Topics in the regularity theory of the Navier–Stokes equations

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Abstract

The dissertation consists of two projects on the regularity of the three-dimensional incompressible Navier–Stokes equations. In the first project, we study Navier–Stokes regularity on the half-space. The existence of minimal blowup-generating initial data, under the assumption that there exists an initial data leading to finite-time singularity, has been studied by Rusin and Šverák (2011), Jia and Šverák (2013), and Gallagher, Koch and Planchon (2013, 2016) in several critical spaces on the whole space. Our aim is to study the influence of the boundary on the existence of minimal blowup data. We introduce a type of weighted critical spaces for the external force that is better-suited for our analysis than the usual Lebesgue spaces. We reestablish regularity theory for the Stokes equations and local-in-time regularity for the Navier–Stokes equations. Our main tools to treat regularity near the boundary are the notion of "split" weak solutions introduced by Seregin and Šverák (2017), the boundary regularity criteria and special decomposition of the pressure near the boundary due to Seregin (2002). Our method works well for both the half-space and the whole space. Our second project is motivated by the work of Li (2014). He introduces a hypothetical relation between the mesh size and the size of the corresponding numerical solution which guarantees the global existence of the exact solution. We formulate this problem for a continuous setting and identify some key difficulties.

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Chapter 1

Introduction

1.1 Statement of the problems

The dissertation consists of two projects: one on Navier–Stokes regularity near the boundary, the other on a type of quantitative criteria for the existence of global strong solutions. ***In the first project,*** we study the initial-boundary value problem of the incompressible Navier–Stokes equations on a domain $\Omega \subset \mathbb{R}^3$

$$(\mathrm{NSE})_\Omega : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= f, \\ \operatorname{div} u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u(\cdot, 0) &= u_0. \end{cases}$$

There are a number of ways to define a solution. Two general tools used to construct solutions are the perturbation theory and energy estimates. Mild (or strong) solutions are those coming from the perturbation of the linear Stokes equations, i.e. the Navier–Stokes equations without the nonlinear term. For quite general domains, mild solutions are known to exist and regular at least over a short period of time. For example, Geissert *et al.* [24] show the local-wellposedness of the Navier–Stokes problem for $f = 0$ and $u_0 \in L^r(\Omega)$, $r > 3$, in any smooth domain Ω that admits Helmholtz decomposition

in L^r. If Ω is the whole space or half-space, the system has scaling symmetry

$$\begin{aligned} u(x,t) &\to \lambda u(\lambda x), \lambda^2 t), \\ p(x,t) &\to \lambda^2 p(\lambda x, \lambda^2 t), \\ f(x,t) &\to \lambda^3 f(\lambda x, \lambda^2 t), \\ u_0(x) &\to \lambda u_0(\lambda x). \end{aligned} \tag{1.1}$$

A function space for a quantity ξ (velocity, pressure, energy,...) is called *critical* if the corresponding norm is invariant under the scaling (1.1). Critical spaces play an important role in the regularity theory. For example, the smallness of data (i.e. initial condition and external force) in many critical spaces implies that the Navier–Stokes problem has a global-in-time mild solution. The existence of global mild solutions given arbitrary data is still not known for any domain, with or without boundaries.

Let X be a critical space of the initial condition u_0, and Y be a critical space of the external force f. Denote by $\rho^{\Omega}_{\max}$ the supremum of all $\rho > 0$ such that $(\mathrm{NSE})_\Omega$ is globally well-posed for every (u_0, f) with $\|(u_0, f)\|_{X\times Y} < \rho$. For $\Omega = \mathbb{R}^3$ and $Y = \{0\}$, whether $\rho^{\Omega}_{\max}$ is finite is essentially the millennium problem of fluid dynamics [18]. Although the global well-posedness is not known, we are interested in the hypothetical situation when $\rho^{\Omega}_{\max}$ is finite. In particular, we are interested in the following question:

(Q1) If $\rho^{\Omega}_{\max}$ is finite, does there exist a data $(u_0, f) \in X\times Y$ with $\|(u_0, f)\| = \rho^{\Omega}_{\max}$, such that the mild solution u of the system $(\mathrm{NSE})_\Omega$ blows up in finite time?

We call such data a *minimal blowup-generating data*, or simply *minimal blowup data*. This question has been addressed in several settings of the initial conditions (with zero force). Affirmative answers are given for $X = \dot{H}^{1/2}(\mathbb{R}^3)$ in [76], for $X = L^3(\mathbb{R}^3)$ in [32] and [21], and for $X = \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$, $3 < p, q < \infty$ in [22]. Our aim is to study the effects of boundaries on this issue.

Physical boundaries are known to complicate the regularity theory. For example, the harmonic pressure may not be smooth near the boundary, thus requiring a careful treatment. Seregin establishes ε-regularity criteria near the boundary using a different decomposition of the pressure [80]. Thus, it is natural to ask if the answer to the above question is still affirmative in the presence of boundaries. We will study the

above problem in the half-space, which is the simplest domain with boundary where the natural scaling still holds. Let us simply write (NSE) and $\rho_{\max}$ when $\Omega = \mathbb{R}^3$, and $(\text{NSE})_+$ and $\rho^+_{\max}$ when $\Omega = \mathbb{R}^3_+$. For simplicity, we will consider the case $u_0 = 0$, so that our data are only represented by f. This situation contains the key difficulties. The generalization to $u_0 \neq 0$ is quite routine. We aim for the following results:

1) If $\rho_{\max} < \infty$ then there exists a minimal blowup data for (NSE).

2) $\rho^+_{\max} \leq \rho_{\max}$.

3) If $\rho^+_{\max} < \rho_{\max}$ then there exists a minimal blowup data for $(\text{NSE})_+$.

4) If $\rho^+_{\max} = \rho_{\max}$ then there does not exist a blowup data for $(\text{NSE})_+$.

Our introduction of the right hand side make the definition of $\rho^\Omega_{\max}$, the threshold of global well-posedness, more stable under the change of equations or domains. The most natural critical space for the force term seems to be $L^{5/3}_{t,x}$. However, this space is not suitable for the persistence of singularities, which is a key step in our analysis. We refer to Section 6.1 for detail. It turns out that it is better to work with certain weighted critical spaces. The spaces we consider in this thesis are of the form

$$Y_q = \{f : \Omega \times (0,\infty) \to \mathbb{R}^3 : \ t^{q_*} f \in L^q(\Omega \times (0,\infty))\}$$

with

$$\|f\|_{Y_q} = \|t^{q_*} f\|_{L^q_{t,x}}, \qquad q_* = \frac{3}{2} - \frac{5}{2q}.$$

The main result in this project is the following.

Theorem 1.1. *For $Y = Y_q$ with $5/2 < q < 3$, the statements 1), 2), and 3) hold true.*

The first statement is consistent with the conclusions in [76, 32, 21, 22] where the force term is assumed to be zero. When $\rho^+_{\max} < \rho_{\max}$, the boundary "helps" the blowup: all singularities (i.e. points (x,t) around which u is unbounded) stay close to the boundary. The main new difficulty (in comparison with previous works on minimal blowup data) is that the presence of the boundary complicates the partial regularity theory which is needed for our proofs. Our main tools in this respect are an adaptation of the notion of "split" weak solutions in [83] to the new setting $f \in Y_q$, and an adaptation

of the results of [80] and [79] where the partial regularity theory near the boundary is developed. The case $\rho^+_{\max} = \rho_{\max}$ happens only when the singularities move away from the boundary. In this situation, the boundary seems to obstruct the existence of minimal blowup data.

In the second project, we consider the initial value problem for the mollified Navier–Stokes equations in the whole space

$$(\mathrm{NSE})_\varepsilon : \begin{cases} \partial_t u - \Delta u + (u * \eta_\varepsilon) \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^3, \end{cases}$$

where $\eta_\varepsilon(x) = \varepsilon^{-3}\eta(x/\varepsilon)$ is the standard mollifier in $\mathbb{R}^3$. Denote by (NSE) the exact Navier–Stokes system. Suppose that the initial condition u_0 belongs to $L^2 \cap L^\infty$. A natural class of solutions in this setting is the so-called strong solutions (or mild solutions), defined in the space $L^\infty_{t,x}$.

In the pioneering work [48], Leray shows that (NSE) is locally well-posed, and $(\mathrm{NSE})_\varepsilon$ is globally well-posed for each $\varepsilon > 0$ (see also [71, Thm. 4.2]). The global well-posedness of (NSE) is still not known. Heuristically, solution u_ε of $(\mathrm{NSE})_\varepsilon$ is more regular than solution u of (NSE) because the mollified nonlinear convection term is dominated by the linear diffusion term. By a careful limit process, Leray constructs a global weak solution to (NSE) as a limit of u_ε. However, the uniqueness of Leray's weak solutions is not known due to possible lack of regularity of this class of solutions (see [18]).

Leray's construction is purely qualitative (using limit, compactness, etc). Without a sufficiently strong *a priori* estimate, many regularity properties are lost in the limit $\varepsilon \to 0$. All *a priori* estimates that have been known so far can be traced back to the energy estimate, which is not strong enough to preserve the boundedness of solutions in the limit process (see [98, Sec. 3.4]). This motivates us to find a reasonable quantitative assumption on u_ε that will imply the existence of a global strong solution. Specifically, we are interested in the question:

> (Q2) For $M > 0$, how large ε can we take so that the following is true: "If u_ε is bounded in $\mathbb{R}^3 \times (0, \infty)$ by M then (NSE) has a global strong solution u which is bounded in $\mathbb{R}^3 \times (0, \infty)$ by $2M$"?

This question is addressed in part by Li [50] from a numerical perspective. Considering a discretized Navier–Stokes system on a polyhedron, he introduces a hypothetical relation between the mesh size and the size of the corresponding numerical solution which guarantees the global existence of the exact solution. He essentially suggests that $\varepsilon \lesssim \exp(-M^{225})$. Although question (Q2) can be formulated for domains with boundaries, system $(\mathrm{NSE})_\varepsilon$ seems to be a more natural model to study because of the scaling symmetry and the absence of boundaries. The "resolution" ε of approximation is analogous to the mesh size. This setting already contains some key difficulties. Like the exact Navier–Stokes system, $(\mathrm{NSE})_\varepsilon$ has scaling symmetry:

$$\begin{aligned} u(x,t) &\to u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \\ p(x,t) &\to p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \\ \varepsilon &\to \varepsilon_\lambda = \lambda^{-1}\varepsilon. \end{aligned}$$

We are interested in obtaining a bound for ε that has the same scaling as ε. In the terminology of Caffarelli–Kohn–Nirenberg [12], each quantity in $(\mathrm{NSE})_\varepsilon$ can be assigned a dimension. The bound $\|u_\varepsilon\|_{L^\infty} \le M$ provides a natural length scale for the problem, which is M^{-1}. The number ε also has dimension length, so the ideal bound for ε would be $\varepsilon \lesssim M^{-1}$. Our goal is to investigate the following conjecture.

Conjecture 1.2. *Let $M > 0$ and $u_0 \in L^2 \cap L^\infty$. There exists a constant $C > 0$ independent of M, possibly dependent on $\|u_0\|_{L^3}$ or other scaling-invariant quantities involving u_0, such that the following is true: Suppose for some $0 < \varepsilon \le CM^{-1}$, the solution u_ε of $(\mathrm{NSE})_\varepsilon$ is bounded by M. Then (NSE) has a global strong solution bounded by $2M$.*

We obtain a partial result that $\varepsilon \le F(M)$ where $F(M)$ has the same scaling as ε and decays as $\exp(-M^3)$ as $M \to \infty$, improving the results of Li [50]. We believe that proving a power decay $F(M) \sim M^{-\alpha}$, for some $\alpha > 0$, would already be significant. One strategy is to choose a different approximate Navier–Stokes system. The key difficulty, however, is that "naive" estimates at global scale, as in our analysis, seems not strong enough to achieve a power decay. A treatment at local scale is perhaps needed to access the strength of the local regularity theory.

1.2 Chapter summaries

The first project is addressed in Chapter 2, 3, 4, 5, 6. The second project is addressed in Chapter 7.

In Chapter 2, we review two ingredients needed for the formulation of solutions to the Stokes system: the Helmholtz decomposition and Stokes semigroup. We summarize (1) several well-known results on Helmholtz decomposition, which is an important tool to formulate strong solutions to the Stokes system and Navier–Stokes system, and (2) *a priori* estimates of the Stokes semigroup $e^{t\mathbb{A}}$. These results have been known for smooth domains with compact boundaries, the half-space and whole space. We present here elementary proofs for the case of half-space and whole space (Proposition 2.5) based on pointwise estimates of Green's matrix due to Solonnikov [91], [92].

In Chapter 3, we study the Stokes system with force term in the critical space Y_q. We obtain *a priori* estimates of the solution in various critical spaces (Proposition 3.1), Solonnikov's coercive estimates (Proposition 3.5), local energy estimates (Proposition 3.7), uniqueness of weak solutions (Proposition 3.8) and compactness theorem (Proposition 3.12). These results are consistent with known results in the setting $f \in L^r_t L^p_x$.

In Chapter 4, we study mild solutions to the Navier–Stokes system with force term in Y_q. We obtain short-time regularity and characterization of blowup (Proposition 4.2) which are consistent with known results in the setting $f \in L^r_t L^p_x$. By a splitting argument due to Seregin and Šverák [83], we show that mild solutions belong to the local energy class (Proposition 4.6). As a consequence, they belong to a class of weak solutions called *sw-solutions* at blowup times.

In Chapter 5, we study the class of sw-solutions. Here s stands for *strong/Stokes/split*, and w for *weak*. This type of weak solutions is introduced by Seregin and Šverák [83] in the setting $u_0 \in L^3(\mathbb{R}^3)$ and $f = 0$ to simplify the proofs of compactness theorem, weak-strong uniqueness, persistence of singularities, etc. and to better adapt with unbounded domains other than the whole space. We present here an adaptation of their treatments to the setting $u_0 = 0$ and $f \in Y_q$ in the whole space and half-space. We obtain similar results: the the compactness theorem (Proposition 5.7), weak-strong uniqueness (Propo-

sition 5.9), ε- regularity criteria (Proposition 5.11 and Proposition 5.14), and persistence of singularities under weak convergence of the data (Proposition 5.21). The key idea to show the persistence of singularities is to apply Seregin's pressure decomposition near the boundary. We notice that sw-solutions are well-suited for boundary regularity and help simplify our regularity-bootstrapping procedure (Proposition 5.18).

In Chapter 6, we treat the problem posed in the introduction. We explain why the critical space $L^{5/3}_{t,x}$ of the force term is not suitable for the persistence of singularities, and why $(\frac{5}{2}, 3)$ is a natural range for q. The main theorems are Proposition 6.3 and Proposition 6.5. Theorem 1.1 is proved by Corollary 6.4 and Proposition 6.5.

In Chapter 7, we give a partial result on Conjecture 1.2 and identify some difficulties. The main theorem is Proposition 7.5.

Appendix A is a collection of several canonical tools in PDE with selective insights apt for this dissertation. These include Young–O'Neil inequality, fractional inequality, anisotropic Sobolev embeddings, Calderón–Zygmund operators, and Fourier multiplier theorems.

1.3 General setup and notations

Let Ω be a domain in $\mathbb{R}^3$, and consider the initial-boundary value problem of the Navier–Stokes equations

$$(\mathrm{NSE})_\Omega : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= f, \\ \operatorname{div} u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0. \end{cases}$$

There are a number of ways to define a solution. Most of the available methods come from either the perturbation theory or the energy methods. The perturbation of the linear Stokes equations gives rise to a class of solutions called *mild solutions*, formally

obtained as follows. First, consider the Stokes system

$$(\mathrm{SE})_\Omega : \begin{cases} \partial_t u - \Delta u + \nabla p &= f, \\ \operatorname{div} u &= 0. \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0. \end{cases}$$

Applying Helmholtz decomposition (Section 2.1) and denoting the Stokes operator $\mathbb{A} = \mathbb{P}\Delta$, we get an evolution system

$$\begin{cases} \partial_t u - \mathbb{A}u &= \mathbb{P}f, \\ u(0) &= u_0, \end{cases}$$

By Duhamel's Principle,

$$u(t) = e^{t\mathbb{A}}u_0 + \int_0^t e^{(t-s)\mathbb{A}}\mathbb{P}f(s)ds \quad \forall\, t > 0. \tag{1.2}$$

We now replace f in the Stokes system by $(f - u\cdot\nabla u)$ to obtain an implicit representation for the solution of the Navier–Stokes system

$$u(t) = e^{t\mathbb{A}}u_0 + \int_0^t e^{(t-s)\mathbb{A}}\mathbb{P}f(s)ds - \int_0^t e^{(t-s)\mathbb{A}}\mathbb{P}\operatorname{div}(u\otimes u)(s)ds. \tag{1.3}$$

The pressure is contributed by the diffusion term Δu, the convection term $u\cdot\nabla u$, and the external force f through Helmholtz decomposition: $p = p_d - p_c + p_e$ where

$$\begin{cases} \Delta u &= \mathbb{P}\Delta u + \nabla p_d, \\ u\cdot\nabla u &= \mathbb{P}(u\cdot\nabla u) + \nabla p_c, \\ f &= \mathbb{P}f + \nabla p_e. \end{cases} \tag{1.4}$$

Note that the pressure is only defined up to a function of t. For estimation purposes, it is often necessary to replace p by $p - [p]_D$, where $[p]_D$ denotes the average of p over some spatial domain D. The component p_d is also called *harmonic pressure* or *boundary pressure* because it is a harmonic function satisfying a Neumann boundary condition

$$\begin{cases} \Delta p_d = 0 & \text{in} \quad \Omega, \\ \dfrac{\partial p_d}{\partial n} = (\Delta u)\cdot n & \text{on} \quad \partial\Omega. \end{cases}$$

When it comes to boundary regularity, this component is most difficult to deal with because it involves second-order derivatives of u. To avoid this issue, a different decomposition of the pressure is needed. It is introduced by Seregin (see Section 5.4).

Denote

$$U(t) = e^{t\mathbb{A}}u_0, \quad F(t) = \int_0^t e^{(t-s)\mathbb{A}}\mathbb{P}f(s)ds, \quad B(u,v) = -\int_0^t e^{(t-s)\mathbb{A}}\mathbb{P}\,\mathrm{div}(u\otimes v)(s)ds.$$

Then $u = U + F + B(u,u)$. This is the problem of finding a fixed point. The following abstract lemma is used to obtain a short-time solution (or *local solution*).

Lemma 1.3. *Let $\mathscr{X}$ be a Banach space and $B : \mathscr{X}\times\mathscr{X} \to \mathscr{X}$ be a continuous bilinear form with $\|B(x,y)\| \le \gamma\|x\|\|y\|$. For $a \in \mathscr{X}$, consider the equation*

$$x = a + B(x,x). \tag{1.5}$$

If $\|a\| < 1/(4\gamma)$ and $0 < r_1 < r_2$ are two roots of the equation $r = \|a\| + \gamma r^2$, then (1.5) has a solution $\bar{x}$ satisfying $\|\bar{x}\| \le r_1$. Moreover, the solution is unique in the ball $\{x \in \mathscr{X} : \|x\| < r_2\}$.

The proof is a direct application of the Picard's contraction mapping principle (see e.g. [95, p. 215]). The solution is obtained as the limit of the sequence

$$\begin{cases} x_0 = a \\ x_{n+1} = a + B(x_n, x_n) \quad \forall\, n \ge 0. \end{cases}$$

We define *mild solutions* as solutions obtained through the iterative process

$$\begin{cases} u^{(0)} = U + F \\ u^{(n+1)} = U + F + B(u^{(n)}, u^{(n)}) \quad \forall\, n \ge 0. \end{cases}$$

Condition $\|U + F\|_{\mathscr{X}} < 1/(4\gamma)$ often amounts to a short-time condition or a smallness condition on the data (u_0, f) in a critical space $X \times Y$. The latter situation results in a global-in-time solution (or *global solution*). The scaling symmetry classifies function spaces into three types: *critical*, *subcritical* and *supercritical*. Caffarelli–Kohn–Nirenberg's notion of dimension [12] provides a simple way to describe these types as follows. To each quantity in the Navier–Stokes equations we assign a number called dimension based on how it is scaled in (1.1):

$$[x] = 1,\ [t] = 2,\ [u] = -1,\ [p] = -2,\ [f] = -3,\ [u_0] = -1.$$

A space X is called *critical/ subcritical/ supercritical* for a quantity ξ if the corresponding norm $\|\xi\|_X$ is of zero/ negative/ positive dimension, respectively. A useful fact is as

follows. Suppose ξ has dimension d and belongs to a mixed-norm Lebesgue space $L^r_t L^q_x$ $(1 \le r, q \le \infty)$. Then $L^r_t L^q_x$ is a critical (respectively subcritical, supercritical) space for ξ if

$$\frac{2}{r} + \frac{3}{q} = -d \qquad (\text{respectively} < -d, \ \ > -d)$$

By computing dimension, one can also check that $L^{5/3}_{t,x}$ and Y_q previously defined are indeed critical spaces for the force term.

The so-called sw-solutions defined in Chapter 5 comes from a mixed point of view of perturbation theory and the energy method. In particular, an sw-solution u is defined as the sum $v+w$ where v solves $(\mathrm{SE})_\Omega$ with the same force term. In this manner, $(\mathrm{NSE})_\Omega$ is decoupled into two systems: one is globally well-posed in critical spaces, the other yields weak solutions in the energy class, which is a supercritical space. Most known *a priori* estimates for solutions of $(\mathrm{NSE})_\Omega$ come from the energy (or local energy) estimates. In 2-dimensional Navier–Stokes problem, energy inequality is sufficient to induce regularity of weak solutions at all times (see e.g. [49, 43, 53]). Unfortunately, this is not the case in 3-dimension.

Notations. Throughout the thesis, we denote by A any positive absolute constant whose value we are not interested to specify. Similarly, $C_{q,r}$ denotes any positive constant only depending on q and r. In so doing, we adopt such operations as $A + A = A$ and $2C_{q,r} = C_{q,r}$. We will use the notation $L^p(\Omega)$ for both $L^p(\Omega, \mathbb{R})$ and $L^p(\Omega, \mathbb{R}^n)$, and use the term "function" and "vector field" interchangeably in case there is no ambiguity.

The fundamental solution to the Laplace equation is denoted by $\Phi(x)$. The fundamental solution to the heat equation is denoted by $\Gamma(x,t)$. We reserve symbol Ψ_0 for the function introduced in Proposition 4.6. We will use Einstein summation convention. That is to write $a_i b_i$ instead of $\Sigma_{i=1}^n a_i b_i$.

A function $u = u(x,t)$ is often regarded as a Banach space-valued function of time $u = u(t)$. The symbols ∇, div, Δ, $\widehat{\ }$ denote the gradient, divergence, Laplacian, Fourier transform with respect to spatial variables. We denote by $\nabla^m u$ the m-tensor of all partial derivatives (with respect to spatial variables) of order m. Other notations are listed in the following chart.

Notation	**Definition**
$\log_+ \alpha$	$\max\{\log \alpha,\, 0\}$
r'	Hölder conjugate of r, i.e. $1/r + 1/r' = 1$
q_*	$\frac{3}{2} - \frac{5}{2q}$
$a \otimes b,\ \operatorname{div} F$	Matrix $(a_i b_j)$, vector $(F_{ij,j})$
$\partial^k u$	$\partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \partial_{x_3}^{k_3} u$ where $k = (k_1, k_2, k_3)$
$F : \nabla u$	$F_{ij} u_{i,j}$
$x = (x', x_n)$	$x' = (x_1, x_2, \ldots, x_{n-1})$
x^*	$(x', -x_n)$, reflection point of x with respect to the plane $\{x_n = 0\}$
$\mathbb{P}$	Helmholtz projection operator
$\mathbb{A}$	$\mathbb{P}\Delta$ (the Stokes operator)
$X^*,\ f^*$	Dual space of X and dual map of f respectively
$[f]_D$	Average of f over domain D, i.e. $\frac{1}{\mathrm{vol}(D)} \int_D f dx$
$I_S,\ \ I_{x_n > 0}$	Indicator (or characteristic) functions of sets S and $\{x :\ x_n > 0\}$
$\tilde{e},\ \ \check{e}$	Extension operators by even and odd reflection respectively
$L_t^m L_x^n$	Mixed-norm Lebesgue space. Also written as $L_{t,x}^m$ if $m = n$.
$C_0^\infty(\Omega)$	Space of smooth functions compactly supported in Ω
$C_{0,\sigma}^\infty(\Omega)$	Subspace of $C_0^\infty(\Omega)$ consisting of divergence-free functions
$W^{m,p}(\Omega)$	Sobolev space with norm $\|u\|_{W^{m,p}(\Omega)} = \sum_{\lvert\alpha\rvert \le m} \|\partial^\alpha u\|_{L^p(\Omega)}$
$W_0^{m,p}(\Omega)$	Completion of $C_0^\infty(\Omega)$ in $\|\cdot\|_{W^{m,p}}$-norm, or equivalently the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$
$W^{-m,p'}(\Omega)$	Dual space of $W_0^{m,p}(\Omega)$
$H^m(\Omega)$	$W^{m,2}(\Omega)$ where $m \in \mathbb{Z}$
$H_0^m(\Omega)$	$W_0^{m,2}(\Omega)$
$\langle f, g \rangle$	Duality between f and g, for example when $f \in W^{-1,3}$ and $g \in W_0^{1,3/2}$. It is equal to $\int_\Omega fg dx$ if the integral is well-defined.
$L_\sigma^r(\Omega)$	Subspace of $L^r(\Omega)$ consisting of divergence-free fields (in weak sense)
$L_\pi^r(\Omega)$	Subspace of $L^r(\Omega)$ consisting of (generalized) gradient fields

Table 1.1: Chart of notations.

Chapter 2

Background

2.1 Helmholtz decomposition

It is well-known that any smooth vector field in $\mathbb{R}^n$, $n \geq 2$, with fast decay at infinity can be expressed as the sum of a divergence-free field and a potential field, i.e. the gradient field of a scalar function. This is known as Helmholtz decomposition. A formulation of this decomposition for general domains and non-smooth vector fields is as follows.

