DECOMPOSITION METHODS FOR ADHERENCE PROBLEMS IN FINITE ELASTICITY

BY

P. LE TALLEC

A. LOTFI

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455

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DECOMPOSITION METHODS FOR ADHERENCE PROBLEMS IN FINITE ELASTICITY

P. LE TALLEC, A. LOTFI

Service de Mathématiques

LABORATOIRE CENTRAL des PONTS & CHAUSSEES
58 Bld Lefebvre, 75732 PARIS Cedex 15 (France)

Summary

The problem discussed in this paper consists in computing the large deformations of incompressible elastic bodies which are glued (not fixed) on part of their boundaries. The proposed numerical technique is based on the augmented Lagrangian approach already used in GLOWINSKI-LE TALLEC [1982] for the numerical solution of two-dimensional equilibrium problems in Finite Elasticity, and is organized as follows :

i) the adhesion problem is first discretized in time, which reduces it to a sequence of contact problems in Finite Elasticity with friction forces ;

ii) each contact problem is then transformed into a saddle-point problem obtained by considering the displacement, the strains and the relative displacement at the contact surface as independent variables ;

iii) these saddle-point problems are finally solved by an iterative technique which considers one variable at a time, thus reducing the global algorithm to the successive solution of a linear elasticity problem with a fixed stiffness matrix, of homogeneous local finite elasticity problems, and of local adhesion problems set on the contact surface.

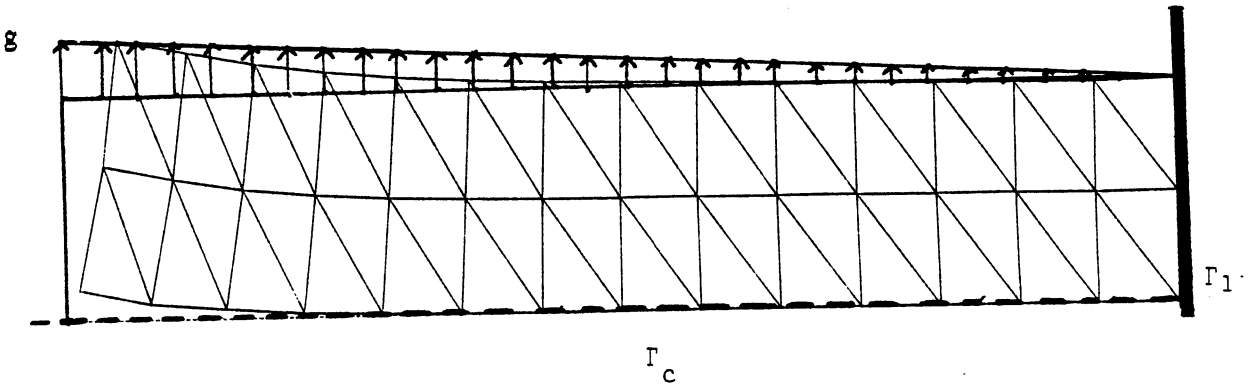
Introduction

The purpose of this paper is to describe numerical methods (namely augmented Lagrangian techniques) for solving nonlinear constrained problems involving several vector fields. The strategy is based on operator's splitting and permits an accurate treatment of the constraints. Compared to early applications of these methods (FORTIN-GLOWINSKI [1982] [1984]) we introduce a more elaborate splitting strategy and deal with a more general and more constrained situation.

Herein, the method will be illustrated in *three-dimensional finite elasticity* with adhesion and the constraints to be satisfied will be *incompressibility, frame-indifference, unilateral contact*. We first recall the equations of incompressible finite elasticity with adhesion forces (§1), discretize them with respect to time (§2), and then introduce an augmented Lagrangian decomposition of this discretized problem (§3). A numerical algorithm for solving this Lagrangian problem is proposed in §4, and solution procedures are presented respectively for the local incompressible problem in strains (§5) and for the adhesion problem on the contact surface (§6). Numerical results are given in conclusion.

1. Equations of the model problem

The problem consists in determining at each time t the position $\mathbf{x} + \mathbf{u}(\mathbf{x},t)$ of any particle $\mathbf{x}$ of a given elastic body. This body occupies a known domain Ω in its stress-free reference configuration, is subjected to a given distribution of external loads, is fixed on a part Γ_1 of its boundary Γ, and is glued to a rigid support on another part Γ_c of its boundary (Fig. 1). For handling the large deformations which are involved, we label any particle $\mathbf{x}$ by its position in the reference configuration (Lagrange coordinates), and we relate both $\mathbf{x}$ and its displacement $\mathbf{u}(\mathbf{x},t)$ to a fixed rectangular Cartesian coordinate system. In addition, on the glued surface, we introduce the adhesion intensity $\beta(\mathbf{x},t)$ which measures the proportion of glue links which are still active. Finally, we denote by $\mathbf{f}$ the density of external body forces measured per unit volume of Ω, we denote by $\mathbf{g}$ the density of external surface tractions applied on $\Gamma_2 = \Gamma-\Gamma_1-\Gamma_c$, and we denote by $\beta_0(\mathbf{x})$ the adhesion intensity at time $t = 0$.



- Figure 1 -
Reference and deformed configuration.

Neglecting inertia effects, assuming the constitutive material to be incompressible and hyperelastic, supposing the loading **f** *and* **g** *independent of the displacement* (dead loading), the evolution of **u** and β is governed by the following energetic principle :

AT ANY TIME t, IN ANY ADMISSIBLE VIRTUAL PERTURBATION OF THE REAL CONFIGURATION, THE ENERGY DISSIPATED BY BREAKING ADHESION MUST BE BIGGER THAN THE POTENTIAL ENERGY RESTITUTED BY THE STRUCTURE.

Here, by construction, $\beta \in [0,1]$ (β is a proportion of active links) $\mathbf{u} = 0$ on Γ_1 (the body is fixed on Γ_1), $\mathbf{u}\cdot\mathbf{n} \leq 0$ on Γ_c (the body cannot penetrate its rigid support), and $\det(1+\nabla\mathbf{u}) = 1$ in Ω (the body is incompressible). Therefore, the set of admissible $\{\mathbf{u},\beta\}$ is finally defined by

$$K = \Big\{\{\mathbf{v},\gamma\} \in H^1(\Omega) \times L^4(\Gamma_c),\ 0 \leq \gamma \leq 1 \text{ on } \Gamma_c,\ \mathbf{v}\cdot\mathbf{n} \leq 0 \text{ on } \Gamma_c,$$
$$\mathbf{v} = 0 \text{ on } \Gamma_1,\ \det(1+\nabla\mathbf{v}) = 1 \text{ a.e. in } \Omega\Big\}. \tag{1.1}$$

Moreover, the potential energy associated to a displacement field **v** and to a distribution of adhesion intensity γ is classically defined by

$$J_{POT}(\mathbf{v},\gamma) = \int_\Omega \mathcal{W}(1+\nabla\mathbf{v})\,d\Omega - \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega - \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{v}\,d\Gamma$$
$$- \int_{\Gamma_c} w\gamma + \frac{1}{2\varepsilon}\int_{\Gamma_c} \gamma^2\,|\mathbf{v}|^2 d\Gamma, \tag{1.2}$$

with $\mathcal{W}(\cdot)$ the stored elastic energy density, w the Dupré superficial energy and ε a small regularizing factor introduced by Frémond [1985] in his modeling of adhesion phenomena. Also following Frémond, we introduce as potential of dissipation the functional

$$J_{DIS}\left(\frac{\partial\beta}{\partial t}\right) = \frac{1}{2}\int_{\Gamma_c}\int_{\Gamma_c} d(\mathbf{x}-\mathbf{y})\,\frac{\partial\beta}{\partial t}(\mathbf{x},t)\,\frac{\partial\beta}{\partial t}(\mathbf{y},t)\,d\mathbf{x}d\mathbf{y} \tag{1.3}$$

whose gradient at value $\frac{\partial\beta}{\partial t}$ will define the energy dissipated by breaking adhesion. From (1.1), (1.2), (1.3), the above energetic principle takes the mathematical form

