\theta_{N-2}}{\theta_{N-1}} < \lambda_{N-1}\left(\frac{\mu(\theta_N)}{\mu(\theta_{N-1})} + \frac{\theta_{N-1}}{\theta_N}\right) \end{aligned}$$

Then, a sufficient condition, albeit very stringent and by no means necessary, is that

$$\frac{\mu(\theta_N)}{\mu(\theta_{N-1})} + \frac{\theta_{N-1}}{\theta_N} \leq 1 \iff \frac{\theta_{N-1}}{\theta_N} \leq \frac{\mu_{N-1} - \mu_N}{\mu_{N-1}} \tag{A.19}$$

which is necessarily met if (i) θ_N is sufficiently large or (ii) $\mu(\theta_N)$ is sufficiently small (as $\mu(\theta_N) \downarrow 0$, the right hand side converges to 1 and consequently the condition is satisfied).

Proof of Lemma 3 Let $v = v_a$. For all $\theta \in \Theta \backslash \{\theta_L\}$ it must be that $c(v_a, \theta) = f(0)$ and $v'(v_a, \theta) = v_a$. In fact, to deliver the value of autarky in a sustainable way, it must be that for all $\theta \in \Theta$:

$$\omega(v_a, \theta) = \theta U\left(c(v_a, \theta)\right) + \beta v'(v_a, \theta) = \theta U\left(f(0)\right) + \beta v_a \tag{A.20}$$

Moreover, for all θ', θ such that $\theta' > \theta$, the sustainability and the incentive compatibility constraints imply that

$$\begin{aligned} \omega(v_a, \theta) &\geq \omega(v_a, \theta') - (\theta' - \theta) U\left(c(v_a, \theta')\right) \\ &= \theta' U\left(f(0)\right) + \beta v_a - (\theta' - \theta) U\left(c(v_a, \theta')\right) \end{aligned} \tag{A.21}$$

Combining (A.20) and (A.21), it follows that

$$\begin{aligned} \theta U\left(f(0)\right) + \beta v_a &\geq \theta' U\left(f(0)\right) + \beta v_a - (\theta' - \theta) U\left(c(v_a, \theta')\right) \\ &\iff (\theta' - \theta) U\left(c(v_a, \theta')\right) \geq (\theta' - \theta) U\left(f(0)\right) \end{aligned}$$

which implies that $c(v_a, \theta') \geq f(0)$. (A.20) and the fact that $v'(\theta) \geq v_a$ imply that $c(v_a, \theta') = f(0)$, as desired. Then, it is only possible that $c(v_a, \theta_L) < f(0)$ for $\theta = \theta_L$.

For θ_L, there are two possibilities: (i) $c(v_a, \theta_L) < f(0)$ and $v'(v_a, \theta_L) > v_a$ or (ii) $c(v_a, \theta_L) = f(0)$ and $v'(v_a, \theta_L) = v_a$. If (ii) is true, v_a is an absorbing state and $B(v_a) = 0$. Suppose for contradiction that we are in case (ii). Consider a two-period variation that decreases current consumption after θ_L by ε and it increases it by ε/K in any states in the next period. That is:

$$\begin{aligned} c_H &= f(0), \quad v'_H = v_a \\ c_L &= f(0) - \varepsilon, \quad v'_L = \sum_{\theta \in \Theta} \mu(\theta)\theta U\left(f(0) + \varepsilon/K\right) + \beta v_a \end{aligned}$$

for some $\varepsilon > 0$ and $K > 0$ such that the variation satisfies the promise keeping constraint at v_a:

$$\theta_L U\left(f(0)\right) + \beta v_a = U\left(f(0) - \varepsilon\right) + \beta \sum_{\theta \in \Theta} \mu(\theta)\theta U\left(f(0) + \varepsilon/K\right) + \beta^2 v_a \tag{A.22}$$

$$\iff \left[U\left(f(0)\right) - U\left(f(0) - \varepsilon\right)\right] = \beta \frac{\mathbb{E}\left(\theta\right)}{\theta_L}\left[U\left(f(0) + \varepsilon/K\right) - U\left(f(0)\right)\right]$$

To get a contradiction, it suffices to show that there exist $\varepsilon > 0$ and $K > 0$ such that (A.22) holds and

$$B(v_a) = 0 < \mu(\theta_L)\left[\varepsilon - q\varepsilon/K\right] = \mu(\theta_L)\left[1 - q/K\right]\varepsilon \iff K > q \tag{A.23}$$

Rewrite (A.22) as

$$\int_0^\varepsilon U'\left(f(0) - e\right) de = \frac{\beta}{K}\frac{\mathbb{E}\left(\theta\right)}{\theta_L}\int_0^\varepsilon U'\left(f(0) + e/K\right) de$$

which implies, for $\varepsilon > 0$ sufficiently close to zero, that

$$\begin{aligned} K &= \beta \frac{\mathbb{E}\left(\theta\right)}{\theta_L}\left[\frac{\int_0^\varepsilon U'\left(f(0) + e/K\right) de}{\int_0^\varepsilon U'\left(f(0) - e\right) de}\right] \approx \beta \frac{\mathbb{E}\left(\theta\right)}{\theta_L}\left[\frac{U'\left(f(0)\right)\varepsilon}{U'\left(f(0)\right)\varepsilon}\right] = \beta \frac{\mathbb{E}\left(\theta\right)}{\theta_L} \\ &> \left(q\frac{\theta_L}{\mathbb{E}\left(\theta\right)}\right)\frac{\mathbb{E}\left(\theta\right)}{\theta_L} = q \end{aligned}$$

where in the last step I use the fact that from Assumption 1 $\beta/q \in \left(\theta_L/\mathbb{E}\left(\theta\right), 1\right]$. Then (A.22) and (A.23) hold. This is a contradiction. Therefore it must be that $c(v_a, \theta_L) < f(0)$ and $v'(v_a, \theta_L) > v_a$.

Proof of Lemma 4 Consider first borrower values in the region with *ex-post* inefficiencies. Let $v \in [v_a, \tilde{v})$. In this interval, it must be that $v'_L(v) \geq \tilde{v}$. Suppose, to the contrary, that $v'_L(v) < \tilde{v}$. By Lemma 8 we know that $c_L > 0$. Consider then the following variation: decrease c_L by ε_c and increase v'_L by ε_v for some $\varepsilon_v > 0$ sufficiently small and $\varepsilon_c(\varepsilon_v) > 0$, defined as the unique solution to

$$\theta'_L U\left(c_L - \varepsilon_c(\varepsilon_v)\right) + \beta v'_L + \varepsilon_v = \theta_L U\left(c_L\right) + \beta v'_L$$

This variation is feasible for v in $(\hat{\mathrm{P}})$[2] and has a positive effect on the objective function:

$$\Delta B(v) = \mu_L \left[qB\left(v'_L + \varepsilon_v\right) - qB\left(v'_L\right) + \varepsilon_c(\varepsilon_v)\right] > 0$$

because it decreases the cost of providing consumption today and it also increases the value of future transfers if $\varepsilon_v > 0$ is sufficiently small. This is a contradiction. Hence, it must be that $v'_L(v) \geq \tilde{v}$ for all $v \in [v_a, \tilde{v})$. Notice how this argument applies for all v. Hence, $v'_L(v) \geq v$ for all v.

Consider now $v \in [\tilde{v}, \bar{v}]$. Let λ_{ic} be the Lagrange multiplier on the incentive compatibility constraint. Combining the necessary fonc with respect to v_L and the envelope condition, the intertemporal condition for v'_L can be written as:

$$B'(v) = \frac{q}{\beta} B(v'_L) + \frac{\lambda_{ic}}{\mu_L} \tag{A.24}$$

Consider first $v = \tilde{v}$. In this case, (A.24) can be written as:

$$0 = \frac{\beta}{q} B'(\tilde{v}) = B(v'_L(\tilde{v})) + \frac{\beta}{q}\frac{\lambda_{ic}}{\mu_L} > B(v'_L(\tilde{v}))$$

Then it must be that $v'_L(\tilde{v}) > \tilde{v}$.

Consider now borrower values in $(\tilde{v}, \bar{v}]$. I have to consider two cases, $\beta = q$ and $\beta < q$. For $\beta = q$, (A.24) specializes to

$$B'(v) = B(v'_L) + \frac{\lambda_{ic}}{\mu_L} > B'(v'_L)$$

Then, by concavity of B, it follows that $v'_L(v) > v$ for all $v \in (\tilde{v}, \bar{v}]$.

[2] Note that the same variation is not feasible for θ_H because it would violate the incentive compatibility constraint. For θ_L instead the proposed variation is actually relaxing a non-binding incentive constraint (type θ_H not reporting θ_L).

For $\beta < q$, rewrite (A.24) as:

$$B'(v) = \frac{q}{\beta}B(v'_L) + \frac{\lambda_{ic}}{\mu_L} = B'(v'_L(v)) + \frac{q-\beta}{\beta}B'(v'_L(v)) + \frac{\lambda_{ic}(v)}{\mu_L}$$

Then $v'_L(v) < v$ if and only if

$$-\frac{q-\beta}{\beta}B'(v'_L(v)) > \frac{\lambda_{ic}(v)}{\mu_L} \tag{A.25}$$

Suppose for contradiction that $v'_L(v) \geq v$ for all $v \in [v_a, \bar{v}]$. Then it must be that for all v condition (A.25) does not hold and $B'(v) \geq B'\left(v'_L(v)\right)$. Therefore it follows that

$$-\frac{q-\beta}{\beta}B'(v) \leq -\frac{q-\beta}{\beta}B'(v'_L(v)) < \frac{\lambda_{ic}(v)}{\mu_L}$$

Since $\lambda_{ic}(v)$ is bounded from above, $B'(v)$ is strictly decreasing for all $v \geq \tilde{v}$, and $\lim_{v\to\bar{v}} B'(v) = \lim_{c\to\infty} -1/U'(c) = -\infty$, for v sufficiently large it must be that $-\frac{q-\beta}{\beta}B'(v) > \frac{\lambda_{ic}(v)}{\mu_L}$. This is a contradiction. Then, for v sufficiently high, condition (A.25) is met. Denote by $\bar{v}_q \in (\tilde{v}, \bar{v})$ the smallest value of promised utility such that (A.25) holds for all $v > \bar{v}_q$.[3] By the above argument, such a $\bar{v}_q$ exists.

Proof of Proposition 4 In light of Lemma 4, I can restrict attention to the compact set $[v_a, \bar{v}_q] \subset [v_a, \bar{v}]$. In fact, starting from any $v \in (\bar{v}_q, \bar{v}]$ the continuation utility is transiting to $[v_a, \bar{v}_q]$ in a finite number of periods because $v > v'_H(v) > v'_L(v)$ for all $v \in (\bar{v}_q, \bar{v}]$. To show that there exists a unique stationary distribution, I will show that the conditions in Theorem 12.12 in SLP are satisfied. In particular, I need to show that Assumption 12.1 in SLP is satisfied. To this end, define the transition $Q : [v_a, \bar{v}_q] \times \mathcal{B}\left([v_a, \bar{v}_q]\right) \to \mathbb{R}$ as

$$Q(v, A) = \sum_{\theta\in\Theta} \mu(\theta) \int_0^1 \mathbb{I}\left\{v'(\theta, v, \xi) \in A\right\} d\xi$$

I need to show that there exists a mixing point $v \in [v_a, \bar{v}_q]$, $K \geq 1$, and $\varepsilon > 0$ such that $Q^K(v_a, [v, \bar{v}_q]) \geq \varepsilon$ and $Q^K(\bar{v}, [v_a, v]) \geq \varepsilon$. Consider $\tilde{v}$ as the mixing point. Because

[3] Notice that I only show that it exists a $\bar{v}_q$ such that $v'_L(v) > v$ for $v \in [\bar{v}_q, \bar{v})$ and $v'_L(\bar{v}_q) = \bar{v}_q$. I haven't shown that for all $v < \bar{v}_q$ it must be that $v'_L(v) > v$. It is however possible to define $v_q \leq \bar{v}_q$ as the largest v such that for all $v \leq v_q$ we have that $v'_L(v) > v$. The support of the limiting distribution (see the next proposition) is a subset of $[v_a, v_q] \subset [v_a, \bar{v}_q]$. In all of my numerical simulations I find that $v_q = \bar{v}_q$.