Let Ω be a domain in $\mathbb{R}^n$. Denote by $C^\infty_{0,\sigma}(\Omega)$ the space of all smooth divergence-free vector fields $\phi : \Omega \to \mathbb{R}^n$ compactly supported in Ω. For $1 < r < \infty$, let $L^r_\sigma(\Omega)$ be the closure of $C^\infty_{0,\sigma}(\Omega)$ in $L^r(\Omega)$ and

$$L^r_\pi(\Omega) = \left\{ \nabla p \in L^r(\Omega) : \ p \in L^r_{\mathrm{loc}}(\bar{\Omega}) \right\}.$$

Whenever the topological direct sum $L^r(\Omega) = L^r_\sigma(\Omega) \oplus L^r_\pi(\Omega)$ holds, we say that Helmholtz decomposition holds in $L^r(\Omega)$. The operator $\mathbb{P}_r$ projecting $L^r(\Omega)$ onto $L^r_\sigma(\Omega)$ is called *Helmholtz projection*. Put $\mathbb{A}_r = \mathbb{P}_r\Delta$. We see that $\mathbb{A}_r$ and $\mathbb{P}_r$ commute with each other. Indeed, let $f = v + \nabla\phi$ be Helmholtz decomposition in L^r of a function $f \in C^\infty_0(\Omega)$. Then

$$\begin{aligned} \mathbb{P}_r\mathbb{A}_r f &= \mathbb{P}_r\mathbb{P}_r(\Delta v + \Delta\nabla\phi) = \mathbb{P}_r\Delta v + \mathbb{P}_r\nabla\Delta\phi = \mathbb{P}_r\Delta v, \\ \mathbb{A}_r\mathbb{P}_r f &= \mathbb{A}_r v = \mathbb{P}_r\Delta v. \end{aligned}$$

From a PDE perspective, decomposing a function $f \in L^r(\Omega)$ as

$$f = v + \nabla\phi, \quad v \in L^r_\sigma(\Omega), \quad \nabla\phi \in L^r_\pi(\Omega) \tag{2.1}$$

is equivalent to solving a Neumann problem

$$\text{(NP)}: \begin{cases} \Delta\phi = \operatorname{div} f & \text{in} \quad \Omega, \\ \dfrac{\partial\phi}{\partial n} = f \cdot n & \text{on} \quad \partial\Omega \end{cases}$$

(see e.g. [20, Lem. III.1.2]). The L^r-theory of Neumann problem (NP) can be found in [54, 55, 101, 37, 87], [24, Sec. 7]. For $r = 2$, (NP) has a unique weak solution in $L^2_\pi(\Omega)$ for any $f \in L^2(\Omega)$ and for domain Ω (see e.g. [88, Ch. II, §2.5] and [20, Thm. III.1.1]). In fact, $L^2_\sigma(\Omega)$ and $L^2_\pi(\Omega)$ are orthogonal complements of each other in the Hilbert space $L^2(\Omega)$. For $r \neq 2$, Helmholtz decomposition may not hold in $L^r(\Omega)$. Bogovskiĭ and Maslennikova [61] give a counterexample: they construct a function $f \in L^{r_\alpha}(\Omega_\alpha)$ where Ω_α is a 2-dimensional obtuse sector

$$\Omega_\alpha = \{(\rho\cos\theta, \rho\sin\theta) : \ \rho > 0, \ 0 < \theta < \alpha\},$$

where

$$\pi < \alpha \leq 2\pi \quad \text{and} \quad r_\alpha = \frac{2}{1 \pm \frac{\pi}{\alpha}}$$

such that (NP) has no weak solutions in $L^{r_\alpha}_\pi(\Omega_\alpha)$. Nevertheless, Helmholtz decomposition holds in $L^r(\Omega)$ for any $1 < r < \infty$ if

- $\Omega = \mathbb{R}^n$, $n \geq 2$: Proposition 2.2,
- $\Omega = \mathbb{R}^n_+$, $n \geq 2$: [63, Lem. A.1], Proposition 2.3,
- Ω is a smooth bounded or exterior domain: [87], [19].

We refer to [30, Sec. 2.2] for a more adequate survey on the domains for which Helmholtz decomposition holds.

Proposition 2.1. *[19], [20, Thm. III.1.2] Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be either a bounded or exterior domain of class C^2, or the whole space, or the half-space.*

(i) $L^r = L^r_\sigma \oplus L^r_\pi$,

(ii) $(L^r_\sigma)^* = L^{r'}_\sigma$,

(iii) $\mathbb{P}^*_r = \mathbb{P}_{r'}$.

(iv) $\mathbb{A}_r^* = \mathbb{A}_{r'}$.

On such domains, each L^r-function admits a unique Helmholtz decomposition determined by (NP). Thus, $\mathbb{P}_r$ coincides $\mathbb{P}_q$ on $L^r \cap L^q$. For this reason, it is safe to drop the subscript from $\mathbb{P}_r$. Since $\mathbb{A}_r = \mathbb{P}_r\Delta$, we can also drop the subscript from $\mathbb{A}_r$. Helmholtz decomposition on certain domains can be computed explicitly. In the following, we give explicit formula of the Helmholtz projection operators $\mathbb{P}$ on the whole space and half-space, emphasizing the fact that they can be written as convolutions. Recall that the fundamental solution of the Laplace equation in $\mathbb{R}^d$ is

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if} \quad n = 2, \\ -\frac{C_n}{|x|^{n-2}} & \text{if} \quad n \geq 3. \end{cases}$$

Proposition 2.2. *The Helmholtz decomposition on $\mathbb{R}^n$, $n \geq 3$, is given by*

$$\mathbb{P}f = f - \nabla^2\Phi * f = (\delta\mathbb{I}_d - \nabla^2\Phi) * f,$$

or in terms of Fourier multiplier

$$\widehat{\mathbb{P}f}(\xi) = \left(\mathbb{I}_d - \frac{\xi \otimes \xi}{|\xi|^2}\right)\hat{f}(\xi).$$

Proof. For $f \in C_0^\infty(\mathbb{R}^n)$, write $f = v + \nabla\phi$ where div $v = 0$. Taking the divergence of both sides, we get $\Delta\phi = \text{div } f$. Thus, $\phi = \Phi * \text{div } f = \partial_j\Phi * f_j$. Then $\partial_k\phi = \partial_k\partial_j\Phi * f_j$. In other words, $\nabla\phi = \nabla^2\Phi * f$. The convolution kernel satisfies that size condition, smoothness condition and cancellation condition (A.6)-(A.8), thus defining a bounded linear map from $L^p(\mathbb{R}^n)$ to itself for every $p \in (1, \infty)$. The Fourier transform of $\nabla^2\Phi$ is obtained from the Fourier transform of Φ, which is $\hat{\Phi}(\xi) = C_n|\xi|^{-2}$. □

The Helmholtz decomposition on the half-space involves extensions of functions to the whole space. We introduce the following notations.

- Restriction: $r(f) = f|_{\mathbb{R}^n_+}$.
- Even reflection:

$$\tilde{e}(f) = \begin{cases} f(x) & \text{if} \quad x_n \geq 0, \\ f(x', -x_n) & \text{if} \quad x_n < 0. \end{cases}$$

- Odd reflection:

$$\check{e}(f) = \begin{cases} f(x) & \text{if} \quad x_n \geq 0, \\ -f(x', -x_n) & \text{if} \quad x_n < 0. \end{cases}$$

- Zero extension:

$$e(f) = \begin{cases} f(x) & \text{if} \quad x_n \geq 0, \\ 0 & \text{if} \quad x_n < 0. \end{cases}$$

Proposition 2.3. *The Helmholtz decomposition on $\mathbb{R}^n_+$, $n \geq 3$, is given by*

$$\mathbb{P}f = f - (\nabla^2\Phi * \bar{f})|_{\mathbb{R}^n_+} = r((\delta\mathbb{I}_d - \nabla^2\Phi) * \bar{f}),$$

where $\bar{f}$ is the following extension of f to $\mathbb{R}^n$.

$$\bar{f}(x) = \begin{cases} f(x) & \textit{if} \quad x_n \geq 0, \\ (\tilde{e}(f_1)(x), ..., \ \tilde{e}(f_{n-1})(x), \ \check{e}(f_n)(x)) & \textit{if} \quad x_n < 0. \end{cases}$$

Proof. For $f \in C_0^\infty(\mathbb{R}^n_+)$, write $f = v + \nabla\phi$ where div $v = 0$ and $v \cdot n = 0$ on $\partial\mathbb{R}^n_+$. Taking the divergence of both sides, we get

$$\begin{cases} \Delta\phi = \operatorname{div} f & \text{in} \quad \mathbb{R}^n_+, \\ \dfrac{\partial\phi}{\partial x_n}(x', 0) = 0. \end{cases}$$

By the definition of $\bar{f}$, the extension by even reflection of div f is exactly div $\bar{f}$. Then $\psi = \tilde{e}(\phi)$ satisfies a Poisson equation on the whole space $\Delta\psi = \operatorname{div} \bar{f}$. We get $\nabla\psi = \nabla^2\Phi * \bar{f}$, the restriction of which on $\mathbb{R}^n_+$ gives $\nabla\phi = (\nabla^2\Phi * \bar{f})|_{\mathbb{R}^n_+}$. This formula also holds for any function $f \in L^p(\mathbb{R}^n_+)$, $1 < p < \infty$, because the kernel is a Calderón–Zygmund operator. □

2.2 Estimates for Stokes system when $f = 0$

The function $U(t) = e^{t\mathbb{A}}u_0$ solves the Stokes problem $(\mathrm{SE})_\Omega$ if $u_0 \in L^r_\sigma(\Omega)$ for some $1 < r < \infty$. If u_0 only belongs to $L^r(\Omega)$, U solves $(\mathrm{SE})_\Omega$ only in a certain sense. For example, when $\Omega = \mathbb{R}^3_+$ the first three equations of $(\mathrm{SE})_\Omega$ are satisfied for all $t > 0$ (where $f = 0$), but $U(t)$ fails to converge to u_0 in $L^r(\Omega)$ as $t \to 0^+$ (see [100, Remark 1.4, 1.5]). Nevertheless, one can still study $e^{t\mathbb{A}}$ as a map between Lebesgue spaces $L^r(\Omega)$ and $L^p(\Omega)$. We are interested in estimates of the form

$$\left\| \partial_t^l \partial_x^k e^{t\mathbb{A}} u_0 \right\|_{L^p(\Omega)} \leq C_{l,k,r,p} t^{-l - \frac{|k|}{2} - \frac{3}{2}\left(\frac{1}{r} - \frac{1}{p}\right)} \|u_0\|_{L^r(\Omega)} \tag{2.2}$$

for $l, |k| = 0, 1, 2, \ldots$ and $1 \le r \le p \le \infty$. For $\Omega = \mathbb{R}^3$, $\mathbb{A} = \Delta$ and $e^{t\mathbb{A}}$ is the heat operator. Estimate (2.2) is well-known.

For $\Omega = \mathbb{R}^3_+$, most published works only address the case $l = 0$, $|k| \le 1$, which is sufficient for many practical purposes. The case $1 < r \le p < \infty$ is studied by Ukai [100]. He derives a formula for $e^{t\mathbb{A}}$ involving the heat operator for the half-space and several pseudo-differential operators of order zero. (Although Ukai only proves for $r < p$, his proof is still valid for $r = p$. See also [10, Prop. 4.1].) McCracken [63] shows that $\{e^{t\mathbb{A}}\}_{t>0}$ is a bounded analytic semigroup on $L^p(\mathbb{R}^3_+)$ by studying the resolvent problem of the Stokes equations. Then estimate (2.2) for $|k| = 0$, $1 < r = p < \infty$ follows as a consequence. The same result for smooth bounded domains is then proved by Giga [26]. (Under what conditions of the domain is the Stokes semigroup bounded analytic is a question of its own interest. See e.g. [1] and [23] for a survey on this topic.) For other borderline cases, (2.2) is proved in [10] for $|k| = 0$, $r = 1$, $p = \infty$, in [25] for $|k| = 1$, $r = p = 1$, and in [86] for $|k| = 1$, $r = p = \infty$. But the estimate does not hold for the case $|k| = 0$, $r = p = 1$ as pointed out in [14, Sec. 5]. Recently, Maekawa, Miura and Prange [58, Thm. 3] show a variation of (2.2) when the L^p- and L^r-norms are replaced by the L^p_{uloc}- and L^r_{uloc}-norms.

The conditions on (r, p) for which (2.2) holds may depend on the domain Ω. For instance, if Ω is an exterior domain, (2.2) is not true for $|k| = 1$, $p > 3$, $r \ge 3/2$ (see e.g. [60, Thm. 1.2]). Geissert *et al.* show that (2.2) holds in appropriate range of r and p depending on the domain for a class of smooth (not necessarily bounded) domains [24, Prop. 3.1]. We refer to [30, Sec. 5] for a recent survey in this respect. If $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$, one can associate $e^{t\mathbb{A}}$ with a Green's matrix

$$e^{t\mathbb{A}} u_0(x) = \int_\Omega G(x, y, t) u_0(y) dy. \tag{2.3}$$

For $\Omega = \mathbb{R}^n$, $n \ge 2$,

$$G_{ij}(x, y, t) = \delta_{ij} \Gamma(x - y, t). \tag{2.4}$$

For $\Omega = \mathbb{R}^n_+$, $n \ge 2$, (see [91, p. 165-169], [90, Sec. 2], [92, Sec. 2.1])

$$\begin{aligned} G_{ij}(x, y, t) &= \delta_{ij}(\Gamma(x - y, t) - \Gamma(x - y^*, t)) \\ &+ 4(1 - \delta_{jn}) \int_0^{x_d} \int_{\mathbb{R}^{n-1}} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x - z) \Gamma(z - y^*, t) dz. \end{aligned} \tag{2.5}$$

Here

$$\Phi(x) = \begin{cases} \frac{1}{2\pi}\log|x| & \text{if} \quad n=2 \\ -\frac{C_n}{|x|^{n-2}} & \text{if} \quad n \geq 3 \end{cases}$$

is the fundamental solution of the Laplace equation in $\mathbb{R}^n$; and $y^* = (y', -y_n)$ is the reflection point of $y = (y', y_n)$ with respect to the plane $\{x_n = 0\}$. To derive (2.5) and an explicit expression for the pressure, Solonnikov invokes Fourier transform in $x' = (x_1, \ldots, x_{n-1})$ and Laplace transform in x_n to the Stokes equations [91], [92, p. 337]. He obtains pointwise estimates for the Green's matrix $G = (G_{ij})$, $i, j = 1, 2, \ldots, n$ and its derivatives [92, Prop. 2.5]. Kang [33, Sec. 2] derives similar estimates by a different approach. Using (2.3) with the Green's matrix given by (2.5) as the definition for $e^{t\mathbb{A}}$, we give an alternative proof for (2.2) without resorting to pseudo-differential operators or semigroup theory (Proposition 2.5). We also describe suitable extensions of u_0 to the whole space such that $e^{t\mathbb{A}}$ is given by a convolution in $\mathbb{R}^n$ (Equation 2.11). Although our proof works well for any $n \geq 3$, we only consider $n = 3$ for simplicity.

Proposition 2.4. *Let $l \geq 0$, multi-indices $k = (k_1, k_2, k_3) = (k', k_3)$ and $m = (m', m_3)$. Denote $|k| = |k'| + k_3 = k_1 + k_2 + k_3$. Then*

(a) For $\Omega = \mathbb{R}^3$,

$$\left|\partial_t^l \partial_x^k \partial_y^m G(x, y, t)\right| \leq t^{-\frac{3}{2} - l - \frac{|k|+|m|}{2}} \Theta_{lkm}\left(\frac{x-y}{\sqrt{t}}\right)$$

for some Θ_{lkm} belonging to $L^\alpha(\mathbb{R}^3)$ for all $1 \leq \alpha \leq \infty$.

(b) For $\Omega = \mathbb{R}^3_+$,

$$\left|\partial_t^l \partial_x^k \partial_y^m G(x, y, t)\right| \leq t^{-\frac{3}{2} - l - \frac{|k|+|m|}{2}} \left(\Theta_{lkm}\left(\frac{x-y}{\sqrt{t}}\right) + \tilde{\Theta}_{lkm}\left(\frac{x-y^*}{\sqrt{t}}\right)\right)$$

where $y^* = (y', -y_3)$. Here $\tilde{\Theta}_{lkm}$ belong to $L^\alpha(\mathbb{R}^3_+)$ for all $1 \leq \alpha \leq \infty$, except for the case $(|k|, |m|, \alpha) = (0, 0, 1)$. Also,

$$\int_{\mathbb{R}^3_+} |G(x, y, t)| dy \leq A \quad \forall\, x \in \mathbb{R}^3_+,\ t > 0 \tag{2.6}$$

where A is an absolute constant.

Proof. (a) The estimates are obtained by noticing that

$$\Gamma(x, t) = At^{-3/2}\Theta_0\left(\frac{x}{\sqrt{t}}\right), \tag{2.7}$$

with $\Theta_0(x) = \exp(-A|x|^2)$.

(b) Consider $\Omega = \mathbb{R}^+$. Let G'_{ij} be the second term on the right hand side of (2.5).

$$G'_{ij}(x,t) = \int_0^{x_3} \int_{\mathbb{R}^2} \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(x-z)\Gamma(z-y^*,t)dz. \tag{2.8}$$

Solonniko [92, Prop. 2.5] shows that

$$\left|\partial_t^l \partial_x^k \partial_y^m G'(x,y,t)\right| \le C_{lkm} t^{-l-\frac{m_3}{2}} (t+x_3^2)^{-k_3/2} \left(|x-y^*|^2+t\right)^{-\frac{3+|k'|+|m'|}{2}} e^{-\frac{Ay_3^2}{t}}. \tag{2.9}$$

Because the exponential function grows faster than any polynomial functions, the following estimates hold

$$\frac{1+(x_3+y_3)^2}{1+x_3^2} \le 2(1+y_3^2) \le C_k e^{\frac{A}{k_3}y_3^2},$$

which lead to $(t+x_3^2)^{-k_3/2} e^{-Ay_3^2/t} \le C_k \Big(t+(x_3+y_3)^2\Big)^{-k_3/2}$. Substituting this estimate into (2.9), we get

$$\begin{aligned} \left|\partial_t^l \partial_x^k \partial_y^m G'(x,y,t)\right| &\le C_{lk} t^{-l-\frac{m_3}{2}} \Big(t+(x_3+y_3)^2\Big)^{-\frac{k_3}{2}} \left(|x-y^*|^2+t\right)^{-\frac{3+|k'|+|m'|}{2}} \\ &= t^{-\frac{3}{2}-l-\frac{|k|+|m|}{2}} \bar{\Theta}_{lkm}\left(\frac{x-y^*}{\sqrt{t}}\right), \end{aligned}$$

where $\bar{\Theta}_{lkm}(x) = C_{lk}\left(1+x_3^2\right)^{-\frac{k_3}{2}} \left(|x|^2+1\right)^{-\frac{3+|k'|+|m'|}{2}}$. The integrability of $\bar{\Theta}_{lkm}$ is equivalent to that of $\theta_{km}(x) = \left(1+x_3^2\right)^{-\frac{k_3}{2}} \left(|x|^2+1\right)^{-\frac{3+|k'|+|m'|}{2}}$. Since $\theta_{km} \in L^\infty$, it suffices to only investigate its L^1-integrability. Consider three following cases.

- $|k| = |m| = 0$: $\theta_{km}(x) = (|x|^2+1)^{-3/2} \in L^\alpha$ for all $1 < \alpha \le \infty$.
- $k_3 = 0$, $|k'| + |m'| \ge 1$: $\theta_{km}(x) \le (|x|^2+1)^{-2} \in L^\alpha$ for all $1 \le \alpha \le \infty$.
- $k_3 \ge 1$: $\theta_{km}(x) \le \left(1+x_3^2\right)^{-\frac{1}{2}} \left(|x|^2+1\right)^{-\frac{3}{2}}$. By Young's inequality,

  $$|x|^2+1 = x_3^2+|x'|^2+1 \ge A\left(1+x_3^2\right)^{\frac{1}{4}} \left(|x'|^2+1\right)^{\frac{3}{4}}.$$

  We obtain a new estimate for θ_{km}

  $$\theta_{k,m}(x) \le \left(1+x_3^2\right)^{-\frac{7}{8}} \left(|x'|^2+1\right)^{-\frac{9}{8}}.$$

  Then

  $$\int_{\mathbb{R}^3_+} \theta_{k,m}(x)dx \le \int_0^\infty \left(1+x_3^2\right)^{-\frac{7}{8}} dx_3 \int_{\mathbb{R}^2} \left(|x'|^2+1\right)^{-\frac{9}{8}} dx' < \infty.$$

Now we show (2.6). It suffices to show (2.6) for G' instead of G. Thanks to the scaling symmetry $G'(x,y,t) = t^{-3/2}G'(xt^{-1/2}, yt^{-1/2}, 1)$, we can assume $t = 1$. By Young's inequality,

$$\begin{aligned} |x-y^*|^2+1 &\geq A\big(|x'-y'|^2+1\big)^{3/4}\Big((x_3+y_3)^2+1\Big)^{1/4} \\ &\geq A\big(|x'-y'|^2+1\big)^{3/4}\big({y_3}^2+1\big)^{1/4}. \end{aligned}$$

Substituting this estimate into (2.9), we get

$$\begin{aligned} \int_{\mathbb{R}^3_+} |G'(x,y,1)|dy &\leq \int_{\mathbb{R}^3_+} \big(|x'-y'|^2+1\big)^{-9/8}\big({y_3}^2+1\big)^{-3/8}e^{-Ay_3^2}dy \\ &= A\int_{\mathbb{R}^2}\big(|y'|^2+1\big)^{-9/8}dy'\int_0^\infty \big({y_3}^2+1\big)^{-3/8}e^{-Ay_3^2}dy_3 = A. \end{aligned}$$

□

Proposition 2.5. *For $\Omega = \mathbb{R}^3$, estimate (2.2) holds for all l, $|k| = 0,1,2,\ldots$ and $1 \leq r \leq p \leq \infty$. For $\Omega = \mathbb{R}^3_+$, the only excluded case is $(|k|,r,p) = (0,1,1)$.*

Proof. For $\Omega = \mathbb{R}^3$, (2.2) is a simple consequence of Part (a) of Proposition 2.4 and Young's inequality for convolution. Now consider $\Omega = \mathbb{R}^3_+$. Part (b) of Proposition 2.4 together with Young's inequality for convolution results in estimate (2.2) for $1 \leq r < p \leq \infty$ and $r = p = \infty$. We also have (2.2) when $|k| \geq 1$ and $1 < r = p < \infty$. The only case that needs special concern is when $|k| = 0$ and $1 < r = p < \infty$. From (2.5), we have

$$\int_{\mathbb{R}^3_+} G(x,y,t)u_0(y)dy = \int_{\mathbb{R}^3}\Gamma(x-y,t)\check{e}(u_0)(y)dy + \int_{\mathbb{R}^3_+} G'(x,y,t)\mathbb{J}u_0(y)dy, \tag{2.10}$$

where $\mathbb{J}$ is the constant matrix

$$\mathbb{J} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Put

$$K(x,t) = (I_{x_3>0}\nabla^2\Phi) * \Gamma(.,t).$$

Then for $x \in \mathbb{R}^3_+$,

$$\begin{aligned}
\int_{\mathbb{R}^3_+} G'(x,y,t)u_0(y)dy &= \int_{\mathbb{R}^3_+}\int_0^{x_3}\int_{\mathbb{R}^2} \nabla^2\Phi(x-z)\Gamma(z-y^*,t)u_0(y)dz'dz_3dy \\
&= \int_0^{x_3}\int_{\mathbb{R}^2} \nabla^2\Phi(x-z)\tilde{u}_0(z)dz'dz_3 \\
&= \int_{\mathbb{R}^3} (I_{x_3>0}\nabla^2\Phi)(x-z)\tilde{u}_0(z)dz = (I_{x_3>0}\nabla^2\Phi) * \tilde{u}_0
\end{aligned}$$

where

$$\begin{aligned}
\tilde{u}_0(x) &= \int_{\mathbb{R}^3_+} \Gamma(x-y^*,t)u_0(y)dy = \int_{\mathbb{R}^3} \Gamma(x-y,t)u_0^*(y)dy = \Gamma(.,t) * u_0^*(x), \\
u_0^*(x) &= \begin{cases} 0 & \text{if} \quad x_3 > 0, \\ u_0(x',-x_3) & \text{if} \quad x_3 < 0. \end{cases}
\end{aligned}$$

We can rewrite (2.10) as

$$\int_{\mathbb{R}^3_+} G(x,y,t)u_0(y)dy = \Gamma(.,t) * \check{e}(u_0) + K(.,t) * \mathbb{J}u_0^*. \tag{2.11}$$

It suffices to show that $\hat{K}$ is a Fourier multiplier (Appendix A.7). By the scaling symmetry of K with respect to t, we can assume $t = 1$. We know that $\widehat{\nabla^2\Phi}(\xi) = \xi \otimes \xi/|\xi|^2$.

$$\begin{aligned}
\hat{I}_{x_3>0}(\xi) &= \int_{\mathbb{R}^3_+} e^{-ix\xi}dx = \left(\int_{\mathbb{R}} e^{-ix_1\xi_1}dx_1\right)\left(\int_{\mathbb{R}} e^{-ix_2\xi_2}dx_2\right)\left(\int_{\mathbb{R}} I_{x_3>0}e^{-ix_3\xi_3}dx_3\right) \\
&= \delta(\xi_1)\delta(\xi_2)\left(\pi\delta(\xi_3) + \frac{1}{i\xi_3}\right) = \pi\delta(\xi) - i\delta(\xi')\frac{1}{\xi_3}.
\end{aligned}$$

Then $\hat{I}_{x_3>0} * \widehat{\nabla^2\Phi}(\xi) = \pi\widehat{\nabla^2\Phi}(\xi) - iQ(\xi)$ where

$$Q(\xi) = \text{p. v.} \int_{-\infty}^{\infty} \frac{(\xi',\zeta_3)\otimes(\xi',\zeta_3)}{|\xi'|^2+\zeta_3^2}\frac{1}{\xi_3-\zeta_3}d\zeta_3.$$

Then $\hat{K}(\xi,1) = \pi\widehat{\nabla^2\Phi}\widehat{\Gamma(.,1)} - iQ(\xi)\widehat{\Gamma(.,1)} = \pi\hat{\Theta}(\xi) - iR(\xi)$, where $R(\xi) = Q(\xi)e^{-|\xi|^2}$. We have $\hat{\Theta}(\xi) = \widehat{\nabla^2\Phi}e^{-|\xi|^2}$. The first factor is a Fourier multiplier because $\nabla^2\Phi$ is a Calderón–Zygmund kernel. The second factor is a Schwarz function. Thus, $\hat{\Theta}$ is a multiplier. It remains to show that $R(\xi)$ is a multiplier. Since $e^{-|\xi|^2}$ is a Schwartz function, it suffices to show that $Q(\xi)$ is a multiplier. One can compute $Q(\xi)$ explicitly thanks to the following Hilbert transform pairs

$$H(1) = 0, \quad H\left(\frac{1}{1+\tau^2}\right) = \frac{A\tau}{1+\tau^2}, \quad H\left(\frac{\tau}{1+\tau^2}\right) = -\frac{A}{1+\tau^2}.$$

Specifically,

$$Q_{kj}(\xi) = Q_{jk}(\xi) = \begin{cases} A\frac{\xi_j\xi_k\xi_3}{|\xi'||\xi|^2} & \text{if} \quad j,k \neq 3, \\ -A\frac{\xi_j\xi_3}{|\xi|^2} & \text{if} \quad k = 3. \end{cases}$$

It is easy to check that each function satisfies the condition of Lizorkin multiplier theorem (Proposition A.14). Thus, $Q(\xi)$ is a multiplier. Note that Hörmander-Mikhlin multiplier theorem (Proposition A.13) is not applicable because the mixed derivative $\partial_{\xi_1}\partial_{\xi_2}Q_{12}(\xi)$ consists of a term of order $|\xi'|^{-1}|\xi|^{-1}$ which is not square-integrable in any spherical shell. □

Remark 2.6. By Proposition 2.4, the constant $C_{l,k,r,p}$ in (2.2) depends on r and p only through the L^α-norms of Θ_{lkm} and $\tilde{\Theta}_{lkm}$ with $1/\alpha = 1+1/p-1/r$. Since $\Theta_{lkm} \in L^1 \cap L^\infty$, $\|\Theta_{lkm}\|_{L^\alpha}$ is bounded by a constant independent of α. Thus, for $\Omega = \mathbb{R}^3$ the subscripts (r,p) can be dropped from the constant. The same is true for $\Omega = \mathbb{R}^3_+$ except in the case $|k| = |m| = 0$. In fact, it can be seen from the proof of Proposition 2.5 that $\tilde{\Theta}_{000}$ does not belong to L^1 because its Fourier transform is discontinuous at the origin. Nevertheless, $\tilde{\Theta}_{l00}$ belongs to L^β for all $1 < \beta \leq \infty$. In applications, when α is known to be bounded from below by an absolute constant $A > 1$, the subscripts (r,p) can also be dropped.

An immediate consequence of Proposition 2.5 is the L^p-estimates of the solution to $(\mathrm{SE})_\Omega$ with vanishing initial condition

$$F(x,t) = \int_0^t e^{(t-s)\mathbb{A}}\mathbb{P}f(s)ds.$$

Corollary 2.7. *For $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$, the estimate*

$$\left\|\partial_x^k F(x,t)\right\|_{L^p_x} \leq \int_0^t \frac{C_{k,\beta,p}}{(t-s)^{\frac{|k|}{2}+\frac{3}{2}\left(\frac{1}{\beta}-\frac{1}{p}\right)}}\|\mathbb{P}f(s)\|_{L^\beta_x}ds$$

holds for all $1 \leq \beta \leq p \leq \infty$ with the exception of $(|k|,p) = (0,1)$ when $\Omega = \mathbb{R}^3_+$. If $1 < \beta < \infty$, $\mathbb{P}$ can be dropped from the right hand side.

