NECESSARY AND SUFFICIENT CONDITIONS FOR

EXPECTED UTILITY MAXIMIZATION:

THE FINITE CASE, WITH A PARTIAL ORDER

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Introduction

An individual is to choose among various possible decisions, each being a mapping from a set of possible states of nature to a set of consequences. If we fix a utility u on the set of consequences and a probability measure ν on the set E of states, the expected utility of a decision f is $\int_E u(f(e))d\nu(e)$. The "expected utility hypothesis"(EUH) is that there exists a utility and a measure such that the choices of the individual are consistent with expected utility maximization.

This hypothesis is a very useful one in modeling choice under uncertainty, and many authors have attempted to justify it by describing reasonable conditions (or axioms) on behavior, which imply EUH. Certain hypotheses are common in these treatments: an infinite, atomless space of states [Savage (1954)] or, in the case of a finite set of events, a complete order (i.e., every pair of events comparable) on the set of all decisions considered (Luce and Krantz, 1971). It is also common to present axioms which are sufficient, but not necessary, to establish EUH.

Here we assume the space of states and of consequences to be finite and do not assume the order to be complete, and our conditions are equivalent to EUH, modulo the finiteness assumptions on states and consequences.

Of course this comes at some cost: Our conditions are not easily stated. They include statements about repetitions of the states, which corresponds to considering states with smaller probabilities than those of the original finite set - however the number of repetitions is finite (it is bounded by the number of consequences), so we do not get into an atomless measure space. Finally, they include a hypothesis that the individual is able to make certain other simple choices, not in the given set of choices, in a consistent way. This feature is common to other treatments and it does not imply an extension of the original partial order to a complete order. In fact these other choices do not involve utility at all.

The key tool in the proof that our conditions are necessary and sufficient is a slight generalization of the "Fourier-Motzkin Elimination Procedure" (Kuhn, 1956).

Since the set of states and consequences are both assumed finite and we assume strict preferences, if a (necessarily finite) set of decisions satisfies EUH for a fixed u and ν then any u', ν' sufficiently close to u and ν will also work. Thus the utilities we describe cannot be cardinal.

1. Definitions, Outline of the Paper

Let X denote a finite set of outcomes, or <u>prizes</u>.[*] We as

assume the set of states, or <u>events</u> E to be finite; for simpli-

city we will further assume it contains two elements, say

$E = \{H,T\}$. An ordered pair $(x,y) \in X^\times X$ is a decision, or

<u>ticket</u>. It is interpreted to yield its owner the prize x if

H occurs and the prize y if T occurs.

We imagine a single <u>subject</u> who cannot influence the events

H and T . He may have some information about each event, e.g.

from having looked carefully at a coin or (if H = rain, T =

sun) reading a barometer, etc. We are given the information

that our subject has made some <u>choices</u> between tickets. We de-

note the fact that he has chosen (x,y) over (z,w) by (x,y) >

(z,w) . For example, if $X = \{x,y,z\}$, the subject may have made

the choices

$(x,z) > (y,z)$ $(w,z) > (y,z)$

$(x,y) > (y,x)$ $(y,w) > (w,y)$.

Note that the subject can make only finitely many choices (since

X is finite) and that he need not make a choice between every

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[*]We do not assume that the elements of X are divisible or com-

parable. Thus if $x \in X$ we cannot require some $y \in X$ to be the

"same as"$(\frac{1}{2})x$.

available pair of tickets. These choices may have arisen as a result of our subject's market behavior (involving a price structure, as in the theory of revealed preference: Richter, 1966) but we adopt a more general approach and ask not whence the information of these choices came.

There are several methods which can be used to make such choices, for example maximizing minimum utility, or expected utility. Given a utility u and probabilities p (of H) and q (of T) , the expected utility of the ticket (x,y) is

(1.1) $E(x,y) = pu(x) + qu(y)$.

(1.2) Definition: Let $\mathcal{S}$ denote a set of choices :

(1.3) $\mathcal{S} = \{(x_{1i}, x_{2i}) > (x_{3i}, x_{4i}) : i = 1,\ldots,n\}$

where $x_{ji} \in X$ for each i and j . We say $\mathcal{S}$ is compatible with EUH (expected utility hypothesis) if there exist p,q positive and a function u from X to the reals $\mathbb{R}$ such that

(1.4) $pu(x_{1i}) + qu(x_{2i}) > pu(x_{3i}) + qu(x_{4i})$ for all $i = 1,\ldots,n$.

Notice that we have required p and q to be positive, rather than just nonnegative. This assumption is merely for convenience, for if the equations (1.4) were solvable with, for example $p=0$, then, since the inequalities involved are strict, they would have a solution for sufficiently small positive values of p .

(1.5) <u>Notation</u>. For simplicity we may delete the u in formulas such as (1.4) so $x \in X$ stands interchangeably for a prize and for the utility of that prize. Thus instead of asking for a utility function $u : X \to R$ satisfying (1.4), we will ask for real numbers x_{ji} , $1 \leq j \leq 4$, $1 \leq i \leq n$, such that

$$px_{1i} + qx_{2i} > px_{3i} + qx_{4i} \qquad \text{for all } i ,$$

where we of course intend that if x_{ji} and $x_{k\ell}$ refer to the same prize, then they should be equal when we are referring to utilities. (In some sense we are viewing the elements of X as variables.) Also we denote q/p by ρ , so ρ is the "odds for T over H". Dividing (1.4) by p and using our variable notation, definition (1.1) becomes: <u>$\mathfrak{S}$ is compatible with EUH if and only if there are real numbers x_{ji} and a positive number</u> ρ <u>such that:</u>

$$(1.6) \qquad x_{1i} + \rho x_{2i} > x_{3i} + \rho x_{4i} \quad \text{for all } i = 1,\ldots,n .$$

Since ρ will always denote these odds and we require $\rho > 0$, it will simplify matters to make the <u>convention</u>: <u>the variable</u> ρ <u>is assumed to take on only positive values</u>, for the remainder of the paper.

We pause for an example . If $\mathfrak{S}$ is given by

(1.7) $\mathfrak{S}$:   (1) $(x,z) > (y,z)$ (3) $(w,z) > (y,z)$

(2) $(x,y) > (y,x)$ (4) $(y,w) > (w,y)$

then $\mathfrak{S}$ can't be compatible with EUH. Intuitively, choice (1)
says x is worth more than y , so (2) says H is more
likely than T . Similarly, (3) and (4) say T is more likely
than H , a contradiction of our intuition. This intuitive
reasoning could be made rigorous, as we do for equations (1) and (2):

$\mathfrak{S}$: (1) $(x,y) > (y,y)$

(2) $(x,y) > (y,x)$

This $\mathfrak{S}$ satisfies EUH iff we can find real numbers x and y ,
and $\rho > 0$, such that

(1') $x + \rho y > y + \rho y$

(2') $x + \rho y > y + \rho x$

These equations are equivalent to

(1") $x - y > 0$

(2") $(1 - \rho)(x - y) > 0$,

and these can be solved if and only if $\rho < 1$, which is equivalent
to $q < p$, i.e. H more likely than T .