$v'_H(v) < v$ for all $v \geq \tilde{v}$, it follows that starting at $\bar{v}_q$ after a sufficiently long (but finite) string of realizations of θ_H, the continuation utility transits to the region with *ex-post* inefficiencies. Thus for some finite K, $Q^K(\bar{v}, [v_a, \tilde{v}]) \geq \mu_H^K > 0$. Furthermore, by Lemma 4, $v'_L(v) \geq \tilde{v}$ for any v. Hence, starting from any v_a after drawing K realizations of θ_L, the continuation value is in the efficient region. Therefore $Q^K(v_a, [\tilde{v}, \bar{v}_q]) \geq \mu_L^K > 0$. Then just let $\varepsilon = \min\{\mu_L^K, \mu_H^K\}$. This shows that $\tilde{v}$ is a mixing point. Therefore, Theorem 12.12 in SLP applies and there exists a unique stationary distribution Ψ^* to which any efficient allocation converges. The fact that Ψ^* is non-degenerate follows from Lemmas 2, 3 and 4.

Proof of Lemma 5 Let Q_L be the space of bounded functions $q_L : [v_a, \bar{v}] \to [0, q/(1-q)]$ and let $T : Q_L \to Q_L$ be defined by the right hand side of (1.49). That is:

$$(Tq_L)(v) = \begin{cases} q \sum_{i=L,H} \mu(\theta_i)\left[1 + q_L(v'_i(v))\right] & \text{if } v \in (v_r, \bar{v}] \\ \frac{q}{1-q}\bar{R}(v) & \text{if } v \in [v_a, v_r] \end{cases}$$

T satisfies the Blackwell's sufficient condition for a contraction mapping, see Theorem 3.3 in SLP. Then, by the contraction mapping theorem, there exists a unique fixed point of T, $\bar{q}_L$. To see that $\bar{q}_L$ is strictly increasing, first notice that $\bar{q}_L$ must be (weakly) increasing. T maps increasing functions into increasing functions. Then, by a corollary of the contraction mapping theorem (see Corollary 3.1 in SLP) it must be that $\bar{q}_L$ is increasing. To see that $\bar{q}_L$ is strictly increasing, first notice that, by definition, $\bar{q}_L$ is strictly increasing over $[v_a, v_r]$. Second, suppose for contradiction that $\bar{q}_L$ is constant over some interval. Let $[v_1, v_3] \subset [v_r, \bar{v}]$ be the first of such intervals so that for all $v \in [v_a, v_1)$, $\bar{q}_L$ is strictly increasing. For all $v_2 \in (v_1, v_3]$ we have that

$$\bar{q}_L(v_1) = q \sum_{i=L,H} \mu(\theta_i)\left[1 + \bar{q}_L(v'_i(v_1))\right] < q \sum_{i=L,H} \mu(\theta_i)\left[1 + \bar{q}_L(v'_i(v_2))\right] \tag{A.26}$$

To see why (A.26) holds, first notice that $v'_L(v_2) > v'_L(v_1)$ and $\bar{q}_L$ is weakly increasing; it follows that $\bar{q}_L(v'_L(v_2)) \geq \bar{q}_L(v'_L(v_1))$. Second, because $v'_H(v_2) > v'_H(v_1)$ and $v'_H(v_1) < v_1$, it must be that $v'_H(v_1) \in [v_a, v_1)$. Since $\bar{q}_L$ is strictly increasing in that region, $\bar{q}_L(v'_H(v_2)) > \bar{q}_L(v'_H(v_1))$. Hence (A.26) holds. This is a contradiction. Then $\bar{q}_L$ is strictly increasing.

Appendix B

Appendix to Chapter 2

Proof of Proposition 1 (i) Consider first a firm that already paid the fixed cost f_x. We can write the dynamic problem of the firm using cash-on-hand as the unique state variable as follows:

$$V^x(a,z) = \max_{d,k',b'} d + q(1-\delta)V^x(a',z)$$

subject to

$$\begin{aligned} d + (1-\delta)qa' &\leq \pi^x(a) \\ d &\geq 0 \end{aligned}$$

It can be shown that V^x is differentiable and concave in a and $V^{x\prime}(a) \geq 1$ with $V^{x'}(a) = 1$ for all $a \geq a^*$ and $V^{x'}(a) > 1$ for $a < a^*$. Letting λ and η be the multiplier associated with the budget constraint and the non-negativity on dividends respectively, the focs for the problem are:

$$d \;:\; 0 = 1 - \lambda + \eta \tag{B.1}$$

$$a' \;:\; 0 = q(1-\delta)\lambda - q(1-\delta)V^{x\prime}(a') \tag{B.2}$$

and the envelope condition:

$$V^{nx\prime}(a) = \lambda\pi^{x\prime}(a)$$

We want to show that if $a' < a^*$ then $\eta > 0$. Suppose for contradiction that $\eta = 0$. Then (B.1) implies that $\lambda = 1$ and in turn (B.2) implies that

$$1 - V^{x\prime}(a') = 0$$

but $V^{x\prime}(a) > 1$ if $a < a^*$ thus $1 - V^{x\prime}(a') < 0$ yielding a contradiction. Then it must be that $\eta > 0$ and $d = 0$.

Consider a non-exporter now. If it is never optimal to export, the same logic we used for an exporter goes through (notice that in this case V^{nx} is concave and differentiable). Instead, if it will be optimal to export at some date, V^{nx} is not necessarily concave and differentiable everywhere. Letting T be the period in which a firm with initial cash on hand a will start to export, we can write the problem as follows:

$$V^{nx}(a) = \max_{\{d_t, a_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t (1-\delta)^t d_t + \beta^{T+1}(1-\delta)^{T+1} V^x(a_{T+1})$$

subject to

$$\begin{aligned} d_t + q(1-\delta)a_{t+1} &\leq \pi^{nx}(a_t) \quad \text{for} \quad t = 0, ..., T-1 \\ d_T + q(1-\delta)a_{T+1} &\leq \pi^{nx}(a_T - f_x) - f_x \\ d_t &\geq 0 \end{aligned}$$

Lettin $\beta^t(1-\delta)^t\lambda_t$ and $\beta^t(1-\delta)^t\eta_t$ be the multiplier associated with the budget constraint and the non-negativity on dividends respectively, the focs for the problem are:

$$0 = 1 - \lambda_t + \eta_t \quad \text{for } t = 0, ..., T \tag{B.3}$$

$$0 = \lambda_t - \lambda_{t+1}\pi'(a_{t+1}) \quad \text{for } t = 0, ..., T-1 \tag{B.4}$$

$$0 = \lambda_T - V^{x'}(a_{T+1}) \tag{B.5}$$

Starting at $T+1$, suppose that $a_{T+1} < a^*$ and for contradiction that $\eta_T = 0$. Then it must be that $\lambda_T = 1$. This and (B.5) imply that

$$1 - V^{x\prime}(a') = 0$$

but $V^{x\prime}(a_{T+1}) > 1$ thus $1 - V^{x\prime}(a') < 0$ yielding a contradiction. Hence $\eta_T > 0$ and $d_T = 0$. Now combine (B.3) at t and $t+1$ with (B.4) at t we obtain:

$$\eta_t = \lambda_t - 1 = \lambda_{t+1}\pi'(a_{t+1}) - 1 \geq \lambda_{t+1} - 1 = \eta_{t+1}$$

Thus, if $\eta_{t+1} > 0$ then $\eta_t > 0$. This is turn implies that as long as any borrowing constraint is binding in the future then there is no dividend distributions as wanted.

When no borrowing constraint in the future are binding then the firms optimal dividend policy is indeterminate. Thus, without loss of generality we can set $d = 0$ to characterize the firm's value and policy functions.

To prove the remaining parts of Proposition 1 we will consider the forward and backward looking case separately.

Forward-Looking Constraint In this case it is convenient to write the problem in (2.9) and (2.11) using their dual formulation. This can be thought of as an optimal contracting problem between the entrepreneur and competitive, risk-neutral financial intermediaries. Financial intermediaries offer the entrepreneur long-term contracts that specify production plans and the value of the dividends paid to the entrepreneur. It is then straightforward to write this problem recursively using the discounted sum of promised dividend payments v as well as the export status of the firm as state variables. Denote the value functions of the financial intermediaries as $W^{nx}(v, z, \phi)$ and $W^{x}(v, z, \phi)$. The problem of the financial intermediary can be written as:

$$W^x(v, z, \phi) = \max -rk + p_d(y_d)y_d + p_x(y_x)y_x - wl - d + q(1-\delta)W^x(v', z, \phi)$$

subject to

$$\begin{aligned} y_d + y_x &\leq zk^{\alpha}l^{1-\alpha} \\ d + q(1-\delta) &= v \\ v &\geq \frac{\theta}{q(1-\delta)}k + \frac{\xi}{q(1-\delta)}v_0 \end{aligned}$$

and for a firm that has not paid the fixed cost already:

$$W^{ns}(v, z, \phi) = -rk + p_d(y_d)y_d - wl - d - xf_x + q(1-\delta)\left[xW^x(v', z, \phi) + (1-x)W^{nx}(v', z, \phi)\right]$$

subject to

$$\begin{aligned} y_d + y_x &\leq zk^{\alpha}l^{1-\alpha} \\ d + q(1-\delta) &= v \\ v &\geq \frac{\theta}{q(1-\delta)}k + \frac{\xi}{q(1-\delta)}v_0 \end{aligned}$$

For notational convenience define

$$\Pi^x(v, z) = \max_{y_d, y_x . l, k} [p_d(y_d)y_d + p_x(y_x)y_x - wl] - rk$$

subject to

$$\begin{aligned} y_d + y_x &\leq zF(k,l) \\ v &\geq \frac{\theta}{q(1-\delta)}k + \frac{\xi}{q(1-\delta)}v_0 \end{aligned}$$

and

$$\Pi^{nx}(v,z) = \max_{y_d,l,k} [p_d(y_d)y_d + p_x(y_x)y_x - wl] - rk$$

subject to

$$\begin{aligned} y_d &\leq zF(k,l) \\ v &\geq \frac{\theta}{q(1-\delta)}k + \frac{\xi}{q(1-\delta)}v_0 \end{aligned}$$

By part (i) we can set $d_t = 0$ without loss for all t and rewrite the intermediary's problem as follows:

$$\begin{aligned} W^{nx}(v,z,\phi) &= \max\{\Pi^{nx}(v,z) + q(1-\delta)W^d\left(\frac{v}{q(1-\delta)}, z, \phi\right); \\ &\quad \Pi^x(v,z) - f_x + q(1-\delta)W^x\left(\frac{v}{q(1-\delta)}, z, \phi\right)\} \\ W^x(v,z,\phi) &= \Pi^x(v,z) + q(1-\delta)W^x\left(\frac{v}{q(1-\delta)}, z\right) \end{aligned}$$

Finally, the minimum equity value for the firm to operate at its efficient scale is given by:

$$v^*(z,\phi) \equiv \min\{\arg\max_v\{\max\{\Pi^{nx}(v,z), \phi\Pi^x(v,z)\}\}\}$$

A firm will eventually reach v^*, because $v_t = \frac{v_0}{((1-\delta)q)^t}$. Then, for $v' \geq v^*$ a domestic firm with inside equity value v' will start exporting iff

$$\frac{\Pi^{x*}(z)}{1-(1-\delta)q} - \frac{\Pi^{nx*}(z)}{1-(1-\delta)q} \geq f_x$$

as in a standard Melitz model. Since the LHS is strictly increasing in z, there exists a cut-off z_x s.t. the above condition holds for all $z \geq z_x$.