Remark 2.8. As noted in Remark 2.6, the subscripts (β,p) can be dropped from the constant $C_{k,\beta,p}$ in most situations. The only exception is when $\Omega = \mathbb{R}^3_+$ and $|k| = 0$. If one replaces $\mathbb{P}f$ by f on the right hand side, the constant is rescaled by the operator norm of $\mathbb{P}$ which depends on β.

For a matrix-valued function $\mathbb{F} = (f_{ij})_{1\leq i,j\leq 3}$, denote $\operatorname{div}\mathbb{F} = (f_{ij,j})$.

Proposition 2.9. *Let $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$. Consider a vector-valued function u_0 and a matrix-valued function $\mathbb{F}$. For $1 < r \le p < \infty$, we have the following inequalities.*

(i) $\left\|e^{t\mathbb{A}}\mathbb{P}u_0\right\|_{L^p(\Omega)} \le C_{r,p}t^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|u_0\|_{L^r(\Omega)}$,

(ii) $\left\|\nabla e^{t\mathbb{A}}\mathbb{P}u_0\right\|_{L^p(\Omega)} \le C_{r,p}t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|u_0\|_{L^r(\Omega)}$,

(iii) $\left\|e^{t\mathbb{A}}\mathbb{P}\operatorname{div}\mathbb{F}\right\|_{L^p(\Omega)} \le C_{r,p}t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\mathbb{F}\|_{L^r(\Omega)}$.

Proof. As mentioned in Section 2.1, $\mathbb{A}$ and $\mathbb{P}$ commute. Hence, $e^{t\mathbb{A}}$ and $\mathbb{P}$ also commute. Recall that the Helmholtz projection $\mathbb{P}$ is a bounded linear operator from $L^p(\Omega)$ to itself. Part (i) and (ii) are simple consequences of Proposition 2.5:

$$\left\|e^{t\mathbb{A}}\mathbb{P}u_0\right\|_{L^p(\Omega)} \le C_{r,p}t^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\mathbb{P}u_0\|_{L^r(\Omega)} \le C_{r,p}t^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|u_0\|_{L^r(\Omega)}.$$

$$\left\|\nabla e^{t\mathbb{A}}\mathbb{P}u_0\right\|_{L^p(\Omega)} \le C_{r,p}t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\mathbb{P}u_0\|_{L^r(\Omega)} \le C_{r,p}t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|u_0\|_{L^r(\Omega)}.$$

For Part (iii), if $\mathbb{F}$ has no generalized divergence then *a priori* $\Psi = e^{t\mathbb{A}}\mathbb{P}\operatorname{div}\mathbb{F}$ (regarded as a function on Ω with $t > 0$ being fixed) is not well-defined. However, one can show that Ψ is a bounded operator from $(C_0^\infty(\Omega), \|\cdot\|_{L^p})$ to $L^r(\Omega)$. Then it uniquely extends to a bounded operator from L^p to L^r. Indeed, for $\mathbb{F} \in C_0^\infty(\Omega)$

$$e^{t\mathbb{A}}\operatorname{div}\mathbb{F} = \int_\Omega G(x,y,t)\operatorname{div}\mathbb{F}(y)dy = -\int_\Omega \nabla_y G(x,y,t) : \mathbb{F}(y)dy.$$

Applying Proposition 2.4 for $(l, |k|, |m|, \alpha) = (0, 0, 1, 1)$ and Young's inequality for convolution, we have

$$\left\|e^{t\mathbb{A}}\operatorname{div}\mathbb{F}\right\|_{L^p(\Omega)} \le C_{r,p}t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\mathbb{F}\|_{L^r(\Omega)}.$$

On the other hand,

$$\left\|e^{t\mathbb{A}}\mathbb{P}\operatorname{div}\mathbb{F}\right\|_{L^p(\Omega)} = \left\|\mathbb{P}e^{t\mathbb{A}}\operatorname{div}\mathbb{F}\right\|_{L^p(\Omega)} \le C_p\left\|e^{t\mathbb{A}}\operatorname{div}\mathbb{F}\right\|_{L^p(\Omega)}.$$

This proves (iii). One can also show (iii) by duality argument, as being done in [24, Prop. 3.1] for general domains: first recall that $\mathbb{P}$ and $\mathbb{A}$ are self-adjoint operators in sense of Proposition 2.1. Then for any $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\left\langle e^{t\mathbb{A}}\mathbb{P}\operatorname{div}\mathbb{F}, \phi\right\rangle = \left\langle \mathbb{P}\operatorname{div}\mathbb{F}, e^{t\mathbb{A}}\phi\right\rangle = \left\langle \operatorname{div}\mathbb{F}, \mathbb{P}e^{t\mathbb{A}}\phi\right\rangle &= -\left\langle \mathbb{F}, \nabla\mathbb{P}e^{t\mathbb{A}}\phi\right\rangle \\ &= -\left\langle \mathbb{F}, \nabla e^{t\mathbb{A}}\mathbb{P}\phi\right\rangle.\end{aligned}$$

By Hölder inequality and Part (ii),

$$\left|\left\langle e^{t\mathbb{A}}\mathbb{P}\,\mathrm{div}\ \mathbb{F},\phi\right\rangle\right| \leq \|\mathbb{F}\|_{L^r}\left\|\nabla e^{t\mathbb{A}}\mathbb{P}\phi\right\|_{L^{r'}} \leq C_{r,p}t^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p'}-\frac{1}{r'}\right)}\|\mathbb{F}\|_{L^r}\|\phi\|_{L^{p'}}.$$

Hence, $e^{t\mathbb{A}}\mathbb{P}\mathrm{div}\ \mathbb{F}$ is a bounded operator from L^r to L^p. $\square$

Chapter 3

Stokes problem with $f \in Y_q$

3.1 Estimates for Stokes system when $u_0 = 0$

For $5/2 < q < 3$, denote

$$q_* = \frac{3}{2} - \frac{5}{2q} \tag{3.1}$$

and $Y_q = \{f : \Omega \times (0,\infty) \to \mathbb{R}^3 : \ t^{q_*} f \in L^q(\Omega \times (0,\infty))\}$. The solution to the Stokes system $(\mathrm{SE})_\Omega$ with vanishing initial condition is

$$F(x,t) = \int_0^t e^{(t-s)\mathbb{A}} \mathbb{P} f(s) ds = \int_0^t \int_\Omega G(x,y,t-s) \mathbb{P} f(s) dy ds. \tag{3.2}$$

Proposition 3.1. *Let $f \in Y_q$ for some $5/2 < q < 3$, and F be given by (3.2). Then*

(i) $F \in L^r_t L^p_x$ *for all* $2/r + 3/p = 1$, $3 \le p \le \infty$. *Moreover,* $\|F\|_{L^r_t L^p_x} \le C_{p,q} \|f\|_{Y_q}$.

(ii) $\sqrt{t} F \in L^\infty_{t,x}$. *Moreover,* $\left\|\sqrt{t} F\right\|_{L^\infty_{t,x}} \le C_q \|f\|_{Y_q}$ *and* $\lim_{t \to 0^+} \sqrt{t} \|F(t)\|_{L^\infty_x} = 0$.

(iii) $F \in C([0,\infty), L^3)$ *and* $\lim_{t \to 0^+} \|F(t)\|_{L^3_x} = 0$.

(iv) $t^{-\alpha} F \in L^r_t L^q_x$ *for all* $1 - 2/q < \alpha < 1 - 1/q$ *and* $2/r + 3/q = 1 + 2\alpha$. *Moreover,* $\|t^{-\alpha} F\|_{L^r_t L^q_x} \le C_{q,\alpha} \|f\|_Y$.

(v) $\nabla F \in L^r_t L^p_x$ *for all* $2/r + 3/p = 2$, $q \le p < 3q/(5-q)$. *Moreover,* $\|\nabla F\|_{L^r_t L^p_x} \le C_{p,q} \|f\|_{Y_q}$.

(vi) $\sqrt{t} \nabla F \in L^\infty_t L^3_x \cap C_t L^3_x$ *with* $\lim_{t \to 0^+} \|\sqrt{t} \nabla F(t)\|_{L^3_x} = 0$ *and* $\left\|\sqrt{t} \nabla F\right\|_{L^\infty_t L^3_x} \le C_q \|f\|_{Y_q}$.

Proof. The main tool for the proofs of (i) and (v) are Corollary 2.7 and Marcinkiewicz Interpolation Theorem (Proposition A.15).

(i) Applying Corollary 2.7 for $\beta = q$, $l = |k| = 0$, and using the boundedness of $\mathbb{P} : L^q \to L^q$, we have

$$\begin{aligned} \|F(t)\|_{L^p_x} &\leq \int_0^t \frac{C_{p,q}}{(t-s)^\gamma} \|f(s)\|_{L^q_x} ds \\ &= \int_0^t \frac{C_{p,q}}{(t-s)^\gamma s^{q_*}} \|s^{q_*} f(s)\|_{L^q_x} ds = \int_0^t \frac{C_{p,q} g(s)}{(t-s)^\gamma s^{q_*}} ds =: G(t) \end{aligned}$$

where $\gamma = 3/(2q) - 3/(2p)$ and $g(s) = \|s^{q_*} f(s)\|_{L^q_x}$. Our goal is to show that $G \in L^r$. Since $g \in L^q(0,T)$ for all $T < \infty$, it follows that $g \in L^{\bar{q}}(0,T)$ for all $\bar{q} \leq q$. By Hölder inequality,

$$G(t) \leq C_{p,q} \left\{ \int_0^t \frac{1}{[(t-s)^\gamma s^{q_*}]^{\bar{q}'}} ds \right\}^{1/\bar{q}'} \|g\|_{L^{\bar{q}}(0,t)} = C_{p,q} C_{\bar{q}} t^{-\frac{1}{\bar{\kappa}}} \|g\|_{L^{\bar{q}}(0,t)} \tag{3.3}$$

where

$$\begin{aligned} C_{\bar{q}} &= \left\{ \int_0^1 \frac{1}{[(1-\sigma)^\gamma \sigma^{q_*}]^{\bar{q}'}} d\sigma \right\}^{1/\bar{q}'}, \\ \frac{1}{\bar{\kappa}} &= \frac{(\gamma + q_*)\bar{q}' - 1}{\bar{q}'} = \frac{1}{2} - \frac{3}{2p} + \frac{1}{\bar{q}} - \frac{1}{q}. \end{aligned} \tag{3.4}$$

If $p = 3$, we choose $\bar{q} = q$ and conclude that $G \in L^\infty$. Consider the case $p > 3$, in which $\kappa < \infty$. By (3.3), the map S_1 assigning g to G is linear continuous from $L^{\bar{q}}$ to the Lorentz space $L^{\bar{\kappa},\infty}$. For any $\bar{q}_1 < \bar{q} < \bar{q}_2 = q$, denote by $\bar{\kappa}_1$, $\bar{\kappa}$, $\bar{\kappa}_2$, κ the respective numbers according to the relation (3.4). Note that $\bar{\kappa}_2 = \kappa = r$. One can consider S_1 as continuous maps from $L^{\bar{q}_1}$ to $L^{\bar{\kappa}_1,\infty}$, and from $L^{\bar{q}_2}$ to $L^{\bar{\kappa}_2,\infty}$. By Marcinkiewicz Interpolation Theorem, S_1 is also a continuous map from $L^{\bar{q}}$ to $L^{\bar{\kappa}}$ for all $\bar{q} < q$. Moreover, the norm of S_1 is bounded by $C_{pq} C_{\bar{q}_1}$. Since $C_{\bar{q}_1} \to C_q$ as $\bar{q}_1 \to q^-$, S_1 is continuous from L^q to L^κ. Hence, $G = S_1 g \in L^\kappa = L^r$ and $\|G\|_{L^r} \leq C_{p,q} C_q \|g\|_{L^q} = C_{p,q} C_q \|f\|_{Y_q}$.

(ii) In (3.3) we take $p = \infty$ and $\bar{q} = q$. In this case, $\bar{\kappa} = 2$. Hence, $\sqrt{t} G(t) \leq \|g\|_{L^q(0,t)} \to 0$ as $t \to 0^+$.

(iv) By Corollary 2.7, $\|F(t)\|_{L^q_x} \leq C_q \int_0^t \|f(s)\|_{L^q_x} ds$. Put $\beta = 1 - 2/q \in (0, \alpha)$. By

Young's inequality, $t^\alpha \geq C_{q,\alpha}(t-s)^\beta s^{\alpha-\beta}$. Thus,

$$t^{-\alpha}\|F(t)\|_{L^q_x} \leq C_{q,\alpha}\int_0^t \frac{\|s^{q_*}f(s)\|_{L^q_x}}{t^\alpha s^{q_*}}ds \leq C_{q,\alpha}\int_0^t \frac{\|s^{q_*}f(s)\|_{L^q_x}}{(t-s)^\beta s^\rho}ds \tag{3.5}$$

where $\rho = q_* + \alpha - \beta \in (0,1)$. Consider the map $S_2 : g \mapsto G$

$$G(t) = \int_0^t \frac{g(s)}{(t-s)^\beta s^\rho}ds.$$

For $\bar{q} > 1$, by Hölder inequality

$$G(t) \leq \left(\int_0^t \frac{1}{(t-s)^{\beta\bar{q}'}s^{\rho\bar{q}'}}ds\right)^{1/\bar{q}'}\|g\|_{L^{\bar{q}}} = \frac{C_{\bar{q},q,\alpha}}{t^{1/\bar{r}}}\|g\|_{L^{\bar{q}}}$$

where

$$\frac{1}{\bar{r}} = \frac{1}{\bar{q}} - \frac{5}{2q} + \frac{1}{2} + \alpha. \tag{3.6}$$

Note that $1/\bar{r} \in (0,1)$ if $\bar{q}$ is close to q. Then $G \in L^{\bar{r},\infty}$ and S_2 is a bounded operator from $L^{\bar{q}}$ to $L^{\bar{r},\infty}$. Let $q_1 < q < q_2$ such that

$$\frac{1}{q} = \frac{1}{2}\frac{1}{q_1} + \frac{1}{2}\frac{1}{q_2}.$$

Let r_1 and r_2 be $\bar{r}$ in (3.6) when $\bar{q}$ is replaced by q_1 and q_2 respectively. By Marcinkiewicz interpolation theorem, S_2 is a bounded operator from L^q to L^r where

$$\frac{1}{r} = \frac{1}{q} - \frac{5}{2q} + \frac{1}{2} + \alpha = -\frac{3}{2q} + \frac{1}{2} + \alpha.$$

Take $g(s) = \|s^{q_*}f(s)\|_{L^q_x} \in L^q$. By (3.5) we get $t^{-\alpha}\|F(t)\|_{L^r_t L^q_x} \leq C_{q,\alpha}S_2 g$. Therefore,

$$\left\|t^{-\alpha}F\right\|_{L^r_t L^q_x} \leq C_{q,r}\|S_2 g\|_{L^r} \leq C_{q,r}\|g\|_{L^q} = C_{q,r}\|f\|_{Y_q}.$$

(v) Applying Corollary 2.7 for $(|k|,\beta,p) = (1,q,p)$, we have

$$H(t) := \|\nabla F(t)\|_{L^p_x} \leq \int_0^t \frac{C_{pq}}{(t-s)^\theta}\|f(s)\|_{L^q_x}ds = \int_0^t \frac{C_{pq}g(s)}{(t-s)^\theta s^{q_*}}ds \tag{3.7}$$

where $\theta = 1/2 - 3/(3p) + 3/(2q)$. Our goal is to show that $H \in L^r$. By Hölder inequality,

$$H(t) \leq C_{pq}\left\{\int_0^t \frac{1}{[(t-s)^\theta s^{q_*}]^{\bar{q}'}}ds\right\}^{1/\bar{q}'}\|g\|_{L^{\bar{q}}(0,t)} = C_{pq}C_{\bar{q}}t^{-\frac{1}{\bar{\eta}}}\|g\|_{L^{\bar{q}}(0,t)} \quad \forall\ \bar{q} \leq q \tag{3.8}$$

where

$$\begin{aligned} C_{\bar{q}} &= \left\{\int_0^1 \frac{1}{[(1-\sigma)^\theta \sigma^{q_*}]^{\bar{q}'}}d\sigma\right\}^{1/\bar{q}'}, \\ \frac{1}{\bar{\eta}} &= \frac{(\theta+q_*)\bar{q}'-1}{\bar{q}'} = 1 - \frac{3}{2p} + \frac{1}{\bar{q}} - \frac{1}{q}. \end{aligned}$$

Note that the specified range for p guarantees that $C_{\bar{q}}$ is finite. By (3.8), the map S_3 assigning g to H is linear continuous from $L^{\bar{q}}$ to the Lorentz space $L^{\bar{\eta},\infty}$. By similar arguments as in Part (i), we conclude that S_3 is also a continuous map from L^q to L^η where $1/\eta = 1 - 3/(2p) = 1/r$.

(iii) The fact that $F \in L^\infty_t L^3_x$ and $\|F(t)\|_{L^3_x} \to 0$ as $t \to 0$ is already proved in Part (i). We only need show to continuity. Let $0 < t_1 < t_2 < \infty$. By (3.2), $F(x,t_2) - F(x,t_1) = I_1 + I_2$ where

$$\begin{aligned} I_1 &= \int_0^{t_1}\int_\Omega [G(x,y,t_2-s) - G(x,y,t_1-s)]\mathbb{P}f(y,s)dyds, \\ I_2 &= \int_{t_1}^{t_2}\int_\Omega G(x,y,t_2-s)\mathbb{P}f(y,s)dyds. \end{aligned}$$

By Fubini's theorem,

$$\begin{aligned} |I_1| &= \left|\int_0^{t_1}\int_\Omega\int_{t_1}^{t_2} \partial_t G(x,y,t-s)\mathbb{P}f(y,s)dtdyds\right| \\ &\le \int_0^{t_1}\int_{t_1}^{t_2}\int_\Omega |\partial_t G(x,y,t-s)\mathbb{P}f(y,s)|dydtds. \end{aligned}$$

By Proposition 2.4 with $(l,|k|,|m|) = (1,0,0)$ and Young's inequality for convolution,

$$\begin{aligned} \|I_1\|_{L^3_x} &\le \int_0^{t_1}\int_{t_1}^{t_2} \frac{C_q}{(t-s)^\rho}\|f(s)\|_{L^q_x}dtds = \int_0^{t_1}\int_{t_1}^{t_2}\frac{C_q g(s)}{(t-s)^\rho s^{q_*}}dtds \\ &= \frac{C_q}{\rho-1}\int_0^{t_1}\left(\frac{1}{(t_1-s)^{\rho-1}s^{q_*}} - \frac{1}{(t_2-s)^{\rho-1}s^{q_*}}\right)g(s)ds \\ &\overset{\text{Hölder}}{\le} \frac{C_q\|g\|_{L^q}}{\rho-1}\left\{\int_0^{t_1}\left(\frac{1}{(t_1-s)^{\rho-1}s^{q_*}} - \frac{1}{(t_2-s)^{\rho-1}s^{q_*}}\right)^{q'}ds\right\}^{1/q'} \end{aligned}$$

where $\rho = 1/2 + 3/(2q) \in (1, 11/10)$. To study the limit $t_2 \downarrow t_1$, we can set $t_1 = 1$ and $t_2 = t > 1$. The integrand of

$$\int_0^1\left(\frac{1}{(1-s)^{\rho-1}s^{q_*}} - \frac{1}{(t-s)^{\rho-1}s^{q_*}}\right)^{q'}ds$$

is less than $(1-s)^{-(\rho-1)q'}s^{-q_*q'}$, which is an L^1-integrable function over interval (0,1). By Lebesgue's Dominated Convergence Theorem, the integral converges to 0 as $t \downarrow 1$. To study the limit $t_1 \uparrow t_2$, we can set $t_2 = 1$ and $t_1 = t < 1$. Using the change of variable $s = t\tau$, we have

$$\int_0^t\left(\frac{1}{(t-s)^{\rho-1}s^{q_*}} - \frac{1}{(1-s)^{\rho-1}s^{q_*}}\right)^{q'}ds = \int_0^1\left(\frac{1}{(1-s)^{\rho-1}s^{q_*}} - \frac{t^{1/q'-q_*}}{(1-t\tau)^{\rho-1}\tau^{q_*}}\right)^{q'}d\tau.$$

The integrand is less than $(1-s)^{-(\rho-1)q'}s^{-q_*q'}$. The conclusion again follows from Lebesgue's Dominated Convergence Theorem. Next, we estimate I_2. By Proposition 2.4 with $(l,|k|,|m|)=(1,0,0)$ and Young's inequality for convolution,

$$\begin{aligned}\|I_2\|_{L^3_x} &\leq \int_{t_1}^{t_2}\frac{C_q}{(t_2-s)^{\delta}}\|f(s)\|_{L^q_x}ds=\int_{t_1}^{t_2}\frac{C_q g(s)}{(t_2-s)^{\delta}s^{q_*}}ds\\ &\overset{\text{Hölder}}{\leq} C_q\left\{\int_{t_1}^{t_2}\frac{1}{(t_2-s)^{\delta q'}s^{q_*q'}}ds\right\}^{1/q'}\|g\|_{L^q}\end{aligned}$$

where $\delta=3/(2q)-1/2$. By the change of variable $s=t_2\tau$, the integral can be written as

$$\int_{t_1}^{t_2}\frac{1}{(t_2-s)^{\delta q'}s^{q_*q'}}ds=\int_{t_1/t_2}^{1}\frac{1}{(1-\tau)^{\delta q'}\tau^{q_*q'}}d\tau$$

which tends to 0 as $t_1\uparrow t_2$ or $t_2\downarrow t_1$.

(vi) In (3.8) we choose $p=3$ and $\bar{q}=q$. In this case, $\bar{\eta}=2$. Hence, $\sqrt{t}H(t)\leq\|g\|_{L^q(0,t)}$. To show the continuity of $\sqrt{t}\nabla F$, we repeat the arguments in Part (iii) with the following adjustments. Let $0<t_1<t_2<\infty$. By (3.2), $\nabla F(x,t_2)-\nabla F(x,t_1)=J_1+J_2$ where

$$\begin{aligned}J_1 &= \int_0^{t_1}\int_\Omega[\nabla_x G(x,y,t_2-s)-\nabla_x G(x,y,t_1-s)]\mathbb{P}f(y,s)dyds,\\ J_2 &= \int_{t_1}^{t_2}\int_\Omega\nabla_x G(x,y,t_2-s)\mathbb{P}f(y,s)dyds.\end{aligned}$$

We have

$$\|J_1\|_{L^3_x}\leq\frac{\|g\|_{L^q}}{\mu-1}\left\{\int_0^{t_1}\left(\frac{1}{(t_1-s)^{\mu-1}s^{q_*}}-\frac{1}{(t_2-s)^{\mu-1}s^{q_*}}\right)^{q'}ds\right\}^{1/q'}$$

where $\mu=1+3/(2q)\in(3/2,8/5)$. Note that $(\mu-1)q'<1$. The integral goes to 0 as $t_1\uparrow t_2$ or $t_2\downarrow t_1$. Similarly,

$$\|J_2\|_{L^3_x}\leq\left\{\int_{t_1}^{t_2}\frac{1}{(t_2-s)^{\gamma q'}s^{q_*q'}}ds\right\}^{1/q'}\|g\|_{L^q}$$

with $\gamma=3/(2q)$ and $\gamma q'<1$. The integral also goes to 0 as $t_1\uparrow t_2$ or $t_2\downarrow t_1$. □

Remark 3.2. Proposition 3.1 with some modifications also holds in the setting $f\in L^\alpha_t L^\beta_x$ with $2/\alpha+3/\beta=3$. For example, Part (i) should be changed to

$$\|F\|_{L^r_tL^p_x}\leq C_{\beta,p}\|f\|_{L^\alpha_tL^\beta_x}\quad\forall\, r\in(\alpha,\infty),\ p\in(\beta,\infty),\ \frac{2}{r}+\frac{3}{p}=3.$$

The proof is an application of Young-O'Neil inequality (Proposition A.3) as follows.

$$\|F(t)\|_{L^p_x} \leq \int_0^t \frac{C_{\beta,p}}{(t-s)^{1/\gamma}} \|f(s)\|_{L^\beta_x} ds$$

where $1/\gamma = 3/(2\beta) - 3/(2p)$. The right hand side is the convolution in $\mathbb{R}$ of $|s|^{-1/\gamma}$ and $\phi(s) = \|f(s)\|_{L^\beta_x} I_{(0,t)}(s)$. The former belongs to weak Lebesgue space $L^{\gamma,\infty}$. By Young-O'Neil inequality,

$$\|F\|_{L^r_t L^p_x} \leq C_{\beta,p} \Big\| |s|^{-1/\gamma} \Big\|_{L^{\gamma,\infty}} \|\phi\|_{L^\alpha} \leq C_{\beta,p} \|f\|_{L^\alpha_t L^\beta_x(\Omega\times(0,t))}.$$

3.2 Coercive and local energy estimates

It is known that the solution to the heat system

$$\begin{cases} \partial_t u - \Delta u = f \quad \text{in} \quad \Omega \times (0,\infty) \\ u|_{\partial\Omega} = 0 \\ u(\cdot, 0) = 0 \end{cases}$$

on the whole space or half-space has maximal regularity in mixed-norm Lebesgue spaces in the following sense ([39, Thm. 2.1], [75, Thm. D12], [44, Thm. 9.1])

$$\|\partial_t u\|_{L^r_t L^q_x} + \|\nabla^2 u\|_{L^r_t L^q_x} \leq C_{r,q,\Omega} \|f\|_{L^r_t L^q_x} \quad \forall\, 1 < r,\, q < \infty.$$

Conceptually, this is the best regularity result because u belongs to the minimal (and most natural) regularity class with respect to the regularity of f. Consider the Stokes system $(\mathrm{SE})_\Omega$ with $u_0 = 0$ and $f \in L^r_t L^q_x$. Solonnikov shows that the solution u given by (1.2) also has maximal regularity in the following sense

$$\|\partial_t u\|_{L^r_t L^q_x} + \|\nabla^2 u\|_{L^r_t L^q_x} + \|\nabla p\|_{L^r_t L^q_x} \leq C_{r,q,\Omega} \|f\|_{L^r_t L^q_x} \quad \forall\, 1 < r,\, q < \infty \tag{3.9}$$

if Ω is the half-space [91, Thm. 3.1], a bounded domain [92, Thm. 1.1] or an exterior domain [92, Thm. 1.2]. When Ω is the whole space, the Stokes system reduces to a heat system after Helmholtz decomposition; thus, (3.9) holds. Estimates of form (3.9) are often referred to as *Solonnikov's coercive estimates*. Solonnikov's proof is based on layer potential theory, relying on estimates of the Green's matrix and singular integrals. Maximal regularity for more general domains is obtained in [24, Thm. 2.1], which we rephrase as follows.

Proposition 3.3. *Let $1 < r, q < \infty$ and $f \in L^r_t L^q_x(\Omega \times (0,\infty))$. Assume $\Omega \subset \mathbb{R}^3$ is a domain with uniform C^3-boundary and that the Helmholtz decomposition exists for $L^q(\Omega)$. Then the solution (u,p) of the Stokes problem (SE)$_\Omega$ with $u_0 = 0$ has the regularity properties*

$$u \in L^r_t L^q_\sigma \cap L^r_t W^{2,q} \cap L^r_t W^{1,q}_0, \quad \partial_t u \in L^r_t L^q_\sigma, \quad \nabla p \in L^r_t L^q_\pi$$

and satisfies (3.9). Here L^q_σ and L^q_π are the subspaces of $L^q(\Omega)$ as defined in Section 2.1.