We finish this section by outlining the rest of the paper.
In section 2 we discuss other approaches to the problem of finding
necessary and sufficient conditions for EUH, and describe criteria
we would like such conditions to meet. None of the approaches

we survey, nor even our own, fully meet these criteria.
Section 3 consists of a detailed example which illustrates the
Fourier-Motzin Elimination Procedure and its extension which is
at the heart of our proofs of necessity and sufficiency. As
mentioned above, our conditions for EUH entail repetitions of
the H - T experiment, and they involve more complicated tickets,
which we call compound tickets. These and related definitions are
given in section 4, where we also extend the definition of EUH to
cover choices between compound tickets, so that ordinary tickets
and the EUH defined above are special cases. We will find it
more convenient to give necessary and sufficient conditions for
EUH in this more general setting. Since our conditions are not
easily stated we devote section 5 to carefully explaining and
motivating them. Section 6 contains the detailed statements of
the conditions, along with some more motivation, and section 7
is the proof of necessity and sufficiency. In section 8 we sur-
vey the success of our necessary and sufficient conditions in
meeting the criteria discussed in section 2.

2. Other Approaches

Given our restatement of EUH as the solvability of the quad-ratic inequalities (1.6), it seems natural to use duality theory. For fixed ρ , the equations (1.6) are linear. In fact they can be written in the form $A(\rho)x > 0$ where $A(\rho)$ is a matrix for each $\rho > 0$ and x is a vector with as many components as prizes. By (Gale, 1960), $A(\rho)x > 0$ has a solution iff there is no vector $y(\rho) \gg 0$ such that $y(\rho)A(\rho) = 0$. Thus $\mathcal{S}$ is compatible with EUH iff there exists some $\rho > 0$ such that for every $y \gg 0$, $yA(\rho) \neq 0$.

This first approach, using duality, is deficient on at least two grounds.

(2.1a)   It is not constructive. Given a specific $\mathcal{S}$, it does not tell us how to go about finding whether or not $\mathcal{S}$ is com-patible with EUH. Of course for a single ρ one can test whether there is a $y(\rho) \gg 0$ with $y(\rho)A(\rho) = 0$ (Kuhn, 1956), but we might have to carry our this procedure for infinitely many $\rho > 0$ before possibly finding one that worked.

(2.1b)   The conditions do not intuitively justify EUH; in fact they are almost a restatement of EUH. We would like to find conditions which are reasonable in the sense that they seem to impose less restrictions on a subject than does EUH. A proto-type of such conditions is the strong axiom of revealed preference. This requires, in the context of choice under certainty, only that a subject not express a cycle of choices (e.g. $x_1 > x_2$

$> x_3 > x_1)$, and eminently reasonable requirement, but (under
certain conditions) it is equivalent to the utility hypothesis,
that a subject acts to maximize the value of a utility function
(Richter, 1966).

A second approach to this problem can be found in the work
of Tarski. Equations such as (1.6) are polynomial equations in
their unknowns which are ρ and the elements of X . The state-
ment that they have a solution is therefore a theorem of the first-
order theory of algebra and by Tarski's theorem on the completeness
of the theory of real-closed fields (Tarski, 1951) its truth or
falsity can be verified in a constructive way. This solution
does satisfy our requirement (2.1a) above, of constructivity, al-
though the methods Tarski uses are not at all simple to apply.
But it does not satisfy our requirement (2.1b) as far as we know:
we have been unable to interpret Tarski's work to yield what we
would call reasonable conditions for a subject's behavior under
uncertainty.

The comments we have made about Tarski's work also apply to
the results in Richter (1975). There another constructive method
does not supply reasonable conditions.

We have previously mentioned the axiom systems of Savage
(1954) and Luce and Krantz (1971). We pointed out that Savage's
axioms presume a nonatomic event space. Luce and Krantz' require
(Axiom 2, p. 265) a complete order on the set of tickets considered.
Actually their theory refers to "decisions", a concept more general
than our "tickets", and their conditions do not guarantee the
existence of a utility function on prizes.

3. Illustration of the Elimination Procedure

We have mentioned that a generalization of the Fourier-Motzkin Elimination Procedure is central to our results. That procedure is used to solve linear inequalities by elimination of variables. To illustrate its use here, we consider another example:

(3.1) S : $(x,y) > (v,x)$

$(v,y) > (y,x)$

$(x,y) > (y,x)$.

First we rewrite the system in inequality form and collect terms to get:

S_0:   (1) $x - v + \rho(y - x) > 0$

(2) $v - y + \rho(y - x) > 0$

(3) $x - y + \rho(y - x) > 0$.

Adding equations (1) and (2) cancels the v and we get

S_1:   (1 + 2) $x - y + 2\rho(y - x) > 0$

(3) $x - y + \rho(y - x) > 0$.

Clearly $S_0 \Rightarrow S_1$ in the sense that if x,y,v and ρ satisfy S_0 then the same values of x,y and ρ satisfy S_1 . Notice

that also $S_1 \Rightarrow S_0$. For if x,y and ρ satisfy S_1 then by $(1 + 2)$ we have

$$x + \rho(y - x) > y - \rho(y - x)$$

so we can choose v between them, i.e.,

$$(*) \qquad x + \rho(y - x) > v > y - \rho(y - x) \quad .$$

The first inequality in (*) shows that x,y,v and ρ satisfy (1) and the second inequality shows (2) . Thus x,y,v and ρ satisfy S_0 . The equivalence $S_0 \Leftrightarrow S_1$ is a special case of the Fourier-Motzkin Elimination Procedure, which can be applied since the variable we eliminated, v , appears linearly. Now to solve the system S_1 we introduce the variable $z = x - y$. Then S_1 is equivalent to

$$S_2: \quad (4) \qquad z(1 - 2\rho) > 0$$

$$(5) \qquad z(1 - \rho) > 0 \quad .$$

We could solve this system in a few simple steps, but it will be instructive to continue an "elimination procedure". We could eliminate z from S_2 if we knew that its coefficients, $1 - 2\rho$ and $1 - \rho$, differed in sign. For example, if $1 - 2\rho < 0$ and $1 - \rho > 0$ then we could multiply (4) by $(1 - \rho) > 0$ and (5) by $-(1 - 2\rho) > 0$, add them together and get $0 > 0$. Similarly, if $1 - 2\rho > 0$ and $1 - \rho < 0$ then $0 > 0$. If either of the coefficients $1 - \rho$ or $1 - 2\rho$ equals 0 , then clearly $0 > 0$.

Summarizing, $S_2 \Rightarrow S_3$ where S_3 is:

S_3 (6) if $1 - 2\rho \leq 0$ and $1 - \rho \geq 0$ then $0 > 0$

(7) if $1 - 2\rho \geq 0$ and $1 - \rho \leq 0$ then $0 > 0$.

As is proved in general below in section 7, S_3 is equivalent to S_2 . Of course $0 > 0$ is impossible, so (6) amounts to requiring that either $1 - 2\rho > 0$ or $1 - \rho < 0$. Similarly, (7) says $1 - 2\rho < 0$ or $1 - \rho > 0$. Summarizing, S_3 is equivalent to: ρ is in the set

$$\lceil \{\rho: 1 - 2\rho > 0\} \cup \{\rho: -(1 - \rho) > 0\} \rceil \cap \lceil \{\rho: -(1 - 2\rho) > 0\}$$

$$\cup \{\rho: 1 - \rho > 0\} \rceil \quad .$$

(The first bracketed term is from equation (6), the other from equation (7)). It is easily checked that this last set is $(0, \frac{1}{2}) \cup (1, \infty)$, which is certainly nonempty, so S_3 has solutions. Therefore so does S_0 , so the S in (3.1) is compatible with EUH.