We now prove part (iii) and (iv). To this end consider

$$\begin{aligned} W^x(v,z) - W^{nx}(v,z) &= \Pi^x(v,z) + q(1-\delta)W^x\left(\frac{v}{q(1-\delta)}, z\right) - \\ &- \max\left\{\Pi^{nx}(v,z) + q(1-\delta)W^{nx}\left(\frac{v}{q(1-\delta)}, z\right); W^x(v,z) - fx\right\} \\ &= \min\{\Pi^x(v,z) - \Pi^x(v,z) + \\ &+ q(1-\delta)\left(W^x\left(\frac{v}{q(1-\delta)}, z\right) - W^{nx}\left(\frac{v}{q(1-\delta)}, z\right)\right);\ f_x\} \\ &= \min\left\{\Delta\Pi(v,z) + q(1-\delta)\left(W^x\left(\frac{v}{q(1-\delta)}, z\right) - W^{nx}\left(\frac{v}{q(1-\delta)}, z\right)\right);\ f_x\right\} \end{aligned}$$

The following lemma shows that the value of becoming an exporter weakly increases with v.

Lemma B.1. *(a) $\forall z$ $W^x(v,z) - W^{nx}(v,z)$ is weakly increasing in v, and (b) $\forall v$ $W^x(v,z) - W^{nx}(v,z)$ is weakly increasing in z.*

Proof B.1. *Define $T : C(\mathbb{R}_+ \times \mathbb{R}_+) \to C(\mathbb{R}_+ \times \mathbb{R}_+)$ as*

$$Tf(v,z) = \min\left\{\Delta\Pi(v,z) + q(1-\delta)f\left(\frac{v}{q(1-\delta)}, z\right);\ f_x\right\}$$

where $C(\mathbb{R}_+ \times \mathbb{R}_+)$ is the space of continuous and bounded functions. T satisfies the Blackwell's sufficient conditions for a contraction mapping. Then T is a contraction, and $W^x - W^{nx}$ is its unique fixed point.

To prove (a), let $C'(\mathbb{R}_+ \times \mathbb{R}_+)$ be the set of continuous, bounded and weakly increasing function in their first argument. $C'(\mathbb{R}_+ \times \mathbb{R}_+)$ is a closed set, hence by Corollary 3.1 in Stokey, Lucas and Prescott (1989) it suffices to show that $\forall f \in C'(\mathbb{R}_+ \times \mathbb{R}_+)$ $Tf \in C'(\mathbb{R}_+ \times \mathbb{R}_+)$ to prove that $W^x - W^{nx}$ is increasing in its first argument. Fix z, let $f \in C'(\mathbb{R}_+ \times \mathbb{R}_+)$ and $v' > v$:

$$\begin{aligned} Tf(v',z) &= \min\left\{\Delta\Pi(v',z) + q(1-\delta)f\left(\frac{v'}{q(1-\delta)}, z\right);\ f_x\right\} \\ &\geq \min\left\{\Delta\Pi(v,z) + q(1-\delta)f\left(\frac{v}{q(1-\delta)}, z\right);\ f_x\right\} = Tf(v,z) \end{aligned}$$

as wanted, because $\Delta\Pi(v,z)$ is increasing in v, and f is weakly increasing by assumption. Then we established (a). The exact same argument can be used to prove (b) noticing that $\Delta\Pi(v,z)$ is increasing in z also.

Thus, if $z \leq z_x$ a firm will never export since for all v $W^x(v,z) - W^{nx}(v,z) \leq W^x(v^*,z) - W^{nx}(v^*,z) < f_x$. Vice versa, if $z \geq z_x$, then the firm will eventually export, proving (ii).

To prove (iii), notice that if $z \geq z_x$ the firm will eventually export, and the fact that $W^x(v,z) - W^{nx}(v,z)$ is increasing in v implies that there exists a unique threshold $\tilde{v}(z)$ such that a firm will export iff $v \geq \tilde{v}(z)$.

Lastly, we prove (iv) by showing that if $z' > z$ then $\tilde{v}(z')/v_0(z') \leq \tilde{v}(z)/v_0(z)$, implying $\tilde{T}(z') \leq \tilde{T}(z)$. Let $z' > z \geq z_x$. The fact that $W^{nx}(v,z)$ is strictly increasing in z for all v implies that $v_0(z') > v_0(z)$, since v_0 is such that $W^{nx}(v_0(z),z) = 0$. To prove the proposition it is sufficient to show that $\tilde{v}(z') < \tilde{v}(z)$. By the previous lemma $\forall v$ $W^x(v,z) - W^{nx}(v,z)$ is weakly increasing in z. Thus, if $W^x(\tilde{v}(z),z) - W^{nx}(\tilde{v}(z),z) = f_x$ then $W^x(\tilde{v}(z),z') - W^{nx}(\tilde{v}(z),z') \geq f_x$ since $z' > z$, therefore $\tilde{v}(z') \leq \tilde{v}(z)$ as wanted.

To relate this to the "cash on hand" formulation, notice that the cash on hand for a firm with value v is given by $W^i(v)$ for $i = x, nx$, which is a monotone relation in v. Hence, all statements about v are also true for a.

Backward-Looking Constraint Proof of part (iii):

Lemma B.2. *Consider a restricted problem in which firms can only choose to either pay the fixed cost in the first or second Let $x(a,z) = 0$ be the decision to not export in the first period, and $x(a,z) = 1$ be the decision to export in the first period. Then $\exists \hat{a} : \forall a < a, x(a,z) = 0,$ and $\forall a \geq a, x(a,z) = 1$.*

Proof B.2. *In this restricted problem, the fact that all firms must be exporters after the second period (and the fact that z does not change) implies that the objective of the firm is equivalent to maximizing third period assets. Then the decision to export today or tomorrow yields the following payouts:*

If the firm exports today (here assuming all constraints are binding to simplify notation):

$$x(a,z) = 1 \implies q(1-\delta)a_x = \pi^x\left(\frac{\pi^x(a-f_x,z)}{1-(1-\delta)q}, z\right)$$

and if they export the next period:

$$x(a,z) = 0 \implies q(1-\delta)a_{nx} = \pi^x\left(\frac{\pi^{nx}(a,z)}{1-(1-\delta)q} - f_x, z\right)$$

The firm then chooses whichever is greater. Define $\Delta(a,z) \equiv q(1-\delta)[a_x - a_{nx}]$. *Let* $F(a,z) \equiv \pi^x(a - f_x, z) - \pi^{nx}(a,z) + (1 - q(1-\delta))f_x$. *Note that* $sign(F(a,z)) = sign(\Delta(a,z))$. *Then any zero of the function* F *is also a zero of the function* Δ. *We can show that* F *is a strictly increasing function of* a*:*

$$F_1(a,z) \equiv \pi_1^x(a - f_x, z) - \pi_1^{nx}(a,z) > 0$$

which is true because π^x *and* π^{nx} *are concave, and* $\forall a, z, \pi_1^x(a,z) > \pi_1^{nx}(a,z)$.

Then notice that $F(f_x, z) < 0$ *and (assuming that* $z \geq z_x$ *)* $F(a^*, z) > 0$. *Therefore,* $\exists \hat{a} \in [f_x, a^*]$ *that has the cutoff properties described in the statement of the lemma.*

To complete the proof, we demonstrate that the cutoff found in the restricted problem corresponds to the cutoff in the general problem.

First, consider firms with asset values $a < \hat{a}(z)$. Our claim is that the firm does not export with that level of assets. For contradition, suppose that they did. Then, by the definition of $\hat{a}$ given in the lemma, we know that the firm could generate strictly greater profits by, instead, delaying their decision to export by one period. Hence, exporting this period is not optimal.

Second, consider firms with asset values $a \geq \hat{a}(z)$. The next lemma shows that for these firms the restriction on the periods when they can export is not binding.

Lemma B.3. *Suppose a firm prefers to export this period instead of one period in the future. Then the firm prefers to export this period rather than any period in the future.*

Proof B.3. *We prove this by induction. The base step is true by hypothesis. Let* $a^k(t)$ *be the asset level of a firm* k *periods in the future who chooses to enter the export market in period* t.

Using the fact that the firm's objective is equivalent to maximizing their assets whenever they are constrained, to complete the proof we need only show that $a^{k+1}(k) < a^{k+1}(1) \implies a^{k+2}(k+1) < a^{k+2}(1)$. *Notice that the fact that* $a'(a,z)$ *is increasing in* a *means that* $a^{k+1}(k) < a^{k+1}(1) \implies a^{k+2}(k) < a^{k+2}(1)$, *so it is sufficient to show that* $a^{k+2}(k+1) < a^{k+2}(k)$. *But this follows immediately from the previous lemma, the fact that* $a \geq \hat{a}(z)$, *and the fact that* $a'(a,z)$ *is increasing in* a. *This completes the proof.*

Therefore, $\forall a \geq \hat{a}(z)$, the fact that they prefer to export this period rather than the following period implies that they prefer to export this period rather than wait until

any other period. Therefore, $\hat{a}(z)$ is the threshold level of assets that determines export status.

Proof of part (iv):

Here we use the fact that $a'(a,z)$ is increasing in z and that $\hat{a}(z)$ is decreasing in z. The fact that $a'(a,z)$ is increasing in z follows immediately from the fact that $\pi^{nx}(a,z)$ is increasing in z. To prove that $\hat{a}(z)$ is decreasing in z we make use of the characterization in the proof to part (iii).

Recall that $\hat{a}(z)$ solves $F(\hat{a}(z),z)=0$. Then the implicit function theorem implies:

$$\frac{d\hat{a}}{dz} = -\frac{\left[\pi_2^{nx}(a,z) - \pi_2^{x}(a-f_x,z)\right]}{\left[\pi_1^{nx}(a,z) - \pi_1^{x}(a-f_x,z)\right]} < 0$$

The sign follows from the fact that for $j \in \{nx, x\}$, π^j is concave in the first argument, $\pi_{21}^j > 0$, $\forall a, z, \pi_1^x(a,z) > \pi_1^{nx}(a,z)$ and $\pi_2^x(a,z) > \pi_2^{nx}(a,z)$.

Therefore, starting from assets a_0, firms with higher productivity both have faster asset growth and a lower asset threshold to enter the export market. Hence, $T(z)$ is decreasing in z.

Proof of Proposition 2 With $f_x = 0$, all firms with $\phi = 1$ always export.Let the aggregate state of the economy be $\mathbf{s} = (y, w, \tau)$ and let $D_0(\tau) = \omega$ and $D_1(\tau) = (\omega^\sigma + (\frac{1-\omega}{1+\tau})^\sigma)^{1/\sigma}$. Let $\Delta y, \Delta w, \Delta D$ and $\Delta(1+\tau)$ be defined by $\Delta x = x\prime/x$, where primes denote post-reform variables.

First we prove some properties for the economy with perfect credit markets. Recall that $k^*(z,\phi;\mathbf{s})$ and $l^*(z,\phi;\mathbf{s})$ are the solution to (2.25).

Lemma B.4. *$k^*(z,\phi;\mathbf{s})$ is homogeneous of degree 1 in y, degree σ in D, degree $\sigma-1$ in z, and degree $(1-\alpha)(1-\sigma)$ in w; $l^*(z,\phi;\mathbf{s})$ is homogeneous of degree 1 in y, degree σ in D, degree $\sigma-1$ in z, and degree $(\alpha-1)\sigma-\alpha$ in w.*

Proof B.4. *Letting λ be the Lagrangian multiplier on the constraint, the first order conditions of the unconstrained firm imply:*

$$k/l = \frac{\alpha}{1-\alpha}\frac{w}{r}$$

and

$$\lambda = w\frac{1}{1-\alpha}\frac{1}{z}\left(\frac{k}{l}\right)^{-\alpha} = w\frac{1}{1-\alpha}\frac{1}{z}\left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{-\alpha} = const \times \frac{w^{1-\alpha}r^{\alpha}}{z}$$

Then, notice that

$$\begin{aligned} y_d &= (1-1/\sigma)^\sigma \omega^\sigma \lambda^{-\sigma} y \propto \omega^\sigma \left(\frac{w^{1-\alpha} r^\alpha}{z}\right)^{-\sigma} y \\ y_x &= \phi(1-1/\sigma)^\sigma \left(\frac{1-\omega}{1+\tau}\right)^\sigma \lambda^{-\sigma} y \propto \left(\frac{1-\omega}{1+\tau}\right)^\sigma \left(\frac{w^{1-\alpha} r^\alpha}{z}\right)^{-\sigma} y \end{aligned}$$

Therefore using the production function:

$$y = z\left(\frac{k}{l}\right)^\alpha l = z\left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^\alpha l$$

it follows that

$$\begin{aligned} l^*(z,\phi;\mathbf{s}) &= y(z,\phi;\mathbf{s})\left[z\left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^\alpha\right]^{-1} \\ &\propto z^{\sigma-1} D_\phi^\sigma y w^{(\alpha-1)\sigma-\alpha} \end{aligned} \tag{B.6}$$

$$k^*(z,\phi;\mathbf{s}) \propto z^{\sigma-1} D_\phi^\sigma y w^{(\alpha-1)\sigma-\alpha+1} = z^{\sigma-1} D_\phi^\sigma y w^{(\alpha-1)(\sigma-1)} \tag{B.7}$$

as wanted.