Remark 3.4. By standard Sobolev embedding arguments, one can show that the right hand side of (3.9) is also an upper bound for u in $L^\infty_t L^q_x(\Omega\times(0,T))$, and ∇u in $L^{2r}_t L^q_x(\Omega\times(0,T))$ where $\Omega_T = \Omega \times (0,T)$ and $T < \infty$. Indeed,

$$\begin{aligned} \|u(t)\|_{L^q_x} &\leq \int_0^t \|\partial_t u(s)\|_{L^q_x} ds \leq t^{1/r'} \|\partial_t u\|_{L^r_t L^q_x}, \\ \|\nabla u\|_{L^{2r}_t L^q_x} &\leq C_\Omega \|u\|^{1/2}_{L^\infty_t L^q_x} \left\|\nabla^2 u\right\|^{1/2}_{L^r_t L^q_x}. \end{aligned}$$

By parabolic Sobolev embeddings (Proposition A.10, [102, Thm. 1.4.1], [75, Appendix D3, D4]), one can infer further regularity properties of u. For example,

- If $2/r + n/q > 2$ then $u \in L^m_t L^p_x(\Omega \times (0,\infty))$ for all

$$r < m < \infty, \quad q < p < \infty, \quad \frac{2}{m} + \frac{n}{p} = \frac{2}{r} + \frac{n}{q} - 2.$$

- If $2/r + n/q < 2$ then u is locally parabolic-Hölder continuous.

Seregin establishes the local versions of (3.9) in the interior [84, Prop. 6.7] and near the boundary [81], [84, Prop. 7.10]. If the force term belongs to the weighted Lebesgue space Y_q, coercive estimates also hold in the following sense.

Proposition 3.5. *Let (u,p) be the solution to (SE)$_\Omega$ with $u_0 = 0$ and $f \in Y_q$ for some $5/2 < q < 3$. Denote $\Omega_T = \Omega \times (0,T)$. Then*

$$\|t^{q_*}\partial_t u\|_{L^q(\Omega_T)} + \left\|t^{q_*}\nabla^2 u\right\|_{L^q(\Omega_T)} + \|t^{q_*}\nabla p\|_{L^q(\Omega_T)} \leq C_{q,\Omega}\|t^{q_*} f\|_{L^q(\Omega_T)}.$$

Moreover, for any $1 < r < 2q/(3q-3)$ and $T < \infty$,

$$\|\partial_t u\|_{L^r_t L^q_x(\Omega_T)} + \left\|\nabla^2 u\right\|_{L^r_t L^q_x(\Omega_T)} + \|\nabla p\|_{L^r_t L^q_x(\Omega_T)} \leq C_{r,q,T,\Omega}\|f\|_{L^r_t L^q_x(\Omega_T)}.$$

Proof. Multiply both sides of the equation $\partial_t u - \Delta u + \nabla p = f$ by t^{q_*},

$$\partial_t v - \Delta v + \nabla \pi = g$$

where $v = t^{q_*} u$, $\pi = t^{q_*} p$ and $g = t^{q_*} f - q_* t^{q_*-1} u$. It is easy to see that div v $= 0$, $v|_{\partial\Omega} = 0$ and $v(\cdot, 0) = 0$. Applying Part (iv) of Proposition 3.1 for $\alpha = 1 - q_*$ and $F = u$, we get $\left\|t^{q_*-1} u\right\|_{L^q} \leq C_q \|f\|_{Y_q}$. Thus, $\|g\|_{L^q} \leq C_q \|f\|_{Y_q}$. By Solonnikov's coercive estimate (Proposition 3.3),

$$\|\partial_t v\|_{L^q} + \left\|\nabla^2 v\right\|_{L^q} + \|\nabla \pi\|_{L^q} \leq C_q \|g\|_{L^q}$$

which yields the first estimate. Now thanks to the range of r, $\|f(t)\|_{L^q_x} = t^{-q_*} \|t^{q_*} f(t)\|_{L^q_x} \in L^r((0,T))$. Thus, $\|f\|_{L^r_t L^q_x(\Omega_T)} \leq C_{r,q,T,\Omega} \|f\|_{Y_q}$. The second inequality in Proposition 3.5 then follows from Solonnikov's coercive estimate. □

Remark 3.6. According to Remark 3.4, $\nabla v \in L^{2q}_t L^q_x(\Omega_T)$. Hence, $t^{q_*} \nabla u \in L^{2q}_t L^q_x(\Omega_T)$.

Before proceeding to some applications of Solonnikov's coercive estimates, we comment on the case $f = 0$ and $u_0 \neq 0$. If u_0 only belongs to $L^p(\Omega)$ for some $1 \leq p \leq \infty$, one cannot expect that $U(t) = e^{t\mathbb{A}} u_0$ satisfies $\nabla^2 U \in L^\alpha_t L^\beta_x$ for some α, $\beta \geq 1$. This can be seen even in the heat equation on the whole space. Indeed, the solution of the heat equation is $U(t) = e^{t\Delta} u_0 = \Gamma(t) * u_0$. Suppose u_0 is the characteristic function of the unit ball. Then $\Delta U(t) = \partial_t U(t) = (\partial_t \Gamma) * u_0$. By simple calculations, one can check that $\|\Delta U(t)\|_{L^1(B_1)} \geq C_n t^{-1}$ which is not integrable in $t \in (0,1)$. The pathology can be overcome if u_0 has some amount of differentiability. For example, Krylov [39] shows that

$$\left\|\nabla^2 U\right\|_{L^r_t L^q_x} \leq C_{n,p,q,T,\varepsilon} \left\|u_0\right\|_{H^{2+\varepsilon-2/r}_q}$$

where $\sigma > 0$ and H^σ_q denotes the Bessel potential space $(Id - \Delta)^{-\sigma/2} L^q$. Also, it is shown in [44, Thm. 9.1] that

$$\left\|\nabla^2 U\right\|_{L^q_{t,x}} \leq C_{n,q} \left\|u_0\right\|_{W^{2-2/q,q}},$$

and in [92, Sec. 1] that

$$\left\|\nabla^2 U\right\|_{L^r_t L^q_x} \leq C_{n,r,q} \left\|u_0\right\|_{B^{2-2/r}_{q,r}}.$$

Note that the fractional Sobolev space $W^{2-2/q,\,q}$ coincides the Besov space $B^{2-2/r}_{q,r}(\Omega)$ when $q=r$. $B^{2-2/r}_{q,r}(\Omega)$ is the trace space at $t=0$ of functions u in the regularity class

$$W^{2,1}_{q,r}(\Omega_T) := \{u:\ \partial_t u,\ u,\ \nabla u,\ \nabla^2 u \in L^r_t L^q_x(\Omega_T)\}$$

(see [11, Thm. 4]). This is a natural space of u_0 coming from the abstract theory of parabolic evolution operators (see e.g. [4, Ch. III, §1, §4.10] for detail). The general idea is as follows. Suppose E_0 and E_1 are Banach spaces such that E_1 is densely injected into E_0. Let $\mathscr{A}: E_1 \to E_0$ be a linear operator (not necessarily bounded) such that $\{e^{t\mathscr{A}}\}_{t\geq 0}$ is a bounded analytic semigroup. From PDE perspective, $\mathscr{A}$ is a differential operator encoding all boundary conditions if there are any. Put

$$\mathbb{E}_0 = L^r((0,T),E_0), \qquad \mathbb{E}_1 = W^{1,r}((0,T),E_0)\cap L^r((0,T),E_1).$$

One can formulate maximal regularity property for the system

$$\begin{cases} u' - \mathscr{A}u = f, & t\in(0,T) \\ u(0)=0 \end{cases}$$

in the setting $f\in\mathbb{E}_0$ and $u_0\in\gamma\mathbb{E}_1$. The latter is called trace space, defined as the image of the trace map $\gamma:\mathbb{E}_1\to\mathbb{E}_0$, $\gamma u = u(0)$, equipped with norm $\|u_0\|_{\gamma\mathbb{E}_1} = \inf\{\|v\|_{\mathbb{E}_1}:\ v\in\mathbb{E}_1,\ \gamma v = u_0\}$. It turns out that $\gamma\mathbb{E}_1 = (E_0,E_1)_{1-1/r,\,r}$, a real interpolation space between E_0 and E_1 (see [4, Thm. 4.10.2]). For $\mathscr{A}=\Delta$, $E_0=L^q$ and $E_1=W^{2,q}$,

$$\gamma\mathbb{E}_1 = (L^q, W^{2,q})_{1-1/r,\,r} = (W^{0,q}, W^{2,q})_{1-1/r,\,r} = B^{2-2/r}_{q,r}.$$

The Stokes system corresponds to $\mathscr{A}=\mathbb{A}$, $\mathbb{E}_0 = L^q_\sigma(\Omega)$ and $\mathbb{E}_1 = W^{2,q}(\Omega)\cap W^{1,q}_0(\Omega)\cap L^q_\sigma(\Omega)$. A consequence of maximal regularity is that solutions to Stokes system belong to local energy class. Specifically, we have

Proposition 3.7. *Let (u,p) be the solution to (SE)$_\Omega$ with $u_0=0$ and $f\in Y_q$ for some $5/2<q<3$.*

(i) For each bounded set $D\subset\Omega$,

$$[\![u]\!]^2_{D\times(0,\infty)} := \operatorname*{ess\ sup}_{t\in(0,\infty)} \int_D |u(x,t)|^2 dx + \int_0^\infty \int_D |\nabla u(x,t)|^2 dxdt \leq C_{D,q}\|f\|^2_{Y_q}.$$

(ii) For any $\phi \in C_0^\infty(\Omega \times (0,\infty))$,

$$\int_\Omega |u(x,t)|^2 \phi dx + 2\int_0^t \int_\Omega |\nabla u|^2 \phi dx ds =$$
$$= \int_0^t \int_\Omega \left[|u|^2 (\partial_t \phi + \Delta \phi) + (|u|^2 + 2p) u \nabla \phi + 2uf\phi \right] dx ds.$$

Proof. (i) We know from Proposition 3.1 that $u \in L_t^\infty L_x^3$ and $\nabla u \in L_t^2 L_x^3$. Put $D_\infty = D \times (0,\infty)$. Then

$$[\![u]\!]_{D_\infty}^2 = \|u\|_{L_t^\infty L_x^2(D_\infty)}^2 + \|\nabla u\|_{L_{t,x}^2(D_\infty)}^2 \le C_D \|u\|_{L_t^\infty L_x^3(D_\infty)}^2 + C_D \|\nabla u\|_{L_t^2 L_x^3(D_\infty)}^2 \le C_{D,q} \|f\|_{Y_q}^2$$

(ii) Multiplying both sides of the equation $\partial_t u - \Delta u + \nabla p = f$ by $u\phi$ and integrating both sides over Ω, we get

$$\int_\Omega (\partial_t u) \cdot u\phi dx - \int_\Omega (\Delta u) \cdot u\phi dx + \int_\Omega (\nabla p) \cdot u\phi dx = \int_\Omega f \cdot u\phi dx. \tag{3.10}$$

Each integral is well-defined because $u \in L_x^\infty L_x^3$ and $\partial_t u$, Δu, ∇p, $f \in L_t^r L_x^q$ with $1 < r < 2q/(3q-3)$. By Poincaré inequality, $p \in L_t^r L_x^q(D \times (0,T))$ for any finite T and bounded $D \subset \Omega$. Thus, one can perform integration by part on the second and third terms on the left hand side of (3.10) to obtain the local energy identity. □

3.3 Weak solutions of Stokes system

The definition, existence and uniqueness of weak solutions to the Cauchy problem of the Stokes equations in the setting $u_0 \in L_\sigma^2$ and $f \in L_t^2 H_x^{-1}$ are given in [99, Ch. III, §1] and [88, Ch. IV, §2]. Here we give an adaptation of the so-called *weak-strong uniqueness* in critical setting $u_0 = 0$ and $f \in Y_q$. A function $u : \Omega \times (0,T) \to \mathbb{R}^3$, $T < \infty$, is said to be a *weak solution* to $(\mathrm{SE})_\Omega$ with $u_0 = 0$ if it satisfies three following conditions.

(a) $u \in L_t^\infty L_\sigma^2 \cap L_t^2 W_0^{1,2}(\Omega \times (0,T))$.

(b) $\|u(t)\|_{L_x^2} \to 0$ as $t \to 0^+$.

(c) For each $t \in (0,T)$ and vector field $\phi \in C_0^\infty(\Omega \times (0,T))$ with div $\phi = 0$,

$$\int_\Omega u(x,t)\phi(x,t)dx + \int_0^t \int_\Omega \nabla u \nabla \phi dx ds = \int_0^t \int_\Omega (u\partial_t \phi + f\phi) dx ds.$$

Proposition 3.8. *Let $u_0 = 0$, $f \in Y_q \cap L^2_t L^{3/2}_x$ for some $5/2 < q < 3$, and u be the solution to (SE)$_\Omega$ defined via Stokes operator. Then u is a weak solution. Moreover, it is the only weak solution.*

Proof. Put $X = L^2_t L^{3/2}_x(\Omega \times (0,T))$. By Solonnikov's coercive estimates, $\|\partial_t u\|_X$, $\|\nabla^2 u\|_X \le A\|f\|_X$. Then

$$\|u(t)\|_{L^{3/2}_x} \le \int_0^t \|\partial_t u(s)\|_{L^{3/2}_x} ds \le t^{1/2}\|\partial_t u(s)\|_{L^2_t L^{3/2}_x} \le At^{1/2}\|f\|_X.$$

By interpolation inequality of derivatives [2, Thm. 4.17],

$$\|\nabla u(t)\|_{L^{3/2}_x} \le A\,\|u(t)\|^{1/2}_{L^{3/2}_x}\left\|\nabla^2 u(t)\right\|^{1/2}_{L^{3/2}_x} \le At^{1/4}\|f\|^{1/2}_X\left\|\nabla^2 u(t)\right\|^{1/2}_{L^{3/2}_x}.$$

For all $0 < s \le t$,

$$\|\nabla u(s)\|_{L^4_t L^{3/2}_x} \le At^{1/4}\|f\|^{1/2}_X\left\|\nabla^2 u(s)\right\|^{1/2}_{L^2_t L^{3/2}_x} \le At^{1/4}\|f\|_X.$$

By Part (v) of Proposition 3.1, $\|\nabla u\|_{L^2_t L^3_x} \le C_q\|f\|_{Y_q}$. By interpolation,

$$\|\nabla u(s)\|_{L^{8/3}_t L^2_x} \le \|\nabla u(s)\|^{1/2}_{L^4_t L^{3/2}_x}\|\nabla u(s)\|^{1/2}_{L^2_t L^3_x} \le C_q t^{1/8}\,\|f\|^{1/2}_X\,\|f\|^{1/2}_{Y_q}.$$

By interpolation,

$$\|u(t)\|_{L^2_x} \le \|u(t)\|^{1/2}_{L^{3/2}_x}\|u(t)\|^{1/2}_{L^3_x} \le C_q t^{1/4}\,\|f\|^{1/2}_X\,\|f\|^{1/2}_{Y_q}. \tag{3.11}$$

Hence,

$$\int_\Omega |u(x,t)|^2 dx + \int_0^t\int_\Omega |\nabla u|^2 dxds \le C_q\sqrt{t}\,\|f\|^{1/2}_X\,\|f\|^{1/2}_{Y_q}.$$

Conditions (a) and (b) are satisfied. To verify (c), we multiply both sides of the Stokes equation $\partial_t u - \Delta u + \nabla p = f$ by ϕ and integrate both sides over $\Omega \times (0,T)$. The regularity of u in Proposition 3.1 and (a) allows us to perform integration by parts.

Let v be an arbitrary weak solution to (SE)$_\Omega$. The weak form of the Stokes equations can be rewritten as

$$\int_\Omega v(x,t)\phi(x,t)dx - \int_0^t\int_\Omega v\partial_t\phi dxds = \int_0^t\int_\Omega(-\nabla v\nabla\phi + f\phi)dxds. \tag{3.12}$$

Let V be the subspace of $H^1_0(\Omega)$ consisting of all divergence-free vector fields, and denote be V' its dual space. Regard v as a function from $(0,T)$ to V. It follows from (3.12) that for a.e. $t \in (0,T)$, the weak derivative $v' = \partial_t v$ is given by

$$\langle \partial_t v, \psi\rangle = \int_\Omega(-\nabla v\nabla\psi + f\psi)dxds \quad \forall\ \psi \in V.$$

Because $H_0^1(\Omega)$ is embedded into $L^3(\Omega)$, taking the duality of each space gives the embedding $L^{3/2}(\Omega) \hookrightarrow H^{-1}(\Omega) \hookrightarrow V'$. Thus, $\|f\|_{L^2V'} \le C_\Omega \|f\|_{L_t^2 L_x^{3/2}}$. Then $\partial_t v \in L_t^2 V'$ and $\|\partial_t v\|_{L_t^2 V'} \le \|\nabla v\|_{L_{t,x}^2} + C_\Omega \|f\|_{L_t^2 L_x^{3/2}}$. For each $t \in (0,T)$,

$$\begin{aligned}
\langle \partial_t v, v-u\rangle &= \int_\Omega [-\nabla u \nabla(v-u) + f(v-u)]dx, \\
\langle \partial_t u, v-u\rangle &= \int_\Omega [-\nabla v \nabla(v-u) + f(v-u)]dx.
\end{aligned}$$

Put $w = v - u$. Then

$$\langle \partial_t w, w\rangle = -\int_\Omega |\nabla w|^2 dx. \tag{3.13}$$

We show that the left hand side is equal to $\frac{1}{2}\frac{d}{dt}\int_\Omega |w(x,t)|^2 dx$ for a.e. $t \in (0,T)$. As a Banach space-valued function on $(0,T)$, w satisfies

$$\begin{aligned}
w &\in L^2((0,T),V) \cap L^\infty((0,T),H), \\
\partial_t w &\in L^2((0,T),V'),
\end{aligned}$$

where $H = L_\sigma^2(\Omega)$, i.e. the subspace of $L^2(\Omega)$ consisting of all divergence-free vector fields. Extend w by 0 outside of the interval $(0,T)$ and denote the extension by $\bar{w}$. Let $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be mollifiers in $\mathbb{R}$. Put $w_\varepsilon = \bar{w} * \eta_\varepsilon$. Then w_ε is a smooth function on $\mathbb{R}$ and

$$\begin{aligned}
w_\varepsilon &\to w && \text{in} \quad L^2((t_1,t_2),V) \cap L^\infty((t_1,t_2),H), \\
\partial_t w_\varepsilon &\to \partial_t w && \text{in} \quad L^2((t_1,t_2),V')
\end{aligned}$$

uniformly on each interval $[t_1,t_2] \subset (0,T)$. As a consequence, $w_\varepsilon(s) \to w(s)$ in H for a.e. $s \in (0,T)$. By the smoothness of w_ε with respect to t and the fact that $\partial_t w_\varepsilon \in L^2(\mathbb{R}, L^2(\Omega))$,

$$\langle \partial_t w_\varepsilon, w_\varepsilon\rangle = \int_\Omega (\partial_t w_\varepsilon) w_\varepsilon dx = \frac{d}{dt}\int_\Omega \frac{|w_\varepsilon(t)|^2}{2} dx.$$

Letting $\varepsilon \to 0$, we obtain

$$\langle \partial_t w, w\rangle = \frac{1}{2}\frac{d}{dt}\int_\Omega |w(x,t)|^2 dx \quad \text{a.e. } t \in (0,T), \tag{3.14}$$

where the right hand side is interpreted as weak derivative of a single-variable function. Together with (3.13), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |w(x,t)|^2 dx = -\int_\Omega |\nabla w|^2 dx.$$

The right hand side is an integrable function. Hence, there exists a constant $c \in \mathbb{R}$ such that

$$\frac{1}{2}\int_\Omega |w(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla w|^2 dx ds = c \quad \text{a.e. } t \in (0,T).$$

As $t \to 0$,one gets $c = 0$. Therefore, $w = 0$ a.e. $t \in (0,T)$. □

Remark 3.9. The smoothing technique in the proof is due to Temam [99, p. 261-264]. It can be used to show that (3.14) is true for more general Hilbert spaces V, H, V' (see the same reference for formulation). Another approximation method is using an orthogonal basis of Hilbert spaces. This is done in [84, Thm. 2.1, 2.2].

Remark 3.10. The weak-strong uniqueness is used to show that the sw-solution introduced in Section 5.1 admits a certain decomposition (Proposition 5.5).

Corollary 3.11. *Under the same assumptions as in Proposition 3.8, u belongs to the Hölder space $C^{1/4}([0,T], L^2(\Omega))$.*

Proof. Similarly to (3.11),

$$\|u(t_2) - u(t_1)\|_{L^2_x} \le C_q (t_2 - t_1)^{1/4} \|f\|_X^{1/2} \|f\|_{Y_q}^{1/2}$$

for all $0 \le t_1 < t_2 \le T$. □

3.4 Compactness theorem

We show another smoothing effect of the Stokes operator: weak convergence of the force terms in Y_q implies strong convergence of the solutions in supercritical spaces.

Proposition 3.12. *Suppose $f_m \rightharpoonup f$ in Y_q for some $5/2 < q < 3$. Let (u_m, p_m) and (u,p) be the solutions to the Stokes system (SE)$_\Omega$ with $u_0 = 0$ and forcing equal to f_m and f respectively. Then for a fixed $1 < r < 2q/(3q-3)$, the sequence (u_m, p_m) has a subsequence, which is still denoted by (u_m, p_m), such that for all $T < \infty$ and bounded set $D \subset \Omega$*

$$\begin{aligned}
&u_m \to u && \text{in} \quad L^\alpha_t L^\beta_x(D \times (0,T)) && \forall\ 2/\alpha + 3/\beta > 1,\ 1 \le \alpha, \beta < \infty,\\
&\nabla u_m \to \nabla u && \text{in} \quad L^\alpha_t L^\beta_x(D \times (0,T)) && \forall\ 2/\alpha + 3/\beta > 2,\ 1 \le \alpha < q_1,\ 1 \le \beta < q_2,\\
&\nabla p_m \rightharpoonup \nabla p && \text{in} \quad L^r_t L^q_x(D \times (0,T)),
\end{aligned}$$

where $q_1 = 2q/(2q-3)$ and $q_2 = 3q/(5-q)$.

Proof of Proposition 3.12. It suffices to show that for each fixed time T and bounded subset D of Ω, there exists a subsequence with the above property (since the rest of the proof follows by Cantor's diagonal argument). The weak convergence of (∇p_m) is due to the coercive estimate (Proposition 3.5). By Proposition 3.5 and Parts (i), (v) of Proposition 3.1,

$$\begin{aligned} &(u_m) \text{ and } (\nabla u_m) && \text{are bounded in } L^2((0,T), L^q(D)), \\ &(\partial_t u_m),\ (\nabla^2 u_m),\ (\nabla p_m) && \text{are bounded in } L^r((0,T), L^q(D)). \end{aligned}$$

Thus, (u_m) is bounded in $L^2((0,T), W^{1,q}(D))$. Applying Aubin-Lions lemma (Proposition A.4) for $W^{1,q}(D) \hookrightarrow L^2(D) \hookrightarrow L^2(D)$ and $\alpha_0 = 2$, $\alpha_1 = r$, we conclude that (u_m) has a convergent subsequence in $L^2(D \times (0,T))$, which we still denote by (u_m). In particular, (u_m) converges strongly in $L^1(D \times (0,T))$. For α, β such that $2/\alpha + 3/\beta > 1$ and $1 \le \alpha, \beta < \infty$, there exist $\gamma > \alpha$ and $\kappa > \beta$ such that $2/\gamma + 3/\kappa = 1$. By Part (i) of Proposition 3.1, (u_m) is bounded in $L^\gamma_t L^\kappa_x(D \times (0,T))$. Then by Lemma A.5, (u_m) converges strongly in $L^\alpha_t L^\beta_x(D \times (0,T))$. On the other hand, we know that the map that maps force term to the solution of Stokes system is a continuous function from Y_q to $L^5_{t,x}$. This implies $u_m \rightharpoonup u$ in $L^5(D \times (0,T))$. Therefore, the strong limit of (u_m) must be equal to u.

By coercive estimates (Proposition 3.5), we know that (∇u_m), $(\nabla^2 u_m)$, (∇p_m) and (f_m) are bounded in $L^r((0,T), L^q(D))$. For each $\phi \in C_0^\infty(D)$,

$$\langle \nabla \Delta u_m, \phi \rangle = -\int_D \Delta u_m \nabla \phi dx \le \left\|\nabla^2 u_m\right\|_{L^q(D)} \|\nabla \phi\|_{L^{q'}(D)} = \left\|\nabla^2 u_m\right\|_{L^q(D)} \|\phi\|_{W^{1,q'}_0(D)},$$

$$\langle \nabla \nabla p_m, \phi \rangle = -\int_D \nabla p_m \nabla \phi dx \le \|\nabla p_m\|_{L^q(D)} \|\nabla \phi\|_{L^{q'}(D)} = \|\nabla p_m\|_{L^q(D)} \|\phi\|_{W^{1,q'}_0(D)},$$

$$\langle \nabla f_m, \phi \rangle = -\int_D f_m \nabla \phi dx \le \|f_m\|_{L^q(D)} \|\nabla \phi\|_{L^{q'}(D)} = \|f_m\|_{L^q(D)} \|\phi\|_{W^{1,q'}_0(D)}.$$

Thus, $\partial_t \nabla u_m = \nabla \Delta u_m - \nabla \nabla p_m + \nabla f_m$ is bounded in $L^r((0,T), W^{-1,q}(D))$. By Aubin-Lions lemma for $W^{1,q}(D) \hookrightarrow L^q(D) \hookrightarrow W^{-1,q}(D)$ and $\alpha_0 = \alpha_1 = r$, we conclude that up to a subsequence (∇u_m) converges strongly in $L^r((0,T), L^q(D))$. In particular, (∇u_m) converges strongly in $L^1(D \times (0,T))$. For α, β such that $2/\alpha + 3/\beta > 2$, $1 \le \alpha < q_1$, $1 \le \beta < q_2$, there exist $\alpha < \gamma \le q_1$ and $\beta < \kappa < q_2$ such that $2/\gamma + 3/\kappa = 2$. By Part (v) of Proposition 3.1, the sequence (∇u_m) is bounded in $L^\gamma_t L^\kappa_x(D \times (0,T))$. Then by

Lemma A.5, (∇u_m) converges in $L^\alpha_t L^\beta_x(D\times(0,T))$. We know that the map mapping the force term to the gradient of the solution of the Stokes system is a continuous function from Y_q to $L^2_t L^3_x$. This implies $\nabla u_m \rightharpoonup \nabla u$ in $L^2_t L^3_x$. Therefore, the strong limit of (∇u_m) must be equal to ∇u. $\square$

Chapter 4

Mild solutions with $f \in Y_q$

4.1 Estimates for the bilinear map $B(u,v)$

Put

$$B(u,v) = -\int_0^t e^{(t-s)\mathbb{A}}\mathbb{P}\,\mathrm{div}(u\otimes v)(s)ds. \tag{4.1}$$

This is the solution to the Stokes system $(\mathrm{SE})_\Omega$ with zero initial condition and force term $-u\cdot\nabla v$. Note that $B(u,v)$ is well-defined even if u and v are not differentiable. The interpretation of $B(u,v)$ in this case is mentioned in Part (iii) of Proposition 2.9. For $0<T\le\infty$ and $3\le p\le\infty$, denote

$$\begin{aligned} X_T &= \{w:\Omega\times(0,T)\to\mathbb{R}^3:\ w\in L^5_{t,x},\ w(t)\in L^5_\sigma \text{ for a.e } t\in(0,T)\},\\ E_p &= \{w:\Omega\times(0,T)\to\mathbb{R}^3:\ t^{\frac12-\frac{3}{2p}}w\in L^\infty_t L^p_x,\ w(t)\in L^p_\sigma,\ \lim_{t\to0^+} t^{\frac12-\frac{3}{2p}}\|w(t)\|_{L^p}=0\}. \end{aligned}$$

Then X_T and E_p are Banach spaces with respective norms

$$\|w\|_{X_T} = \left(\int_0^T\int_\Omega |w|^5dxdt\right)^{1/5},\quad \|w\|_{E_p} = \left\|t^{\frac12-\frac{3}{2p}}w\right\|_{L^\infty_t L^p_x(\Omega\times(0,T))}.$$

The space $E=\{w\in X_T:\ t^{1/8}\nabla w\in L^{8/3}_t L^3_x\}$ has a semi-norm $[w]_E=\|t^{1/8}\nabla w\|_{L^{8/3}_t L^3_x}$.