Find $\{\mathbf{u},\beta\} : [0,T] \to K$ such that :

(i) $\beta(\mathbf{x},0) = \beta_0(\mathbf{x})$ on Γ_c ;

(ii) for almost any t in $[0,T]$, there exists a neighborhood $\mathcal{V}(\mathbf{u},\beta)$ of $\{\mathbf{u},\beta\}$ in K such that, for any $\{\mathbf{v},\gamma\}$ in $\mathcal{V}(\mathbf{u},\beta)$ we have

$$\nabla J_{DIS}\left(\frac{\partial\beta}{\partial t}\right)\cdot(\gamma-\beta) \geq J_{POT}(\mathbf{u},\beta) - J_{POT}(\mathbf{v},\gamma). \tag{1.4}$$

Remark. Typical stored energy functions $\mathcal{W}$ and coupling functions d are of the form

$$\mathcal{W}(\mathbf{F}) = C_1(|\mathbf{F}|^2-3) + C_2(|\mathrm{adj}\ \mathbf{F}|^2-3), \tag{1.5}$$

$$d(\mathbf{y}) = d_0 \exp(-d_0|\mathbf{y}|). \tag{1.6}$$

In (1.5) $\mathcal{W}$ corresponds to the so-called Mooney-Rivlin materials and adj $\mathbf{F}$ denotes the transpose of the cofactor matrix of $\mathbf{F}$.

2. Implicit time discretization

Let us introduce particular times $t_0 = 0 < t_1 < \ldots < t_P = T$ and solve (1.4) only at those instants, replacing the time derivatives $\frac{\partial\beta}{\partial t}(\mathbf{x},t_p)$ by the finite differences

$$[\beta(\mathbf{x},t_p) - \beta(\mathbf{x},t_{p-1})] \,/\, (t_p - t_{p-1}).$$

In other words, let us proceed to an implicit (backward) time discretization of the time dependent problem (1.4). From the convexity of J_{DIS}, after discretization, and denoting by $\{u_p, \beta_p\}$ the values of the discretized solution at time t_p, (1.4) becomes

$$\boxed{\forall 1 \le p \le P, \text{ \underline{with} } \beta_{p-1} \text{ \underline{known}, \underline{find a local minimum} } \{u_p, \beta_p\} \text{ \underline{of} } J_p \text{ \underline{over the set} } K \text{ \underline{of admissible solutions.}}} \tag{2.1}$$

In (2.1), K is the set introduced in (1.1), and the functional J_p is defined by

$$J_p(\mathbf{v},\gamma) = (t_p - t_{p-1})\, J_{DIS}\left[\frac{\gamma-\beta_{p-1}}{t_p - t_{p-1}}\right] + J_{POT}(\mathbf{v},\gamma) \tag{2.2}$$

that is, from (1.2) and (1.3)

$$J_p(\mathbf{v},\gamma) = \overbrace{\int_\Omega \mathcal{W}(1+\nabla\mathbf{v})\,d\Omega}^{\text{elastic energy}} - \overbrace{\int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\Omega - \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{v}\,d\Gamma}^{\text{dead loading}} + \overbrace{\frac{1}{2\varepsilon}\int_{\Gamma_c} \gamma^2 |\mathbf{v}|^2 d\Gamma}^{\text{friction}}$$

$$- \overbrace{\int_{\Gamma_c} w\gamma\,d\Gamma + \frac{1}{2(t_p - t_{p-1})}\int_{\Gamma_c}\int_{\Gamma_c} d(\mathbf{x}-\mathbf{y})[\gamma-\beta_{p-1}](\mathbf{x})[\gamma-\beta_{p-1}](\mathbf{y})\,d\mathbf{x}d\mathbf{y}}^{\text{adhesion}}.$$

As written, the discretized problem (2.1) consists in *minimizing the nonconvex functionals* J_p *over the nonconvex set* K. In view of its numerical solution, (2.1) is first written in an abstract form by introducing

$$V = \left\{\mathbf{v} \in H^1(\Omega),\ \mathbf{v} = 0 \text{ on } \Gamma_1\right\}, \tag{2.3}$$

$$\mathscr{H} = \left\{ H = \{\mathbf{E}, \mathbf{e}, \gamma\} \in L^2(\Omega)^9 \times L^2(\Gamma_c) \times L^2(\Gamma_c) \right\}, \tag{2.4}$$

$$\mathscr{G} : \begin{cases} V \to \mathbb{R}, \\ \mathscr{G}(\mathbf{v}) = \displaystyle\int_\Omega C_1 \, |1+\nabla \mathbf{v}|^2 d\Omega - \int_\Omega \mathbf{f}\cdot\mathbf{v} \, d\Omega - \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{v} \, d\Gamma, \end{cases} \tag{2.5}$$

$$\mathscr{F} = \begin{cases} \mathscr{H} \to \mathbb{R} \cup \{+\infty\}, \\ \mathscr{F}(H) = + \displaystyle\int_\Omega \left(\mathscr{W}(\mathbf{E}) - C_1 \, |\mathbf{E}|^2 \right) d\Omega - \int_{\Gamma_c} w\gamma \, d\Gamma + \frac{1}{2\varepsilon} \int_{\Gamma_c} \gamma^2 \, |\mathbf{e}|^2 d\Gamma \\ \qquad + \dfrac{1}{2(t_p - t_{p-1})} \displaystyle\int_{\Gamma_c}\int_{\Gamma_c} d(\mathbf{x}-\mathbf{y}) [\gamma - \beta_{p-1}](\mathbf{x}) [\gamma - \beta_{p-1}](\mathbf{y}) d\mathbf{x} d\mathbf{y} \\ \qquad \text{if } \det \mathbf{E} = 1, \ 0 \le \gamma \le 1 \text{ and } \mathbf{e}\cdot\mathbf{n} \le 0, \\ \mathscr{F}(H) = +\infty \text{ if not} \end{cases} \tag{2.6}$$

$$B : \begin{cases} V \times \mathscr{H} \to L^2(\Omega)^9 \times L^2(\Gamma_c), \\ B(\mathbf{v}, H) = \left\{ 1 + \nabla \mathbf{v} - \mathbf{E}, \mathbf{v}\big|_{\Gamma_c} - \mathbf{e} \right\}. \end{cases} \tag{2.7}$$

With the above notations, (2.1), that is the time-discretized adhesion problem in finite elasticity, takes the usual form

$$\boxed{\begin{array}{l} \forall 1 \le p \le P, \ \underline{\text{with}} \ \beta_{p-1} \ \underline{\text{known}}, \ \underline{\text{find a local minimum of}} \ \mathscr{F}(H) + \mathscr{G}(\mathbf{v}) \\ \quad \underline{\text{over the space}} \ V \times \mathscr{H} \ \underline{\text{under the linear constraint}} \ B(\mathbf{v}, H) = 0. \end{array}} \tag{2.8}$$

3. Augmented Lagrangian decomposition of (2.8)

Two strategies can be employed for the numerical solution of (2.8) :

(i) the first one (Penalty + Newton) makes the functional $\mathscr{F}$ differentiable by penalizing the constraints involved in its definition, eliminates the variables F and e through the linear constraint $B(\mathbf{v}, H) = 0$, and obtains afterwards an unconstrained minimization problem with respect to the variable $\{\mathbf{v}, \gamma\}$, problem to be solved by a general purpose Newton's method ;

(ii) the second one, to be presented hereafter, treats the linear constraint $B(\mathbf{v},\mathbf{H}) = 0$ by an augmented Lagrangian technique (POWELL [1969], HESTENES [1969]). The problem in $\mathbf{v}$ (displacements), the problem in $\mathbf{F}$ (strains) and the problem in $\{\mathbf{e},\gamma\}$ (adhesion) can then be treated independently.