4. Compound Lottery Tickets

Ideally, a set of necessary and sufficient conditions for sets $\mathcal{S}$ of choices to satisfy EUH should involve only the sets $\mathcal{S}$ themselves, e.g. requiring that no cycles (appropriately defined) exist. We are unable to find such conditions which are also constructive and reasonable, as defined in (2.1). Our conditions, though constructive and (we think) reasonable, require that some "choices", other than those in $\mathcal{S}$, be made. These choices, finite in number, concern tickets involving a <u>repeating</u> of the experiment with outcomes H and T . Thus these results are not strictly in the spirit of the subjectivist school, which claims expected utility maximization even in the case of unrepeatable experiments. See section 8 for more information on this point.

The main result of Richter-Shapiro (1976) is that the set of ρ's which appear in solutions to (1.6) can be an arbitrary polynomial set, i.e., polynomials of arbitrarily high order are needed to describe such sets. This result suggests that repetitions of events will be needed in any necessary and sufficient conditions.

In describing our necessary and sufficient conditions it will be helpful to use tickets which involve positive integer multiples of the prizes in X , as well as repetitions of the H-T experiment. (We can use a fair coin to interpret these multiples, as suggested by Ramsey and described below in (4.6)). It will also be convenient to use notation such as

(4.1) $(x - z, y - w) > 0$,

instead of

(4.2) $(x,y) > (z,w)$.

Even with more complicated tickets one can always go from the form (4.1) to (4.2) by merely putting all the "negative" terms on the right.

One might interpret (4.1) in its own right, not just as a notation for (4.2), by viewing it as a ticket which yields $x - z$ (i.e., the subject gets x but must pay z) in case of H , and $y - w$ in case of T . But this interpretation is quite different than (4.2). For example, a subject who did not have a "z" might make the choice (4.2) but not want to risk having to pay "z", as in this interpretation of (4.1). Even though this interpretation differs from the one we have in mind, it may be the appropriate one in some circumstances.

We now give a rigorous description of the compound tickets we have been mentioning, which involve repetitions of the H-T experiment and positive integer multiples of the prizes in X .

For fixed $K \geq 1$, define

$$E_K = \{e_0, e_1, \ldots, e_K\}$$

to be a certain set of outcomes of the H-T experiment repeated K times, namely e_k denotes the outcome of H occuring on the first k trials, T on the other K-k trials. Notice that

$e_k \in E_K$ differs from $e_k \in E_{K'}$ if $K \neq K'$, but the appropriate meaning will be clear from the context. E_1 is just the outcomes of the H-T experiment: e_0 corresponds to H and e_1 to T . The set E_K is not the set of <u>all</u> outcomes; for example "T then H" is not an outcome in E_2 .

We number the elements of X as $x_1, \ldots, x_L$ and we expand the set of prizes to G , the free abelian group on the original prize set X . The group G consists of expressions of the form

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_L x_L$$

where for $1 \leq \ell \leq L$, α_ℓ is an integer (possibly negative or zero) . Note that the zero prize (all $\alpha_\ell = 0$) is in G , and one adds or subtracts elements in G in the obvious way.

A <u>compound lottery ticket</u> is a function from E_K into G , i.e., it associates with each "compound event" in E_K a "compound prize" in G . If h is a compound lottery ticket we can denote h by its values: $(h(e_0), h(e_1), \ldots, h(e_K))$. For example, $(x, 2y - z, 0)$ is a compound lottery ticket which awards x for the outcome e_0 , $2y - z$ for the outcome e_1 , and nothing for the outcome e_2 . (See (4.6) for the interpretation of $2y - z$.)

A <u>linear utility on G</u> is a real-valued function u on G satisfying

$$u(x + y) = u(x) + u(y) \quad \text{and} \quad u(\alpha x) = \alpha u(x) \quad \text{for} \quad x, y \in G$$

and integers α .

For a fixed linear utility u on G and positive numbers p and q we define the expected utility function E for compound lottery tickets by

$$(4.3) \qquad E(h) = \sum_{k=0}^{K} p^{K-k} q^{k} u(h(e_{k})) \quad \text{for tickets} \quad h \quad \text{on the event}$$

$$\text{set} \quad E_{K} \ .$$

As in section 2, we will denote q/p by ρ . Thus

$$(4.4) \qquad E(h) > 0 \quad \text{iff} \quad \sum_{k=0}^{K} \rho^{k} u(h(e_{k})) > 0 \ .$$

We call h a simple lottery ticket if h = (x,y) where x and y are in X . Note that if h is a simple lottery ticket then (4.3) reduces to the definition of expected utility given in (1.1). We use the phrase "h is desirable" to mean $h > 0$, as in (4.1).

In section 1, EUH meant that there existed a utility u and probabilities p,q such that a preference for (x,y) over (z,w) implied $E(x,y) > E(z,w)$, where E was defined (1.1) in terms of u , p and q . Here it means that u, p, q exist such that the desirability of h, $h > 0$, implies $E(h) > 0$, where now E is defined by (4.3).

(4.5) Definition: Let $\mathfrak{J}$ denote a set of the form

$$\mathfrak{J} = \{h_{j} > 0: \quad j = 1,2,\ldots,J\}$$

where each h_{j} is a compound lottery ticket. Then $\mathfrak{J}$ is compatible with EUH if there exist a linear utility u on G and $p,q > 0$ such that, for the E defined by (4.3),

$$E(h_j) > 0 \quad \text{for} \quad j = 1,2,\ldots,J \;.$$

Since (4.3) is a special case of (1.1), as remarked above, this definition is a generalization of (1.2). Thus, our goal of finding necessary and sufficient conditions for EUH will be reached if we use definition 4.5.

(4.6) To interpret multiples and sums of a prize x we give an example, from which it should be easy to generalize. Consider the statement

(4.7) $\quad$ (x + w, 3y - z) > 0 , $\quad$ or $\quad$ (x + w, 3y) > (0,z) ,

and consider a fair 3-sided "coin" with sides marked a, b and c, so each side is expected to come up with equal probability equal to $\frac{1}{3}$. Define the ticket

(4.8) $\quad$ (x,y) $\oplus$ (w,y) $\oplus$ (0,y)

as follows: The fair coin is flipped. If a comes up the subject gets the ticket (x,y) , he gets (w,y) in case b , and (0,y) in case c . The ticket

(4.9) $\quad$ (0,z) $\oplus$ (0,0) $\oplus$ (0,0)

is interpreted similarly. Then it is easy to check that E(x + w, 3y - z) > 0 iff the expected utility of (4.8) is greater than that of (4.9). Thus we can interpret (4.7) as the statement that (4.8) is preferred over (4.9).

If one objects to the zero prize, it is possible to go further and write the choice of (4.8) over (4.9) as a choice between lottery tickets not involving the zero prize. We omit the details.