Consider now an economy with limited enforcement. Define

$$v_t(z,\phi;\mathbf{s}) = \sum_{s=0}^{\infty} (q(1-\delta))^s d_{t+s}(z,\phi;\varphi) = \frac{v_0(z,\phi;\varphi)}{(q(1-\delta))^t}$$

be the present value of future dividends for a firm of age t. The second equality comes from Proposition 1: whenever the borrowing constraint is binding there are no dividends paid. Hence, v_t grows at rate $1/(q(1-\delta))$. Let $k(v,z,\phi;\mathbf{s})$ and $l(v,z,\phi;\mathbf{s})$ be the solution to:

$$\pi(v,z,\phi;\mathbf{s}) = \max_{y_x,y_d,l,k} \omega y^{1/\sigma} y_d^{1-1/\sigma} + \phi\frac{(1-\omega)}{1+\tau} y^{1/\sigma} y_x^{1-1/\sigma} - wl - rk \tag{B.8}$$

subject to (2.26) and

$$q(1-\delta)v \geq \theta k + \xi v_0(z,\phi;\mathbf{s}) \tag{B.9}$$

For v sufficiently high, the enforcement constraint is not binding and the firm operates at its optimal scale $k^*(z,\phi;\mathbf{s})$ and makes profits $\pi^*(z,\phi;\mathbf{s})$. Define $v^*(z,\phi;\mathbf{s}) = \theta k^* + \xi v_0$ as the smallest value of v needed to sustain optimal scale, and $T^*(z,\phi;\mathbf{s}) =$

$\lceil \log(v_0/v^*)/\log(q(1-\delta)) \rceil$ as the number of periods it takes the firm to reach optimal scale.

Given that the financial sector makes zero expected profits, in equilibrium the initial value of the firm v_0 is the solution to:

$$v_0(z,\phi;\mathbf{s}) = \sum_{t=0}^{\infty} (q(1-\delta))^t \pi\left(\frac{v_0(z,\phi;\mathbf{s})}{(q(1-\delta))^t}, z, \phi; \mathbf{s}\right)$$

We now prove a series of Lemmas that we will use in the proof of the Proposition.

Lemma B.5. *$k(v,z,\phi;\mathbf{s})$ is given by*

$$k(v,z,\phi;\mathbf{s}) = \min\left\{k^*(z,\phi;\mathbf{s}), \frac{q(1-\delta)v - \xi v_0(z,\phi;\mathbf{s})}{\theta}\right\} \tag{B.10}$$

and the indirect profits function is given by

$$\pi(v,z,\phi;s) = Cw^{\frac{(\alpha-1)(\sigma-1)}{1+\alpha(\sigma-1)}} D_\phi^{\frac{\sigma}{1+\alpha(\sigma-1)}} y^{\frac{1}{1+\alpha(\sigma-1)}} z^{\frac{\sigma-1}{1+\alpha(\sigma-1)}} k(v,z,\phi;\mathbf{s})^{\frac{\alpha(\sigma-1)}{1+\alpha(\sigma-1)}} - rk(v,z,\phi;\mathbf{s}) \tag{B.11}$$

where C is a constant.

Proof B.5. *The solution for k is given by*

$$k(v,z,\phi;\mathbf{s}) = \min\left\{k^*(z,\phi;\mathbf{s}), \frac{q(1-\delta)v - \xi v_0(z,\phi;\mathbf{s})}{\theta}\right\} \tag{B.12}$$

Combining first order conditions, the solution to the problem is given by the solution to the following equations:

$$y_d = \omega^\sigma (1-1/\sigma)^\sigma y \lambda^{-\sigma} = \omega^\sigma (1-1/\sigma)^\sigma y \left[\frac{(1-\alpha)z}{w}\left(\frac{k}{l}\right)^\alpha\right]^\sigma \tag{B.13}$$

$$y_x = \left(\frac{1-\omega}{1+\tau}\right)^\sigma (1-1/\sigma)^\sigma y \lambda^{-\sigma} = \left(\frac{1-\omega}{1+\tau}\right)^\sigma (1-1/\sigma)^\sigma y \left[\frac{(1-\alpha)z}{w}\left(\frac{k}{l}\right)^\alpha\right]^\sigma \tag{B.14}$$

Then I can use the production function to solve for l:

$$\begin{aligned} l &= \left(\frac{(1-1/\sigma)^\sigma \left[\omega^\sigma + \left(\frac{1-\omega}{1+\tau}\right)^\sigma\right] y \left[\frac{(1-\alpha)z}{w}\left(\frac{k}{l}\right)^\alpha\right]^\sigma}{zk^\alpha}\right)^{\frac{1}{1-\alpha}} \\ &= const \times D_\phi^{\frac{\sigma}{1+\alpha(\sigma-1)}} y^{\frac{1}{1+\alpha(\sigma-1)}} z^{\frac{\sigma-1}{1+\alpha(\sigma-1)}} w^{\frac{-\sigma}{1+\alpha(\sigma-1)}} k^{\frac{\alpha(\sigma-1)}{1+\alpha(\sigma-1)}} \end{aligned} \tag{B.15}$$

Plugging these solutions back in the objective function, it follows that when the enforcement constraint is binding we have that:

$$\pi_\phi(v,z;s) = D_\phi(\tau) y_d^{1/\sigma} \left[z k^\alpha l^{1-\alpha} \right]^{1-1/\sigma} - wl - rk \tag{B.16}$$

Plugging in the above yields

$$\pi_\phi(v,z;s) = C w^{\frac{(\alpha-1)(\sigma-1)}{1+\alpha(\sigma-1)}} D_\phi^{\frac{\sigma}{1+\alpha(\sigma-1)}} y^{\frac{1}{1+\alpha(\sigma-1)}} z^{\frac{\sigma-1}{1+\alpha(\sigma-1)}} k(v,z,\phi;\mathbf{s})^{\frac{\alpha(\sigma-1)}{1+\alpha(\sigma-1)}} - rk(v,z,\phi;\mathbf{s})$$

where C is a constant.

Lemma B.6. *With perfect credit market we have that*

$$\Delta_{k,\phi} = \Delta_{D,\phi}^\sigma \Delta_y \Delta_w^{(\alpha-1)(\sigma-1)} \tag{B.17}$$

Proof B.6. *It follows directly from (B.7).*

Lemma B.7. *$v_0(z,\phi;\mathbf{s})$ is homogeneous of degree $\sigma-1$ in z, σ in D_ϕ, 1 in y, and $(\alpha-1)(\sigma-1)$ in w. That is, $\exists$ a scalar $\tilde{v}_0$ such that $\forall(z,\phi,\mathbf{s})$*

$$v_0(z,\phi;\mathbf{s}) = \tilde{v}_0 z^{\sigma-1} D_\phi^\sigma y w^{(\alpha-1)(\sigma-1)} \tag{B.18}$$

Proof B.7. *We now proceed by guess and verify. Suppose that $v_0(z,\phi;\mathbf{s})$ takes the form in (B.18). Then, given the guess $v^*(w) = \theta\tilde{k}w^{(\alpha-1)(\sigma-1)} + \xi\tilde{v}_0 w^{(\alpha-1)(\sigma-1)} = \tilde{v}^* w^{(\alpha-1)(\sigma-1)}$. Hence it follows that*

$$k(v,z,\phi;\mathbf{s}) = \min\left\{ k^*(z,\phi;\mathbf{s}), \frac{q(1-\delta)v - \xi v_0(z,\phi;\mathbf{s})}{\theta} \right\} \propto z^{\sigma-1} D_\phi^\sigma y w^{(\alpha-1)(\sigma-1)} \tag{B.19}$$

Lastly, it can be shown that $\forall t \geq 0$

$$\pi(v_t(w);w) = \pi\left(\frac{\tilde{v}_0}{(q(1-\delta))^t}; 1 \right) w^{(\alpha-1)(\sigma-1)} \tag{B.20}$$

by combining (B.11) and (B.19). Thus, using (B.20) in the definition of v_0 it follows that $v_0(z,\phi;\mathbf{s})$ is homogeneous of degree $\sigma-1$ in z, σ in D_ϕ, 1 in y, and $(\alpha-1)(\sigma-1)$ in w as wanted.

Thus, the above lemmas imply that if the path $\{k_t(z,\phi), b_t(z,\phi), d_t(z,\phi)\}_{t=0}^\infty$ for a firm of type (z,ϕ) with aggregate state $\mathbf{s}$, then $\{\Delta_k k_t(z,\phi), \Delta_k b_t(z,\phi), \Delta_k d_t(z,\phi)\}_{t=0}^\infty$

is optimal for the aggregate state $s' = (\Delta_y y, \Delta_w w, \tau')$. We are now left to show that labor and good market clear. We denote variables from the perfect credit markets environment with superscript PC and from the forward-looking environment FL. First we prove a lemma that shows that the financial friction induces a distortion across age, but not across productivity levels.

Lemma B.8. *Suppose s^{PC} is an equilibrium in the perfect credit markets environment and s^{FL} is an equilibrium in the forward looking environmet. Then $\forall z, \phi$,*

$$l^{PC}(z, \phi; s^{PC}) = \delta \sum_t (1-\delta)^t l^{FL}\left(\frac{v_0(z,\phi)}{((1-\delta)q)^t}, z, \phi; s^{FL}\right)$$

$$y_d^{PC}(z, \phi; s^{PC})/y^{PC} = \delta \sum_t (1-\delta)^t y_d^{FL}\left(\frac{v_0(z,\phi)}{((1-\delta)q)^t}, z, \phi; s^{FL}\right)/y^{FL}$$

and

$$y_x^{PC}(z, \phi; s^{PC})/y^{PC} = \delta \sum_t (1-\delta)^t y_x^{FL}\left(\frac{v_0(z,\phi)}{((1-\delta)q)^t}, z, \phi; s^{FL}\right)/y^{FL}$$

Proof B.8. *We only show the case with l as the other cases are analogous. Combining (B.18) and (B.15),*

$$\forall t, z, \phi, l^{FL}\left(\frac{v_0(z,\phi)}{((1-\delta)q)^t}, z, \phi; \mathbf{s}\right) \propto z^{\sigma-1} D_\phi^\sigma y w^{(\alpha-1)\sigma-\alpha}$$

As above, we already know that

$$\forall z, \phi, l^{PC}(z, \phi; s) \propto z^{\sigma-1} D_\phi^\sigma y w^{(\alpha-1)\sigma-\alpha}$$

Then the fact that labor markets clear in both cases implies

$$\begin{aligned}
1 &= \rho \int_Z l^{PC}(z, 0; s^{PC}) d\Gamma(z) + (1-\rho) \int_Z l^{PC}(z, 1; s^{PC}) d\Gamma(z) \\
1 &= \delta \int_Z \sum_t (1-\delta)^t \left[\rho l^{FL}\left(\frac{v_0(z,0)}{((1-\delta)q)^t}, z, 0; s^{FL}\right)\right] d\Gamma(z) \\
&\quad + \delta \int_Z \sum_t (1-\delta)^t \left[(1-\rho) l^{FL}\left(\frac{v_0(z,1)}{((1-\delta)q)^t}, z, 1; s^{FL}\right)\right] d\Gamma(z)
\end{aligned}$$

Using the above facts yields,

$$\begin{aligned}
1 &= l^{PC}(1,0;s^{PC})\left(\rho\int_Z z^{\sigma-1}d\Gamma(z)+\left(\frac{D_1}{D_0}\right)^{\sigma}(1-\rho)\int_Z z^{\sigma-1}d\Gamma(z)\right)\\
1 &= \delta\sum_t(1-\delta)^t l^{FL}\left(\frac{v_0(1,0)}{((1-\delta)q)^t},1,0;s^{FL}\right)\times\\
&\quad\times\left(\rho\int_Z z^{\sigma-1}d\Gamma(z)+\left(\frac{D_1}{D_0}\right)^{\sigma}(1-\rho)\int_Z z^{\sigma-1}d\Gamma(z)\right)
\end{aligned}$$

Combining these equations and multiplying through by any value of $z^{\sigma-1}$ or D_ϕ yields the result.