Classically, augmented Lagrangian methods replace the linearly constrained minimization problem (2.8) by the saddle-point problem

$$\boxed{\begin{array}{c}\forall 1 \le p \le P, \ \underline{\text{with}}\ \beta_{p-1}\ \underline{\text{known}},\ \underline{\text{solve}} \\ \displaystyle\sup_{\mu\in L^2(\Omega)^9\times L^2(\Gamma_c)} \left\{ \inf_{(\mathbf{v},\mathbf{H})\in \mathbf{V}\times\mathcal{H}} \mathcal{L}_R(\mathbf{v},\mathbf{H};\mu) \right\}\end{array}} \tag{3.1}$$

In (3.1), the supremum and the infimum are local, the space V and $\mathcal{H}$ are as defined in (2.3) and (2.4), and the augmented Lagrangian $\mathcal{L}_R$ is defined by

$$\mathcal{L}_R(\mathbf{v},\mathbf{H};\mu) = \overbrace{\mathcal{F}(\mathbf{H}) + \mathcal{G}(\mathbf{v})}^{\text{functional}} + \overbrace{\frac{R}{2}\|B(\mathbf{v},\mathbf{H})\|^2}^{\text{penalty}} - \overbrace{\langle \mu, B(\mathbf{v},\mathbf{H})\rangle}^{\text{duality}}, \tag{3.2}$$

with :

$$\|B(\mathbf{v},\mathbf{H})\|^2 = \int_\Omega q(\mathbf{x})\ |1+\nabla\mathbf{v}-\mathbf{E}|^2 d\Omega + \int_{\Gamma_c} q_c\,|\mathbf{v}-\mathbf{e}|^2 d\Gamma, \tag{3.3}$$

$$\langle \mu, B(\mathbf{v},\mathbf{H})\rangle = \int_\Omega \mu\cdot(1+\nabla\mathbf{v}-\mathbf{E})\,d\Omega + \int_{\Gamma_c} \mu_c\cdot(\mathbf{v}-\mathbf{e})\,d\Gamma. \tag{3.4}$$

Above, $R, q(\mathbf{x})$ and $q_c(\mathbf{x})$ are arbitrary strictly positive numbers whose choice has no effect on the values of the solution of (3.1) but determines the speed of convergence of the algorithm used for its numerical solution.

Obviously, if $\{u_p, G_p; \lambda_p\}$ is a solution of (3.1), then $\frac{\partial \mathcal{L}_R}{\partial \mu}(u_p, G_p; \lambda_p)$ is equal to zero, thus, from (3.2), $B(u_p, G_p)$ is equal to zero and by definition of (3.1), $\{u_p, G_p\}$ minimizes locally $\mathcal{L}_R(\mathbf{v},\mathbf{H},\lambda_p\}$ over the kernel of B ; in other words $\{u_p, G_p\}$ is a solution of (2.8). Moreover, formally the converse is also true. Therefore, from now on, *instead of solving the adhesion problem in Finite*

Elasticity under its variational formulation (2.8) (or (2.1)), *we solve it under its saddle-point* (augmented Lagrangian) *formulation* (3.1).

4. Uzawa Algorithm

It consists in solving (3.1) with respect to the (dual or stress) variable μ by a *steepest descend method*. Taking the step parameter ρ equal to R, and computing the quantity $\underset{\{v,H\}}{\mathrm{INF}}\ \mathcal{L}_R(v,H;\mu)$ by block-relaxation, the application of this algorithm to (3.1) leads to the following :

β_0 := adhesion intensity at time t = 0 ;
choose λ_0 in $L^2(\Omega)^9 \times L^2(\Gamma_c)$;
choose $\{F_0, h_0\}$ in $L^2(\Omega)^9 \times L^2(\Gamma_c)$;
$G_0 := \{F_0, h_0, \beta_0\}$;

<u>for time</u> p <u>equal</u> 1 <u>to</u> P, <u>do</u>

$\{G^0, \lambda^0\} = \{G_{p-1}, \lambda_{p-1}\}$;

<u>for iteration</u> n <u>equal</u> 1 <u>until</u> $(\|G^n - G^{n-1}\| + \|B(u^n, G^n)\|)$ <u>small</u>, <u>do</u>

bloc relaxation

$$\begin{cases} \text{solve } \mathcal{L}_R(u^n, G^{n-1};\ \lambda^{n-1}) \le \mathcal{L}_R(v, G^{n-1};\ \lambda^{n-1}), \\ \forall v \in V,\ u^n \in V\ ; & (4.1) \\ \text{solve } \mathcal{L}_R(u^n, G^n; \lambda^{n-1}) \le \mathcal{L}_R(u^n, H, \lambda^{n-1}), \\ \forall H \in \mathcal{H},\ G^n \in \mathcal{H}; & (4.2) \end{cases}$$

gradient update

$$\lambda^n := \lambda^{n-1} - R\rho B(u^n, G^n)\ ; \qquad (4.3)$$

<u>end loop</u> n

$$\{u_p, G_p; \lambda_p\} = \{u^n, G^n; \lambda^n)\ ; \qquad (4.4)$$

<u>end loop</u> p.

Obviously, many variants can be imagined for such algorithms, some of them being described in FORTIN-GLOWINSKI [1982, 1984 : chap. 3]. Moreover, they can be interpreted as alternating directions time-integrators of a parabolic

evolution equation in λ, in the sense introduced by PEACEMAN-RACHFORD [1955] and studied by LIONS-MERCIER [1979]. But, what is more important is to understand to which numerical operators correspond the different steps of algorithm (4.1)-(4.4).

First, from the definition of $\mathcal{G}$, $\mathcal{L}_R$ and B, (4.1) is

$$a(\mathbf{u}^n,\mathbf{v}) = L(\mathbf{v}),\ \forall \mathbf{v} \in V,\ \mathbf{u}^n \in V, \tag{4.5}$$

with

$$a(\mathbf{u}^n,\mathbf{v}) = \int_\Omega 2C_1\ \nabla\mathbf{u}^n\cdot\nabla\mathbf{v}\ d\Omega + R\int_\Omega q\nabla\mathbf{u}^n\cdot\nabla\mathbf{v}\ d\Omega + R\int_{\Gamma_c} q_c\mathbf{u}^n\cdot\mathbf{v}\ d\Gamma,$$

$$L(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\ d\Omega + \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{v}\ d\Gamma - \int_\Omega 2C_1\ \mathrm{div}\ \mathbf{v}\ d\Omega$$

$$+ \int_\Omega \{Rq(\mathbf{F}^{n-1}-1) + \lambda^{n-1}\}\cdot\nabla\mathbf{v}\ d\Omega + \int_{\Gamma_c} \{Rq_c\ \mathbf{e}^{n-1} + \lambda_c^{n-1}\}\cdot\mathbf{v}\ d\Gamma.$$

This is the variational formulation of a linear, strongly elliptic, non-homogeneous Poisson equation which, after finite element discretization of V, reduces to a well-posed linear system in $\mathbb{R}^{\dim V_h}$. The associated matrix is sparse, positive definite, uncoupled in each direction, does not change during the iterative process, and can thus be constructed and factorized once for all. The linear system can then be solved very efficiently at each step in $O(\dim V_h)$ operations.