5. Motivation

If any conditions claim to justify EUH (cf.(2.1) above) they must in particular explain why the subject who made the choices in example (1.7), repeated below, is unreasonable. Let us look at that example carefully. There the subject made the choices $(x-y, y-x) > 0$ and $(x-y, 0) > 0$, which implied $p > q$ (H is more likely than T). Call these two choices $\underset{\sim}{s}^1$. He made two other choices, which we'll call $\underset{\sim}{s}^2$, which implied $q > p$, so $\underset{\sim}{s} = \underset{\sim}{s}^1 \cup \underset{\sim}{s}^2$ was incompatible with EUH. But what does the term "implied" mean in this context? On the mathematical level it means that, for example, if $\underset{\sim}{s}^1$ were compatible with EUH then the p and q involved would satisfy $p > q$. But our conditions should justify, as well as be mathematically equivalent to, EUH. So there is another sense in which we might intend the word "implied" -- that a subject who made the choices in $\underset{\sim}{s}^1$ should believe that H is more likely than T. We use the word "reveal" in place of "imply" for this latter, subjective, meaning.

Even if we agree that $\underset{\sim}{s}^1$ reveals H is more likely than T, and $\underset{\sim}{s}^2$ reveals T is more likely than H, we must also agree that these revelations are inconsistent, and not the behavior we expect of a "reasonable" subject.

That is, our conditions for EUH should have two parts:

(i)   Showing how ticket-choices reveal information about

the relative likelihood of H and T ;

(ii) ensuring that this information is consistent.

Let us look at (i) more carefully, in the case

$$S^1 = \{(x\text{-}y, \ y\text{-}x) > 0 \ , \ \ (x\text{-}y, \ 0) > 0\} \ \ .$$

The choice $(x\text{-}y, \ y\text{-}x) > 0$ potentially reveals that either

$p > q$ or $p < q$, for if the two events H and T were equally

likely then the ticket $(y\text{-}x, \ x\text{-}y)$ would also be desirable, and

the sum of the two tickets, $(0,0)$, would be desirable -- a con-

tradiction. If we suppose $p < q$ (H less likely than T) to

be what is actually revealed then $(x\text{-}y, \ y\text{-}x) > 0$ also reveals

that y is more useful than x , so $(y\text{-}x,0)$ should be desirable.

Now add this latter ticket to the other ticket in S^1 , namely

$(x\text{-}y, \ 0)$ to get that $(0,0)$ is desirable -- again a contradiction.

So we reject $p < q$ and conclude that $p > q$ is actually revealed.

Generalizing from this example, we can split (i) into two

parts, first that choices (like $(x\text{-}y, \ y\text{-}x) > 0$) potentially re-

veal some things about p and q (like $p < q$ or $p > q$) , and

second that these potential revelations are actual when they do

not allow us to derive a contradiction (like $(0,0) > 0$) .

In summary, we would like our conditions for EUH to involve

three steps:

S1 Show how choices reveal (potential) information about
the relative likelihood of H and T .

S2 Ensure that the information is consistent.

S3 Show how this information can be used to derive new
choices and, possibly, a contradiction.

Notice that (i) has become S1   and S3 .

In general, new choices found in S3 must be fed into S1
to reveal new potential information, but this process stops
after L (= the number of prizes in X) stages.

The condition S2 is analogous to the axiom of revealed
preference (Richter, 1965). There the "new" choices are derived
by applying transitivity, and a contradiction is a choice of the
form $x > x$, analogous to our $(0,0) > 0$.

We need a few definitions in order to clarify S1`. From here
on we number the prizes in X :

$$X = \{x_1, x_2, \ldots, x_L\} .$$

Let h be a compound lottery ticket on E_K , then h is given
by

$$h(e_k) = \Sigma_{\ell=1}^{L} \alpha_\ell^k x_\ell \quad \text{for} \quad 1 \leq k \leq K ,$$

for some integers α_ℓ^k . We define the <u>restriction</u> of h to a
prize x_ℓ , denoted $h|x_\ell$, to be the ticket defined by

$$(h|x_\ell)(e_k) = \alpha_\ell^k x_\ell \quad \text{for} \quad 1 \le k \le K \quad .$$

Thus $h|x_\ell$ is the game h with all prizes ignored except x_ℓ . Given positive probabilities p and q , and a linear utility u , the expected value of $h|x_\ell$ is clearly

$$E(h|x_\ell) = \Sigma_{k=0}^{K} \; p^{K-k} q^k \alpha_\ell^k u(x_\ell) \quad .$$

Setting $\rho = q/p$, we obtain

$$(5.1) \qquad E(h|x_\ell) > 0 \quad \text{iff} \quad u(x_\ell) \; \Sigma_{k=0}^{K} \; \alpha_\ell^k \rho^k > 0 \quad .$$

The polynomial $\Sigma_{k=0}^{K} \; \alpha_\ell^k \rho^k$ is called the <u>characteristic polynomial</u> of h , and it is denoted $P[h|x_\ell]$. Note that $P[h|x_\ell]$ is just the coefficient of $u(x_\ell)$ in the formula for $E(h)$. The intuitive meaning of $P[h|x_\ell]$ is this: it is positive iff x_ℓ is more likely to be gained than lost, from using the ticket h .

Given the previous definitions, we can explain S1 :

each ticket-choice $h > 0$ potentially reveals, for each $x \in X$, that $P[h|x]$ is positive, zero, or negative.

Next we explain the term "consistent" in S2 .

(5.2) <u>Definition.</u> A set of choices of signs of characteristic polynomials $P[h_j|x_\ell]$ (i.e., choices of their being $>$, $<$, or $=$, 0) for various j and ℓ is <u>consistent</u> if there is a positive value of ρ , say $\bar{\rho}$, which gives all those polynomials their chosen sign, e.g. if we have chosen $P[h_j|x_\ell]$ to be positive for some j and ℓ then we must have $P[h_j|x_\ell](\bar{\rho}) > 0$

for that j and ℓ .

Step S3 , deriving new choices, is complicated and we postpone it for now. But a <u>contradiction</u> is as hinted above -- it is the choice (as desirable) of a ticket h having all prizes equal to zero, i.e., $h(e_k) = 0$ for all k , for example $(0,0,0) > 0$.

In general the steps S1 - S3 get quite involved -- there are potential revelations for each ticket and each prize in X , and these lead to new tickets, then new potential revelations, and so on ad nauseam (but not ad infinitum -- everything is finite). Meanwhile we must keep track of which "potential" revelations lead to contradictions, so they can be discarded (the remaining revelations are "actual").

It will simplify matters greatly if we state our conditions contrapositively -- that is, we require a set of "actual" revealed information (revelations which do not lead to a contradiction) to exist. We emphasize three points:

(1) This revealed information involves whether $Pr[h|x]$ is positive, negative, or zero, where $x \in X$ and h is either a ticket from the original set or a derived new ticket (see S3).

(2) The number of "new" tickets created is finite, so in deciding whether a set of revelations is actual -- i.e., does not lead to a contradiction -- one needs go through only a finite number of steps.

(3) It is possible to express the decision as to whether $P[h|x]$ is $>$, $<$, or $= 0$ in terms of choices between tickets -- we explain this at the end of section 6.