Proposition 4.1. *The following statements are true.*

(i) $\|B(u,v)\|_{L^r_t L^p_x}\le A\|u\|_{X_T}\|v\|_{X_T}$ *for all* $u,v\in X_T$ *and* $2/r+3/p=1$, $3<p<15$.

(ii) $\|B(u,v)\|_{E_p}\le A\|u\|_{X_T}[v]_E$ *for all* $u\in X_T$, $v\in E$, $3\le p<4$.

(iii) $\|B(u,v)\|_{E_p}\le C_{\bar p p}\|u\|_{E_{\bar p}}\|v\|_{E_{\bar p}}$ *for all* $u,v\in E_{\bar p}$, $3\le p$, $\bar p\le\infty$, $1/p>2/\bar p-1/3$.

(iv) $B(u,v) \in C([0,T), L^3)$ *for all* $u \in E_{24}$, $v \in E$.

(v) $\|\sqrt{t}\nabla B(u,v)\|_{L^\infty_t L^3_x} \le A\|u\|_{E_p}[v]_E$ *for all* $u \in E_p$, $v \in E$, $15 < p < \infty$.

Proof. (i) By Part (iii) of Proposition 2.9,

$$\|B(u,v)\|_{L^p_x} \le A \int_0^t \frac{\|u \otimes v(s)\|_{L^{5/2}_x}}{(t-s)^{\frac{11}{10}-\frac{3}{2p}}}. \tag{4.2}$$

The specified range for p guarantees that $0 < 1/r < 2/5$. This allows the use of the fractional inequality (Proposition A.16), which yields

$$\|B(u,v)\|_{L^r_t L^p_x} \le A\|u \otimes v\|_{L^{5/2}_{t,x}} \le A\|u\|_X \|v\|_X.$$

(ii) By Corollary 2.7,

$$\begin{aligned}
\|B(u,v)\|_{L^p_x} &\le A \int_0^t \frac{\|u \cdot \nabla v(s)\|_{L^{15/8}_x}}{(t-s)^{\frac{4}{5}-\frac{3}{2p}}} ds \le A \int_0^t \frac{\left\|u \cdot s^{1/8}\nabla v(s)\right\|_{L^{15/8}_x}}{(t-s)^{\frac{4}{5}-\frac{3}{2p}} s^{\frac{1}{8}}} ds \\
&\le A\left\|u \cdot s^{1/8}\nabla v(s)\right\|_{L^{40/23}_t L^{15/8}_x} \left(\int_0^t \frac{1}{(t-s)^{\left(\frac{4}{5}-\frac{3}{2p}\right)\frac{40}{17}} s^{\frac{1}{8}\cdot\frac{40}{17}}} ds \right)^{17/40} \\
&\le C_p \|u\|_{L^5_{t,x}} \left\|s^{1/8}\nabla v(s)\right\|_{L^{8/3}_t L^3_x} \frac{1}{t^{\frac{1}{2}-\frac{3}{2p}}}.
\end{aligned}$$

Note that the condition $3 \le p < 4$ guarantees that the constant C_p is finite.

(iii) By Part (iii) of Proposition 2.9,

$$\|B(u,v)\|_{L^p_x} \le A \int_0^t \frac{\|u \otimes v(s)\|_{L^{\bar{p}/2}_x}}{(t-s)^{2-\frac{3}{2\alpha}}} ds \le A \int_0^t \frac{\|u\|_{L^{\bar{p}}_x}\|v\|_{L^{\bar{p}}_x}}{(t-s)^{2-\frac{3}{2\alpha}}} ds$$

where $1/\alpha = 1 - 2/\bar{p} + 1/p$. Therefore,

$$\|B(u,v)\|_{L^p_x} \le A\|u\|_{E_{\bar{p}}}\|v\|_{E_{\bar{p}}} \int_0^t \frac{ds}{(t-s)^{\frac{1}{2}+\frac{3}{2}\left(\frac{2}{\bar{p}}-\frac{1}{p}\right)} s^{1-\frac{3}{\bar{p}}}} = C_{\bar{p}p} \frac{\|u\|_{E_{\bar{p}}}\|v\|_{E_{\bar{p}}}}{t^{\frac{1}{2}-\frac{3}{2p}}}.$$

(iv) Put $f = \operatorname{div}(u \otimes v) = u \cdot \nabla v$ and

$$\bar{q} = \frac{8}{3} \in \left(\frac{5}{2},\ 3\right), \quad \bar{q}_* = \frac{3}{2} - \frac{5}{2\bar{q}} = \frac{9}{16}.$$

Then

$$t^{\bar{q}_*} f = \underbrace{(t^{7/16} u)}_{L_t^\infty L_x^{24}} \cdot \underbrace{(t^{1/8} \nabla v)}_{L_t^{8/3} L_x^3} \in L_{t,x}^{8/3} = L_{t,x}^{\bar{q}}.$$

Applying Part (iii) of Proposition 3.1 for $F = B(u,v)$, we conclude that $B(u,v) \in C([0,T), L^3)$.

(v) For $12 < p < \infty$, the constant C_p is finite. Put $f = u \cdot \nabla v$ and $\tilde{p} = 3p/(3+p)$. Note that $\tilde{p} \in (5/2, 3)$ for $p > 15$. Then for any $T_1 \leq T$, $T_1 < \infty$,

$$t^{\bar{p}_*} f = t^{\frac{1}{24} + \frac{3}{2p}} \underbrace{(t^{\frac{1}{2} - \frac{3}{2p}} u)}_{L_t^\infty L_x^p} \cdot \underbrace{(t^{1/8} \nabla v)}_{L_t^{8/3} L_x^3} \in L_t^{8/3} L_x^{\tilde{p}} \subset L_{t,x}^{\tilde{p}}(\Omega \times (0, T_1)).$$

Thus, $f \in Y_{\tilde{p}, T_1}$. By Part (vi) of Proposition 3.1, $\sqrt{t} \nabla B(u,v) \in C_t L_x^3(\Omega \times [0, T_1])$. □

4.2 Mild solutions

In the subcritical setting $f = 0$ and $u_0 \in L^r(\Omega)$ for some $r > 3$, it is known that mild solutions exist for any domain Ω with uniform C^3-boundary that admits Helmholtz decomposition for $L^r(\Omega)$ (see [24, Thm. 3.2], [77]). In this section, we show the existence, uniqueness, and several regularity properties of mild solutions in the critical setting $f \in Y_q$ and $u_0 = 0$ for $\Omega = \mathbb{R}^3_+$ or $\mathbb{R}^3$. Mild solution u of the $(\text{NSE})_\Omega$ is given by $u = F + B(u,u)$ where F is given by (3.2) and B given by (4.1).

Proposition 4.2. *Let $f \in Y_q$ for some $5/2 < q < 3$. There exists $T \in (0, \infty)$ such that $(\text{NSE})_\Omega$ has a unique mild solution $u \in X_T$. Moreover,*

(i) $u \in L_t^r L_x^p(\Omega \times (0,T))$ *for all* $2/r + 3/p = 1$, $3 < p < 15$.

(ii) $t^{1/8} \nabla u \in L_t^{8/3} L_x^3(\Omega \times (0,T))$.

(iii) $u \in E_p$ *for all* $3 \leq p \leq \infty$. *Moreover,* $u \in C([0,T], L^3)$.

(iv) $\nabla u \in L_t^r L_x^p(\Omega \times (0,T))$ *for all* $2/r + 3/p = 2$, $q \leq p < 3q/(5-q)$.

(v) $\sqrt{t} \nabla u \in L_t^\infty L_x^3 \cap C_t L_x^3(\Omega \times [0,T])$ *and* $\sqrt{t} \|\nabla u\|_{L_x^3} \to 0$ *as* $t \to 0^+$.

(vi) u can be extended via Picard's iterative process to a maximal time interval $(0, T_*)$. The properties (i)-(v) hold for any $T < T_*$. Moreover, if T_* is finite then $\|u\|_{X_T} \to \infty$ as $T \to T_*^-$.

Proof. Denote $Y_{q,T} = \left\{ f : \Omega \times (0,T) \to R^3 : t^{q_*} f \in L^q_{t,x} \right\}$. By Part (i) of Proposition 3.1,

$$\|F\|_{X_T} \le C_q \|f\|_{Y_{q,T}} = C_q \|t^{q_*} f\|_{L^q(\Omega \times (0,T))}. \tag{4.3}$$

By Part (i) of Proposition 4.1, $\|B(u,v)\|_{X_T} \le A\|u\|_{X_T}\|v\|_{X_T}$. By Lemma 1.3, the equation $u = F + B(u,u)$ has a unique solution in X_T provided that $\|F\|_{X_T}$ is less than an constant depending on q. Thanks to (4.3), this condition is satisfied when T is sufficiently small. To show the properties of u, we first show that (i)-(iv) hold for the Stokes equations, i.e. when u is substituted by F. In this situation, Part (i) and (iv) are already proved in Proposition 3.1. Part (ii) and (iii) are consequences of the interpolation inequalities. Indeed,

$$\int_\Omega |t^{1/8}\nabla F|^3 dx = \int_\Omega |t^{1/2}\nabla F|^{3/4} |\nabla F|^{9/4} dx \overset{\text{Hölder}}{\le} \left\{ \int_\Omega |t^{1/2}\nabla F|^3 dx \right\}^{\frac{1}{4}} \left\{ \int_\Omega |\nabla F|^3 dx \right\}^{\frac{3}{4}}.$$

Thus, $\left\|t^{1/8}\nabla F\right\|_{L_t^{8/3} L_x^3} \le \left\|\sqrt{t}\nabla F\right\|^{1/4}_{L_t^\infty L_x^3} \|\nabla F\|^{3/4}_{L_t^2 L_x^3} \le C_q \|f\|_{Y_{q,T}}$. Similarly,

$$\left\|t^{1/2-3/(2p)} F\right\|_{L_t^\infty L_x^p} \le \left\|\sqrt{t} F\right\|^{1-3/p}_{L^\infty_{t,x}} \|F\|^{3/p}_{L_t^\infty L_x^3} \le C_{pq} \|f\|_{Y_T}.$$

(i) This is a consequence of Part (i) of Proposition 4.1.

(ii) Suppose T is sufficiently small such that the mild solution u obtained by the iterative process

$$\begin{cases} u^{(0)} = F, \\ u^{(n+1)} = F + B(u^{(n)}, u^{(n)}) \quad \forall\, n \ge 0. \end{cases}$$

satisfies $\|u\|_{X_T} < \varepsilon$, where $\varepsilon > 0$ is an absolute constant to be chosen later.

$$\nabla B(u,v) = -\int_0^{t/2} e^{(t-s)\mathbb{A}} \mathbb{P}\mathrm{div}(u(s) \otimes v(s)) ds - \int_{t/2}^t e^{(t-s)\mathbb{A}} \mathbb{P}\mathrm{div}(u(s) \otimes v(s)) ds.$$

By Corollary 2.7 and Part (iii) of Proposition 2.9,

$$\begin{aligned} \left\|\nabla B(u^{(n)}, u^{(n)})\right\|_{L_x^3} &\le \int_0^{t/2} \frac{A \left\|u^{(n)}(s)\right\|^2_{L_x^5}}{(t-s)^{11/10}} ds + \int_{t/2}^t \frac{A \left\|\nabla u^{(n)}\right\|_{L_x^3} \left\|u^{(n)}\right\|_{L_x^5}}{(t-s)^{4/5}} ds. \\ &\le \frac{A}{t^{1/8}} \int_0^{t/2} \frac{\left\|u^{(n)}(s)\right\|^2_{L_x^5}}{(t-s)^{39/40}} ds + \frac{A}{t^{1/8}} \int_{t/2}^t \frac{\left\|s^{1/8}\nabla u^{(n)}\right\|_{L_x^3} \left\|u^{(n)}(s)\right\|_{L_x^5}}{(t-s)^{4/5}} ds. \end{aligned}$$

Applying the fractional inequality (Proposition A.16) after multiplying both sides by $t^{1/8}$, we get

$$\left\|t^{1/8}\nabla B(u^{(n)},u^{(n)})\right\|_{L_t^{8/3}L_x^3} \le A\left\|u^{(n)}\right\|_{X_T}^2 + A\left\|u^{(n)}\right\|_{X_T}\left\|s^{1/8}\nabla u^{(n)}\right\|_{L_s^{8/3}L_x^3}.$$

Put $a_n = \|t^{1/8}\nabla u^{(n)}\|_{L_t^{8/3}L_x^3}$. Then

$$\begin{aligned} a_{n+1} = \left\|t^{1/8}\nabla u^{(n+1)}\right\|_{L_t^{8/3}L_x^3} &\le \left\|t^{1/8}\nabla F\right\|_{L_t^{8/3}L_x^3} + \left\|t^{1/8}\nabla B(u^{(n)},u^{(n)})\right\|_{L_t^{8/3}L_x^3} \\ &\le a_0 + A\varepsilon^2 + A\varepsilon a_n. \end{aligned}$$

Choose $\varepsilon > 0$ such that $A\varepsilon$, $A\varepsilon^2 < 1/2$. Then $a_n \le 2(a_0+1)$ for all $n \in \mathbb{N}$. The sequence $t^{1/8}\nabla u^{(n)}$ converges weakly in $L_t^{8/3}L_x^3$. Therefore, $t^{1/8}\nabla u \in L_t^{8/3}L_x^3$.

(iii) For $3 \le p < 4$, by Part (ii) of Proposition 4.1,

$$\|B(u,u)\|_{E_p} \le A\|u\|_{X_T}\left\|s^{1/8}\nabla u\right\|_{L_s^{8/3}L_x^3}.$$

Hence, $u = F + B(u,u) \in E_p$. Applying Part (iii) of Proposition 4.1 for $\bar{p} = 15/4$, we get $u \in E_p$ for all $3 \le p < 5$. We then can take $\bar{p} = 9/2$, and obtain $u \in E_p$ for all $3 \le p < 9$. With $\bar{p} = 8$, we have $u \in E_p$ for all $3 \le p \le \infty$.

(iv) Put $\alpha = 3q/(3-q) > 15$. Because $u \in E_\alpha \cap E$, we have $u \cdot \nabla u \in Y_q$ by the argument in the proof of Part (v) of Proposition 4.1. Then by Part (v) of Proposition 3.1, we have $\nabla B(u,u) \in L_t^r L_x^p$. Then $\nabla u = \nabla F + \nabla B(u,u) \in L_t^r L_x^p$.

(v) Similarly to Part (iv), we have $u \cdot \nabla u \in Y_q$. Then by Part (vi) of Proposition 3.1, $\sqrt{t}\nabla B(u,u) \in L_t^\infty L_x^3 \cap C_t L_x^3$. Then $\sqrt{t}\nabla u = \sqrt{t}\nabla F + \sqrt{t}\nabla B(u,u) \in L_t^\infty L_x^3 \cap C_t L_x^3$.

(vi) Regard T as the starting time. Put $g(x,t) = f(x,t+T)$ and $v_0(x) = u(x,T)$. Then $g \in L_{t,x}^q$ and $v_0 \in W_0^{1,3}(\Omega)$ by Part (iii) and (v). Consider the Navier–Stokes problem with initial condition v_0 and external force g

$$(\mathrm{S}) : \begin{cases} \partial_t v - \Delta v + v \cdot \nabla v + \nabla \pi &= g, \\ \operatorname{div} v &= 0, \\ v|_{\partial\Omega} &= 0, \\ v(\cdot,0) &= v_0. \end{cases}$$

For extension purpose, it suffices to show that (S) has a short-time mild solution in the setting $L^5_{t,x}$. However, for the continuation of regularity we show that (S) has a short-time mild solution in a more regular space Z_τ defined as follows.

$$Z_\tau = \left\{ v \in L^\infty_t L^5_\sigma(\Omega \times (0,\tau)) : \ \nabla v \in L^\infty_t L^3_x \right\}, \qquad \|v\|_{Z_\tau} = \|v\|_{L^\infty_t L^5_x} + \|v\|_{L^\infty_t L^3_x}.$$

This is a subcritical setting. Put $V(x,t) = e^{t\mathbb{A}} v_0$ and $G_0(x,t) = \int_0^t e^{(t-s)\mathbb{A}} \mathbb{P} g(s)$. We first show that V, $G_0 \in Z_\tau$ for any $\tau < \infty$. By Corollary 2.7,

$$\|G_0(t)\|_{L^5_x} \le \int_0^t \frac{C_q}{(t-s)^{\frac{3}{2q}-\frac{3}{10}}} \|g(s)\|_{L^q_x} ds \le C_q \left\{ \int_0^t (t-s)^{-\left(\frac{3}{2q}-\frac{3}{10}\right)q'} ds \right\}^{1/q'} \|g\|_{L^q_{t,x}},$$

which is finite for all $t > 0$ because $(\frac{3}{2q} - \frac{3}{10})q' < 1$. By Corollary 2.7,

$$\|\nabla G_0(t)\|_{L^3_x} \le \int_0^t \frac{C_q}{(t-s)^{\frac{3}{2q}}} \|g(s)\|_{L^q_x} ds \le C_q \left\{ \int_0^t (t-s)^{-\frac{3}{2q}q'} ds \right\}^{1/q'} \|g\|_{L^q_{t,x}}$$

which is also finite for all $t > 0$ because $\frac{3}{2q}q' < 1$. Since $v_0 \in W^{1,3}_0(\Omega) \hookrightarrow L^5$, we have $\|V(t)\|_{L^5_x} \le A\|v_0\|_{L^5}$ for all $t > 0$. For $\Omega = \mathbb{R}^3$, $V(t) = \Gamma(t) * v_0$ and

$$\|\nabla V(t)\|_{L^3_x} \le \|\Gamma(t) * \nabla v_0\|_{L^3_x} \le A\|\nabla v_0\|_{L^3} \quad \forall t > 0.$$

For $\Omega = \mathbb{R}^3_+$, we obtain in the proof of Proposition 2.5 that modulo proper extensions of v_0 to the whole space, $V(x,t)$ is given by a convolution. Specifically, we derived that

$$V(t) = e^{t\mathbb{A}} v_0 = \int_{\mathbb{R}^3_+} G(x,y,t) v_0(y) dy = \Gamma(.,t) * \check{e}(v_0) + K(.,t) * \mathbb{J} v_0^*,$$

where $\check{e}(v_0)$ is the extension of v_0 by odd reflection

$$v_0^*(x) = \begin{cases} 0 & \text{if} \quad x_3 > 0, \\ v_0(x', -x_3) & \text{if} \quad x_3 < 0, \end{cases}$$

$$K(x,t) = (I_{x_3>0} \nabla^2 \Phi) * \Gamma(.,t),$$

and $\mathbb{J}$ is a constant 3×3 matrix. Because $v_0 \in W^{1,3}_0(\mathbb{R}^3_+)$, both $\check{e}(v_0)$ and v_0^* belong to $W^{1,3}(\mathbb{R}^3)$. Then

$$\nabla V(t) = \Gamma(.,t) * \check{e}(v_0) + K(.,t) * \mathbb{J} \nabla v_0^*.$$

Note that K has a scaling property $K(x,t) = t^{-3/2} K(\frac{x}{\sqrt{t}}, 1)$. As shown in the proof of Proposition 2.5, $\widehat{K(.,1)}$ is a Fourier multiplier. Hence, $\|\nabla V(t)\|_{L^3_x} \le A\|\nabla v_0\|_{L^3}$ for

all $t > 0$. We have showed that V, $G_0 \in Z_\tau$ for all $\tau < \infty$. For any $v, w \in Z_\tau$, by Corollary 2.7

$$\|B(v,w)\|_{L^5_x} \le \int_0^t A\frac{\|v\cdot\nabla w\|_{L^{15/8}_x}}{(t-s)^{1/2}}ds \le \int_0^t A\frac{\|v\|_{L^5_x}\|\nabla w\|_{L^3_x}}{(t-s)^{1/2}}ds \le A\tau^{1/2}\|v\|_{Z_\tau}\|w\|_{Z_\tau},$$

$$\|\nabla B(v,w)\|_{L^3_x} \le \int_0^t A\frac{\|v\cdot\nabla w\|_{L^{15/8}_x}}{(t-s)^{4/5}}ds \le \int_0^t A\frac{\|v\|_{L^5_x}\|\nabla w\|_{L^3_x}}{(t-s)^{4/5}}ds \le A\tau^{1/5}\|v\|_{Z_\tau}\|w\|_{Z_\tau}.$$

This shows that $B : Z_\tau \times Z_\tau \to Z_\tau$ is a continuous bilinear map. Moreover, $\|B\| \le A\tau^{1/5}$ provided that $\tau < 1$. By Lemma 1.3, if $A\tau^{1/5}(\|v_0\|_{L^3} + \|\nabla v_0\|_{L^3} + \|g\|_{L^q}) < 1/4$ then the problem (S) has a mild solution in Z_τ defined by Picard's iterative process. This solution clearly belongs to $L^5_{t,x}$. Thus, we can extend mild solution (u,p) to the interval $(0, T+\tau]$ by defining $u(x, T+t) = v(x,t)$ and $p(x, T+t) = \pi(x,t)$ for all $t \in [0,\tau]$. Property (ii) holds for the extended solution because $\nabla v \in L^\infty_t L^3_x$. Properties (iii), (iv), (v) are consequences of (ii).

This extension method can be repeated as long as u remains in $L^5_{t,x}$. It ceases at some finite time T_* only if $\|u\|_{L^5(\Omega\times(0,T))} \to \infty$ as $t \to T_*^-$. □

Proposition 4.3. *Let $5/2 < q < 3$. There exists $\varepsilon_0 > 0$, possibly depending on q, such that for $f \in Y_q$ satisfying $\|f\|_{Y_q} < \varepsilon_0$, (NSE)$_\Omega$ has a unique global solution $u \in X_\infty$. Moreover, Parts (i)-(v) of Proposition 4.2 hold for $T = \infty$.*

Proof. By Proposition 3.1, $\|F\|_{X_\infty} \le C_q\|f\|_{Y_q} < C_q\varepsilon_0$. Applying Lemma 1.3 for $X = X_\infty$, we conclude that there exists a unique solution $u \in X_\infty$ to the equation $u = F + B(u,u)$ if ε_0 is sufficiently small. The proof of Part (i) of Proposition 4.2 is still valid for $T = \infty$. The proof of Part (ii) shows that $t^{1/8}\nabla u \in L^{8/3}_t L^3_x(\Omega\times(0,\tau))$ for all $0 < \tau < \infty$. For $0 < 2\tau_1 < t < \tau_2 < \infty$,

$$\|\nabla B(u,u)\|_{L^3_x} \le \frac{A}{t^{1/8}}\int_0^{t/2}\frac{\|u(s)\|^2_{L^5_x}}{(t-s)^{39/40}}ds + \frac{A}{t^{1/8}}\int_{t/2}^t\frac{\left\|s^{1/8}\nabla u\right\|_{L^3_x}\|u\|_{L^5_x}I_{(\tau_1,\tau_2)}(s)}{(t-s)^{4/5}}ds.$$

By the fractional inequality,

$$\left\|s^{1/8}\nabla B(u,u)\right\|_{L^{8/3}_s L^3_x(\Omega\times(\tau_1,\tau_2))} \le A\,\|u\|^2_{X_\infty} + A\|u\|_{L^5(\Omega\times(\tau_1,\tau_2))}\left\|s^{1/8}\nabla u\right\|_{L^{8/3}_s L^3_x(\Omega\times(\tau_1,\tau_2))}.$$

If τ_1 and τ_2 are sufficiently large, $\|u\|_{L^5(\Omega\times(\tau_1,\tau_2))} < 1/2$. In this case,

$$\left\|s^{1/8}\nabla u\right\|_{L^{8/3}_s L^3_x(\Omega\times(\tau_1,\tau_2))} \le C_q\|f\|_{Y_q} + A\,\|u\|^2_{X_\infty}.$$

By letting $\tau_2 \to \infty$, we obtain $s^{1/8}\nabla u \in L_s^{8/3}L_x^3(\Omega \times (0,\infty))$. Parts (iii)-(v) are consequences of Part (ii). $\square$

A useful consequence of Proposition 4.2 is the regularity of the nonlinear convection term $u \cdot \nabla u$ in critical spaces.

Corollary 4.4. *Let f, T, u be as in Proposition 4.2, except that T is now allowed to be ∞. Then*

(i) $u \cdot \nabla u \in Y_{q,T}$,

(ii) $u \cdot \nabla u \in L_t^r L_x^p(\Omega \times (0,T))$ *for all* $2/r + 3/p = 3$ *with*

$$\frac{5}{3q} - \frac{4}{15} < \frac{1}{p} < \frac{1}{q} + \frac{1}{3}.$$

Proof. (i) Pick $\beta \in \left(\frac{1}{8}, \frac{5}{q} - \frac{3}{2}\right)$. Then

$$|t^{q_*} u \cdot \nabla u| \le |t^\beta \nabla u||t^{1/2}u|^{2q_*-2\beta}|u|^{1+2\beta-2q_*}.$$

Write $|t^\beta \nabla u| = |t^{1/2}\nabla u|^\theta |t^{1/8}\nabla u|^{1-\theta}$ where $\theta = (8\beta - 1)/3 \in (0,1)$. By Part (ii) and (v) of Proposition 4.2 and Hölder inequality, $t^\beta \nabla u \in L_t^{\gamma_1}L_x^3$ where $\gamma_1 = \frac{2}{1-2\beta}$. Put $\lambda = 1 + 2\beta - 2q^*$ and $\frac{1}{\alpha} = \frac{1}{q} - \frac{1}{3}$. The range of β guarantees that $\alpha\lambda \in (3,15)$. By Part (i) of Proposition 4.2, $|u|^\lambda \in L_t^{\lambda_1}L_x^\alpha$ where $\frac{2}{\gamma_2} + \frac{3}{\alpha} = \lambda$. By Hölder inequality,

$$|t^{q_*} u \cdot \nabla u| \le \underbrace{|t^\beta \nabla u|}_{L_t^{\gamma_1}L_x^3} \underbrace{|t^{1/2}u|}_{L_{t,x}^\infty}^{2q_*-2\beta} \underbrace{|u|^\lambda}_{L_t^{\gamma_2}L_x^\alpha} \in L_t^r L_x^q$$

where

$$\frac{1}{r} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{q}.$$

Therefore, $t^{q_*} u \cdot \nabla u \in L_{t,x}^q$.

(ii) By Part (i) and (iv) of Proposition 4.2, $u \in L_t^{r_1}L_x^{p_1}$ for all $2/r_1 + 3/p_1 = 1$ with

$$\frac{1}{15} < \frac{1}{p_1} < \frac{1}{3},$$

$\nabla u \in L_t^{r_2}L_x^{p_2}$ for all $2/r_2 + 3/p_2 = 2$ with

$$\frac{5-q}{3q} < \frac{1}{p_2} < \frac{1}{q}.$$

By Hölder inequality, $u \cdot \nabla u \in L^r_t L^p_x$ where $1/p = 1/p_1 + 1/p_2$ takes all values strictly between $\frac{1}{15} + \frac{5-q}{3q}$ and $\frac{1}{3} + \frac{1}{q}$. $\square$

Remark 4.5. Due to Part (i), u solves the Stokes system $(\mathrm{SE})_\Omega$ with zero initial condition and force term $f - u \cdot \nabla u \in Y_{q,T}$. By Proposition 3.5, (u, p) has maximal regularity, i.e. $\partial_t u$, $\nabla^2 u$, $\nabla p \in Y_{q,T}$.