Next, from the construction of $\mathcal{F}$, $\mathcal{L}_R$ and B, (4.2) splits into

$$\int_\Omega J_E(\mathbf{F}^n)\,d\Omega \le \int_\Omega J_E(\mathbf{E})\,d\Omega,\ \forall \mathbf{E} \in K_E,\ \mathbf{F}^n \in K_E, \tag{4.6}$$

$$\int_{\Gamma_c} J_c(\mathbf{h}^n,\beta^n)\,d\Gamma \le \int_{\Gamma_c} J_c(\mathbf{e},\gamma)\,d\Gamma,\ \forall\{\mathbf{e},\gamma\} \in K_c,\ \{\mathbf{h}^n,\beta^n\} \in K_c. \tag{4.7}$$

The constrained minimization problems (4.6) and (4.7) are local and will be studied respectively in §5 and §6, under the notations

$$J_E(\mathbf{E}) = \left\{\mathcal{W}(\mathbf{E}) - C_1|\mathbf{E}|^2 + \frac{R}{2}\cdot q|\mathbf{E}-1-\nabla\mathbf{v}^n|^2 + \lambda^{n-1}\cdot\mathbf{E}\right\}, \tag{4.8}$$

$$K_E = \{\mathbf{E} \in (L^2(\Omega))^9,\ \det \mathbf{E} = 1\}, \tag{4.9}$$

$$J_c(\mathbf{e},\gamma) = \left\{\frac{R}{2} q_c |\mathbf{u}^n - \mathbf{e}|^2 + \lambda_c^{n-1}\cdot\mathbf{e} + \frac{\gamma^2}{2\varepsilon}|\mathbf{e}|^2 - w\gamma \right.$$
$$\left. + \frac{1}{2(t_p - t_{p-1})} \int_{\Gamma_c} d(\mathbf{x}-\mathbf{y})[\gamma-\beta_{p-1}](\mathbf{x})[\gamma-\beta_{p-1}](\mathbf{y})\,d\mathbf{y}\right\}, \tag{4.10}$$

$$K_c = \{\{\mathbf{e},\gamma\},\ 0 \le \gamma \le 1,\ \mathbf{e}\cdot\mathbf{n} \le 0\}. \tag{4.11}$$

Finally, (4.3) is an explicit gradient update of λ and (4.4) simply expresses that if $\mathbf{G}^n = \mathbf{G}^{n-1}$ is a local minimizer of $\mathcal{L}_R(\mathbf{u}^n,\cdot;\lambda^{n-1})$, if $\mathbf{u}^n$ is a local minimizer of $\mathcal{L}_R(\cdot,\mathbf{G}^{n-1};\lambda^{n-1})$ and if $B(\mathbf{u}^n,\mathbf{G}^n) = 0$, then $\{\mathbf{u}^n,\mathbf{G}^n\}$ will be the desired solution of the finite elasticity problem (2.8) at time-step p.

From these remarks, algorithm (4.1)-(4.4) reduces to the flow chart represented in Fig. 2.

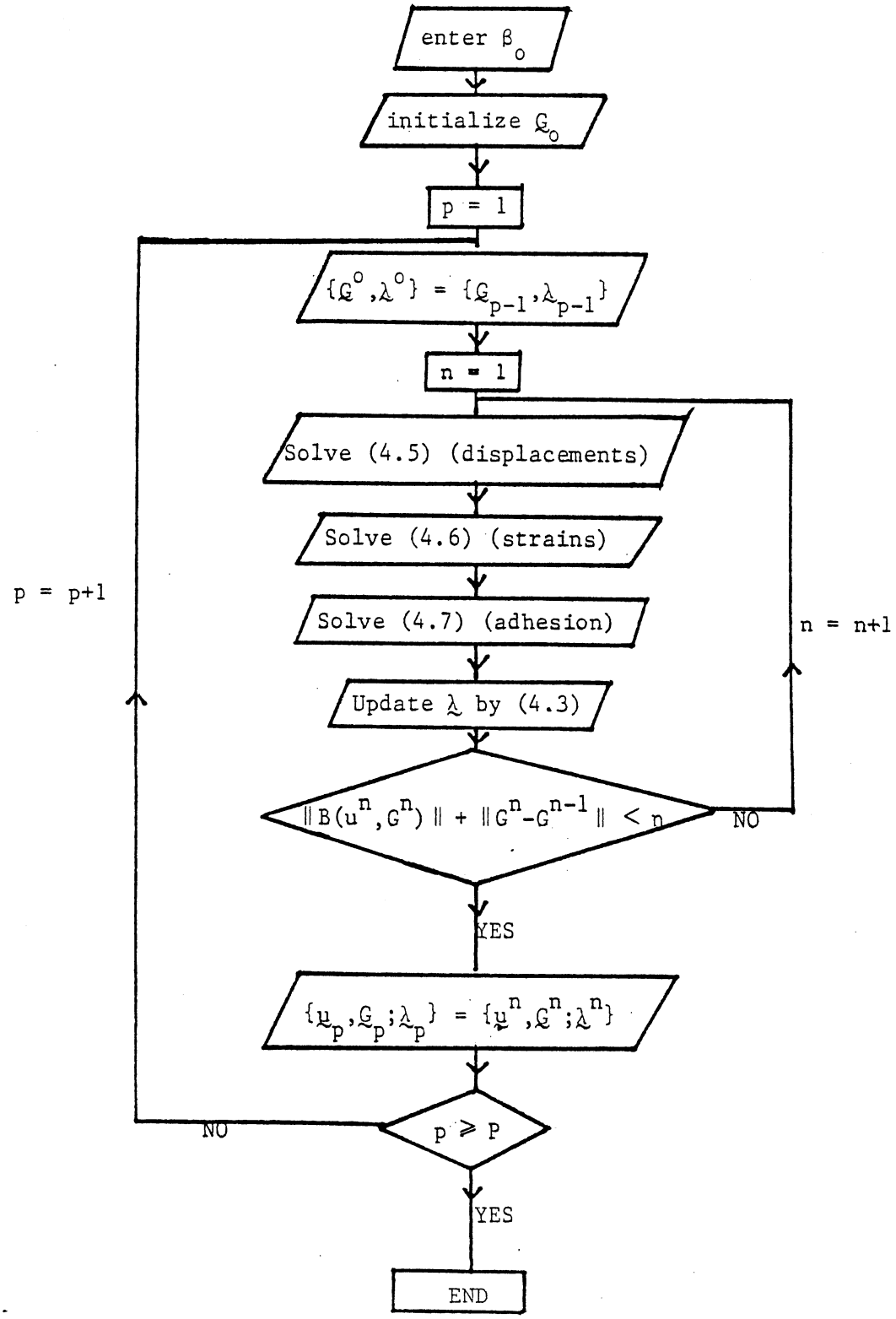


Fig. 2. Flow Chart of Uzawa algorithm (4.1)-(4.4)