6. Statement of Necessary and Sufficient Conditions

- The Main Theorem

(6.0) <u>Theorem</u>: <u>Let $\mathcal{S}$ be a set of simple ticket-choices,</u>

$$\mathcal{S} = \{(x_j, y_j) > (z_j, w_j) : \quad j = 1,2,\ldots,J\}$$

<u>Define</u> $h_j = (x_j - z_j, y_j - w_j)$ <u>for $1 \leq j \leq J$ and define</u>

$$\mathcal{J}_1 = \{h_j > 0 : \quad j = 1,\ldots,J\} \quad .$$

<u>Then $\mathcal{S}$ is compatible with EUH iff $\mathcal{J}_1$ is compatible with EUH,</u>
<u>iff for each $\ell = 1,2,\ldots,L$ it is possible to choose signs of</u>
<u>characteristic polynomials</u> $P[h|x_\ell]$, <u>for</u> $h \in \mathcal{J}_\ell$, <u>such that</u>

    a) <u>The choices are consistent i.e. there exists</u> $\bar{\rho} > 0$
        <u>such that each number</u> $P[h|x_\ell](\bar{\rho})$, <u>for $1 \leq \ell \leq L$ and</u>
        $h \in \mathcal{J}_\ell$, <u>has a sign equal to that chosen for</u> $P[h|x_\ell]$.

    b) <u>The sets</u> $\mathcal{J}_\ell$ <u>defined (inductively - see below) by</u>
        <u>these choices do not contain a contradiction, i.e., a</u>
        <u>member $h > 0$ where</u> $h(e_k) = 0$ <u>for all k</u> .

<u>Remarks</u>:

    (i) The sets $\mathcal{J}_\ell$, $\ell = 1,2,\ldots,L+1$, will be defined in the
remainder of this section. For $1 \leq \ell \leq L$, $\mathcal{J}_{\ell+1}$ will be con-
structed from the tickets in $\mathcal{J}_\ell$ and the choices of signs of the
restricted polynomials $P[h|x_\ell]$ as h ranges over $\mathcal{J}_\ell$.

(ii)   A similar theorem could be stated where $\mathbf{S}$ contains arbitrary compound tickets, of arbitrary degree, and the proof would differ only in notation.

(iii)   That $\mathbf{S}$ is compatible with EUH iff $\mathfrak{I}_1$ is, was shown in the comments following (4.5); the rest of the proof of 6.0 is in section 7.

We will now define the sets $\mathfrak{I}_\ell$, $\ell = 2,\ldots,L+1$, mentioned in the Theorem.   Each $\mathfrak{I}_\ell$ will consist of tickets on E_K , $K = 2^{\ell-1}$.   Intuitively, $\mathfrak{I}_{\ell+1}$ will be built from T_ℓ by "eliminating" the prize x_ℓ ; it consists of the "new" ticket-choices mentioned in S3.

Fix ℓ , $1 \le \ell \le L$.   Assume $\mathfrak{I}_\ell$ has been defined so that it consists of tickets on E_K , $K = 2^{\ell-1}$, and each ticket in $\mathfrak{I}_\ell$ involves only the prizes $x_\ell,x_{\ell+1},\ldots,x_L$.   (Clearly these hypotheses are satisfied for $\ell = 1$ by the Theorem's $\mathfrak{I}_1$.) As in the Theorem, assume we are given signs of the characteristic polynomials $P[h|x_\ell]$ for all $h \in \mathfrak{I}_\ell$; these signs will be used to define $\mathfrak{I}_{\ell+1}$.   The set $\mathfrak{I}_{\ell+1}$ is the smallest set of tickets satisfying N1, N2 and N3 below:

N1)   <u>Suppose</u> $h \in \mathfrak{I}_\ell$ <u>and</u> $h|x_\ell(e_k) = 0$ <u>for all</u> k .   <u>Then</u> $h^o \in \mathfrak{I}_{\ell+1}$ <u>where</u> $h^o(e_k) = h(e_k)$ <u>for</u> $1 \le k \le K$ <u>and</u> $h^o(e_k) = 0$ <u>for</u> $K < k \le 2K$,

N2) <u>Suppose</u> $h \in \mathcal{J}_\ell$ <u>and</u> $h|x_\ell(e_k) \neq 0$ <u>for some</u> k . <u>If</u> $P[h|x_\ell] = 0$, <u>then</u> $h^{\#} \in \mathcal{J}_{\ell+1}$ <u>where</u> $h^{\#}$ <u>is defined as follows: If</u>

$$h(e_k) = \sum_{m=\ell}^{L} \alpha_{km} x_\ell \quad \text{for} \quad 0 \leq k \leq K ,$$

<u>then</u>

$$h^{\#}(e_k) = \sum_{m=\ell+1}^{L} \alpha_{km} x_\ell \quad \text{for} \quad 0 \leq k \leq K$$

<u>and</u>

$$h^{\#}(e_k) = 0 \quad \text{for} \quad K < k \leq 2K .$$

Conditions N1 and N2 say essentially "if $h \in \mathcal{J}_\ell$ and x_ℓ is irrelevant in h , then put h , properly extended, in $\mathcal{J}_{\ell+1}$." The meaning of "properly extended" is that, regardless of the value of p , q and the utility u , we do not change the sign of the expected utility of h . The following lemma, whose proof is a simple computation, is a mathematical version of the relevant part of this paragraph.

(6.1) <u>Lemma.</u> <u>Fix</u> p, q <u>and a utility</u> u , <u>and let</u> E <u>be the expected utility defined by them.</u>

    a) <u>If</u> $h|x_\ell(e_k) = 0$ <u>for all</u> k <u>then</u> $E(h) > 0$ <u>iff</u>

        $E(h^o) > 0$.

    b) <u>If</u> $E(h|x_\ell) = 0$ <u>then</u> $E(h) > 0$ <u>iff</u> $E(h^{\#}) > 0$.

The third method of elimination (N3) will assume there are h ,

$h' \in \mathcal{J}_\ell$ satisfying $P[h|x_\ell] > 0$, $P[h'|x_\ell] < 0$ and will put a new ticket

$h*_\ell h'$ in $\mathcal{J}_{\ell+1}$. It is analogous to the following elimination

method for inequalities: Suppose we are to eliminate x from the

inequalities

(6.2a) $R_1 x + R_2 y + \ldots > 0$

(6.2b) $Q_1 x + Q_2 y + \ldots > 0$

where the R_i and Q_i are real numbers. If $R_1 > 0$ and

$Q_1 < 0$ we can multiply (6.2a) by $-Q_1$ and (6.2b) by R_1 to

get

(6.3a) $-Q_1 R_1 x - Q_1 R_2 y - \ldots > 0$

(6.3b) $R_1 Q_1 x + R_1 Q_2 y + \ldots > 0$

Adding these two inequalities, the x terms cancel:

(6.4) $-Q_1 R_2 y + R_1 \ Q_2 y - \ldots > 0$.

Note that if the coefficients R_i , Q_i in (6.2) are polynomials of

degree K then those in (6.3) are of degree $2K$. (That's why

if $\mathcal{J}_\ell$ contains tickets on E_K , $\mathcal{J}_{\ell+1}$ contains tickets on

E_{2K} .) The conditions $R_1 > 0$ and $Q_1 < 0$ will be

analogues of $P[h|x_\ell] > 0$ and $P[h'|x_\ell] < 0$.