Now we check the labor market clearing condition. Again, let $\mathbf{s}=(y,w,\tau)$. The post-reform labor market clearing condition is:

$$\begin{aligned}
1 &= \rho\int_Z l^{PC}(z,0;s^{PC\prime})d\Gamma(z)+(1-\rho)\int_Z l^{PC}(z,1;s^{PC\prime})d\Gamma(z)=\\
&= \frac{\Delta_y}{\Delta_w^{1+(1-\alpha)(\sigma-1)}}\left[\rho\int_Z l^{PC}(z,0;s^{PC})d\Gamma(z)+\Delta_D^{\sigma}(1-\rho)\int_Z l^{PC}(z,1;s^{PC})d\Gamma(z)\right]
\end{aligned}$$

Applying the above lemma we get:

$$\begin{aligned}
1 &= \frac{\Delta_y\delta}{\Delta_w{}^{1+(1-\alpha)(\sigma-1)}}\int_Z\sum_t(1-\delta)^t\times\\
&\quad\times\left[\rho l^{FL}\left(\frac{v_0(z,0)}{((1-\delta)q)^t},z,0;s^{FL}\right)+\Delta_D^{\sigma}(1-\rho)l^{FL}\left(\frac{v_0(z,1)}{((1-\delta)q)^t},z,1;s^{FL}\right)\right]d\Gamma(z)
\end{aligned}$$

which implies

$$\begin{aligned}
1 &= \delta\int_Z\sum_t(1-\delta)^t\times\\
&\quad\times\left[\rho l^{FL}\left(\frac{v_0(z,0)}{((1-\delta)q)^t},z,0;s^{FL\prime}\right)+(1-\rho)l^{FL}\left(\frac{v_0(z,1)}{((1-\delta)q)^t},z,1;s^{FL\prime}\right)\right]d\Gamma(z)
\end{aligned}$$

Hence, labor market clearing is satisfied for the forward-looking case. The perfect credit markets goods market clearing condition is equivalent to:

$$1=\int_Z\left[\omega\left(\frac{y_d^{PC}(z,0;s^{PC\prime})}{y^{PC}}\right)^{\gamma}+(1-\omega)\rho\left(\frac{y_x^{PC}(z,1;s^{PC\prime})}{y^{PC}}\right)^{\gamma}\right]d\Gamma(z)$$

which implies

$$\begin{aligned}
1 &= \omega\Delta_w^{(1-\sigma)(1-\alpha)}\int_Z\left(\frac{y_d^{PC}(z,0;s^{PC})}{y^{PC}}\right)^{\gamma}d\Gamma(z)+\\
&\quad+(1-\omega)\Delta_w^{(1-\sigma)(1-\alpha)}\Delta_D^{\sigma-1}\rho\int_Z\left(\frac{y_x^{PC}(z,1;s^{PC})}{y^{PC}}\right)^{\gamma}d\Gamma(z)
\end{aligned}$$

Applying the above lemma yields:

$$\begin{aligned} 1 &= \omega\delta\Delta_w^{(1-\sigma)(1-\alpha)}\left[\int_Z \sum_t (1-\delta)^t \left(\frac{y_d^{FL}(v_0(z,0)/(q(1-\delta))^t, z, \phi; s^{FL})}{y^{FL}}\right)^\gamma d\Gamma(z) + \right. \\ &\left. + \frac{1-\omega}{\omega}\Delta_D^{\sigma-1}\rho\int_Z \sum_t (1-\delta)^t \left(\frac{y_x^{FL}(v_0(z,1)/(q(1-\delta))^t, z, 1; s^{FL})}{y^{FL}}\right)^\gamma d\Gamma(z)\right] \end{aligned}$$

which implies

$$\begin{aligned} 1 &= \omega\delta\left[\int_Z \sum_t \delta^t \left(\frac{y_d^{FL}(v_0(z,0)/(q(1-\delta))^t, z, 0; s^{FL\prime})}{y^{FL\prime}}\right)^\gamma d\Gamma(z) + \right. \\ &\left. + \frac{1-\omega}{\omega}\rho\int_Z \sum_t (1-\delta)^t \left(\frac{y_x^{FL}(v_0(z,1)/(q(1-\delta))^t, z, 1; s^{FL\prime})}{y^{FL\prime}}\right)^\gamma d\Gamma(z)\right] \end{aligned}$$

which is goods market clearing in the forward-looking environment. Hence, the changes in prices from the perfect credit markets environment are also the equilibrium changes in prices in the forward-looking environment. This is equivalent to the statement of Proposition 2.

Small Open Economy Framework The small open economy model differs from the two country framework in three ways.

First, there is an imported good M with price P that enters the production function of the domestic final good as follows:

$$y = \left[\omega\int_I y_d(i)^{\frac{\sigma-1}{\sigma}} di + (1-\omega)M^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

Second, we derive the inverse demand functions for domestic firms that sell abroad as the solution to the following problem of the final goods producer in the rest of the world:

$$\max_{Y,M,y_x} P^*Y - PM - (1+\tau_x)\int_{I_x} p_x^f(i)y_x(i)di$$

subject to

$$Y = \left((1-\mu)\int_{I_x} y_x(i)^{\frac{\sigma-1}{\sigma}} di + \mu M\right)^{\frac{\sigma-1}{\sigma}}$$

The solution to this implies an inverse demand function:

$$p_x^f(i) = \frac{1-\mu}{1+\tau_x}P^*Y^{1/\sigma}y_x(i)^{-1/\sigma}$$

The rest of the world is large, so domestic exports have no impact on the foreign price level. Formally, we consider parameter values for μ that satisfy $(1-\mu) = Y^{-1/\sigma}$, then consider the limit as Y goes to infinity. Then the foreign price level converges to $P^* = P$ and the inverse demand function limits to

$$p_x^f(i) = \frac{P}{1+\tau_x} y_x(i)^{-1/\sigma}$$

Lastly, as in the data, trade is not balanced. Therefore, we are assuming that bonds are internationally traded with a gross interest rate of $1/\beta$, and that the economy is initially endowed with a net foriegn asset position such that their net payments on debt are exactly equal to domestic net exports.

Besides these three features, the model is exactly the same as the two country case.

Appendix C

Appendix to Chapter 3

Proof of Lemma 1 ($\Rightarrow$) Let $x \equiv \{c_t, n_t, k_{t+1}, k^*_{t+1}, a_{t+1}, g_t\}_{t=0}^{\infty}$ be part of a competitive equilibrium. Then it must satisfy the necessary focs for the stand-in household's problem:

$$\begin{aligned} U_{ct} &= \beta U_{ct+1}(1-\tau^k_{t+1})F_{kt+1} \\ U_{ct} &= \beta U_{ct+1}(1-\tau^a_{t+1})F_{kt+1} \\ -U_{nt} &= U_{ct}(1-\tau^n_t)F_{nt} \end{aligned}$$

and the TVCs

$$\begin{aligned} \lim_{T\to\infty} \beta^T U_{cT}(1-\tau^k_T)F_{kT}k_T &= 0 \\ \lim_{T\to\infty} \beta^T U_{cT}(1-\tau^a_T)R^* a_T &= 0 \end{aligned}$$

and from the foreign investors:

$$\begin{aligned} R^*_t &= (1-\tau^*_t)F_{kt} \\ q_{t+1} &= \delta_{t+1}/R^*_{t+1} \end{aligned}$$

Therefore, substituting out for prices and taxes in the household's budget constraint at $t = 0$ and iterating forward (invoking the TVCs), I obtain the implementability condition (3.5). In the same way $\forall t \geq 1$ I obtain that

$$\sum_{s=0}^{\infty} \beta^s \left[U_{ct+s}c_{t+s} + U_{nt+s}n_{t+s}\right] = U_{ct}\left[(1-\tau^k_t)F_{kt}k_t + (1-\tau^a_t)R^*_t a_t\right] \geq 0$$

where the fact that the rhs is greather than zero comes from the fact that in any equilibrium $k_t, a_t, U_{ct}, (1-\tau_t^k)F_{kt}, (1-\tau_t^a)R_t^* \geq 0$. Hence $\forall\ t \geq 1$ (3.6) must hold in any competitive equilibrium. Finally, since the budget constraint for the government

$$\delta_t b_t + g_t = \tau_t^n w_t n_t + \tau_t^k R_t k_t + \tau_t^a R^* a_t + \tau_t^* R_t k_t^* + q_{t+1} b_{t+1}$$

and the domestic household

$$\begin{aligned} c_{t+1} + k_{t+1} + a_{t+1} &= (1-\tau_t^k)R_t k_t + (1-\tau_t^a)R^* a_t + (1-\tau_t^n)w_t n_t \\ &= -\left[\tau_t^n w_t n_t + \tau_t^k R_t k_t + \tau_t^a R^* a_t\right] + R_t k_t + w_t n_t + R_t^* a_t \end{aligned}$$

must hold with equality in any equilibrium, I can combine them substituting out for prices to obtain:

$$\delta_t b_t + g_t = \left[\tau_t^n w_t n_t + \tau_t^k R_t k_t + \tau_t^a R^* a_t\right] + \tau_t^* R_t k_t^* + q_{t+1} b_{t+1}$$

$$= [R_t k_t + w_t n_t + R_t^* a_t - (c_{t+1} + k_{t+1} + a_{t+1})] + \tau_t^* R_t k_t^* + \frac{\delta_{t+1}}{R_{t+1}^*} b_{t+1}$$

$$= [F_{kt} k_t + F_{nt} n_t + R_t^* a_t - (c_{t+1} + k_{t+1} + a_{t+1})] + \tau_t^* F_{kt} k_t^* + \frac{\delta_{t+1}}{R_{t+1}^*} b_{t+1}$$

$$= [F_{kt} k_t + F_{nt} n_t + R_t^* a_t - (c_{t+1} + k_{t+1} + a_{t+1})] + (F_{kt} - R_t^*)\, k_t^* + \frac{\delta_{t+1}}{R_{t+1}^*} b_{t+1}$$

$$= F(k_t + k_t^*, n_t) + R_t^* a_t - (c_{t+1} + k_{t+1} + a_{t+1}) - R_t^* k_t^* + \frac{\delta_{t+1}}{R_{t+1}^*} b_{t+1}$$

$$\iff F(k_t + k_t^*, n_t) + R_t^* a_t - \delta_t b_t = g_t + c_t + k_{t+1} + R_t^* k_t^* + a_{t+1} - \frac{\delta_{t+1}}{R_{t+1}^*} b_{t+1}$$

Then, iterating forward I obtain:

$$\sum_{t=0}^{\infty} Q_t \left[g_t + c_t + k_{t+1} + R^* k_t^* - F(K_t, n_t)\right] - R^* a_0 + (1-\tau_0^*)F_k(K_0, n_0)k_0^* = -b_0 \delta_0$$

Hence the consolidated budget constraint for the country (3.7) must also hold, as desired.