5. Solving problem (4.6) in strains (isotropic case)

By construction, the spatial derivatives of the unknown strains **F** do not appear in our problem. Therefore, it is legitimate to use piecewise constant finite elements to approximate **F**, that is to replace in (4.6) the set K_E by

$$K_E^h = \left\{ \mathbf{E},\ \det \mathbf{E} = 1,\ \mathbf{E}\Big|_{\Omega^e} = \text{constant},\ \forall e = 1,\ N_h \right\}, \tag{5.1}$$

where the family $(\Omega^e)_{e=1,N_h}$ forms a regular triangulation of the domain Ω. In this case, since the minimal value of the sum of independent terms

$$\int_\Omega J_E(\mathbf{E})\,d\Omega = \sum_{e=1}^{N_h} \int_{\Omega^e} J_E(\mathbf{E})\,d\Omega = \sum_{e=1}^{N_h} J_E(\mathbf{E})\ \text{meas}(\Omega^e)$$

is equal to the sum of the minimal value of each term, (4.6) reduces to

$$\forall e = 1,N_h,\ J_E(\mathbf{F}) \le J_E(\mathbf{E}),\ \forall \mathbf{E} \in \mathbb{R}^9 \text{ with } \det \mathbf{E} = 1,\ \det \mathbf{F} = 1, \tag{5.2}$$

with

$$J_E(\mathbf{E}) = \left(\frac{R}{2}\,q - C_1\right) |\mathbf{E}|^2 + \mathcal{W}(\mathbf{E}) - [Rq(1+\nabla u^n) - \lambda^{n-1}]\cdot\mathbf{E}, \tag{5.3}$$

q, C_1, ∇u^n and λ^n being averaged over Ω^e. Moreover, if the constitutive hyperelastic material is isotropic, the stored elastic energy density $\mathcal{W}$ only depends on the singular values of **E** (frame indifference + symmetry).

From this frame indifference and symmetry, and from the incompressibility, (5.2) can in turn be solved through a change of variables which reduces it to *unconstrained minimization problems in two scalar variables*. Indeed, let us diagonalize $Rq(1+\nabla u^n)-\lambda^{n-1}$ into the product **Q D R**, **Q** and **R** being orthogonal 3×3 matrices with positive determinant and **D** being diagonal, and let us set

$$\mathbf{F} = \mathbf{Q}\ \mathbf{T}\ \mathbf{R}. \tag{5.4}$$

With respect to the new variable **T**, (5.2) becomes

$$\forall e = 1,N_h,\ J_T(\mathbf{T}) \le J_T(\mathbf{E}),\ \forall \mathbf{E} \in \mathbb{R}^9 \text{ with } \det \mathbf{E} = 1,\ \det \mathbf{T} = 1, \tag{5.5}$$

$$J_T(\mathbf{E}) = \left(\frac{R}{2} q - C_1\right) |\mathbf{E}|^2 + \mathcal{W}(\mathbf{E}) - \sum_{i=1}^{3} D_{ii} E_{ii} . \tag{5.6}$$

It is a simple exercise to observe that, in the isotropic case, (5.5) is well posed and has diagonal matrices $\mathbf{T}$ as solutions. In other words, we can restrict (5.5) to the set of diagonal matrices with determinant equal to 1, set parametrized by $\mathbb{R}^2$ through the mapping

$$\mathbf{T}(t_1, t_2) = \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & 1/t_1 t_2 \end{pmatrix} . \tag{5.7}$$

Then, finally, (5.2) is equivalent to

$$\boxed{\forall e = 1, N_h ,\ J_T(\mathbf{T}(t_1, t_2)) \le J_T(\mathbf{T}(e_1, e_2)),\ \forall \{e_1, e_2\} \in \mathbb{R}^2} , \tag{5.8}$$

with J_T and $\mathbf{T}$ respectively given by (5.6) and (5.7).

In summary, based on (5.8), *the numerical procedure for solving problem* (4.6) *in strains for the isotropic case proceeds element by element, and on each element* Ω^e *does the following :*

. *diagonalise* $Rq(1+\nabla u^n) - \lambda^{n-1}$ *into* $\mathbf{Q}\ \mathbf{D}\ \mathbf{R}$;

. *solve locally the unconstrained two-dimensional minimization problem* (5.8) by a Newton's method (very simple to code and efficient since there are only two unknowns t_1 and t_2) ;

. *set* $\mathbf{F} = \mathbf{Q}\ \mathbf{T}(t_1, t_2)\mathbf{R}$ (which automatically respects symmetry, frame-indifference and incompressibility, since $\mathbf{F}$ has the same singular vectors as the local loading $Rq(1+\nabla u^n)-\lambda^{n-1}$ and since $\mathbf{T}$ was constructed with det $\mathbf{T} = 1$).

6. Solving problem (4.7) for adhesion

Let us approximate the adhesion intensity β and the displacement $\underline{h}$ of the contact surface by piecewise constant finite elements Γ^e defined on Γ_c. If we proceed by relaxation, that is if we iteratively suppose that the solution

$\{\mathbf{h},\beta\}$ is known on each finite element except on the finite element Γ^i, problem (4.7) for adhesion reduces to a sequence of minimization problems of the form

$$\boxed{J_c(\mathbf{h},\beta) \le J_c(\mathbf{e},\gamma),\ \forall\{\mathbf{e},\gamma\} \in K_c \cap \mathbb{R}^4,\ \{\mathbf{h},\beta\} \in K_c \cap \mathbb{R}^4}. \tag{6.1}$$

In (6.1), K_c has been defined in (4.11) and we set

$$J_c(\mathbf{e},\gamma) = \frac{R}{2} q_c |\mathbf{e}|^2 - \mathbf{a}\cdot\mathbf{e} + \frac{1}{2\varepsilon}\gamma^2|\mathbf{e}|^2 + \frac{aa}{2}\gamma^2 - b\gamma, \tag{6.2}$$

$$\mathbf{a} = (Rq_c\mathbf{u}^n - \lambda_c^{n-1}), \tag{6.3}$$

$$aa = d(0)\ \mathrm{meas}(\Gamma^i)\ /\ (t_p - t_{p-1}), \tag{6.4}$$

$$\begin{aligned} b = w - \sum_{e\neq i} d(\mathbf{x}_i - \mathbf{x}_e)\ [\beta_p(\mathbf{x}_e) - \beta_{p-1}(\mathbf{x}_e)]\ \mathrm{meas}(\Gamma^e)\ /\ (t_p - t_{p-1}) \\ + d(0)\ \beta_{p-1}(\mathbf{x}_i)\ \mathrm{meas}(\Gamma^i)\ /\ (t_p - t_{p-1}). \end{aligned} \tag{6.5}$$

As in §5, we can simplify (6.1) and reduce it to a one-dimensional local minimization problem over the closed interval [0,1]. Indeed, if we write the optimality conditions associated to (6.1), we observe that, at the solution, $\mathbf{h}$ is given from β through the relation

$$\mathbf{h} = \varepsilon\mathbf{a}^-\ /\ (\beta^2 + Rq_c\varepsilon), \tag{6.6}$$

$$\begin{aligned} \mathbf{a}^- &= \mathbf{a} \quad \text{if} \quad \mathbf{a}\cdot\mathbf{n} \le 0, \\ &= \mathbf{a} - \mathbf{a}\cdot\mathbf{n}\ \mathbf{n} \quad \text{if} \quad \text{not}. \end{aligned} \tag{6.7}$$

Plugging this value of $\mathbf{h}$ back in (6.1), the local adhesion problem (6.1) becomes

$$\boxed{\begin{array}{c} \textit{Find a local minimum of } \left\{aa\gamma^2 - 2b\gamma - \varepsilon|\mathbf{a}^-|^2\ /\ (\gamma^2 + Rq_c\varepsilon)\right\} \\ \textit{over the closed interval } [0,1]. \end{array}} \tag{6.8}$$

In summary, based on (6.8), *the numerical procedure for solving Problem*

(4.7) *for adhesion proceeds by relaxation on the finite elements* Γ^e *and solves each local problem* (6.1) by

* *computing* $\mathbf{a}^-$ by (6.7),

* *minimizing* (6.8) over [0,1] by a scalar Newton's method, which gives $\beta^n(\mathbf{x}_i)$,

* *computing* $\mathbf{h}$ by (6.6).