We now define the analogues of (6.3a) and (6.3b), $h*$ and $h'*$: If

(6.5) $h|x_\ell(e_k) = \alpha_k x_\ell$ and $h'|x_\ell(e_k) = \beta_k x_\ell$

then

$$h^*(e_k) = - \sum_{m+n=k} \beta_m \, h(e_n) \, , \quad 0 \le k \le 2K$$

(6.6)

$$h'^*(e_k) = \sum_{m+n=k} \alpha_m h'(e_n) \, , \quad 0 \le k \le 2K \, .$$

Then, as (6.4) is the sum of (6.3a) and (6.3b), we define $h* \underset{\ell}{} h'$ to be the sum of h^* and h'^* :

(6.7) $\quad h* \underset{\ell}{} h'(e_k) = h^*(e_k) + h'^*(e_k) \quad$ for $\quad 1 \le k \le 2K \quad$.

N3) <u>If h and h' are in</u> $\mathcal{J}_\ell$ <u>and</u> $P[h|x_\ell] > 0$ <u>and</u> $P[h'|x_\ell] < 0$ <u>then</u> $h* \underset{\ell}{} h' \in \mathcal{J}_{\ell+1}$.

More motivation for the definition of $h* \underset{\ell}{} h'$ can be found in the next lemma, especially if one sees $E(h)$, $E(h')$, $P[h|x_\ell]$ and $P[h'|x_\ell]$ as analogues of (6.2a), (6.2b), R_1 and Q_1 respectively.

(6.8) <u>Lemma:</u> <u>Fix positive probabilities p and q, and a linear utility u on G, and set $\rho = q/p$</u> .

    a)   <u>Using the definitions of h^*, h'^* in (6.6), we have</u>

$$p^K E(h^*) = -P[h'|x_\ell](\rho)E(h)$$

$$p^K E(h'^*) = P[h|x_\ell](\rho)E(h')$$

    b)   <u>If $E(h)$, $E(h')$, $P[h|x_\ell](\rho)$ and $-P[h'|x_\ell](\rho)$ are positive, so is $E(h* \underset{\ell}{} h')$</u> .

c) <u>The ticket</u> $h* \underset{\ell}{h'}$ <u>does not include any non-zero multiple</u>

<u>of</u> x_ℓ <u>in any of its prizes</u>.

<u>Proof:</u>a) From the various definitions involved we get

$$P[h' \, | \, x_\ell](\rho) = \Sigma_{m=0}^{K} \beta_m \rho^m$$

$$E(h) = p^K \Sigma_{n=0}^{K} u(h(e_n)) \rho^n$$

The formula for multiplication of polynomials gives

$$P[h' \, | \, x_\ell](\rho) E(h) = p^K \Sigma_{k=0}^{2K} \rho^k \underset{m+n=k}{\Sigma} \beta_m u(h(e_n)) \quad .$$

Now use the fact that u is linear:

$$P[h' \, | \, x_\ell](\rho) E(h) = p^K \Sigma_{k=0}^{2K} \rho^k u \left(\underset{m+n=k}{\Sigma} \beta_m h(e_n) \right) \quad .$$

The right-hand side is clearly $-p^K E(h*)$. The proof of the other

half of (a) is similar.

b) Since $p > 0$, the hypotheses and (a) imply that $E(h*)$

and $E(h'*)$ are positive, so $E(h* \underset{\ell}{h'}) = E(h*) + E(h'*)$

is positive.

c) The terms involving x_ℓ in $h*$ and $h'*$ are

$\underset{m+n=k}{\Sigma} \beta_m \alpha_n x_\ell$ and $-\underset{m+n=k}{\Sigma} \alpha_m \beta_n x_\ell$, respectively, and these

clearly add up to zero.

Lemma 6.8 also provides the key to an intuitive explanation of N3: Let us say you are presented with h and h' , both desirable tickets, both involving a certain prize x , and in h you feel you are likely to get x (i.e., $P[h|x] > 0$), in h' you are likely to lose x (i.e., $P[h'|x] < 0$). Now suppose that for some reason the prize x is unavailable. You should be willing to exchange h and h' for some new ticket which is a combination of h and h' , but which does not involve x . The ticket $h *_? h'$ is just such a ticket, if $x = x_\ell$.

(6.10) <u>Stating the choice</u> "$P[h|x]$ is $>$, $<$, or $= 0$" <u>as a ticket choice</u>.

The key to this is (5.1). Suppose for the moment that $x \in X$ and $u(x)$ is positive. Then (5.1) becomes

(6.11) $E(h|x) > 0$ iff $P[h|x](\rho) > 0$.

Thus, instead of asking whether $P[h|x]$ were positive, we could ask a subject whether the restricted ticket $h|x$ were desirable. By (6.11), desirability of $h|x$ corresponds to positivity of $P[h|x]$. Similarly, desirability of $-h|x$ corresponds to $P[h|x] < 0$. As in (4.1, 4.2) desirability of tickets can be stated in terms of choices between tickets not involving "negative" prizes. Our assumption $u(x) > 0$ is not justifiable, but one can get around it by creating a new prize <u>a</u> , which the subject agrees is itself desirable, then replacing x by <u>a</u> in $h|x$, and asking whether the resulting ticket is desirable.

7. Proofs of Necessity and Sufficiency
    in the Main Theorem

Proof of Necessity in (6.0) Assume that $\mathfrak{J}_1$ is compatible with EUH and

let $\bar{\rho}$, $u(x_1),\ldots,u(x_\ell)$ denote the values of ρ and u which

define the expected utility function E such that

$$h > 0 \in \mathfrak{J}_1 \quad \text{iff} \quad E(h) > 0 \ .$$

For this proof, E is to be computed using those values. The

choices of signs of $P[h|x]$ can be made consistently if one

chooses $P[h|x] > 0$ iff $E(h|x) > 0$, etc., proving the condition

6.9a. We claim that if $g \in \mathfrak{J}_\ell$ for any ℓ then $E(g) > 0$.

This will imply that some outcome of g is nonzero, proving

that 6.9b holds. By induction, assume $E(h) > 0$ for every h

in some $\mathfrak{J}_\ell$, $\ell \le L$, and let $g \in \mathfrak{J}_{\ell+1}$. If g arises from (N1)

then $g = h^o$ for some $h \in \mathfrak{J}_\ell$, and $E(h) > 0$. Thus by lemma

(6.1a), $E(h^o) > 0$, and $E(g) > 0$. A similar argument is

used in cases (N2) and (N3), appealing to Lemmas (6.1b) and

(6.8b), respectively.

Proof of Sufficiency in (6.0)

   We assume the choices made of signs of $P[h|x_\ell]$ are con-

sistent and we let $\bar{\rho}$ be a value of ρ which shows that they

are, as in (5.2). We will isolate the main step of the proof in

a definition and a lemma.

<u>Definition</u>: Let $\mathfrak{J}$ be a set of compound lottery tickets. Then $\rho(\mathfrak{J})$ is the set of q/p such that for some real numbers $u(x_i)$, $i = 1,\ldots,L$, the expected utility function E defined by p, q and $\{u(x_i)\}$ satisfies $E(h) > 0$ if $h \in \mathfrak{J}$.

Clearly, $\mathfrak{J}$ is compatible with EUH iff $\rho(\mathfrak{J}) \neq \phi$.