($\Leftarrow$) Suppose $x \equiv \{c_t, n_t, k_{t+1}, k_{t+1}^*, a_{t+1}, g_t\}_{t=0}^{\infty}$ satisfies (3.5), (3.6) and (3.7). Let taxes be such that $\forall\ t \geq 0$

$$\begin{aligned} \tau_{t+1}^p &: \quad R_t^* = (1-\tau_t^*)F_{kt} \\ \tau_{t+1}^k &: \quad U_{ct} = \beta U_{ct+1}(1-\tau_{t+1}^k)F_{kt+1} \\ \tau_{t+1}^a &: \quad U_{ct} = \beta U_{ct+1}(1-\tau_{t+1}^a)R_{t+1}^* \\ \tau_t^n &: \quad -U_{nt} = U_{ct}(1-\tau_t^n)F_{nt} \end{aligned}$$

and $\delta_t = 1\ \forall t \geq 1$. Let factor prices be $w_t = F_{nt}$ and $R_t = F_{kt}$ and $q_{t+1} = 1/R^*_{t+1}$ $\forall t \geq 0$. Then the allocation satisfies the (sufficient) focs for the household optimization problem at the constructed prices and taxes and it is affordable due to (3.5). It also satisfies the optimality conditions for the foreign investors.

I am then left to show that the allocation, constructed taxes and prices satisfy the government budget constraint. To this end, I construct government debt $\{b_{t+1}\}_{t=0}^{\infty}$ recursively as follows:

$$R^*_t b_t = \tau^n_t w_t n_t + \tau^k_t R_t k_t + \tau^a_t R^* a_t + \tau^*_t R_t k^*_t + b_{t+1} - g_t$$

to make the government consumption $\{g_t\}_{t=0}^{\infty}$ affordable. Notice that $\lim_{t\to\infty} b_t$ is going to be bounded because of (3.7). □

Sketch of Proof of Lemma 3

Claim C.1. *Can construct strategy to support as an equilibrium outcome the allocation* $\underline{x}$ *that solves the problem above. Call this strategy* $(\underline{\sigma}, \underline{f}, \underline{f^*}) \in SE(k, k^*, a, b)$*.Then* $\underline{V}(k, k^*, a, b) = \underline{V} = W(\underline{\sigma}, \underline{f}, \underline{f^*})$*.*

Claim C.2. *If* x *is an equilibrium outcome path for the game thenn it is an equilibrium outcome path for a similar game when the government can also tax investment.*

Claim C.3. $\underline{V} = \min v$ *such that* $\exists(\sigma^i, f^i, f^{*i}) \in SE^i(k, k^*, a, b)\ :\ v = W^i(\sigma^i, f^i, f^{*i})$ *where a superscript* i *refers to the game where the government can tax investment also*

Claim C.4. $\underline{V} = \min v$ *such that* $\exists(\sigma, f, f^*) \in SE(k, k^*, a, b)\ :\ v = W(\sigma, f, f^*)$*.*

This final claim is implied by the previous two claims. □

Proof of Proposition 2 First consider the case with $\beta = 1/R^*_t\ \forall t \geq 1$. The Ramsey problem can be written as:

$$V^R_0(K_0, a_0 - b_0) = \max_{\{c_t, n_t, g_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[U(c_t, n_t) - G(g_t)\right]$$

subject to

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \left[U_{ct} c_t + U_{nt} n_t\right] &\geq 0 \\ \sum_{t=0}^{\infty} \beta^t \left[c_t + g_t + K_{t+1} - F(K_t, n_t)\right] &= a_0 - b_0 \end{aligned}$$

Notice that for $a_0 - b_0$ sufficiently large, the implementability constraint doesn't bind. In fact, for $a_0 - b_0 \geq \bar{A}'$ sufficiently high, it is possible to finance the "first best" level of public expenditure - the solution to the above problem dropping the implementability constraint, denoted by a superscript FB - without levying any taxes:

$$a_0 - b_0 \geq \sum_{t=0}^{\infty} \beta^t g_t^{FB}(K_0, a_0 - b_0) = \frac{g^{FB}(K_0, a_0 - b_0)}{1 - \beta}$$

Hence, for $a_0 - b_0 \geq \bar{A}'$ $V_0^R(K_0, a_0 - b_0) = V_0^{FB}(K_0, a_0 - b_0)$ and $V_1^R(K_0, a_0 - b_0) = V_1^{FB}(K_0, a_0 - b_0)$, where

$$V_1^R(K_0, a_0 - b_0) \equiv \sum_{t=1}^{\infty} \beta^{t-1} \left[U(c_t, n_t) - G(g_t)\right]$$

Then, since $\underline{V}_1 < V_1^{FB}$, since V_1^R is continuous in $a_0 - b_0$, $\underline{V}_1 \leq V_1^R$ for $a_0 - b_0 \geq \bar{A} < \bar{A}'$, proving part (i).

Consider now the case with $\beta R^* < 1$. The solution with commitment is such that

$$\{c_t^R\} \downarrow 0, \ \ \{g_t^R\} \downarrow 0, \ \{n_t^R\} \to \bar{N}, \ \{K_t^R\} \to K^R > 0$$

That is, it is optimal to front-load consumption and leisure. It is easy to see that for t sufficiently high and for $\varepsilon > 0$ sufficiently small

$$V_t^R = \sum_{s=0}^{\infty} \beta^s \left[u(c_{t+s}^R) - v(n_{t+s}^R) + G(g_{t+s}^R)\right] \leq \frac{u(0) - v(\bar{N}) + G(0)}{1 - \beta} + \varepsilon < \Omega(K_t)$$

Then the Ramsey plan cannot be supported as a SE if $\beta R^* < 1$, proving the second part of the proposition. □

Proof of Proposition 3: Under assumption A2 the implementability condition (3.5) in (P) can be written as

$$\sum_{t=0}^{\infty} \beta^t \left(c_t^{1-\eta} - \chi n_t^{1+\gamma}\right) = c_0^{-\eta} A_0$$

The constraint set in (P) is not convex. Moreover, the plan

$$x = \{c_t, n_t, k_{t+1}, k_{t+1}^*, a_{t+1}, b_{t+1}, g_t\}_{t=0}^{\infty}$$

that solves (P) is not necessarily interior. In fact, it is likely a_{t+1} and b_{t+1} are at a corner as shown in Proposition 4. Thus, using Lagrangian methods to characterize the full problem (P) is not feasible. However, if x is the solution to (P), then it must be that $\{c_t, n_t, k_{t+1}, k^*_{t+1}, g_t\}_{t=0}^{\infty}$ solves

$$\max_{\{c_t,n_t,k_{t+1},k^*_{t+1},g_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\eta}}{1-\eta} - \chi \frac{n_t^{1+\gamma}}{1+\gamma} + G(g_t) \right] \tag{P'}$$

subject to

$$(\lambda) \quad \sum_{t=0}^{\infty} \beta^t \left(c_t^{1-\eta} - \chi n_t^{1+\gamma} \right) = c_0^{-\eta} A_0 \tag{C.1}$$

$$(\beta^t \psi_t) \quad \sum_{s=0}^{\infty} \beta^s \left(c_{t+s}^{1-\eta} - \chi n_{t+s}^{1+\gamma} \right) \geq 0 \quad \forall t \geq 1 \tag{C.2}$$

$$(Q_t \phi_t) \quad F(k_t + k^*_t, n_t) + R^*_t (a_t - b_t) \geq g_t + c_t + k_{t+1} + R^*_t k^*_t + (a_{t+1} - b_{t+1}) \quad \forall t \geq 0 \tag{C.3}$$

$$(\beta^t \mu_t) \quad \sum_{s=0}^{\infty} \beta^s \left[U(c_{t+s}, n_{t+s}) + G(g_{t+s}) \right] \geq \Omega(k_t + k^*_t, a_t + \max\{-b_t, 0\}) \quad \forall t \geq 1 \tag{C.4}$$

$$k_t, k^*_t \geq 0 \quad \forall t \geq 1$$

given the optimal path for $\{a_{t+1}, b_{t+1}\}_{t=0}^{\infty}$. Thus, if the optimal $\{c_t, n_t, k_{t+1}, k^*_{t+1}\}_{t=0}^{\infty}$ is interior, then it must satisfy the following necessary foncs wrt $c_t, n_t, g_t, k_{t+1}, k^*_{t+1}$ $\forall t \geq 1$:

$$0 = \beta^t \left[1 + (1-\eta)\lambda + M_t + (1-\eta) P_t \right] c_t^{-\eta} - Q_t \phi_t \tag{C.5}$$

$$0 = \beta^t \left[1 + (1+\gamma)\lambda + M_t + (1+\gamma) P_t \right] \chi n_t^{\gamma} - Q_t \phi_t F_{nt} \tag{C.6}$$

$$0 = \beta^t \left(1 + M_t \right) G'(g_t) - Q_t \phi \tag{C.7}$$

$$0 = -Q_t \phi_t - \beta^{t+1} \mu_{t+1} \Omega_{k,t+1} + Q_{t+1} \phi_{t+1} F_{kt+1} \tag{C.8}$$

$$0 = Q_{t+1} \phi_{t+1} \left[F_{kt+1} - R^*_{t+1} \right] - \beta^{t+1} \mu_{t+1} \Omega_{k,t+1} \tag{C.9}$$

where λ, $\beta^t \psi_t$, $Q_t \phi_t$, and $\beta^t \mu_t$ are the Lagrange multipliers associated with (C.1), (C.2), (C.3), (C.4) respectively, and $M_t \equiv \sum_{s=1}^{t} \mu_s$ $\forall t \geq 1$ and $P_t \equiv \sum_{s=1}^{t} \psi_s$. For a proof for the fact that multiplier for this class of problems are in ℓ_1 see Rustichini (1998).

The fonc wrt k^*_{t+1}, (C.9), can be rearranged as follows:

$$R^*_{t+1} = F_{kt+1} - \frac{\beta^{t+1}\mu_{t+1}}{Q_t\phi_t}\Omega_{k,t+1}$$

Therefore, since in any equilibrium it must be that $R^*_{t+1} = (1-\tau^*_{t+1})F_{kt+1}$, we have that $\forall t \geq 1$

$$\tau^*_{t+1} = \frac{\beta^{t+1}\mu_{t+1}}{Q_t\phi_t}\frac{\Omega_{k,t+1}}{F_{k,t+1}} \geq 0$$

strictly if $\mu_{t+1} > 0$, establishing part 1 of the proposition.

To establish part 2, notice that I can combine (C.8) and (C.9) to obtain:

$$\begin{aligned}
\beta^{t+1}\mu_{t+1}\Omega_{k,t+1} &= Q_{t+1}\phi_{t+1}\left[F_{kt+1} - R^*_{t+1}\right] = -Q_t\phi_t + Q_{t+1}\phi_{t+1}F_{kt+1} \\
&\Rightarrow Q_t\phi_t = Q_{t+1}\phi_{t+1}R^*_{t+1} \Rightarrow \phi_t = \phi_{t+1}
\end{aligned}$$

Then, using (C.5) to substitute for $Q_t\phi_t$ and $Q_{t+1}\phi_{t+1}$ I get:

$$\beta^t\left[1+(1-\eta)\lambda + M_t + (1-\eta)P_t\right]c_t^{-\eta} = \beta^t\left[1+(1-\eta)\lambda + M_{t+1} + (1-\eta)P_{t+1}\right]c_{t+1}^{-\eta}R^*_{t+1}$$

$$\begin{aligned}
\Rightarrow \frac{c_t^{-\eta}}{\beta R^*_{t+1}c_{t+1}^{-\eta}} &= \left(\frac{1+(1-\eta)\lambda + M_{t+1} + (1-\eta)P_{t+1}}{1+(1-\eta)\lambda + M_t + (1-\eta)P_t}\right) \\
&= 1 + \frac{\mu_{t+1} + (1-\eta)\psi_{t+1}}{1+(1-\eta)\lambda + M_t + (1-\eta)P_t}
\end{aligned}$$

Therefore, since in any competitive equilibrium it must be that $c_t^{-\eta} = \beta c_{t+1}^{-\eta}(1-\tau^k_{t+1})F_{kt+1}$, it follows that $\forall t \geq 1$

$$\begin{aligned}
(1-\tau^k_{t+1}) &= (1-\tau^*_{t+1})\left(1 + \frac{\mu_{t+1} + (1-\eta)\psi_{t+1}}{1+(1-\eta)\lambda + M_t + (1-\eta)P_t}\right) \\
&\Rightarrow \tau^k_{t+1} = \tau^*_{t+1} - (1-\tau^*_{t+1})\frac{\mu_{t+1} + (1-\eta)\psi_{t+1}}{1+(1-\eta)(\lambda + P_t) + M_t} \leq \tau^*_{t+1}
\end{aligned}$$

strictly if $\mu_{t+1} > 0$, since $(1-\tau^*_{t+1}) \in (0,1]$ as $\tau^* \leq 1$. This establish part 2 of the Proposition. □