7. Relations with the numerical treatment of the Signorini problem proposed by Bermudez-Moreno.

The Signorini problem considers a linearly elastic solid in frictionless contact with a rigid support and takes the abstract from (2.8) within the notations

$$\begin{cases} \mathcal{G}(\mathbf{v}) = \dfrac{1}{8}\displaystyle\int_\Omega a_{ijk\ell}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\left(\frac{\partial v_k}{\partial x_\ell} + \frac{\partial v_\ell}{\partial x_k}\right) d\Omega \\ \qquad - \displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v}\, d\Omega - \int_{\Gamma_2} \mathbf{g}\cdot\mathbf{v}\, d\Gamma, \end{cases} \tag{7.1}$$

$$\mathcal{H} = L^2(\Gamma), \tag{7.2}$$

$$\begin{cases} \mathcal{F}(H) = 0 \quad \text{if} \quad H \le 0 \quad \text{on} \quad \Gamma_c, \\ \qquad\quad = +\infty \quad \text{if not}, \end{cases} \tag{7.3}$$

$$B(\mathbf{v},H) = \mathbf{v}\cdot\mathbf{n}\Big|_{\Gamma_c} - H = B^*\mathbf{v} - H. \tag{7.4}$$

Rewriting this problem as

$$0 \in \partial\mathcal{G}(u) + {}^tB^*\partial\mathcal{F}(B^*u), \tag{7.5}$$

introducing the operator

$$\mathbb{H}^{\omega}_{1/2\omega} = 2\omega\left(\mathrm{Id} - \left(\frac{\mathrm{Id}}{2} + \frac{\partial\mathcal{F}}{2\omega}\right)^{-1}\right), \tag{7.6}$$

which is the Yosida approximation of $\partial\mathcal{F} - \omega\mathrm{Id}$, BERMUDEZ and MORENO [1981] have proposed the following algorithm for solving (7.5) :

<u>for iteration</u> n <u>equal</u> 1-<u>until satisfied, with</u> q^n <u>known in</u> $\mathcal{H}$ <u>solve successively</u>

$$\begin{cases} 0 \in (\partial\mathcal{G}(u^n) + \omega^t B^* B^* u^n + {}^t B^* q^n),\ u^n \in V, \\ q^{n+1/2} = \mathbb{H}^{\omega}_{1/2\omega}\,(B^* u^n + q^n/2\omega), \\ q^{n+1} = \rho q^{n+1/2} + (1-\rho) q^n. \end{cases}$$

If we set $R = \omega$, $\lambda^{n-1} = -q^n - \omega G^{n-1}$, and if we introduce the augmented Lagrangian $\mathcal{L}_R$ defined in (3.2), this algorithm can be rewritten

$$\mathcal{L}_R(u^n, G^{n-1}; \lambda^{n-1}) \le \mathcal{L}_R(v, G^{n-1}; \lambda^{n-1}),\ \forall v \in V,\ u^n \in V,$$

$$\lambda^{n-1/2} = -q^n - \omega B^* u^n = \lambda^{n-1} - \omega(B^* u^n - G^{n-1}),$$

$$\frac{r^{n+1/2}}{2} + \frac{\partial\mathcal{F}r^{n+1/2})}{2\omega} = q^n/(2\omega) + B^* u^n = (\omega B^* u^n - \lambda^{n-1/2}) \ / \ (2\omega),$$

$$q^{n+1/2} = (\omega B^* u^n - \lambda^{n-1/2}) - 2\omega r^{n+1/2},$$

$$q^{n+1} = \rho q^{n+1/2} + (1-\rho) q^n.$$

Within the identification $G^n = r^{n+1/2}$, this algorithm is finally

$$\mathcal{L}_R(u^n, G^{n-1}; \lambda^{n-1}) \le \mathcal{L}_R(v, G^{n-1}; \lambda^{n-1}),\ \forall v \in V,\ u^n \in V\ ; \tag{7.7}$$

$$\lambda^{n-1/2} = \lambda^{n-1} - RB(u^n, G^{n-1}), \tag{7.8}$$

$$\mathcal{L}_R(u^n, G^n; \lambda^{n-1/2}) \leq \mathcal{L}_R(u^n, H; \lambda^{n-1/2}), \ \forall H \in \mathcal{H}, \ G^n \in \mathcal{H}; \tag{7.9}$$

$$\begin{aligned} \lambda^n &= -q^{n+1} - \omega G^n = -\rho q^{n+1/2} + (1-\rho)(\lambda^{n-1} + \omega G^{n-1}) - \omega G^n \\ &= \lambda^{n-1/2} - (2\rho-1) RB(u^n, G^n). \end{aligned} \tag{7.10}$$

Therefore this BERMUDEZ-MORENO algorithm appears, within a small variant, as a particular case of algorithm (4.1)-(4.4).

8. Numerical Results

We first begin by testing the numerical method presented in §4 to §6 on a problem solved in small strains by GUIDOUCHE and POINT [1985]. The considered body is a hyperelastic compressible bar, occupying the domain $[0,1] \times [0,0.20] \times \mathbb{R}$ in its reference configuration, initially glued on the surface $x_2 = 0$ ($\beta_0 = 1$), and subjected to a vertical surface traction of density $g_2 = 10(1-x_1)$kpa (Fig. 1). The other physical data are

$$w = 0.01 \text{ SI},$$

$$d(\mathbf{y}) = d_0 \exp(-d_0 |\mathbf{y}|), \ d_0 = 1\,000.0 \text{ SI},$$

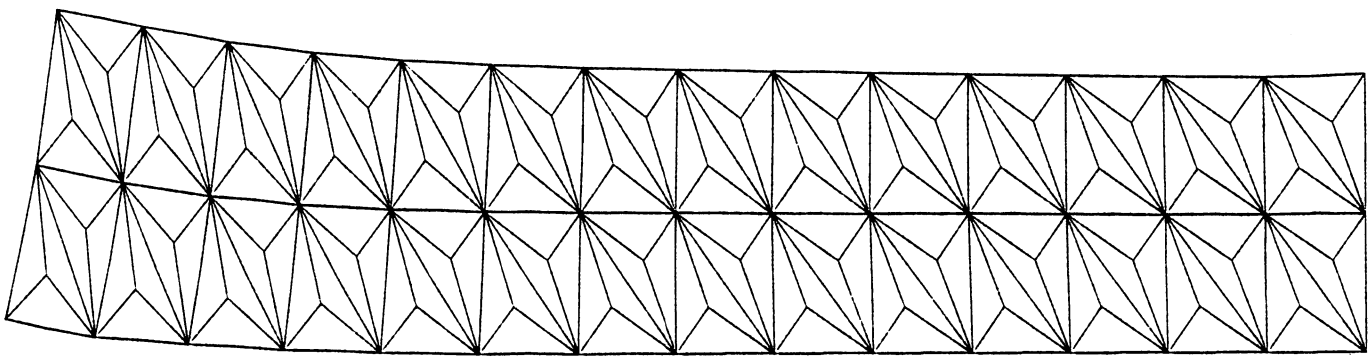
$$\varepsilon = 10^{-9},$$

$$\mathcal{W}(\mathbf{F}) = C_1(|\mathbf{F}|^2 - 3) + C_3(\det \mathbf{F})^2 - C_3 - 2(C_1 + C_3)\ln(\det \mathbf{F}),$$