4.3 Decomposition of mild solutions

For $D \times (a, b) \subset \Omega \times (0, T)$, a function u is said to belong to the *energy class* on $D \times (a, b)$ if

$$[\![u]\!]^2_{D\times(a,b)} := \operatorname*{ess\,sup}_{t\in(a,b)} \int_D |u(x,t)|^2 dx + \int_a^b \int_D |\nabla u(x,t)|^2 dxdt < \infty. \tag{4.4}$$

It is said to belong to the *local energy class* on $\Omega \times (0, T)$ if $[\![u]\!]_{D\times(a,b)} < \infty$ for every bounded subset $D \times (a, b) \subset \Omega \times (0, T)$. Any inequality of form (4.4) is generally referred to as energy inequality or local energy inequality (depending on whether $D \times (a, b) = \Omega \times (0, T)$). As a rule of thumb, the mild solution of $(\mathrm{NSE})_\Omega$ blows up after finite time if and only if one of its critical norms blows up. On the other hand, blowup solutions typically remain in the local energy class near the blowup time. This is because under very weak assumptions on the regularity of solutions one can obtain *a priori* local energy estimates.

Global energy inequalities impose certain restrictions on the data (u_0, f); for example, $u_0 \in L^2(\Omega)$ and $f \in L^2_t H^{-1}_x(\Omega \times (0, T))$ is the required setting to define Leray-Hopf solutions [48, Para. 26-31], [31], [71, Sec. 4.2]. These spaces are supercritical with respect to scaling, and are preferred in physics since the solutions have finite kinetic and dissipation of energy. A Hilbert-space framework for Leray-Hopf solutions can be found in [99]. However, local energy inequalities are available for a much larger class of data (u_0, f); for example, u_0 is required to be only locally square-integrable. Scheffer [78] introduces the concept of local energy solutions. That is a type of weak solutions satisfying

$$2\int_0^T \int_\Omega |\nabla u|^2 \phi dxdt \le \int_0^T \int_\Omega \left[|u|^2 (\partial_t \phi + \Delta\phi) + (|u|^2 + 2p) u \nabla\phi + 2uf\phi\right] dxdt \tag{4.5}$$

for all smooth compactly supported $\phi = \phi(x,t) \geq 0$. Later, Caffarelli–Kohn–Nirenberg [12] show several local regularity criteria, known as ε-regularity criteria, which play an important role in local regularity theory. Results in this theory include: (1) estimates on the size of the set of singularities (called *partial regularity*) in the interior in [12, 45, 41], on the boundary of a 3-dimensional domain in [82, 64], 4-dimensional domain in [15], 5-dimensional domain (steady-state) in [34], 6-dimensional domain (steady-state) in [16, 15], (2) weak-strong uniqueness, i.e. the fact that weak solutions coincide the mild solution up to the blowup time, [47, Thm. 14.7, 15.3, 15.7] and (3) the existence of singularities of mild solutions at blowup time [76, 32].

For $\Omega = \mathbb{R}^3$ or a bounded domain, Caffarelli–Kohn–Nirenberg construct a local energy solution with $u_0 \in L^2_\sigma(\Omega)$ and $f \in L^2_t H^{-1}_x(\Omega \times (0,T))$ that exists globally in time [12, Thm. A.1]. For $\Omega = \mathbb{R}^3$, Lemarié-Rieusset constructs a global-in-time local energy solution with $f = 0$ and $u_0 \in L^2_{\text{uloc}}$ [46, Thm. 33.1]. A global-in-time local energy solution on the half-space is recently constructed by Maekawa, Miura and Prange [58, 73].

Seregin and Šverák [83] split the mild solution into two parts: one solves the linear Stokes system and belongs to local energy class, the other solves a nonlinear system and belongs to the energy class. Specifically, $u = v + w$ and $p = p_1 + p_2$ where

$$(\mathrm{I}) : \begin{cases} \partial_t v - \Delta v + \nabla p_1 &= f, \\ \operatorname{div} v &= 0. \\ v|_{\partial\Omega} &= 0, \\ v(0) &= 0. \end{cases}$$

$$(\mathrm{II}) : \begin{cases} \partial_t w - \Delta w + \nabla p_2 &= -u \cdot \nabla u, \\ \operatorname{div} w &= 0. \\ w|_{\partial\Omega} &= 0, \\ w(0) &= 0. \end{cases}$$

For $\Omega = \mathbb{R}^3$, they show that w exists globally in weak sense, thus obtain a global weak solution $u = v + w$. We will refer to this type of solutions as *sw-solutions*, where s stands for *strong/Stokes/split*, and w for *weak*. Since sw-solutions are also local energy solutions, ε-regularity criteria hold. This class of solutions has a number of good

properties, for example the weak-strong uniqueness and compactness theorem [83, Prop. 1.7, 1.8]. It works well with domains with boundaries because the boundary condition is treated separately in the Stokes system. Although Seregin and Šverák only consider the critical setting $u_0 \in L^3$, $f = 0$, their treatments can be adapted to the setting $u_0 = 0$, $f \in Y_q$ (Proposition 5.7 and Proposition 5.9).

Because v is global and solves the linearized problem, solutions u of $(\mathrm{NSE})_\Omega$ and w of (II) have essentially the same regularity up to blowup time. However, system (II) is more convenient to deal with because the forcing $-u \cdot \nabla u$ belongs to more diverse critical spaces (Corollary 4.4) compare to the forcing f of $(\mathrm{NSE})_\Omega$. Whether a global-in-time sw-solution exists on the half-space is an interesting question which we will not address in this thesis. Precise definition for sw-solutions in the half-space will be given in the next chapter. In the following, we show that mild solutions admit a decomposition $u = v + w$ as mentioned. Hence, we can at least conclude that sw-solutions exist locally in time.

Consider $\Omega = \mathbb{R}^3_+$ or $\mathbb{R}^3$. Let $u \in L^5(\Omega \times (0,T))$ be the mild solution to $(\mathrm{NSE})_\Omega$ with $u_0 = 0$ and $f \in Y_q$ for some $5/2 < q < 3$. Decompose $(u,p) = (v,p_1) + (w,p_2)$ where (v,p_1) solves (I) and (w,p_2) solves (II).

Proposition 4.6. *The following estimates hold.*

(i) For all $t \in (0,T)$,

$$\frac{1}{2}\int_\Omega |w(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla w|^2 dx ds = \int_0^t \int_\Omega v \otimes u : \nabla w dx ds. \tag{4.6}$$

(ii) For all $t \in (0,T)$,

$$\int_\Omega |w(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla w(x,t)|^2 dx ds \le \sqrt{t}\Psi_0(q, \|f\|_{Y_q}) \tag{4.7}$$

where $\Psi_0(q,\tau) = C_q \tau^4 \exp(C_q \tau^5)$.

(iii) For any function $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$,

$$\int_\Omega \phi(x,t)|w(x,t)|^2 dx + 2\int_0^t \int_\Omega \phi |\nabla w|^2 dx ds =$$

$$\int_0^t \int_\Omega \left[2v \otimes u : \phi \nabla w + |w|^2(\Delta\phi + \partial_t \phi) + u \cdot \nabla\phi(|w|^2 + 2q_2 + 2v \cdot w)\right] dx ds. \tag{4.8}$$

Proof. (i) By the regularity of u and v in Proposition 3.1 and Proposition 4.2, $w = u - v$ satisfies $w \in C([0,T], L^3)$, $\nabla w \in L^2_t L^3_x$, $\sqrt{t}\nabla w \in L^\infty_t L^3_x$. By Part (ii) of Corollary 4.4, $u \cdot \nabla u \in L^2_t L^{3/2}_x$. By the coercive estimate (Proposition 3.3), $\partial_t w$, $\nabla^2 w$, $\nabla p_2 \in L^2_t L^{3/2}_x(\Omega \times (0,T))$. By Poincaré–Sobolev inequality, $p_2 \in L^2((0,T), L^3_{\text{loc}}(\bar{\Omega}))$. The right hand side of (4.6) is well-defined because

$$\begin{aligned}\int_0^t \int_\Omega |v \otimes u : \nabla w| \, dxds &\leq \int_0^t \|v\|_{L^3_x} \|u\|_{L^3_x} \|\nabla w\|_{L^3_x} ds \\ &\leq \|v\|_{L^\infty_s L^3_x} \|u\|_{L^\infty_s L^3_x} \left\|\sqrt{s}\nabla w\right\|_{L^\infty_s L^3_x} \int_0^t \frac{ds}{\sqrt{s}} < \infty.\end{aligned}$$

For each $i = 1, 2, 3$, w_i satisfies the differential equation

$$\partial_t w_i - \Delta w_i + u_j v_{i,j} + u_j w_{i,j} + p_{2,i} = 0. \tag{4.9}$$

Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a nonnegative smooth function supported in $B_2(0)$ and equal to 1 in $B_1(0)$. For each $R > 0$, define $\phi_R(x) = \phi\left(\frac{x}{R}\right)$. Multiplying both sides of (4.9) by $w_i \phi_R$ and integrating over $\Omega \times (0,t)$, we get

$$\begin{aligned}&\frac{1}{2}\int_\Omega |w(t)|^2 \phi_R dx + \int_0^t \int_\Omega |\nabla w|^2 \phi_R dxds \\ =\ & \int_0^t \int_\Omega \frac{|w|^2}{2} \Delta \phi_R dxds + \int_0^t \int_\Omega p_2 w \cdot \nabla \phi_R dxds \\ +\ & \int_0^t \int_\Omega \left(v \cdot w + \frac{|w|^2}{2}\right) u \cdot \nabla \phi_R dxds + \int_0^t \int_\Omega v \otimes u : \nabla w \phi_R dxds. \end{aligned} \tag{4.10}$$

The first and third term on the right hand side approach 0 as $R \to \infty$. Let $\bar{p}_2$ be the average of p_2 over the ball B_{2R}. Then

$$\begin{aligned}\left|\int_0^t \int_\Omega p_2 w \cdot \nabla \phi_R dxds\right| &= \left|\int_0^t \int_\Omega (p_2 - \bar{p}_2) w \cdot \nabla \phi_R dxds\right| \\ &\leq \int_0^t \|p_2 - \bar{p}_2\|_{L^{3/2}_x} \|w\|_{L^3_x} \|\nabla \phi_R\|_{L^\infty_x} ds.\end{aligned}$$

We have $\|\nabla \phi_R\|_{L^\infty_x} \leq AR^{-1}$ and $\|p_2 - \bar{p}_2\|_{L^{3/2}_x} \leq AR\|\nabla p_2\|_{L^{3/2}_x}$ (Poincaré–Sobolev inequality). Hence,

$$\left|\int_0^t \int_\Omega p_2 w \cdot \nabla \phi_R dxds\right| \leq A \int_0^t \|w(s)\|_{L^3(\Omega \setminus B_{2R})} ds \leq At^{2/3} \|w\|_{L^3(\Omega \setminus B_{2R} \times (0,t))}$$

which converges to 0 as $R \to \infty$. As $R \to \infty$, (4.10) becomes (4.6).

(ii) By Schwarz inequality,

$$\begin{aligned}\int_0^t \int_\Omega v \otimes v : \nabla w dx ds &\leq \int_0^t \|v\|_{L_x^4}^2 \|\nabla w\|_{L_x^2} ds \\ &\leq \int_0^t \left(4 \|v\|_{L_x^4}^4 + \frac{1}{4} \|\nabla w\|_{L_x^2}^2\right) ds \\ &\leq 4\sqrt{t} \|v\|_{L_t^8 L_x^4}^4 + \frac{1}{4} \int_0^t \|\nabla w\|_{L_x^2}^2 ds. \end{aligned} \tag{4.11}$$

By Hölder inequality,

$$\begin{aligned}\int_0^t \int_\Omega v \otimes w : \nabla w dx ds &\leq \int_0^t \|v\|_{L_x^5} \|w\|_{L_x^{10/3}} \|\nabla w\|_{L_x^2} ds \\ &\leq \int_0^t \|v\|_{L_x^5} \|w\|_{L_x^2}^{2/5} \|w\|_{L_x^6}^{3/5} \|\nabla w\|_{L_x^2} ds. \end{aligned} \tag{4.12}$$

By Part (ii) of Corollary 4.4, $u \cdot \nabla u \in L_t^{4/3} L_x^2$. This is the forcing term in system (II). By the coercive estimate (Proposition 3.3), $w \in L_t^{4/3} W_0^{1,2}$. By Gargliardo-Nirenberg inequality, $\|w(s)\|_{L_x^6} \leq A \|\nabla w(s)\|_{L_x^2}$. Substituting this estimate into (4.12), we get

$$\begin{aligned}\int_0^t \int_\Omega v \otimes w : \nabla w dx ds &\leq \int_0^t \|v\|_{L_x^5} \|w\|_{L_x^2}^{2/5} \|\nabla w\|_{L_x^2}^{8/5} ds \\ &\leq \int_0^t \left(4 \|v\|_{L_x^5}^5 \|w\|_{L_x^2}^2 + \frac{1}{4} \|\nabla w\|_{L_x^2}^2\right) ds. \end{aligned} \tag{4.13}$$

Summing (4.11) and (4.13), we get

$$\int_0^t \int_\Omega v \otimes u : \nabla w dx ds \leq 8\sqrt{t} \|v\|_{L_t^8 L_x^4}^4 + 4 \int_0^t \|v\|_{L_x^5}^5 \|w\|_{L_x^2}^2 ds + \frac{1}{2} \int_0^t \|\nabla w\|_{L_x^2}^2 ds$$

By Part (i) of Proposition 3.1, $\|v\|_{L_t^8 L_x^4} \leq C_q \|f\|_{Y_q}$. Then (4.6) implies

$$\int_\Omega |w(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla w|^2 dx ds \leq \sqrt{t} C_q \|f\|_{Y_q}^4 + 8 \int_0^t \|v\|_{L_x^5}^5 \|w\|_{L_x^2}^2 ds \tag{4.14}$$

for all $t \in (0,T)$. By Grönwall inequality,

$$\|w(t)\|_{L_x^2}^2 \leq C_q \sqrt{t} \|f\|_{Y_q}^4 \exp\left(8 \int_0^t \|v(s)\|_{L_x^5}^5 ds\right) \leq C_q \sqrt{t} \|f\|_{Y_q}^4 \exp\left(C_q \|f\|_{Y_q}^5\right).$$

Substituting this estimate into (4.14), we obtain (4.7).

(iii) Let $\phi \in C_0^\infty(\Omega \times (0,T))$. Identity (4.8) is obtained by multiplying both sides of (4.9) by $w_i \phi$, integrating over $\Omega \times (0,t)$, and using integration by part. □

We further decompose

$$\begin{cases} w &= w^{(1)} + w^{(2)} + w^{(3)} + w^{(4)} \\ p_2 &= p^{(1)} + p^{(2)} + p^{(3)} + p^{(4)} \end{cases} \tag{4.15}$$

where (w^j, p^j) is the solution of the Stokes system

$$(\mathrm{III})_j : \begin{cases} \partial_t w^{(j)} - \Delta w^{(j)} + \nabla p^{(j)} &= f^{(j)}, \\ \operatorname{div} w^{(j)} &= 0. \\ w^{(j)}|_{\partial\Omega} &= 0, \\ w^{(j)}(0) &= 0. \end{cases}$$

where

$$f^{(1)} = -w \cdot \nabla w, \qquad f^{(2)} = -w \cdot \nabla v,$$

$$f^{(3)} = -v \cdot \nabla w, \qquad f^{(4)} = -v \cdot \nabla v.$$

Proposition 4.7. *Let $u \in L^5(\Omega \times (0,T))$ be a mild solution to the Navier–Stokes system with $u_0 = 0$ and $f \in Y_q$ for some $5/2 < q < 3$. We have the following a priori estimates.*

(i) For all $2/\alpha + 3/\beta = 4$ with $1 < \beta < 3/2$,

$$\left\|\partial_t w^{(1)}\right\|_{L_t^\alpha L_x^\beta} + \left\|\nabla^2 w^{(1)}\right\|_{L_t^\alpha L_x^\beta} + \left\|\nabla p_2^{(1)}\right\|_{L_t^\alpha L_x^\beta} \le C_{q,\beta}\sqrt{T}\Psi_0(q, \|f\|_{Y_q}).$$

(ii) For all $2/\alpha + 3/\beta = 7/2$ with $2q/(2+q) \le \beta < 6q/(10-q)$,

$$\left\|\partial_t w^{(2)}\right\|_{L_t^\alpha L_x^\beta} + \left\|\nabla^2 w^{(2)}\right\|_{L_t^\alpha L_x^\beta} + \left\|\nabla p_2^{(2)}\right\|_{L_t^\alpha L_x^\beta} \le C_{q,\beta}T^{1/4}\Psi_0(q, \|f\|_{Y_q}).$$

(iii) For all $2/\alpha + 3/\beta = 7/2$ with $6/5 \le \beta \le 2$,

$$\left\|\partial_t w^{(3)}\right\|_{L_t^\alpha L_x^\beta} + \left\|\nabla^2 w^{(3)}\right\|_{L_t^\alpha L_x^\beta} + \left\|\nabla p_2^{(3)}\right\|_{L_t^\alpha L_x^\beta} \le C_{q,\beta}T^{1/4}\Psi_0(q, \|f\|_{Y_q}).$$

(iv) For all $2/\alpha + 3/\beta = 3$ with $3q/(3+q) \le \beta < 3q/(5-q)$,

$$\left\|\partial_t w^{(4)}\right\|_{L_t^\alpha L_x^\beta} + \left\|\nabla^2 w^{(4)}\right\|_{L_t^\alpha L_x^\beta} + \left\|\nabla p_2^{(4)}\right\|_{L_t^\alpha L_x^\beta} \le C_{q,\beta}\|f\|_{Y_q}.$$

(v) For any bounded set $D \subset \Omega$, put $D_T = D \times (0,T)$. Then

$$\|\partial_t w\|_{L_t^{3/2}L_x^{9/8}(D_T)} + \left\|\nabla^2 w\right\|_{L_t^{3/2}L_x^{9/8}(D_T)} + \|\nabla p_2\|_{L_t^{3/2}L_x^{9/8}(D_T)} \le C_{q,D,T}\Psi_0\left(q, \|f\|_{Y_q}\right).$$

Proof. By Part (i) and (v) of Proposition 3.1,

$$\|v\|_{L_t^{\alpha_1} L_x^{\beta_1}} \leq C_{\beta_1} \|f\|_{Y_q} \quad \forall\ 2/\alpha_1 + 3/\beta_1 = 1, \quad 3 \leq \beta \leq \infty,$$

$$\|\nabla v\|_{L_t^{\alpha_2} L_x^{\beta_2}} \leq C_{\beta_2} \|f\|_{Y_q} \quad \forall\ 2/\alpha_2 + 3/\beta_2 = 2, \quad q \leq \beta < 3q/(5-q).$$

By Part (ii) of Proposition 4.6 and the Sobolev embedding $W_0^{1,2} \hookrightarrow L^6$,

$$\|w\|^2_{L_t^{\alpha_3} L_x^{\beta_3}} \leq C_{\beta_3} \sqrt{T} \Psi_0(q, \|f\|_{Y_q}) \quad \forall\ 2/\alpha_3 + 3/\beta_3 = 3/2, \quad 2 \leq \beta \leq 6,$$

$$\|\nabla w\|^2_{L^2_{t,x}} \leq \sqrt{T} \Psi_0(q, \|f\|_{Y_q}).$$

By Hölder inequality, we get the bounds for each $f^{(j)}$ in mixed-norm Lebesgue spaces. The estimates in Parts (i)-(iv) follow from Solonnikov's coercive theorem (Proposition 3.3). As a consequence,

$$\left\|\nabla p_2^{(1)}\right\|_{L_t^{3/2} L_x^{9/8}}, \left\|\nabla p_2^{(2)}\right\|_{L_t^{3/2} L_x^{18/13}}, \left\|\nabla p_2^{(3)}\right\|_{L_t^{3/2} L_x^{18/13}}, \left\|\nabla p_2^{(4)}\right\|_{L_t^{3/2} L_x^{9/5}} \leq C_q \Psi_0(q, \|f\|_{Y_q}).$$

This leads to Part (v). □

Chapter 5

Sw-solutions

5.1 Definition

The concept of sw-solutions has been mentioned in Section 4.3. We now give a precise definition. Let $u_0 = 0$, $f \in Y_q$ for some $5/2 < q < 3$, and (v, p_1) be the mild solution of the Stokes system $(\mathrm{SE})_\Omega$.

Definition 5.1 (sw-solution)**.** A function u defined in $\Omega_T = \Omega \times (0,T)$, $T < \infty$, is called a *sw-solution* if $u = v + w$ and $p = p_1 + p_2$ where (w, p_2) satisfies five following properties.

(a) They have regularity properties

$$\begin{aligned} &w \in L^\infty_t L^2_\sigma \cap L^2_t W^{1,2}_0(\Omega_T), \\ &\partial_t w,\ \nabla^2 w,\ \nabla p_2 \in L^{3/2}((0,T), L^{9/8}_{\mathrm{loc}}(\bar{\Omega})). \end{aligned}$$

(b) They satisfy the equations

$$\begin{cases} \partial_t w - \Delta w + \nabla p_2 = -w \cdot \nabla w - w \cdot \nabla v - v \cdot \nabla w - v \cdot \nabla v \\ w(\cdot, 0) = 0 \end{cases}$$

in sense of distribution in $\Omega \times [0,T)$. That is, for any vector field $\phi \in C^\infty_0(\Omega_T)$,

$$\begin{aligned} &\int_\Omega w(x,t)\phi(x,t)dx + \int_0^t \int_\Omega \nabla w : \nabla \phi dx ds = \\ &= \int_0^t \int_\Omega \left[w \partial_t \phi - p_2 \operatorname{div} \phi + (v+w) \otimes (v+w) : \nabla \phi \right] dx ds. \end{aligned} \tag{5.1}$$

(c) They satisfy a local energy inequality: for any nonnegative function $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ and $t \in (0,T)$,

$$\int_\Omega \phi(x,t)|w(x,t)|^2 dx + 2\int_0^t \int_\Omega \phi|\nabla w|^2 dxds \leq$$

$$\leq \int_0^t \int_\Omega \left[2v \otimes u : \phi\nabla w + |w|^2(\Delta\phi + \partial_t\phi) + u \cdot \nabla\phi(|w|^2 + 2p_2 + 2v \cdot w)\right] dxds. \tag{5.2}$$

(d) For any $\phi \in L^2(\Omega)$, the map $t \mapsto \int_\Omega w(x,t)\phi(x)dx$ is continuous on $[0,T)$. Moreover, $\|w(t)\|_{L^2(\Omega)} \to 0$ as $t \to 0^+$.

(e) For all $t \in (0,T)$,

$$\frac{1}{2}\int_\Omega |w(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla w|^2 dxds \leq \int_0^t \int_\Omega v \otimes u : \nabla w dxds. \tag{5.3}$$

Remark 5.2. The right hand side of (5.2) is well-defined because $v \in L_t^{20}L_x^{10/3} \cap L_{t,x}^5$, $w \in L_{t,x}^{10/3}$, $\nabla w \in L_{t,x}^2$ and $p_2 \in L^{3/2}((0,T), L_{\text{loc}}^{3/2}(\bar{\Omega}))$.

Remark 5.3. Conditions (d) and (e) can be dropped from the definition because they are consequences of (a), (b), (c). Indeed, one can derive the first part of (d) from (a) as follows. For $\phi \in C_0^\infty(\Omega)$ and $D = \text{supp } \phi$,

$$\int_D w(x,t)\phi(x)dx = \int_D \int_0^t \partial_t w(x,s)\phi(x)dsdx.$$

This is a continuous function of t because the integrand $(\partial_t w)\phi$ is an L^1-function in $D \times (0,T)$. Condition (e) can be derived from (c) as follows. Choose test functions of the form $\phi(x,t) = \psi_R(x)\rho_T(t)$ where ρ_T is a smooth function supported in $(-1, T+1)$, equal to 1 in $(0,T)$, and ψ_R is supported in the ball of radius $2R$, equal to 1 in the ball of radius R, having decay derivatives as $R \to \infty$. Then take the limit of both sides of (5.2) as $R \to \infty$. Finally, the second part of Condition (d) is a consequence of (e).

Remark 5.4. By Proposition 4.6 and Part (i) of Proposition 4.7, the mild solution to the Navier–Stokes system is also an sw-solution.

Proposition 5.5. *Let $u = v + w$ be an sw-solution to (NSE)$_\Omega$ with $u_0 = 0$. Then w and p_2 satisfy a priori estimates*

$$\int_\Omega |w(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla w(x,t)|^2 dxds \leq \sqrt{t}\Psi_0(q, \|f\|_{Y_q}), \tag{5.4}$$

$$\|\partial_t w\|_{L_t^{3/2}L_x^{9/8}(D_T)}+\left\|\nabla^2 w\right\|_{L_t^{3/2}L_x^{9/8}(D_T)}+\|\nabla p_2\|_{L_t^{3/2}L_x^{9/8}(D_T)} \le C_{q,D,T}\Psi_0\left(q,\|f\|_{Y_q}\right) \quad (5.5)$$

for all bounded $D \subset \Omega$. Here $D_T = D\times(0,T)$. Moreover, (w,p_2) admits a decomposition (4.15) where each $(w^{(j)},p^{(j)})$ is the solution to (III)$_j$ given by Stokes operator as in (1.2). They satisfy the estimates in Proposition 4.7.

Proof. The proof of (4.7) in Proposition 4.6 is still valid for any w satisfying Conditions (a) and (e) in Definition 5.1. The proofs of Parts (i)-(v) in Proposition 4.7 are still valid because we only use the energy norm of w in the estimates. Put $\bar{w} = w-w^{(1)}-w^{(2)}-w^{(3)}$. Then $\bar{w}$ is a weak solution to the system (SE)$_\Omega$ with force term $f^{(4)} = -v\cdot\nabla v$. By Corollary 4.4, $f^{(4)} \in Y_q \cap L_t^2 L_x^{3/2}$. By Proposition 3.8, $\bar{w} = w^{(4)}$. □

Combining the *a priori* local energy estimates for v and w, we obtain *a priori* local energy estimate for sw-solutions.

Corollary 5.6. *Let (u,p) be an sw-solution to (NSE)$_\Omega$ with $u_0 = 0$ and $f \in Y_q$ for some $5/2 < q < 3$. Then for any bounded set $D \subset \Omega$ and $t \in (0,T)$,*

$$\int_\Omega |u(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla u|^2 dxds \le C_{D,q}\|f\|_{Y_q} + \sqrt{t}\Psi_0(q,\|f\|_{Y_q}).$$

Proof. This is a direct consequence of Proposition 3.7 and Proposition 5.5. □

5.2 Compactness theorem

Let $f_m \rightharpoonup f$ in Y_q for some $5/2 < q < 3$. Suppose (u_m,p_m) defined on $\Omega\times(0,T)$, $T<\infty$, be an sw-solution to (NSE)$_\Omega$ with $u_0 = 0$ and force term f_m. Let $u_m = v_m + w_m$ be the decomposition of u_m according to Definition 5.1.

Proposition 5.7. *(u_m) has a subsequence, which is still denoted by (u_m), such that*

$$u_m \rightharpoonup u, \quad \nabla u_m \rightharpoonup \nabla u \quad \text{in } L^2(\Omega\times(0,T)),$$

$$w_m \to w \quad \text{in } L_t^\alpha L_x^\beta(D\times(0,T)) \quad \forall\ 1\le\beta<6,\ 2/\alpha+3/\beta>3/2,$$

$$\nabla w_m \to \nabla w \quad \text{in } L^\gamma(D\times(0,T)) \quad \forall\ 1\le\gamma<2,$$

$$\nabla p_{2,m} \rightharpoonup \nabla p_2 \quad \text{in } L_t^{3/2}L_x^{9/8}(D\times(0,T))$$

for all bounded set $D \subset \Omega$, where u is an sw-solution to (NSE)$_\Omega$ with force form f, and w is the second component of u according to Definition 5.1. Moreover, (v_m) converges to the first component v in the spaces described in Proposition 3.12.