7.1 <u>Lemma</u>. <u>Suppose</u> $\bar{\rho}$ <u>is a value of</u> ρ <u>which satisfies all</u> <u>the choices of signs of</u> $P[h|x_\ell]$ <u>for all</u> $h \in \mathcal{J}_\ell$ <u>and all</u> ℓ , $1 \le \ell \le L$, <u>and</u> $\bar{\rho} \in \rho(\mathcal{J}_{\ell+1})$ <u>for some</u> ℓ , $1 \le \ell \le L$. <u>Then</u> $\bar{\rho} \in \rho(\mathcal{J}_\ell)$.

<u>Proof</u>: Let

$$\mathcal{J}_\ell = \{h_j > 0 : \quad j = 1,2,\ldots,J\}$$

$$\mathcal{J}_{\ell+1} = \{g_i > 0 : \quad i = 1,2,\ldots,I\} \quad ,$$

By construction, the tickets in $\mathcal{J}_\ell$ involve prizes $x_\ell, x_{\ell+1}, \ldots, x_L$ and those in $\mathcal{J}_{\ell+1}$ involve $x_{\ell+1}, \ldots, x_L$. (If $\ell = L$ then $\mathcal{J}_{\ell+1}$ is the empty set or has one ticket, all of whose outcomes are zero.)

That $\bar{\rho} \in \rho(\mathcal{J}_{\ell+1})$ means there exist real numbers $u(x_{\ell+1}), \ldots, u(x_L)$ such that the resulting expected utility function E satisfies $E(g) > 0$ for all $g \in \mathcal{J}_{\ell+1}$. Our task is to find a value for $u(x_\ell)$ so that the E defined by $\bar{\rho}$, $u(x_\ell)$, $u(x_{\ell+1}), \ldots, u(x_L)$, which we call E' , satisfies $E'(h) > 0$ for all $h \in \mathcal{J}_\ell$. Each of the J conditions $E'(h_j) > 0$ will impose a condition on $u(x_\ell)$ and we must show they all can be satisfied. We denote h_j by just h and see what conditions $E'(h) > 0$ imposes. As in (4.4), $E'(h) > 0$ iff

$$\sum_{k=\ell}^{K} (\bar{\rho})^k u(h(e_k)) > 0 \quad .$$

Now use the linearity of u to rearrange the left side:

(7.2) $E'(h) = \sum\limits_{m=\ell}^{L} P_m(\bar{\rho})u(x_m) > 0$,

where each $P_m(\rho)$ is an integer polynomial in ρ . Notice that $E'(h|x_\ell) = P_\ell(\bar{\rho})u(x_\ell)$ and $P[h|x_\ell] = P_\ell$.

Case 1. $P_\ell(\rho)$ is the zero polynomial. Then the ticket h doesn't involve the prize x_ℓ , so by (N1) the ticket h^o is in $\mathbb{J}_{\ell+1}$. Thus $E(h^o) > 0$, and lemma 6.1a

implies $E'(h) > 0$, regardless of the value of $u(x_\ell)$. So in this case $E'(h) > 0$ imposes no restriction on $u(x_\ell)$.

Case 2. P_ℓ not identically zero, but $P_\ell(\bar{\rho}) = 0$. Since $E'(h|x_\ell) = P_\ell(\bar{\rho})u(x_\ell)$, in this case $E'(h|x_\ell) = 0$,regardless of the value of $u(x_\ell)$. Since $\bar{\rho}$ is a value of ρ which makes all the choices of signs of $P[h|x_\ell]$ consistent, we cannot have $P[h|x_\ell] > 0$ or $P[h|x_\ell] < 0$. Thus this case corresponds to (N2), and $h^\# \in \mathbb{J}_{\ell+1}$. Again $E(h^\#) > 0$, so $E'(h) > 0$, regardless of the value of $u(x_\ell)$, and no restriction is imposed.

Case 3. $P_\ell(\bar{\rho}) \neq 0$. Since we have eliminated all other cases, we can without loss of generality assume that all the h_j's in $\mathbb{J}_\ell$ fall into this case. For each h_j we rewrite $E'(h_j) > 0$, as in (7.2):

(7.3) $E'(h_j) = \sum\limits_{m=\ell}^{L} P_m^j(\bar{\rho})u(x_m) > 0$.

By renumbering, we can assume that $P_\ell^j(\bar{\rho}) > 0$ for $1 \le j \le J'$ and $P_\ell^{j'}(\bar{\rho}) < 0$ for $J' < j' \le J$. We can also assume $1 < J' < J$ since if, for example,

$P^j_\ell(\bar{\rho}) > 0$ for all j, $1 \le j \le J$, then we can achieve (7.3) by just making $u(x_\ell)$ sufficiently positive. We can rearrange (7.3) as:

$$u(x_\ell) > - (P^j_\ell(\bar{\rho}))^{-1} \sum_{m=\ell+1}^{L} P^j_m(\bar{\rho})u(x_m) , \quad 1 \le j \le J'$$

(7.4)

$$-(P^{j'}_\ell(\bar{\rho}))^{-1} \sum_{m=\ell+1}^{L} P^{j'}_m(\bar{\rho})u(x_m) > u(x_\ell) , \quad J' < j' \le J .$$

It is clear that, given $u(x_m)$ for $m \ge \ell + 1$ and $\bar{\rho}$, there exists a real number $u(x_\ell)$ satisfying (7.4) iff for every j and j' with $1 \le j \le J'$ and $J' < j' \le J$,

$$(7.5) \quad -(P^{j'}_\ell(\bar{\rho}))^{-1} \sum_{m=\ell+1}^{L} P^{j'}_m(\bar{\rho})u(x_m) > - (P^j_\ell(\bar{\rho}))^{-1} \sum_{m=\ell+1}^{L} P^j_m(\bar{\rho})u(x_m)$$

Thus the lemma will be proved if we can show: for every h_j and $h_{j'}$ in $\mathcal{J}_\ell$ with $P^j_\ell(\bar{\rho}) > 0$, $P^{j'}_\ell(\bar{\rho}) < 0$,

$$(7.6) \quad (P^{j'}_\ell(\bar{\rho}))^{-1} \sum_{m=\ell+1}^{L} P^{j'}_m(\bar{\rho})u(x_m) < (P^j_\ell(\bar{\rho}))^{-1} \sum_{m=\ell+1}^{L} P^j_m(\bar{\rho})u(x_m) .$$

Recall that $P^j_\ell(\bar{\rho}) = P[h_j|x_\ell](\bar{\rho})$. Thus $P^j_\ell(\bar{\rho}) > 0$, $P^{j'}_\ell(\bar{\rho}) < 0$ imply $P[h_j|x_\ell] > 0$, $P[h_{j'}|x_\ell] < 0$ and the pair h_j, $h_{j'}$ falls into case (N3). From now on we leave off the "j"s in our formulas, and denote $h_{j'}$ by h'. By N3 the ticket $g = h*h'_\ell$

is in $\mathcal{J}_{\ell+1}$, so $E(g) > 0$. Recall $g = h* + h'*$, as defined in (6.7), so $E(g) > 0$ implies

$$E'(h'*) > - E'(h*) \quad ,$$

regardless of the value of $u(x_\ell)$. By lemma 6.8a

$$P[h|x_\ell](\bar{\rho})E(h') > P[h'|x_\ell](\bar{\rho})E(h) \quad .$$

Since $P[h|x_\ell] = P_\ell$ and similarly for h' and P'_ℓ , this implies

$$P_\ell(\bar{\rho})E(h') > P'_\ell(\bar{\rho})E(h) \quad .$$

Now expanding $E(h')$ and $E(h)$, we get

$$P_\ell(\bar{\rho})[P'_\ell(\bar{\rho})u(x_\ell) + \sum_{m=\ell+1}^{L} P'_m(\bar{\rho})u(x_m)] > P'_\ell(\bar{\rho})[P_\ell(\bar{\rho})u(x_\ell) + \sum_{m=\ell+1}^{L} P_m(\bar{\rho})u(x_m)] \quad .$$