$$C_1 = C_3 = 187.5 \text{ kpa}.$$

In small strains, the above hyperelastic material corresponds to a classical linearly elastic material with a Young modulus of 1000 kpa and a Poisson coefficient of 0.33. Our computation is performed in plane strains with a variant of algorithm (4.1)-(4.4), taking $R = 8.0$, $q = 2C_1$, $q_c = 1/\varepsilon$, $t_p - t_{p-1} = 5$s, using 201 nodes for constructing the finite element approximation of $\mathbf{u}$, 168 triangles for constructing the piecewise constant approximation of $\mathbf{F}$ and 28 segments for constructing the piecewise constant approximation of β. The variant of algorithm

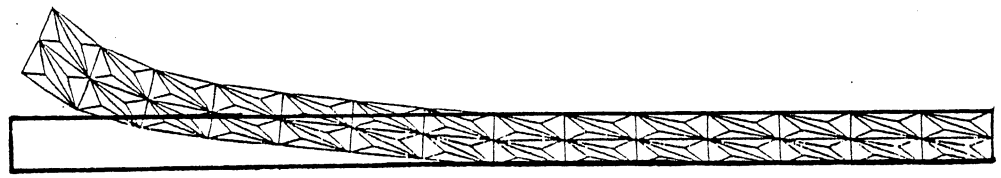
(4.1)-(4.4) consists in using for each n two iterations of block relaxation (4.1)-(4.2) instead of one. The finite elements used for approximating displacements are the asymmetric elements of RUAS [1981] which are well-adapted to incompressible materials. The obtained numerical solution is represented in Fig. 3, is quite similar to the small strains one of GUIDOUCHE and POINT, and corresponds to roughly 1h of CPU time on a VAX 750. The use of the BERMUDEZ--MORENO variant of §7 did not bring any speed-up of the algorithm.



RAPPORT= 0.2000E02

Figure 3

Next, to observe large strains, we consider the same numerical example, but we reduce the thickness of the bar to 0.05m and use a truly incompressible Mooney-Rivlin material with C_1 = 187.5 kpa and C_2 = 0. The adhesion intensity and the vertical displacement of the contact surface are indicated on Fig. 4 for a time t = 800 s.



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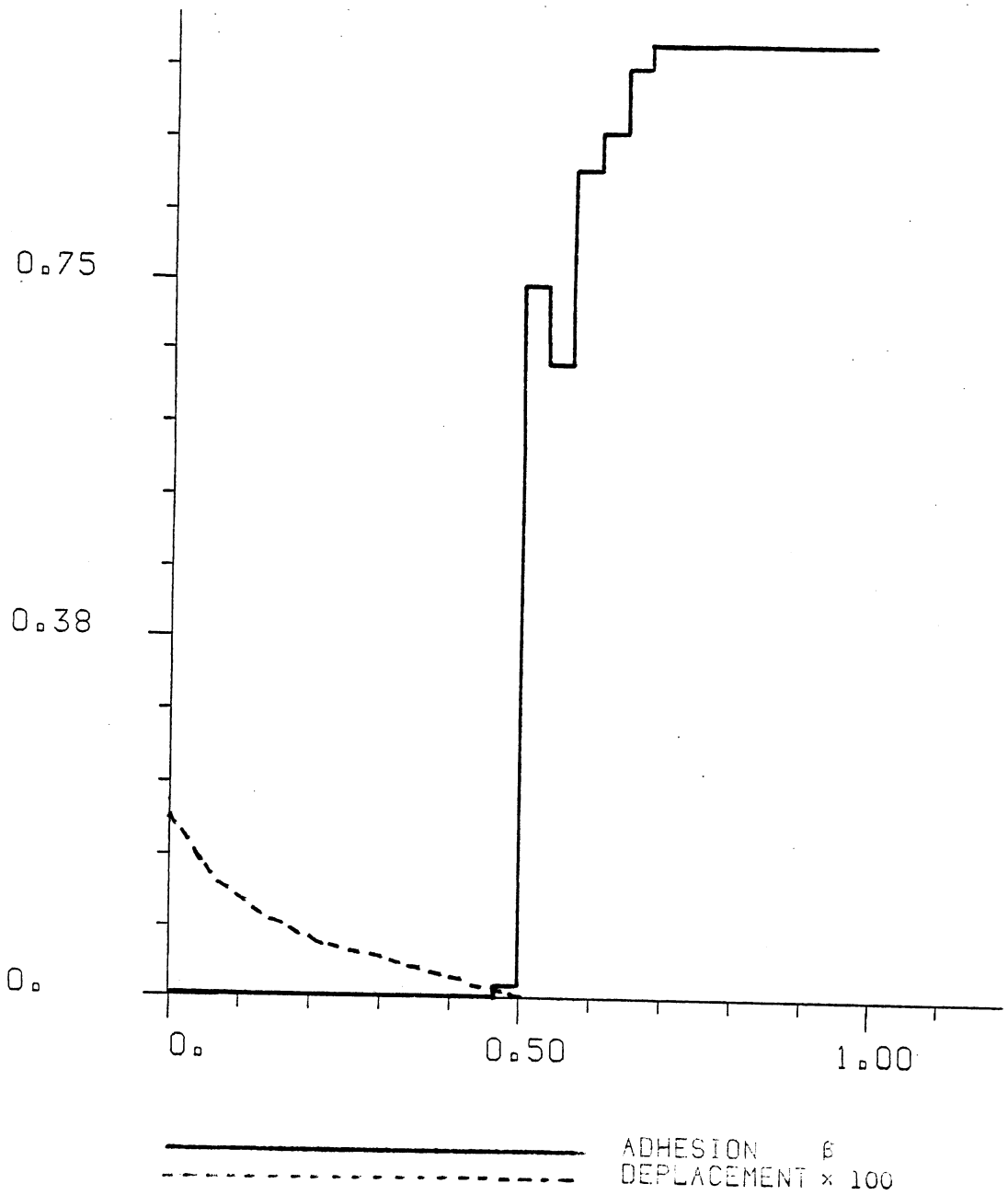
deformed configuration



- Figure 4 -

The last two numerical results deal with cases of pure contact. They correspond to the same mathematical and numerical model, in which we set

$$w = 1/\varepsilon = d(\mathbf{x}) = 0, \; P = 1.$$

The first one, described on Fig. 5, considers a cylinder of radius unity, made of a Mooney-Rivlin material, with C_1 = 435 kPA, C_2 = 65kPa, at rest on a flat plane, and subjected to a vertical body force of 400 kN/m^3. The result was obtained after 40 iterations of algorithm (4.1)-(4.4) with R = 5, $q = q_c$ = 1000. and with 120 finite elements of Ruas type for approximating half the cylinder.