Remark 5.8. Statements of this type are sometimes referred to as *compactness theorems* because the weak convergence of the data implies strong convergence of the solutions. Our proof uses Aubin-Lions lemma (Proposition A.4), following the idea of Lin [52, Thm. 2.2].

Proof of Proposition 5.7. The proof is twofold. First we show that there exists a subsequence of (u_m) such that u_m, ∇u_m, w_m, ∇w_m converge in certain spaces. Then we show that the limit u must be an sw-solution.

To show the convergence, it suffices to show that there exists a convergent subsequence for each fixed bounded subset D of Ω. The rest of the proof would follow by Cantor's diagonal argument. By Corollary 5.6, the sequence (u_m) is bounded in the local energy space $L^\infty_t L^2_x \cap L^2_t H^1_x(D_T)$. This implies that up to a subsequence, u_m and ∇u_m converge weakly in $L^2(D_T)$. By (5.4), the energy norm of w_m is bounded. Up to a subsequence w_m converges weakly in the energy space $L^\infty_t L^2_x \cap L^2_t H^1_x(\Omega_T)$. Then w_m and ∇w_m converge weakly in $L^2(\Omega_T)$. Together with the estimate (5.5), we have

$$\begin{aligned} &(w_m) && \text{is bounded in} \quad L^{3/2}((0,T), W^{1,9/8}(D)), \\ &(\partial_t w_m) && \text{is bounded in} \quad L^{3/2}((0,T), L^{9/8}(D)). \end{aligned}$$

Applying Aubin-Lions lemma for $W^{1,9/8}(D) \hookrightarrow L^{3/2}(D) \hookrightarrow L^{9/8}(D)$ and $\alpha_0 = \alpha_1 = 3/2$, we conclude that up to a subsequence (w_m) converges strongly in $L^{3/2}(D_T)$. In particular, (w_m) converges strongly in $L^1(D_T)$. For α, β such that $2/\alpha + 3/\beta > 3/2$, $1 \le \beta < 6$, there exist $\theta > \alpha$ and $\beta < \kappa < 6$ such that $2/\theta + 3/\kappa = 3/2$. By the embedding $L^\infty_t L^2_x \cap L^2_t L^6_x \hookrightarrow L^\theta_t L^\kappa_x$, the sequence (w_m) is bounded in $L^\theta_t L^\kappa_x(D_T)$. Therefore, by Lemma A.5 it converges strongly in $L^\alpha_t Ł^\beta_x(D_T)$. For each $\phi \in C^\infty_0(D)$,

$$\langle \nabla \Delta w_m, \phi \rangle = -\int_D \Delta w_m \nabla \phi dx \le \left\|\nabla^2 w_m\right\|_{L^{9/8}(D)} \|\nabla \phi\|_{L^9(D)} \le \left\|\nabla^2 w_m\right\|_{L^{9/8}} \|\phi\|_{W^{2,9}_0},$$

$$\langle \nabla \nabla p_{2,m}, \phi \rangle = -\int_D \nabla p_{2,m} \nabla \phi dx \le \|\nabla p_{2,m}\|_{L^{9/8}(D)} \|\nabla \phi\|_{L^9(D)} \le \|\nabla p_{2,m}\|_{L^{9/8}} \|\phi\|_{W^{2,9}_0},$$

$$\langle \nabla \mathrm{div}(u_m \otimes u_m), \phi \rangle = -\int_D u_m \otimes u_m \nabla^2 \phi dx \quad \leq \quad \|u_m\|^2_{L^{9/4}(D)} \left\|\nabla^2 \phi\right\|_{L^9(D)}$$
$$\leq \quad \|u_m\|^2_{L^{9/4}(D)} \|\phi\|_{W_0^{2,9}(D)}.$$

Note that $u_m = v_m + w_m$ is bounded in $L^{10/3}(D_T)$. Thus, $\partial_t \nabla w_m = \nabla \Delta w_m - \nabla \nabla p_{2,m} - \nabla \mathrm{div}(u_m \otimes u_m)$ is bounded in $L^{3/2}((0,T), W^{-2,9}(D))$. Applying Aubin-Lions lemma for $W^{1,9/8}(D) \hookrightarrow L^{3/2}(D) \hookrightarrow W^{-2,9}(D)$ and $\alpha_0 = \alpha_1 = 3/2$, we conclude that up to a subsequence (∇w_m) converges strongly in $L^{3/2}(D_T)$. Because the sequence (∇w_m) is bounded in $L^2(D_T)$, it converges strongly in $L^\gamma(D_T)$ for all $1 \leq \gamma < 2$.

Now we show that the limit function u is indeed an sw-solution to $(\mathrm{NSE})_\Omega$ with forcing term f. We do so by verifying each condition in Definition 5.1. By Remark 5.3, it suffices to check Conditions (a), (b), (c). Since $u_m = v_m + w_m$ for all m, it follows that $u = v + w$. Because $w_m \in L^\infty_t L^2_\sigma \cap L^2_t W^{1,2}_0(\Omega_T)$ and $w_m \rightharpoonup w$ in the energy space $L^\infty_t L^2_x \cap L^2_t H^1_x(\Omega_T)$, w also belongs to $w_m \in L^\infty_t L^2_\sigma \cap L^2_t W^{1,2}_0(\Omega \times (0,T))$. Condition (a) holds.

For each vector field $\phi \in C_0^\infty(\Omega \times (0,T))$,

$$\int_\Omega w_m(x,t)\phi(x,t)dx = \int_0^t \int_\Omega \left[w_m(\Delta\phi + \partial_t \phi) - p_{2,m} \,\mathrm{div}\, \phi + u_m \otimes u_m : \nabla \phi\right] dxds.$$

The limit of each term as $m \to \infty$ is attained because of the strong convergence of (u_m) in $L^3_{\mathrm{loc}}(\Omega_T)$ and weak convergence of (w_m) in $L^\infty_t L^2_x$. Hence, Condition (b) holds. We now verify Condition (c). Let $\phi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$.

$$\int_\Omega \phi(x,t)|w_m(x,t)|^2 dx + 2\int_0^t \int_\Omega \phi |\nabla w_m|^2 dxds \leq$$

$$\leq \int_0^t \int_\Omega \big[2v_m \otimes u_m : \phi \nabla w_m + |w_m|^2(\Delta\phi + \partial_t\phi) + \\ + u_m \cdot \nabla\phi(|w_m|^2 + 2p_{2,m} + 2v_m \cdot w_m)\big] \, dxds. \tag{5.6}$$

The left hand side shrinks though the limit due to weak convergence of w_m in the energy space. We only need to show that each term on the right hand side converges to the corresponding term without index m. Put $S = \Omega \cap \mathrm{supp}\ \phi$. Other than the first term, the convergence of other term can be seen as a result of strong convergence of v_m, w_m, u_m and their derivatives. Indeed, because $w_m \to w$ in $L^2(S)$,

$$\int_0^t \int_\Omega |w_m|^2(\Delta\phi + \partial_t\phi)dxds \to \int_0^t \int_\Omega |w|^2(\Delta\phi + \partial_t\phi)dxds.$$

Because $v_m \to v$, $w_m \to w$, $u_m \to u$ in $L^3(S)$, and $q_{2,m} \rightharpoonup q_2$ in $L^{3/2}(S)$,

$$\int_0^t \int_\Omega |w_m|^2 u_m \cdot \nabla\phi |w_m|^2 dxds \to \int_0^t \int_\Omega |w_m|^2 u_m \cdot \nabla\phi |w_m|^2 dxds.$$

$$\int_0^t \int_\Omega q_{2,m} u_m \cdot \nabla\phi dxds \to \int_0^t \int_\Omega q_{2,m} u_m \cdot \nabla\phi dxds.$$

$$\int_0^t (u_m \cdot \nabla\phi)(v_m \cdot w_m) dxds \to \int_0^t (u_m \cdot \nabla\phi)(v_m \cdot w_m) dxds.$$

The first term on RHS (5.6) is treated by integration by part and using the divergence-free property

$$\begin{aligned} \int_0^t v_m \otimes u_m : \phi \nabla w_m dxds &= -\int_0^t v_m \otimes u_m : \phi \nabla w_m dxds + \int_0^t \frac{|w_m|^2}{2} v_m \cdot \nabla\phi dxds \\ &- \int_0^t v_m \otimes u_m : (\nabla\phi) \otimes w_m dxds. \end{aligned} \tag{5.7}$$

The convergence of the first term on RHS (5.7) is due to $v_m \to v$ in $L^4(S)$. The convergence of the second and third term is due to $v_m \to v$, $w_m \to w$, $u_m \to u$ in $L^3(S)$. □

5.3 Weak–strong uniqueness

Proposition 5.9. *Let u, $\tilde{u} : \Omega \times (0,T) \to \mathbb{R}^3$ be two sw-solutions to (NSE)$_\Omega$ with $u_0 = 0$ and $f \in Y_q$ for some $5/2 < q < 3$. Suppose u is a mild solution. Then $\tilde{u} = u$.*

Proof. Let $u = v + w$ and $\tilde{u} = v + \tilde{w}$ be the decomposition of u and $\tilde{u}$ according to Definition 5.1. Denote

$$\begin{aligned} V &= \left\{\psi \in H_0^1(\Omega) : \text{div } \psi = 0\right\}, \\ H &= \left\{\psi \in L^2(\Omega) : \text{div } \psi = 0\right\}, \\ V_1 &= \left\{\psi \in W_0^{1,3}(\Omega) : \text{div } \psi = 0\right\}. \end{aligned}$$

Let V' and V_1' be the dual spaces of V and V_1 respectively. It follows from the weak form (5.1) of (NSE)$_\Omega$ that for a.e. $t \in (0,T)$,

$$\langle \partial_t w, \psi \rangle = \int_\Omega (-\nabla w + u \otimes u) : \nabla\psi dx \quad \forall\ \psi \in V, \tag{5.8}$$

$$\langle \partial_t \tilde{w}, \psi \rangle = \int_\Omega (-\nabla \tilde{w} + \tilde{u} \otimes \tilde{u}) : \nabla\psi dx \quad \forall\ \psi \in V. \tag{5.9}$$

Since u is a mild solution, $u \in L_t^8 L_x^4$ by Part (i) of Proposition 4.2. Then $u \otimes u \in L_t^4 L_x^2$. It follows from (5.8) that $\partial_t w \in L^2((0,T), V')$ with

$$\|\partial_t w\|_{L_t^2 V'} \leq \|\nabla w\|_{L_{t,x}^2} + \|u\|_{L_{t,x}^4}^2 \leq \|\nabla w\|_{L_{t,x}^2} + T^{1/4} \|u\|_{L_t^8 L_x^4}^2 .$$

Thus, for a.e. $T \in (0,T)$,

$$\langle \partial_t w, \tilde{w} \rangle = \int_\Omega (-\nabla w + u \otimes u) : \nabla \tilde{w} dx.$$

Because $v \in L_t^8 L_x^4 \cap L_{t,x}^5$ and $\tilde{w} \in L_{t,x}^{10/3} \cap L_t^4 L_x^3$,

$$\tilde{u} \otimes \tilde{u} = v \otimes v + v \otimes \tilde{w} + \tilde{w} \otimes v + \tilde{w} \otimes \tilde{w} \in L_{t,x}^2 + L_t^2 L_x^{3/2}.$$

Hence, $\partial_t \tilde{w} \in L^2((0,T), V') + L^2((0,T), V_1')$ with

$$\|\partial_t \tilde{w}\|_{L_t^2 V' + L_t^2 V_1'} \leq \|\nabla w\|_{L_{t,x}^2} + T^{1/4} \|v\|_{L_t^8 L_x^4}^2 + \|v\|_{L_{t,x}^5} \|\tilde{w}\|_{L_{t,x}^2} + \|\tilde{w}\|_{L_t^4 L_x^3}^2 .$$

Note that $w \in L^2((0,T), V) \cap L^2((0,T), V_1)$. By (5.9),

$$\langle \partial_t \tilde{w}, w \rangle = \int_\Omega (-\nabla \tilde{w} + \tilde{u} \otimes \tilde{u}) : \nabla w dx.$$

Therefore,

$$\langle \partial_t w, \tilde{w} \rangle + \langle \partial_t \tilde{w}, w \rangle = \int_\Omega [(-\nabla w + u \otimes u) : \nabla \tilde{w} + (-\nabla \tilde{w} + \tilde{u} \otimes \tilde{u}) : \nabla w] dx. \tag{5.10}$$

By regularizing w and $\tilde{w}$ by mollifiers in $\mathbb{R}$ as in the proof of Proposition 3.8, we have the identity

$$\langle \partial_t w, \tilde{w} \rangle + \langle \partial_t \tilde{w}, w \rangle = \frac{d}{dt} \langle w, \tilde{w} \rangle$$

where the right hand side is understood as weak derivative of a single-variable function. Because the right hand side of (5.10) is integrable with respect to t and that

$$\langle w, \tilde{w} \rangle = \int_\Omega w(x,t) \tilde{w}(x,t) dx \to 0 \quad \text{as } t \to 0,$$

we obtain

$$\int_\Omega w(x,t) \tilde{w}(x,t) dx = \int_0^t \int_\Omega [(-\nabla w + u \otimes u) : \nabla \tilde{w} + (-\nabla \tilde{w} + \tilde{u} \otimes \tilde{u}) : \nabla w] dx ds. \tag{5.11}$$

By Condition (e) of Definition 5.1,

$$\frac{1}{2}\int_\Omega |w(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla w|^2 dxds \leq \int_0^t \int_\Omega v \otimes u : \nabla w dxds \tag{5.12}$$

$$\frac{1}{2}\int_\Omega |\tilde{w}(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla \tilde{w}|^2 dxds \leq \int_0^t \int_\Omega v \otimes \tilde{u} : \nabla \tilde{w} dxds. \tag{5.13}$$

By (5.11), (5.12), (5.13), we get

$$\begin{aligned}\frac{1}{2}\int_\Omega |\bar{w}(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla \bar{w}|^2 dxds \leq \int_0^t \int_\Omega [v \otimes \bar{w} : \nabla \bar{w} - w \otimes v : \nabla \tilde{w} \\ - w \otimes w : \nabla \tilde{w} - \tilde{w} \otimes v : \nabla w - \tilde{w} \otimes \tilde{w} : \nabla w] dxds\end{aligned} \tag{5.14}$$

where $\bar{w} = w - \tilde{w}$. By the divergence-free properties,

$$\int_\Omega w \otimes v : \nabla \tilde{w} dx = -\int_\Omega w \otimes v : \nabla \bar{w} dx, \quad \int_\Omega w \otimes w : \nabla \tilde{w} dx = -\int_\Omega w \otimes w : \nabla \bar{w} dx,$$

$$\int_\Omega \tilde{w} \otimes v : \nabla w dx = -\int_\Omega \tilde{w} \otimes v : \nabla \bar{w} dx, \quad \int_\Omega \tilde{w} \otimes \tilde{w} : \nabla w dx = -\int_\Omega w \otimes \tilde{w} : \nabla \bar{w} dx.$$

Substituting these identities into the right hand side of (5.14), we get

$$\begin{aligned}\frac{1}{2}\int_\Omega |\bar{w}(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla \bar{w}|^2 dxds &\leq \int_0^t \int_\Omega (v \otimes \bar{w} + \bar{w} \otimes v + w \otimes \bar{w}) : \nabla \tilde{w} dxds \\ &= \int_0^t \int_\Omega u \otimes \bar{w} : \nabla \bar{w} dxds.\end{aligned}$$

The rest of the proof follows the same techniques we employ to verify Part (ii) of Proposition 4.6. Specifically, the right hand side of the above inequality is estimated as follows.

$$\begin{aligned}\int_0^t \int_\Omega u \otimes \bar{w} : \nabla \bar{w} dxds &\leq \int_0^t \|u\|_{L^5_x} \|\bar{w}\|^{2/5}_{L^2_x} \|\nabla \bar{w}\|^{8/5}_{L^2_x} ds \\ &\leq \int_0^t \left(4 \|u\|^5_{L^5_x} \|\bar{w}\|^2_{L^2_x} + \frac{1}{4} \|\nabla \bar{w}\|^2_{L^2_x}\right) ds.\end{aligned}$$

This implies

$$\int_\Omega |\bar{w}(x,t)|^2 dx + \int_0^t \int_\Omega |\nabla \bar{w}|^2 dxds \leq 8 \int_0^t \|u\|^5_{L^5_x} \|\bar{w}\|^2_{L^2_x} ds.$$

By Grönwall inequality, $\|\bar{w}(t)\|_{L^2_x} = 0$ a.e. $t \in (0,T)$. Therefore, $\bar{w} = 0$ a.e. in $\Omega \times (0,T)$. □

5.4 ε-regularity criteria and applications

For $x_0 \in \bar{\Omega}$, $t_0 > 0$, $z_0 = (x_0, t_0)$ and $r > 0$, denote

$$\begin{aligned} B_r(x_0) &= \left\{x \in \mathbb{R}^3 : |x - x_0| < r\right\}, \quad B_r^{\Omega}(x_0) = B_r(x_0) \cap \Omega, \\ Q_r(z_0) &= B_r(x_0) \times (t_0 - r^2, t_0), \quad Q_r^{\Omega}(z_0) = B_r^{\Omega}(x_0) \times (t_0 - r^2, t_0). \end{aligned}$$

Suppose u is a function defined on $\Omega \times (0, T)$. We call $z_0 \in \bar{\Omega} \times (0, T]$ a *singular point*, or *singularity*, if u is unbounded in any parabolic neighborhood of z_0. That is,

$$\sup_{Q_r^{\Omega}} |u| = \infty \quad \forall\, r > 0.$$

If z_0 is not a singular point, it is called a regular point. Caffarelli, Kohn and Nirenberg [12] establish two criteria of regularity, known as ε-regularity, for suitable weak solutions. These are a type of weak solutions defined locally in space and time and satisfying a local energy inequality.

Definition 5.10 (Suitable weak solution)**.** [12], [52] Consider $D \subset \mathbb{R}^3$ and an interval $I \subset \mathbb{R}$. A pair (u, p) is called *suitable weak solution* on $D \times I$ to the Navier–Stokes problem with force term f if they satisfy the following conditions.

(a) They have regularity properties $u \in L_t^{\infty} L_{\sigma}^2 \cap L_t^2 H_x^1(D \times I)$ and $p \in L^{3/2}(D \times I)$.

(b) They satisfy the equation $\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f$ in sense of distribution on $D \times I$.

(c) They satisfy a local energy inequality: for all $t \in I$ and $0 \le \phi \in C_0^{\infty}(D \times I)$,

$$\int_{\Omega} |u(x,t)|^2 \phi dx + 2 \int_0^t \int_{\Omega} |\nabla u|^2 \phi dx ds \le$$
$$\le \int_0^t \int_{\Omega} \left[|u|^2 \left(\partial_t \phi + \Delta \phi\right) + (|u|^2 + 2p) u \nabla \phi + 2 u f \phi \right] dx ds.$$

To study the interior regularity of a solution to the Navier–Stokes problem with initial-boundary conditions, one can regard it as a local solution. For each $x_0 \in \Omega$, there exists $r > 0$ such that $B_r(x_0) \subset \Omega$. If the solution is a suitable weak solution in $Q_r(x_0, t_0)$, one can apply the following regularity criterion.

Proposition 5.11 (First interior ε-regularity)**.** *[12, Prop. 1], [52, Thm. 3.1], [45, Prop. 2.6] There are absolute constants ε_1, $C_1 > 0$ and a constant $\varepsilon_2(q) > 0$ with the following property. Suppose (u, p) is a suitable weak solution on $Q_r(z_0)$ with force term $f \in L^q(Q_r(z_0))$, for some $q > 5/2$. Suppose further that*

$$\frac{1}{r^2}\int_{Q_r(z_0)} (|u|^3 + |p - [p]_{B_r(x_0)}|^{3/2})dz \le \varepsilon_1 \tag{5.15}$$

and

$$r^{3q-5}\int_{Q_r(z_0)} |f|^q dz \le \varepsilon_2. \tag{5.16}$$

Here $[p]_{B_r(x_0)}$ denotes the average of p over the ball $B_r(x_0)$. Then $|u(x,t)| \le C_1 r^{-1}$ almost everywhere in $Q_{r/2}(z_0)$. In particular, u is regular in $Q_{r/2}(z_0)$.

The lower bound $5/2$ of q is sharp: if f only belongs to L^α_{loc} for $\alpha < 5/2$, the solution does not necessarily belong to L^∞_{loc}. This phenomenon can be seen even in the heat equation: suppose g is the source term of the heat equation in $\mathbb{R}^n$

$$\begin{cases} \partial_t v - \Delta v &= g, \\ v(\cdot, 0) &= 0. \end{cases}$$

It is natural to expect an *a priori* estimate of the form $\|v\|_{L^q_t L^p_x} \le C_{q,p,m,r}\|g\|_{L^m_t L^r_x}$. In order for this estimate to be invariant under the natural scaling

$$v(x,t) \to v(\lambda x, \lambda^2 t), \quad g(x,t) \to \lambda^2 g(\lambda x, \lambda^2 t)$$

it is necessary that

$$\frac{2}{q} + \frac{n}{p} = \frac{2}{m} + \frac{n}{r} - 2.$$

This is also the relation required by the fractional inequality (Proposition A.16, see also [75, Appendix D3]). For $q = p = \infty$ and $m = r$, one gets $m = r = (2+n)/2$, which is equal to $5/2$ when $n = 3$. Thus, $5/2$ is a natural threshold coming from the scaling symmetry of the system. If m and r are under this threshold, one can construct a counterexample of the form $v(x,t) = (|x|^2 + t)^{-\alpha}\chi(x,t)$, where $0 < \alpha \ll 1$ and χ is a smooth cutoff function.

Return to Proposition 5.11. Seregin and Ladyzhenskaya [45] show that u is Hölder continuous when the force term belongs to parabolic Morrey spaces (the spaces L^q with

$q > 5/2$ are particular cases). Kukavica [40, Thm. 2.7], [41, Thm. 6.3.3, 6.4.1] gives a version of this criterion when $f \in L^q$ with $q \geq 5/3$. The case $q = 5/3$ is particularly interesting because $L^{5/3}_{t,x}$ is a critical space for the force term. However, one cannot conclude that $u \in L^\infty(Q_{r/2}(z_0))$, only $u \in L^5(Q_{r/2}(z_0))$. If $f = 0$, the regularity of u can bootstrap itself. For example, Nečas *et al.* [67, Prop. 2.1] show that u and its spatial derivatives are bounded. Note that Condition (5.16) is satisfied whenever r is sufficiently small. Thanks to the local energy inequality, one can obtain a sufficient condition for (5.15) that is free of pressure as follows.

Proposition 5.12 (Second interior ε-regularity)**.** *There exists an absolute constant $\varepsilon_3 > 0$ such that if a suitable weak solution (u, p) satisfies*

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r(z_0)} |\nabla u|^2 dz \leq \varepsilon_3$$

then

$$\liminf_{r\to 0} \frac{1}{r^2} \int_{Q_r(z_0)} (|u|^3 + |p - [p]_{B_r(x_0)}|^{3/2}) dz \leq \frac{\varepsilon_1}{2}.$$

For a proof, see e.g. [12, Prop. 2], [52, Thm. 3.3], [84, Thm. 1.4], [45, Prop. 2.9]. By Proposition 5.12 and Vitali covering lemma, Caffarelli, Kohn and Nirenberg show that the set of interior singular points of a suitable weak solution has one-dimensional Hausdorff measure zero [12, p. 777, 807].

Local boundary regularity is studied by Seregin [79, 80] for flat boundaries, and Mikhailov [64, 65] for curved C^2-boundaries. They define suitable weak solutions near a portion of the boundary as follows.

Definition 5.13 (Suitable weak solution near the boundary)**.** [80] Consider $D \subset \mathbb{R}^3$, $\Gamma \subset \partial D$ and an interval $I \subset \mathbb{R}$. A pair (u, p) is called a suitable weak solution on $D \times I$ near the boundary $\Gamma \times I$ to the Navier–Stokes problem with force term f if they satisfy the following conditions.

(a) They have regularity properties

$$u \in L^\infty_t L^2_\sigma \cap L^2_t H^1_x(D \times I), \quad p \in L^{3/2}(D \times I), \quad \nabla^2 u,\ \nabla p \in L^{3/2}_t L^{9/8}_x(D \times I).$$

(b) The equation $\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f$ is satisfied in sense of distribution on $D \times I$, with u vanishing on $\Gamma \times I$ in sense of trace. That is, for all $\phi \in C^\infty_0(\mathbb{R}^3 \times I)$

vanishing in a neighborhood of $(\partial D \backslash \Gamma) \times I$,

$$\int_D w(x,t)\phi(x,t)dx + \int_a^t \int_D \nabla w : \nabla \phi dxds =$$
$$= \int_a^t \int_D \left[w\partial_t \phi - p \operatorname{div} \phi + u \otimes u : \nabla \phi \right] dxds,$$

where a is the left endpoint of I.

(c) They satisfy a local energy inequality: for all $t \in I$ and $0 \leq \phi \in C_0^\infty(\mathbb{R}^3 \times I)$ vanishing in a neighborhood of $(\partial D \backslash \Gamma) \times I$,

$$\int_D |u(x,t)|^2 \phi dx + 2\int_a^t \int_D |\nabla u|^2 \phi dxds \leq$$
$$\leq \int_a^t \int_D \left[|u|^2 \left(\partial_t \phi + \Delta \phi \right) + (|u|^2 + 2p) u \nabla \phi + 2uf\phi \right] dxds.$$

The main difficulty with boundary regularity is to obtain local estimates for the harmonic pressure near the boundary. Instead of decomposing $p = p_d - p_c + p_e$ as in (1.4), Seregin [80, Lem. 7.2] splits the pressure into two parts: one is the pressure coming from a Stokes system on a local domain with the right hand side $-u \cdot \nabla u$; the other comes from a Stokes system with zero forcing. Specifically, $u = u^{(1)} + u^{(2)}$ and $p = \pi^{(1)} + \pi^{(2)}$ where

$$\begin{cases} \partial_t u^{(1)} - \Delta u^{(1)} + \nabla \pi^{(1)} &= -u \cdot \nabla u \qquad \text{on } D \times I, \\ \operatorname{div} u^{(1)} &= 0, \\ u^{(1)}|_{\partial D} &= 0, \\ u^{(1)}(0) &= 0. \end{cases} \tag{5.17}$$

$$\begin{cases} \partial_t u^{(2)} - \Delta u^{(2)} + \nabla \pi^{(2)} &= 0 \qquad \text{on } D \times I, \\ \operatorname{div} u^{(2)} &= 0, \\ u^{(2)}|_{\Gamma} &= 0. \end{cases} \tag{5.18}$$

Here D is a smooth, local domain contained in Ω such that $\Gamma \subset \partial D$. Pressure $\pi^{(1)}$ can be estimated in $D \times I$ via Solonnikov's coercive estimates. Pressure $\pi^{(2)}$ has better integrability on any subdomain $D' \times I' \subset\subset D \times I$ than on $D \times I$. An analog of this idea for the heat equation is well-known: if v satisfies

$$\begin{cases} \partial_t v - \Delta v = 0 \quad \text{in} \quad Q_1^+ \\ v|_\Gamma = 0 \end{cases}$$

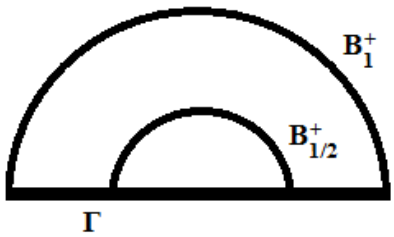

Figure 5.1: The half-balls.

where Γ is the flat portion of ∂B_1^+, then $\|v\|_{L^\infty(Q_{1/2}^+)} \leq C\|v\|_{L^1(Q_1^+)}$ (see e.g. [51, Lem. 4.12]). The precise statement for $\pi^{(2)}$ is given in Lemma 5.22. Seregin obtains boundary ε-regularity criteria analogous to the interior regularity criteria as follows.