If we multiply out, the first terms $P_\ell(\bar{\rho})P'_\ell(\bar{\rho})u(x_\ell)$ cancel. Then recall $P_\ell(\bar{\rho}) > 0$, $P'_\ell(\bar{\rho}) < 0$ so we can divide through by $(P_\ell(\bar{\rho})P'_\ell(\bar{\rho})) < 0$ to get

$$(P'_\ell(\bar{\rho}))^{-1} \sum_{m=\ell+1}^{L} P'_m(\bar{\rho})u(x_m) < (P_\ell(\bar{\rho}))^{-1} \sum_{m=\ell+1}^{L} P_m(\bar{\rho})u(x_m) \quad ,$$

which is exactly (7.6), QED .

Now for the proof of sufficiency: Assume conditions 6.9a and 6.9b. Consider $\mathcal{J}_{L+1}$. Since all prizes have been eliminated at that stage (see 6.8c) the only possible ticket in $\mathcal{J}_{L+1}$ is the zero ticket. This is excluded by assumption 6.9b. Therefore

$\mathfrak{J}_{L+1} \neq \phi$ and, by default, $\rho(\mathfrak{J}_{L+1}) = \{\rho : \rho > 0\}$. Now let $\bar{\rho}$ be a positive value of ρ satisfying all the sign-choices -- such a $\bar{\rho}$ exists by 6.9a. Since $\bar{\rho} \in \rho(\mathfrak{J}_{L+1})$, by the previous lemma $\bar{\rho} \in \rho(\mathfrak{J}_L)$. For similar reasons $\bar{\rho} \in \rho(\mathfrak{J}_{L-1})$, etc. until we get $\bar{\rho} \in \rho(\mathfrak{J}_1)$. Since $\rho(\mathfrak{J}_1) \neq \phi$, $\mathfrak{J}_1$ is compatible with EUH.

This proof may seem a bit too slick, although it is valid. A more complex and perhaps more believable argument can be made using $\mathfrak{J}_L$ instead of $\mathfrak{J}_{L+1}$: Since all prizes but x_L have been eliminated from $\mathfrak{J}_L$, each $h_j \in \mathfrak{J}_L$ has expected utility equal to

$$P_j(\rho) u(x_L) \quad ,$$

for some polynomial P_j (in fact $P_j = P[h_j | x_L]$). Thus $\bar{\rho} \in \rho(\mathfrak{J}_\ell)$ iff

$$(7.7) \qquad P_j(\bar{\rho}) u(x_\ell) > 0 \quad \text{for all} \quad j \ .$$

If the numbers $P_j(\bar{\rho})$ are all of the same sign, one picks $u(x_\ell)$ to be of that sign to achieve (7.7), and $\bar{\rho} \in \rho(\mathfrak{J}_L)$, then induct using the lemma as above to prove $\mathfrak{J}_1$ compatible with EUH . But this must be the case since otherwise for some $h_j, h_{j'} \in \mathfrak{J}_L$, $P_j(\bar{\rho}) > 0$ and $P_{j'}(\bar{\rho}) < 0$, and one can show that then $h_j *_L h_{j'}$ has all outcomes equal to zero, violating 6.9b.

8. Criticisms

We have mentioned certain criteria that a theory of expected utility might satisfy:

(A) It should apply to a partial order on the set of decisions and to finite sets of events and consequences.

(B) The axioms, or conditions, should be constructive and reasonable (cf. 2.1), and should not assume repetitions of events.

Before discussing the extent to which our theory has met each of these criteria, we must admit that if our theory is to be judged by this yardstick then it is surely a failure. But we would like to view it as an attempt to answer the question: Can any theory satisfying (A) also satisfy (B)? Note that (A) contains some of the most fundamental criteria for an adequate theory put forth by the objectivist school, while (B) includes claims made by the subjectivist school in favor of its theories. Since some of the criteria in (B) cannot be rigorously stated, we cannot hope to answer this question yes or no. But, since our conditions are necessary (as well as sufficient) it gives evidence that any theory satisfying (A) cannot satisfy (B) any better than does ours. Of course this is not conclusive evidence -- someone might design totally different conditions which, although they would have to imply ours, would better meet the criteria of (B). But it seems to be the best available way to attack the question. If one is an objectivist, it is probably the closest

one can come to answering it in the negative, i.e., to showing that any theory satisfying (A) cannot satisfy (B).

Now we turn to the question of whether our theory does meet the criteria (A) and (B).

Partial order on decisions: We meet this criteria unless one interprets the statement "there exist choices of signs ..." in 6.9 as requiring the subject to make other choices than those in the original partial order (cf. 6.10). Even then, these other choices do not require an extension of the partial order to a complete one, or even an extension to choices between constant tickets. Luce and Krantz (1971, pg. 258) make this comment about requiring such other choices:

> "The measurer must be prepared to present for serious consideration by the decision maker some rather artificial alternatives, and the decision maker must be induced to make realistic decisions among them."

Finite set of events: We have assumed that the set of events contains only two elements, but this was merely for simplicity; our methods could equally well apply to any finite set. They do not, however, apply directly to an infinite event set.

Finite set of consequences: We have explicitly assumed this. Again, our methods do not apply to an infinite set of consequences.

Constructive: Our method seems on the surface to be con-
structive, since everything is finite. There are finitely many
choices of signs of characteristic polynomials $P[h|x_\ell]$ for
each $\mathfrak{I}_\ell$, thus finitely many possible $\mathfrak{I}_{\ell+1}$'s given $\mathfrak{I}_\ell$.
Each $\mathfrak{I}_\ell$ is finite so it is easy to check if it contains a
zero ticket. But checking whether a choice of signs is consistent
is not so simple. In general it involves verifying that a certain
set of polynomials in one variable (the characteristic poly-
nomials) have a common point of positive value, i.e., $P(\bar\rho) > 0$
for all P in that set. This problem is decidable in the mathe-
matical sense, since it is a special case of the solvability of
polynomial inequalities discussed in section 2, but so is the
original problem of solving the inequalities (1.6). So by some
simple steps we have reduced the problem of deciding solvability
of the polynomial inequalities (1.6) to the solvability of some
polynomial inequalities in one unknown ρ. The main result of
Richter-Shapiro (1977) is that given any set $\mathfrak{J}$ of polynomial
inequalities in one unknown, there is a set $\mathfrak{S}$ of simple (i.e.,
as in section 1) ticket-choices such that $\mathfrak{S}$ has a solution with
$p/q = \rho$ iff ρ is a solution to $\mathfrak{J}$.

Reasonability is a purely subjective judgement and our method
does require repetitions of events.

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