The second one, described in Fig.6, considers a three dimensional gutter, of length 20., of thickness ·2, of internal radius 1·, hanged over the solid plane y = -3. It is made of a Mooney-Rivlin material, with C_1 = 50 kPa, C_2 = 0, and is subjected on its above face to a vertical surface loading of 0.5 kPa. For symmetry reasons, only one fourth of the body was computed, using 270 H8 hexaedra, and 70 iterations of algorithm (4.1)-(4.4) for a total CPU time of 108 mn on a DPS 8/Multics.

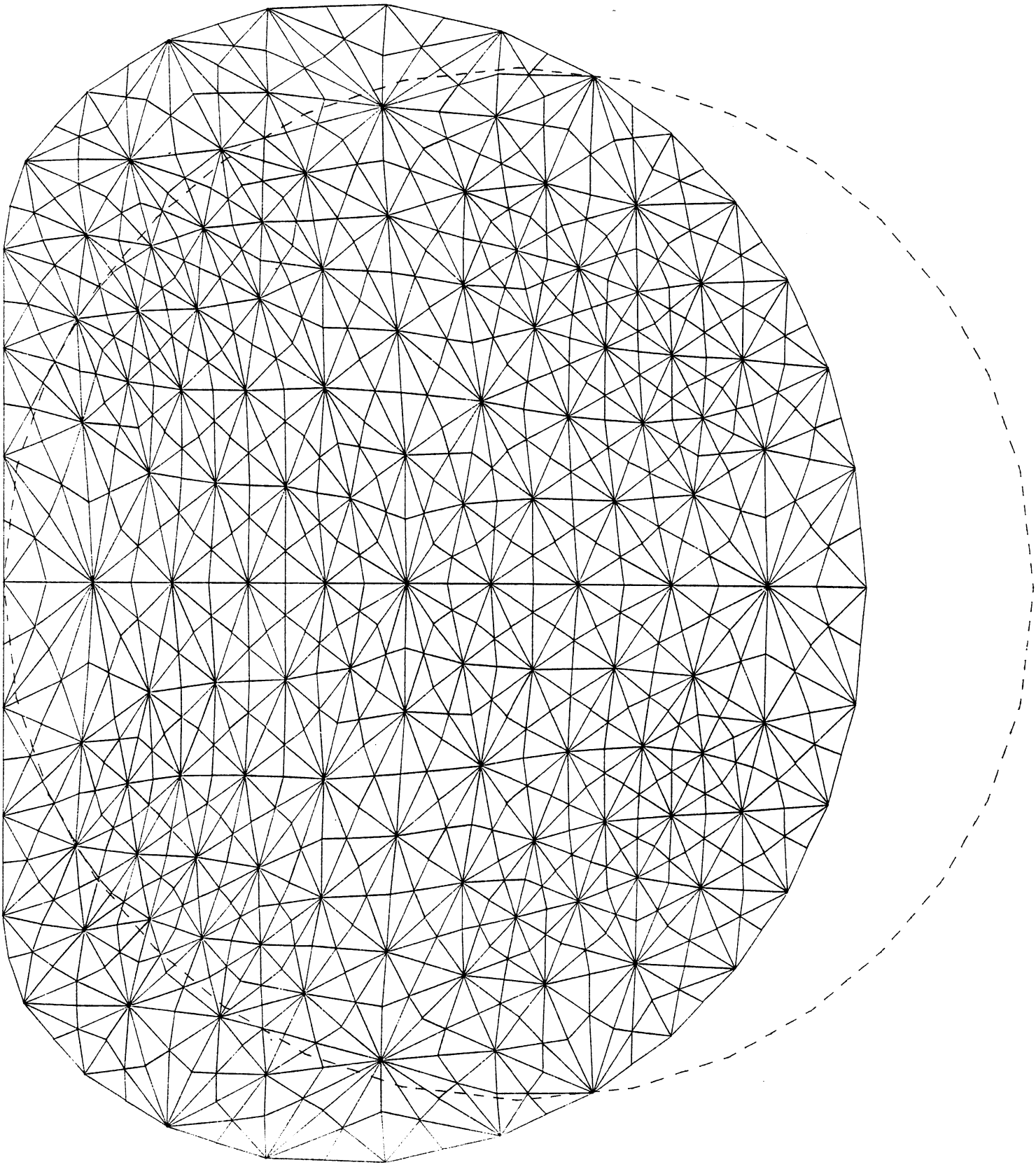


Figure 5

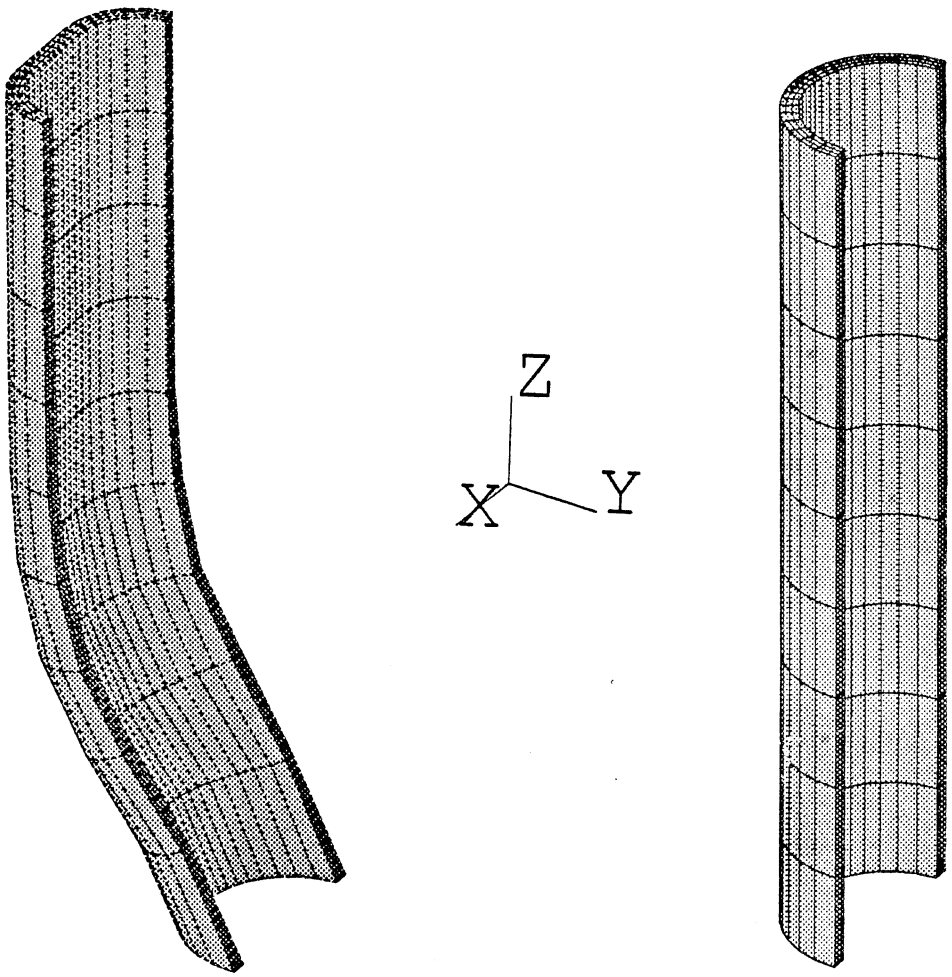


Figure 6

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