Proof of Proposition 4 Let $\hat{x}$ be the optimal solution to (P). I will concentrate with public NFA position, the argument for the private position is identical. Suppose for contradiction that there exists a $t \geq 1$ such that the sustainability constraint is binding,

$\hat{k}_t < \hat{K}_t$, and that is the government is saving abroad ($b_t < 0$). Consider the alternative plan x constructed from $\hat{x}$ and some $\varepsilon > 0$ sufficiently small as follows:

$$\begin{aligned}
b_t &= \hat{b}_t + \varepsilon \leq 0, \quad k_t = \hat{k}_t + \varepsilon > 0, \quad k_t^* = \hat{k}_t^* - \varepsilon + \xi(\varepsilon) \geq 0, \quad g_t = \hat{g}_t + \zeta(\varepsilon) > \hat{g}_t \\
b_s &= \hat{b}_s, \quad k_s = \hat{k}_s, \quad k_s^* = \hat{k}_s^*, \quad g_s = \hat{g}_s \quad \forall s \neq t \\
c_s &= \hat{c}_s, \quad n_s = \hat{n}_s, \quad a_s = \hat{a}_s \quad \forall s \geq 0
\end{aligned}$$

where $\xi(\varepsilon), \zeta(\varepsilon) > 0$ are defined as follows. $\xi(\varepsilon) > 0$ is chosen such that

$$\Omega(\hat{K}_t, \hat{b}_t + \hat{a}_t) = \Omega(\hat{K}_t + \xi, \hat{b}_t - \varepsilon + \hat{a}_t) \tag{C.10}$$

Notice that $\forall \varepsilon > 0$ such a $\xi > 0$ exists by continutity of Ω and the fact that it is strictly increasing in both arguments. Finally, $\zeta(\varepsilon) > 0$ is defined by

$$c_t + \hat{g}_t + \zeta(\varepsilon) \ + k_{t+1} + a_{t+1} - b_{t+1} + R_t^* k_t^* = F(K_t, n_t) + R_t^* (a_t - b_t)$$

Because $\xi(\varepsilon) > 0$, it is evident that x attains higher utility than $\hat{x}$. Then, to prove that $\hat{x}$ cannot be an optimal plan, it is sufficient to show that x is in the constraint set of (P). First, notice that if x satisfies the implementability constraint then x does too since c and n are unchanged. Consider now the sequence of sustainability constraints. $\forall s \geq t$ they are unchanged. At t instead

$$\begin{aligned}
\sum_{s=0}^{\infty} \beta^s \left[U(c_{t+s}, n_{t+s}) + G(g_{t+s})\right] &> \sum_{s=0}^{\infty} \beta^s \left[U(\hat{c}_{t+s}, \hat{n}_{t+s}) + G(\hat{g}_{t+s})\right] = \Omega(\hat{K}_t, \hat{b}_t + \hat{a}_t) \\
&= \Omega(\hat{K}_t + \xi, \hat{b}_t - \varepsilon + \hat{a}_t) = \Omega(K_t, b_t + a_t)
\end{aligned}$$

by definition of $\xi(\varepsilon)$. Then the sustainability constraint holds at t, and so it does $\forall s < t$ because the RHS is unchanged and the LHS is increased. So, finally, I must check the consolidated budget constraint at $t-1$, since $\forall s \neq t, t-1$ the consolidated budget constraint is the same under $\hat{x}$ and x, and at t it holds by definition of $\zeta(\varepsilon)$. To this end, notice that at $t-1$

$$\hat{c}_{t-1} + \hat{g}_{t-1} + \hat{k}_t + \hat{a}_t - \hat{b}_t + R^* \hat{k}_{t-1} = F(\hat{k}_{t-1} + \hat{k}_{t-1}^*, \hat{n}_{t-1}) + R^* \left(\hat{a}_{t-1} - \hat{b}_{t-1}\right)$$

$$\hat{c}_{t-1} + \hat{g}_{t-1} + (k_t - \varepsilon) + a_t - (b_t - \varepsilon) + R_{t-1}^* \hat{k}_{t-1} = F(\hat{k}_{t-1} + \hat{k}_{t-1}^*, \hat{n}_{t-1}) + R_{t-1}^* \left(\hat{a}_{t-1} - \hat{b}_{t-1}\right)$$

$$c_{t-1} + g_{t-1} + k_t + a_t - b_t + R_{t-1}^* k_{t-1}^* = F(k_{t-1} + k_{t-1}^*, n_{t-1}) + R_{t-1}^* (a_{t-1} - b_{t-1})$$

Then x is feasible and attains higher utility. Hence $\hat{x}$ cannot be optimal. $\square$

Proof of Proposition 5: Using the fact that $\beta R^* = 1$, I can then rewrite the foncs wrt c_t, g_t and n_t as follows:

$$c_t^{\eta} = \frac{\beta^t}{Q_t}\frac{1+\lambda(1-\eta)+M_t}{\phi} = \frac{1+\lambda(1-\eta)+M_t}{\phi} \tag{C.11}$$

$$1/G'(g_t) = \frac{\beta^t}{Q_t}\frac{1+M_t}{\phi} = \frac{1+M_t}{\phi} \tag{C.12}$$

$$\frac{F_{n,t}}{\chi n_t^{\gamma}} = \frac{(1-\alpha)K_t^{\alpha}n_t^{-\alpha}}{\chi n_t^{\gamma}} = \frac{\beta^t}{Q_t}\frac{1+\lambda(1+\gamma)+M_t}{\phi} = \frac{1+\lambda(1+\gamma)+M_t}{\phi}$$

$$\iff n_t^{-(\gamma+\alpha)} = \frac{\chi}{(1-\alpha)K_t^{\alpha}}\frac{1+\lambda(1+\gamma)+M_t}{\phi} \geq \frac{\chi}{(1-\alpha)\bar{K}^{\alpha}}\frac{1+\lambda(1+\gamma)+M_t}{\phi} \tag{C.13}$$

Recall that $M_t \equiv \sum_{s=1}^{t}\mu_s$. Suppose for contradiction that $\mu_t \nrightarrow 0$, implying that $M_t \to +\infty$. Then, since it must be that if $k_0 + k_0^* \in [0, \bar{K}]$ then $K_t \in [0, \bar{K}]$ $\forall t \geq 1$, the RHS of the three equations above is converging to $+\infty$. Thus, it must be that also the LHS must converge to $+\infty$. Therefore we have that $c_t \to +\infty$, $n_t \to 0$, and $g_t \to +\infty$. Moreover, notice that the necessary foc wrt k_t^* is

$$F_{Kt} - 1/\beta = \frac{\mu_t}{\phi}\Omega_{Kt} \iff \beta\alpha\left(\frac{n_t}{K_t}\right)^{1-\alpha} = 1 + \beta\frac{\mu_t}{\phi}\Omega_{Kt} \geq 1 \iff \beta\alpha n_t^{1-\alpha} \geq K_t^{1-\alpha} \tag{C.14}$$

Since $n_t \to 0$ then also $K_t \to 0$. Now, because $n_t \to 0$ and $K_t \to 0$, $\forall\varepsilon > 0\ \exists\ T(\varepsilon) \in \mathbb{N}$ sufficiently large such that $\forall t \geq T(\varepsilon)$ $K_t \leq \varepsilon$ and $n_t \leq \varepsilon$ and then $y_t \leq \varepsilon$. I will now show that the plan $x = \{c_t, n_t, g_t, k_t, k_t^*, b_t\}_{t=0}^{\infty}$ cannot be optimal. To this end, consider an alternative plan $\hat{x}(\varepsilon) = \{\hat{c}_t, \hat{n}_t, \hat{g}_t, \hat{k}_t, \hat{k}_t^*, \hat{b}_t\}_{t=0}^{\infty}$ defined as follows:

$$\begin{aligned}
\hat{x}_t &= x_t \quad \forall t = 0, 1, ..., T(\varepsilon) \\
\hat{c}_t &= \hat{c}, \quad \hat{n}_t = \hat{n}, \quad \hat{g}_t = \hat{g}, \quad \hat{k}_t = k_{T(\varepsilon)+1}, \quad \hat{k}_t^* = k_{T(\varepsilon)+1} \quad \forall t \geq T(\varepsilon)+1
\end{aligned}$$

where $\hat{c}$, $\hat{n}$, $\hat{g}$ are defined as

$$\frac{\hat{c}^{1-\eta}}{1-\beta} = \sum_{t=1}^{\infty}\beta^{t-1}c_t^{1-\eta} \tag{C.15}$$

$$\frac{\hat{n}^{1+\gamma}}{1-\beta} = \sum_{t=1}^{\infty}\beta^{t-1}n_t^{1+\gamma} \tag{C.16}$$

$$\frac{\hat{g}}{1-\beta} = \frac{1}{\beta}FA_{T+1} + \frac{F(\hat{K},\hat{n}) - R^*\hat{K}}{1-\beta} - \frac{\hat{c}}{1-\beta} \tag{C.17}$$

where

$$\begin{aligned} FA_{T+1} &\equiv Y_T - (c_t + g_T) \\ Y_T &\equiv F(K_T, n_T) + R^* a_0 + (1-\tau_0^*) F_k(K_0, n_0) k_0^* - R^* b_0 \\ \hat{K} &= \hat{k}_t + \hat{k}_t^* \forall t \geq T(\varepsilon) + 1 \end{aligned}$$

Finally, $\{\hat{a}_t, \hat{b}_t\}_{t \geq T(\varepsilon)+1}^{\infty}$ can be obtained residually. Notice that by construction $\hat{x}(\varepsilon)$ satisfies the implementability constraint (3.5), and the consolidated budget constraint of the country (3.7). To prove that x cannot be optimal for (P), it suffices to show that $\hat{v}_{T(\varepsilon)+1} \equiv \frac{u(\hat{c})-v(0)+G(\hat{g})}{1-\beta} > v_{T(\varepsilon)+1}$. In fact, this implies that $\hat{x}(\varepsilon)$ satisfies the sustainability constraint $\forall t \geq 0$ as $\forall t = 0, 1, ..., T(\varepsilon)$

$$\begin{aligned} \hat{v}_t &= \sum_{s=0}^{T(\varepsilon)-t} \beta^s \left(u(c_{t+s}) - v(n_{t+s}) + G(g_{t+s})\right) + \beta^{T(\varepsilon)+1} \hat{v}_{T(\varepsilon)+1} > \\ &> v_t \geq \Omega(K_t, FA_t) = \Omega(\hat{K}_t, F\hat{A}_t) \end{aligned}$$

and $\forall t \geq T(\varepsilon) + 1$

$$\hat{v}_t = \frac{u(\hat{c}) - v(\hat{n}) + G(\hat{g})}{1-\beta} > v_{T(\varepsilon)+1} = \Omega(K_{T(\varepsilon)+1}, FA_{T(\varepsilon)+1}) = \Omega(\hat{K}_t, F\hat{A}_t) = \Omega(\hat{K}, F\hat{A})$$

Thus if x is feasible in (P) so is $\hat{x}(\varepsilon)$. Moreover, if $\hat{v}_{T(\varepsilon)+1} \equiv \frac{u(\hat{c})-v(\hat{n})+G(\hat{g})}{1-\beta} > v_{T(\varepsilon)+1}$ then x cannot be an optimal plan since $x(\varepsilon)$ is feasible for (P) and it attains a higher utility: $\hat{v}_0 > v_0$. First, notice that from (C.15) and (C.16) we have by strict concavity of u that

$$\frac{\hat{c}^{1-\eta}}{1-\beta} = \sum_{t=1}^{\infty} \beta^{t-1} c_t^{1-\eta} \Rightarrow \frac{\hat{c}}{1-\beta} + \Delta(\varepsilon) = \sum_{t=1}^{\infty} \beta^{t-1} c_t$$

for some $\Delta(\varepsilon) > 0$. Second, notice that (C.17) implies that

$$\begin{aligned}
\frac{\hat{g}}{1-\beta} &= \frac{1}{\beta} F A_{T+1} + \frac{F(\hat{K},\hat{n}) - R^*\hat{K}}{1-\beta} - \frac{\hat{c}}{1-\beta} \qquad \text{(C.18)} \\
&= \left[\sum_{t=1}^{\infty} \beta^{t-1}\left(g_{T(\varepsilon)+t} + c_{T(\varepsilon)+t}\right) - \sum_{t=1}^{\infty} \beta^{t-1}\left(F(K_{T(\varepsilon)+t}, n_{T(\varepsilon)+t}) - R^* K_{T(\varepsilon)+t}\right)\right] \\
&\quad + \frac{F(\hat{K},\hat{n}) - R^*\hat{K}}{1-\beta} - \frac{\hat{c}}{1-\beta} \\
&= \sum_{t=1}^{\infty} \beta^{t-1} g_{T(\varepsilon)+t} + \Delta(\varepsilon) - \sum_{t=1}^{\infty} \beta^{t-1}\left(F(K_{T(\varepsilon)+t}, n_{T(\varepsilon)+t}) - R^* K_{T(\varepsilon)+t}\right) \\
&\quad + \frac{F(\hat{K},\hat{n}) - R^*\hat{K}}{1-\beta} \\
&\geq \sum_{t=1}^{\infty} \beta^{t-1} g_{T(\varepsilon)+t} - \sum_{t=1}^{\infty} \beta^{t-1} F(K_{T(\varepsilon)+t}, n_{T(\varepsilon)+t}) - \frac{R^*\hat{K}}{1-\beta} \\
&\geq \sum_{t=1}^{\infty} \beta^{t-1} g_{T(\varepsilon)+t} - \frac{(1+R^*)\varepsilon}{1-\beta}
\end{aligned}$$

Then I can pick $\varepsilon > 0$ sufficiently close to zero such that (C.18) and strict concavity of $G(.)$ imply that

$$\frac{G(\hat{g})}{1-\beta} > \sum_{t=1}^{\infty} \beta^{t-1} G(g_{T(\varepsilon)+t}) \qquad \text{(C.19)}$$

which implies that $\hat{v}_{T(\varepsilon)+1} \equiv \frac{u(\hat{c})-v(0)+G(\hat{g})}{1-\beta} > v_{T(\varepsilon)+1}$. Hence $\hat{x}(\varepsilon)$ is feasible for (P) and it attains a higher utility than x. Hence x cannot be an optimal plan.Therefore it must be that $M_t \to M_\infty < +\infty$ and (a necessary condition for this) $\mu_t \to \mu_\infty = 0$. Therefore, from (C.11)-(C.13), it follows that $c_t \to c_\infty$, $n_t \to n_\infty$, $g_t \to g_\infty$ and from (C.14) we have that

$$F_{kt+1} \to R^* \qquad \text{(C.20)}$$

All these results imply that $\tau_t^k, \tau_t^{k*}, \tau_t^a \to 0$ and finally

$$\tau_t^n = \frac{\lambda(\eta+\gamma)}{1+\lambda(1+\gamma)+M_t} \to \frac{\lambda(\eta+\gamma)}{1+\lambda(1+\gamma)+M_\infty} > 0 \qquad \text{(C.21)}$$

as wanted. □

Proof of Proposition 6: The foncs wrt c_t, g_t and n_t can be rewritten as follows:

$$c_t^{\eta} = \frac{\beta^t}{Q_t}\frac{1+\lambda(1-\eta)+M_t}{\phi} = \frac{\beta^t}{Q_t}\frac{1+\lambda(1-\eta)}{\phi} + \frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to \frac{\beta^t}{Q_t}\frac{M_t}{\phi} \quad \text{(C.22)}$$

$$n_t^{-(\gamma+\alpha)} = \frac{\chi}{(1-\alpha)K_t^{\alpha}}\frac{\beta^t}{Q_t}\frac{1+\lambda(1+\gamma)+M_t}{\phi} \quad \text{(C.23)}$$

$$1/G'(g_t) = \frac{\beta^t}{Q_t}\frac{1+M_t}{\phi} \to \frac{\beta^t}{Q_t}\frac{M_t}{\phi} \quad \text{(C.24)}$$

Since $M_t \equiv \sum_{s=0}^{\infty}\mu_s$ with $\mu_t \geq 0$ $\forall t \geq 0$, then $\{M_t\}_{t=0}^{\infty}$ is an increasing sequence. Thus, either $M_t \to M < \infty$ or $M_t \to \infty$. Consider now the behavior of $\left\{\frac{\beta^t}{Q_t}\frac{M_t}{\phi}\right\}_{t=0}^{\infty}$ as $t \to \infty$. There are three possibilities: (i) $\frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to 0$, (ii) $\frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to \tilde{M}_{\infty} \in (0,\infty)$, or (iii) $\frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to +\infty$. Case (iii) can be ruled out using an argument similar to the one used in Proposition 3. I can then consider only case (i) and (ii).

Consider first case (ii). Suppose that $\frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to \tilde{M}_{\infty} < \infty$. Then (C.22) and (C.24) directly imply that $c_t \to c_{\infty} = \tilde{M}_{\infty}^{1/\eta} \in (0, M^{1/\eta}]$ and $g_t \to g_{\infty} = \ell(\tilde{M}_{\infty}) \in (0, \ell(M)]$, where ℓ is the inverse of $1/G'(g)$. In order for $\frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to \tilde{M}_{\infty}$, it must be that $\mu_t \to \mu_{\infty} = \left(\frac{1-\beta R^*}{\beta R^*}\right)\tilde{M}_{\infty} > 0$. Hence, from the fonc wrt k_t^*

$$F_{Kt} - R^* = \frac{\mu_t}{\phi}\Omega_{Kt} \to \frac{\mu_{\infty}}{\phi}\Omega_{K\infty} > 0 \quad \text{(C.25)}$$

and (C.23) it follows that $K_t \to K_{\infty} > 0$ and $n_t \to n_{\infty} \in (0, \bar{N}]$. Moreover, combining (C.25) with the Euler equation for the foreingn investors, $R^* = (1-\tau_t)F_{kt}$, I obtain that as $t \to \infty$

$$\tau_t^* \to \frac{F_{K\infty} - R^*}{F_{K\infty}} = \frac{\mu_t}{\phi}\frac{\Omega_{Kt}}{F_{K\infty}} > 0 \quad \text{(C.26)}$$

Combining the household Euler equation with the one for the foreign investors I can write $\forall t$

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta R^* \frac{1-\tau_{t+1}^k}{1-\tau_{t+1}^*} \to 1 \iff \lim_{t\to\infty}\tau_t^k = \tau_{\infty}^k < \tau_{\infty}^* \quad \text{(C.27)}$$

Finally, from (C.22) and (C.23) I have that $\forall t$

$$\begin{aligned} v'(n_t) &= u'(c_t)F_{nt}\frac{1+\lambda(1+\gamma)+M_t}{1+\lambda(1-\eta)+M_t} \\ &= u'(c_t)F_{nt}\left(1 - \frac{\lambda(\gamma+\eta)}{1+\lambda(1-\eta)+M_t}\right) \end{aligned}$$

then labor income taxes are converging to zero as $t \to \infty$:

$$\tau_t^n = \frac{\lambda(\eta+\gamma)}{1+\lambda(1-\eta)+M_t} \to \frac{\lambda(\eta+\gamma)}{1+\lambda(1-\eta)+M_{\infty}} = 0$$

Consider now case (i). If $\frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to 0$ then from (C.22) and (C.24) one gets that $c_t \to 0$ and $g_t \to 0$. Hence the LHS of the sustainability constraint it is converging to its lower bound with $n_t \to 0$ and $K_t \to 0$. In fact, suppose that $n_t \to n > 0$, then $\forall K \geq 0$

$$\frac{u(0)}{1-\beta} - \frac{v(n)}{1-\beta} + \frac{G(0)}{1-\beta} < \Omega(K,0) \leq \Omega(K,A) \quad \forall A \geq 0$$

then the sustainability constraint is violated. Then it must be that $n_t \to 0$, and this implies that also $K_t \to 0$, otherwise the sustainability constraint won't hold. Therefore, if $\frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to 0$ then the economy converges to a steady state such that $(c_\infty, n_\infty, g_\infty, K_\infty) = (0,0,0,0)$. Call this the "immiseration" steady state.

Finally, I want to argue that it cannot be that $\frac{\beta^t}{Q_t}\frac{M_t}{\phi} \to 0$. To this end, I will show that I can find an alternative plan, $\hat{x}$, that starting from the "immiseration" steady state - i.e. zero assets - attains strictly higher utility ($\varpi > \Omega(0,0)$, the value of the immiseration steady state), and it is sustainable, i.e. it attains higher utility than the worst SE $\forall t$, $\varpi \geq \Omega(k,0)$, where k is the aggregate capital prescribed by the alternative strategy. To define $\hat{x}$, let $\{\underline{c}_t, \underline{n}_t, \underline{g}_t\}_{t=0}^{\infty}$ be the optimal plan associated with $\Omega(\kappa,0)$, the worst SE starting from some $\kappa > 0$ and zero foreign assets. Then define $\hat{x}$ as follows: $\forall t = 0,1,...,T$ $\hat{x}_t = x_t$, and for $t \geq T$ let

$$\hat{c}_t = \underline{c}_t, \quad \hat{n}_t = \underline{n}_t, \quad \hat{a}_t = \hat{k}_t = 0, \quad \hat{k}_t^* = \kappa > 0, \quad \hat{b}_t = \underline{b}_t \tag{C.28}$$

$$\hat{g}_t = \underline{g}_t + (F(\kappa, \hat{n}_t) - R^*\kappa) - F\left((1-\delta)^t\kappa, \underline{n}_t\right) \tag{C.29}$$

Notice that since

$$\sum_{t=0}^{\infty} \beta^t \left(\hat{c}_t^{1-\eta} - \chi \hat{n}_t^{1+\gamma}\right) = \sum_{t=0}^{\infty} \beta^t \left(\underline{c}_t^{1-\eta} - \chi \underline{n}_t^{1+\gamma}\right) = 0$$

then $\hat{x}$ satisfies (3.5) for T sufficiently high. This is due to the fact that $c_t, n_t \to 0$ implies that $\lim_{T\to\infty} \sum_{t=0}^{\infty} \beta^t \left(c_{T+t}^{1-\eta} - \chi n_{T+t}^{1+\gamma}\right) \to 0$[1] . By construction (3.7) must also hold. It is then left to show that (i) $\hat{x}$ is sustainable and (ii) it is an improvement over x. Since $\forall n > 0$ $\lim_{K\to 0} F_K(K,n) = \infty$, from (C.29) it follows that for κ small enough, $\hat{g}_t > \underline{g}_t$. This is because

$$\begin{aligned}
(F(\kappa,\hat{n}_t) - R^*\kappa) &\approx F\left((1-\delta)^t\kappa, \underline{n}_t\right) + F_K\left((1-\delta)^t\kappa, \underline{n}_t\right)\left[1-(1-\delta)^t\right]\kappa - R^*\kappa \\
&= F\left((1-\delta)^t\kappa, \underline{n}_t\right) + \left\{F_K\left((1-\delta)^t\kappa, \underline{n}_t\right)\left[1-(1-\delta)^t\right] - R^*\right\}\kappa \\
&> F\left((1-\delta)^t\kappa, \underline{n}_t\right)
\end{aligned}$$

[1] To be precise, I should add some $\varepsilon > 0$ sufficiently small to $\hat{c}_T = \underline{c} + \varepsilon$.

for $\kappa > 0$ sufficiently close to zero. Then it follows that

$$\hat{v}_T = \sum_{t=0}^{\infty} \beta^t \left[u(\hat{c}_{T+t}) - v(\hat{n}_{T+t}) + G(\hat{g}_{T+t}) \right] > \Omega(\kappa, 0) > \Omega(0, 0) = v_\infty = v_T - \varepsilon$$

Thus $\hat{x}$ is sustainable and it attains higher utility than x, which, therefore, cannot be optimal. Therefore, it must be true that the economy is in case (ii). $\square$