Proposition 5.14 (First boundary ε-regularity)**.** *[79], [80, Prop. 2.4] There are absolute constants ε_4, $C_2 > 0$ and a constant $\varepsilon_5(q) > 0$ with the following property. Suppose (u, p) is a suitable weak solution on $B_r^\Omega(z_0) \times I$ near the boundary $\Gamma \times I$, where Γ is the flat portion of the boundary of $B_r^\Omega(x_0)$ and $z_0 = (x_0, t_0)$, $x_0 \in \Gamma$, $t_0 \in I$, with force term $f \in L^q(Q_r^\Omega(z_0))$, for some $q > 5/2$. Suppose further that*

$$\frac{1}{r^2}\int_{Q_r^\Omega(z_0)} (|u|^3 + |p - [p]_{B_r^\Omega(x_0)}|^{3/2})dz \leq \varepsilon_4 \tag{5.19}$$

and

$$r^{3q-5}\int_{Q_r^\Omega(z_0)} |f|^q dz \leq \varepsilon_5. \tag{5.20}$$

Then $|u(x,t)| \leq C_2 r^{-1}$ almost everywhere in $Q_{r/2}^\Omega(z_0)$. In particular, u is regular in $Q_{r/2}^\Omega(z_0)$.

Proposition 5.15 (Second boundary ε-regularity)**.** *[80, Prop. 2.7] There exists an absolute constant $\varepsilon_6 > 0$ such that if a suitable weak solution (u, p) near the boundary satisfies*

$$\limsup_{r\to 0} \frac{1}{r}\int_{Q_r^\Omega(z_0)} |\nabla u|^2 dz \leq \varepsilon_6$$

then

$$\liminf_{r\to 0} \frac{1}{r^2}\int_{Q_r^\Omega(z_0)} (|u|^3 + |p - [p]_{B_r^\Omega(x_0)}|^{3/2})dz \leq \frac{\varepsilon_4}{2}.$$

Proposition 5.16. *Every sw-solution is locally a suitable weak solution in sense of Definition 5.10 and Definition 5.13.*

Proof. Let $u = v + w$ be an sw-solution. The only property we need to check is the local energy inequality of u. Note that v satisfies a local energy identity (Proposition 3.7) and w satisfies a local energy inequality (by the definition of sw-solutions). Adding them together, we get the local energy inequality that u satisfies. □

Proposition 5.17. *Let (u, p) be an sw-solution on $\Omega \times (0, T)$. There exists $R > 0$ such that $u \in L^\infty(\Omega \backslash B_R^\Omega \times (\tau, T))$ for all $\tau \in (0, T)$.*

Proof. We assume $T = 5$ for simplicity. By the weak-strong uniqueness theorem (Proposition 5.9), u coincides the mild solution $\tilde{u}$ of the Navier–Stokes system on a short period of time, which we assume to be (0,4). By Part (iii) of Proposition 4.2, $u \in L^\infty(\Omega \times (\tau, 4])$ for all $0 < \tau < 4$. It suffices to find $R > 0$ such that $u \in L^\infty(\Omega \backslash B_R^\Omega \times (4, 5))$. Let $u = v + w = v + w^{(1)} + w^{(2)} + w^{(3)} + w^{(4)}$ be the decomposition of u as an sw-solution. Because $v \in L_t^\infty L_x^3$ and $w \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$, $u = v + w \in L^3(\Omega \times (0, 1))$. Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_4\}$ where ε_1 and ε_4 are the constants in Proposition 5.11 and Proposition 5.14. For each $r > 0$ and $x_0 \in \bar{\Omega}$, let us simplify the notation by denoting $B_r(x_0) = B_r^\Omega(x_0)$ and $Q_r(z_0) = Q_r^\Omega(x_0, 5) = B_r^\Omega(x_0) \times (5 - r^2, 5)$. For each $\delta > 0$, there exists $R_1 = R_1(\delta) > 0$ such that

$$\|u\|_{L^3(Q_2(z_0))} < \delta \quad \forall\, x_0 \in \bar{\Omega},\ |x_0| > R_1. \tag{5.21}$$

Now we estimate the pressure $p = p_1 + p_2 = p_1 + p_2^{(1)} + p_2^{(2)} + p_2^{(3)} + p_2^{(4)}$. By the coercive estimate (Proposition 3.5), $t^{q_*} \nabla p_1 \in L^q(\Omega \times (0, 5))$. There exists $R_2 = R_2(\delta) > 0$ such that $\|\nabla p_1\|_{L^q(Q(x_0))} < \delta$ for all $x_0 \in \bar{\Omega}$ with $|x_0| > R_2$. By Proposition 5.5, $\nabla p_2^{(1)}$ belongs to $L_t^{3/2} L_x^{9/8}(\Omega \times (0, 5))$, $\nabla p_2^{(2)}$ and $\nabla p_2^{(3)}$ belong to $L_t^{3/2} L_x^{18/13}(\Omega \times (0, 5))$, and $\nabla p_2^{(4)}$ belongs to $L_t^{3/2} L_x^{9/5}(\Omega \times (0, 5))$. There exists $R_3 = R_3(\delta) > 0$ such that

$$\left\|\nabla p_2^{(1)}\right\|_{L_t^{\frac{3}{2}} L_x^{\frac{9}{8}}(Q_2(z_0))}, \left\|\nabla p_2^{(2)}\right\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}(Q_2(z_0))}, \left\|\nabla p_2^{(3)}\right\|_{L_t^{\frac{3}{2}} L_x^{\frac{18}{13}}(Q_2(z_0))}, \left\|\nabla p_2^{(4)}\right\|_{L_t^{\frac{3}{2}} L_x^{\frac{9}{5}}(Q_2(z_0))} < \delta$$

for all $x_0 \in \bar{\Omega}$ with $|x_0| > R_3$. Before using Poincaré–Sobolev inequality, it is important to note that both the whole space and half-space satisfy

$$(\text{diam } B_2(x_0))^3 \leq A\ |B_2(x_0)| \quad \forall\, x_0 \in \bar{\Omega}$$

where A does not depend on x_0. Thus, the Poincaré constant of $B_2(x_0)$ is independent of

x_0 (see Proposition A.2 and the remark preceding it). By Poincaré–Sobolev inequality,

$$\begin{aligned}
\left\|p-[p]_{B_2(x_0)}\right\|_{L^{3/2}(Q_2(z_0))} &\leq A\|\nabla p\|_{L_t^{3/2}L_x^{9/8}(Q_2(x_0))} \\
&\leq A\|\nabla p_1\|_{L_t^{3/2}L_x^{9/8}(Q_2(z_0))} + A\sum_{j=1}^{4}\left\|\nabla p_2^{(j)}\right\|_{L_t^{3/2}L_x^{9/8}(Q_2(z_0))} \\
&\leq A\delta.
\end{aligned}$$

Put $R=\max\{R_1,R_2,R_3\}$. Together with (5.21), we have

$$\int_{Q_2(z_0)}(|u|^3+|p-[p]_{B_2(x_0)}|^{3/2})dz < A\delta \quad \forall x_0\in\bar{\Omega},\ |x_0|>R.$$

By choosing δ such that $4A\delta<\varepsilon$ and using ε-regularity criteria (Proposition 5.11 and Proposition 5.14), we conclude that u is bounded in $Q_1(z_0)$ by a constant C, independent of x_0, for all $x_0\in\bar{\Omega}$, $|x_0|>R$. □

Proposition 5.18. *Suppose $\Omega=\mathbb{R}^3$. Let $f\in Y_q$ and (u,p) be an sw-solution of the Navier–Stokes problem on $\mathbb{R}^3\times(0,T)$. Then there exists $R>0$ such that*

(i) $\nabla u\in L^{10/3}(\mathbb{R}^3\backslash B_R\times(\tau,T))$ for all $\tau\in(0,T)$.

(ii) $\partial_t u$, $\nabla^2 u$, $\nabla p\in L^q(\mathbb{R}^3\backslash B_R\times(\tau,T))$ for all $\tau\in(0,T)$.

Proof. Assume $T=3$. Let $R_0>0$ be the number found in Proposition 5.17. We show that R can be chosen as $6R_0$.

(i) Take $\tau=2$ for simplicity. Let $(u,p)=(v,p_1)+(w,p_2)$ be the decomposition of (u,p) as an sw-solution. Let $\eta=\eta(t)$ be a smooth function equal to 0 on $[0,1]$, and equal to 1 on $[2,3]$. Put $\tilde{v}=v\eta$ and $\tilde{p}_1=p_1\eta$. Then $(\tilde{v},\tilde{p}_1)$ solves the Stokes system

$$\begin{cases}
\partial_t\tilde{v}-\Delta\tilde{v}+\nabla\tilde{p}_1 &= f\eta+v\eta', \\
\operatorname{div}\tilde{v} &= 0, \\
\tilde{v}(0) &= 0.
\end{cases}$$

By Remark 3.4, we know that $v\in L_t^\infty L_x^q$. Thus, the new force term belongs to $L_{t,x}^q$. By Solonnikov's coercive estimates (Proposition 3.3), $\nabla\tilde{p}_1\in L_{t,x}^q$. Put $g=f\eta+v\eta'-\nabla\tilde{p}_1\in L_{t,x}^q$. Then $\tilde{v}$ solves the heat equation

$$\begin{cases}
\partial_t\tilde{v}-\Delta\tilde{v} &= g, \\
\tilde{v}(0) &= 0,
\end{cases}$$

having an explicit form

$$\tilde{v}(t) = \int_0^t \Gamma(t-s) * g(s)ds.$$

By the fractional inequality (Proposition A.16), $\nabla v \in L_t^n L_x^m$ for all $q < n, m < \infty$ such that

$$\frac{2}{n} + \frac{3}{m} = \frac{5}{q} - 1.$$

In particular, $\nabla \tilde{v} \in L^{10/3}(\mathbb{R}^3 \times (0,3))$. Therefore, $\nabla v \in L^{10/3}(\mathbb{R}^3 \times (1,3))$. We now consider w, which solves the equation

$$\partial_t w - \Delta w + \nabla p_2 = -u \cdot \nabla u. \tag{5.22}$$

By the choice of R_0, $u \in L^\infty(\mathbb{R}^3 \backslash B_{R_0} \times (1,3))$. Let $\chi : \mathbb{R}^3 \to \mathbb{R}$ be smooth function equal to 0 inside the ball B_{3R_0} and equal to 1 outside the ball B_{6R_0}. Put $\tilde{w} = w\chi(x)\eta(t)$ and $\tilde{p}_2 = p_2\chi(x)\eta(t)$. Then

$$\partial_t \tilde{w} - \Delta \tilde{w} + \nabla \tilde{p}_2 = -\underbrace{u \cdot \nabla u \chi \eta}_{\{1\}} + \underbrace{w\chi\eta'}_{\{2\}} - \underbrace{\nabla w \nabla \chi \eta}_{\{3\}} - \underbrace{w \Delta \chi \eta}_{\{4\}} - \underbrace{p_2 \nabla \chi \eta}_{\{5\}}. \tag{5.23}$$

We show that each term on the right hand side belongs to $L^2(\mathbb{R}^3 \times (0,3))$. We have $v \in L_t^\infty L_x^3 \subset L_{t,x}^3$ and $w \in L_t^4 L_x^3 \subset L_{t,x}^3$. Thus, $u \in L^3(\mathbb{R}^3 \times (0,3))$. Put $\mathscr{O} = \mathbb{R}^3 \backslash B_{R_0} \times (1,3)$. The boundedness of u on $\mathscr{O}$ implies that $u \in L^5(\mathscr{O})$. Then $u \cdot \nabla v \in L^2(\mathscr{O})$ because $\nabla v \in L^{10/3}(\mathscr{O})$ as shown earlier. Also, $u \cdot \nabla w \in L^2(\mathscr{O})$ because $u \in L^\infty$ and $\nabla w \in L^2$. Therefore, $\{1\} \in L^2$. The terms $\{2\}$, $\{3\}$, $\{4\}$ belong to $L_{t,x}^2$ because w and ∇w belong to $L_{t,x}^2$. Taking the divergence of both sides of (5.22), we get

$$\Delta p_2 = -\text{div div } (u \otimes u).$$

Let $\Phi(x) = -A|x|^{-1}$ be the fundamental solution to the Laplace equation in $\mathbb{R}^3$. Then

$$p_2 = -\Phi * \text{div}\,\text{div}(u \otimes u) = -\nabla^2 \Phi * (u \otimes u).$$

For estimation purposes, it is convenient for us to denote $\nabla(u \otimes u)$ by $u\nabla u$. Taking gradient of the above equation, we get $\nabla p_2 = -\nabla^2 \Phi * (u \nabla u)$. The kernel of the convolution

is $k : \mathbb{R}^3\backslash\{0\} \to \mathbb{R}^9$, $k(x) = -\nabla^2\Phi(x)$. It follows that $|k(x)| \le A|x|^{-3}$. We have

$$\begin{aligned}\nabla p_2 = \int_{\mathbb{R}^3} k(x-y)u\nabla u(y)dy &= \underbrace{\int_{|y|<R_0} k(x-y)u\nabla u(y)dy}_{\{6\}} \\ &+ \underbrace{\int_{|y|>R_0} k(x-y)u\nabla u(y)dy}_{\{7\}}.\end{aligned}$$

For $|x| > 3R_0$,

$$\begin{aligned}|\{6\}| \le \int_{|y|<R_0} A\frac{|u\nabla u(y)|}{|x-y|^3}dy &\le AR_0^{-3}\int_{|y|<R_0}|u\nabla u(y)|dy \\ &\le AR_0^{-3}\|u\|_{L^2_x(B_{R_0})}\|\nabla u\|_{L^2_x(B_{R_0})}.\end{aligned}$$

Denote by D the spherical shell with inner radius $3R_0$ and outer radius $6R_0$. Then

$$\|\{6\}\|_{L^2(D\times(0,3))} \le AR_0^{-3/2}\|u\|_{L^\infty_t L^2_x(B_{R_0}\times(0,3))}\|\nabla u\|_{L^2(B_{R_0}\times(0,3))} < \infty.$$

Because k is a Calderón–Zygmund kernel, $\|\{7\}\|_{L^2(\mathbb{R}^3)} \le A\big\|uI_{|y|>R_0}\nabla u\big\|_{L^2_x(\mathbb{R}^3)}$ where I_E denotes the characteristic function of a set E. Since $\{1\} \in L^2(\mathscr{O})$,

$$\|\{7\}\|_{L^2(\mathbb{R}^3\times(1,3))} \le A\big\|uI_{|y|>R_0}\nabla u\big\|_{L^2(\mathscr{O})} < \infty.$$

Replace p_2 by $p_2 - [p_2]_D$. By Poincaré inequality and the fact that $\nabla p_2 = \{6\} + \{7\}$, we get

$$\|\{5\}\|_{L^2(\mathscr{O})} \le A\|p_2 - [p_2]_D\|_{L^2(D\times(1,3))} \le AR_0\|\nabla p_2\|_{L^2(D\times(1,3))} < \infty.$$

Let $\tilde{g}$ be the right hand side of (5.23). We have showed that $\tilde{g} \in L^2(\mathbb{R}^3 \times (0,3))$. Projecting both sides of (5.23) onto the divergence-free vector fields, we get $\partial_t\tilde{w} - \Delta\tilde{w} = \mathbb{P}\tilde{g} \in L^2_{t,x}$. By the theory of heat equations (see e.g. [17, Sec. 7.1.3]), we have $\tilde{w} \in L^\infty_t H^1_x \cap L^2_t H^2_x(\mathbb{R}^3 \times (0,3))$. This implies $\nabla\tilde{w} \in L^\infty_t L^2 \cap L^2_t H^1_x$. By interpolation, $\nabla\tilde{w} \in L^{10/3}_{t,x}$. Note that the power $10/3$ can be seen from the fractional inequality as in our proof for $\nabla\tilde{v} \in L^{10/3}_{t,x}$. Therefore, $\nabla u = \nabla v + \nabla w \in L^{10/3}(\mathbb{R}^3\backslash B_R \times (2,3))$ with $R = 6R_0$.

(ii) By Part (i), $\nabla u \in L^{10/3}(\mathbb{R}^3\backslash B_R \times (\tau,3))$ for all $\tau \in (0,3)$. With the same cutoff functions χ and η, we show that the right hand side of (5.23) belongs to $L^{10/3}(\mathbb{R}^3) \times (0,3)$.

It is straightforward that the first four terms belong to $L^{10/3}(\mathbb{R}^3)\times(0,3)$. We refine the estimate of the fifth term (involving pressure) as follows. Let ρ be a smooth function supported in B_{2R_0} and equal to 1 in B_{R_0} with $\|\nabla\rho\|_{L^\infty}\le AR_0^{-1}$.

$$\nabla p_2=\underbrace{\int_{\mathbb{R}^3}k(x-y)\rho(y)\nabla(u\otimes u)(y)dy}_{\{8\}}+\underbrace{\int_{\mathbb{R}^3}k(x-y)(1-\rho(y))\nabla(u\otimes u)(y)dy}_{\{9\}}.$$

Term $\{9\}$ is treated similarly to term $\{7\}$ in Part (i): since k is a Calderón–Zygmund kernel, $\|\{9\}\|_{L_x^{10/3}(\mathbb{R}^3)}\le A\|u(1-\rho)\nabla u\|_{L_x^{10/3}(\mathbb{R}^3)}$. Then

$$\|\{9\}\|_{L^{10/3}(\mathbb{R}^3\times(1,3))}\le A\|u\|_{L^\infty(\mathscr{O})}\|\nabla u\|_{L^{10/3}(\mathscr{O})}<\infty. \tag{5.24}$$

By integration by parts, term $\{8\}$ is split as follows.

$$\{8\}=\underbrace{\int_{\mathbb{R}^3}\nabla k(x-y)\rho(y)u\otimes u(y)dy}_{\{11\}}-\underbrace{\int_{\mathbb{R}^3}k(x-y)u\otimes u(y)\nabla\rho(y)dy}_{\{12\}}.$$

For $|x|>3R_0$,

$$\begin{aligned}|\{11\}| &\le \int_{|y|<2R_0}A\frac{|u\otimes u(y)|}{|x-y|^4}dy\le AR_0^{-4}\int_{|y|<2R_0}|u(y)|^2dy=AR_0^{-4}\,\|u\|^2_{L^2_x(B_{2R_0})},\\ |\{12\}| &\le \int_{|y|<2R_0}A\frac{|u\otimes u(y)|}{|x-y|^3}|\nabla\rho(y)|dy\le AR_0^{-4}\,\|u\|^2_{L^2_x(B_{2R_0})}.\end{aligned}$$

Hence,

$$\|\{8\}\|_{L^{10/3}(D\times(0,3))}\le AR_0^{-31/10}\,\|u\|^2_{L^\infty_tL^2_x(B_{2R_0}\times(0,3))}<\infty. \tag{5.25}$$

By (5.24) and (5.25), $\nabla p_2\in L^{10/3}(\mathscr{O})$. Replace p_2 by $p_2-[p_2]_D$. By Poincaré inequality,

$$\|\{5\}\|_{L^{10/3}(\mathscr{O})}\le A\|p_2-[p_2]_D\|_{L^{10/3}(D\times(1,3))}\le AR_0\|\nabla p_2\|_{L^{10/3}(D\times(1,3))}<\infty.$$

Now for simplicity we show Part (ii) for $\tau=2$. Because the right hand side of (5.23) belongs to $L^{10/3}(\mathbb{R}^3\times(0,3))$, we have $\partial_t\tilde{w}$, $\nabla^2\tilde{w}$, $\nabla\tilde{p}_2\in L^{10/3}(\mathbb{R}^3\times(0,3))$ by Solonnikov's coercive estimates (Proposition 3.3). Hence, $\partial_t w$, $\nabla^2 w$, $\nabla p_2\in L^{10/3}(\tilde{\mathscr{O}})$ with $\tilde{\mathscr{O}}=\mathbb{R}^3\backslash B_R\times(2,3)$. In the proof of Part (i), we showed that these terms also belong to $L^2(\tilde{\mathscr{O}})$. By interpolation, they belong to $L^q(\tilde{\mathscr{O}})$. On the other hand, by coercive estimate (Proposition 3.5), $\partial_t v$, $\nabla^2 v$, $\nabla p_1\in L^q(\mathbb{R}^3\times(2,3))$. Therefore, $\partial_t u$, $\nabla^2 u$, $\nabla p\in L^q(\tilde{\mathscr{O}})$.

□

Remark 5.19. By parabolic Sobolev embedding theorem (see e.g. Proposition A.10, [102, Thm. 1.4.1] and [75, Appendix D3, D4]), u belongs to the parabolic Hölder space $C^{\alpha,\alpha/2}_{\text{loc}}(\mathbb{R}^3\backslash B_R \times (\tau, T))$ with $\alpha = 1/2$. If f has better regularity, one can continue to bootstrap the regularity of the right hand side of (5.23).

We know from Proposition 4.2 that a mild solution always exists locally in time, and that it blows up after finite time if and only if its L^5-norm (in space and time) blows up. Before the blowup time, the solution has no singular points since it is bounded (Part (iii) of Proposition 4.2). The following is an application of ε-regularity criteria to show that the blowup of a mild solution is only caused by the formation of a singular point in finite time. This result is known in the setting $u_0 \in L^3(\mathbb{R}^3)$, $f = 0$ [32, Lem. 3.1], and $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$, $f = 0$ [76, p. 886]. Here we follow their techniques of proof.

Proposition 5.20. *Let u be a mild solution to (NSE)$_\Omega$ with $u_0 = 0$ and $f \in Y_q$ for some $5/2 < q < 3$. Suppose the maximal time of existence is $T_* < \infty$. Then there exists $x_0 \in \bar{\Omega}$ such that $z_0 = (x_0, T_*)$ is a singular point of u.*

Proof. Assume $T_* = 1$ for simplicity. By Proposition 5.17, there exists $R > 0$ such that $u \in L^\infty(\Omega\backslash B^\Omega_R \times (1/2, 1))$. Suppose by contradiction that u has no singular points in $\bar{\Omega} \times \{1\}$. Then each $z_0 = (x_0, 1)$ with $|x_0| \leq R$ is a regular point of u. In other words, u is bounded in a parabolic cylinder centered at z_0. By compactness, u is bounded in $B^\Omega_R \times (T_1, 1)$ for some $T_1 \in (3/4, 1)$. Therefore, u is bounded in $\Omega \times (T_1, 1)$. By the fact that $u \in L^3(\Omega \times (0, 1))$, we get $u \in L^5(\Omega \times (T_1, 1))$. Then $u \in L^5(\Omega \times (0, 1))$. This is a contradiction because u blows up at $T_* = 1$. □

5.5 Persistence of singularities

Let $f_m \rightharpoonup f$ in Y_q for some $5/2 < q < 3$. Suppose (u_m, p_m) defined on $\Omega \times (0, T)$, $T < \infty$, be an sw-solution to (NSE)$_\Omega$ with $u_0 = 0$ and force term f_m. We know that (u_m, p_m) converges to (u, p), an sw-solution to (NSE)$_\Omega$ with $u_0 = 0$ and force term f, in the sense described in Proposition 5.7.

Proposition 5.21. *Suppose each u_m has a singularity at $z_m = (x_m, T_1)$, with $x_m \in \bar{\Omega}$, $T_1 \in (0, T]$, and that the sequence (z_m) converges to $z_0 = (x_0, T_1)$. Then z_0 is a singular*

point of u.

Proof. We assume $T_1 = 5$ for simplicity. Put $M = \sup_m \|f_m\|_{Y_p} < \infty$. By ε-regularity criteria, there exists $\varepsilon > 0$ and $\delta = \delta(M) > 0$ such that

$$\int_{Q_r^\Omega(z_m)} (|u_m|^3 + |p_m - [p_m]_{B_r^\Omega(x_m)}|^{3/2})dz \geq \varepsilon \quad \forall\, m \in \mathbb{N},\ \forall\, r \in (0, \delta). \tag{5.26}$$

The goal is to be able to pass through the limit as $m \to \infty$. By the compactness theorem (Proposition 5.7), $u_m \to u$ in $L^3(Q_1^\Omega(z_0))$. For every $r \in (0, 1)$, we have

$$\int_{Q_r^\Omega(z_m)} |u_m|^3 dz \to \int_{Q_r^\Omega(z_0)} |u|^3 dz \quad \text{as } m \to \infty. \tag{5.27}$$

Similar convergence of the pressure term is not obvious, probably not true, because p_m only converges weakly to p. In the following, we distinguish the cases $x_0 \in \Omega$ and $x_0 \in \partial\Omega$ to apply different decompositions of the pressure. As shown below, if $x_0 \in \Omega$ the decomposition $p = p_d - p_c + p_e$ as mention in (1.4) works well. The harmonic pressure p_d enjoys rich regularity away from the boundary (see estimate (5.36)). This is the decomposition used in [76] and [32]. With regard to the decoupling $p = p_1 + p_2$ of an sw-solution, one can see that $p_1 = p_e$ and $p_2 = p_d - p_c$. If $x_0 \in \partial\Omega$, the decomposition $p_2 = p_d - p_c$ is replaced by $p_2 = \pi^{(1)} + \pi^{(2)}$ as in (5.17) and (5.18).

In the rest of the proof, we assume $x_m = x_0$ for all m. The method itself is not affected by this assumption. Put $z_0 = (x_0, 5)$.To simplify the notation, we write B_r for $B_r^\Omega(x_0)$, and Q_r for $Q_r^\Omega(z_0)$ when $x_0 \in \Omega$, and write B_r^+ for $B_r^\Omega(x_0)$, and Q_r^+ for $Q_r^\Omega(z_0)$ when $x_0 \in \partial\Omega$. To any function g, denote

$$C_r(g) = \frac{1}{r^2} \int_{Q_r^\Omega(z_0)} |g|^3 dz, \qquad D_r(g) = \frac{1}{r^2} \int_{Q_r^\Omega(z_0)} |g - [g]_{B_r^\Omega(x_0)}|^{3/2} dz.$$

It is easy to check that

$$C_r(g + h) \leq 4(C_r(g) + C_r(h)), \qquad D_r(g + h) \leq \sqrt{2}(D_r(g) + D_r(h)).$$

Because our method is mainly qualitative, absolute constants such as 4 and $\sqrt{2}$ will not be specified. We also denote by δ an unspecified number such that certain estimates are valid for all $r \in (0, \delta)$, or "for all sufficiently small r". With these notations, (5.26) and (5.27) become

$$C_r(u_m) + D_r(p_m) \geq \varepsilon \quad \forall\, m \in \mathbb{N},\ \forall\, r \in (0, \delta). \tag{5.28}$$

$$\lim_{m\to\infty} C_r(u_m) = C_r(u) \quad \forall\, r \in (0,\delta). \tag{5.29}$$

Let $(u_m, p_m) = (v_m, p_{1,m}) + (w_m, p_{2,m})$ and $(u,p) = (v,p_1) + (w,p_2)$ be the decoupling of sw-solutions. By (5.28),

$$C_r(u_m) + D_r(p_{1,m}) + D_r(p_{2,m}) \geq A\varepsilon \quad \forall\, m \in \mathbb{N},\ \forall\, r \in (0,\delta). \tag{5.30}$$

By the coercive estimates of the Stokes system (Proposition 3.5),

$$\|\nabla p_{1,m}\|_{L^q(\Omega\times(4,5))} \leq C\|f_m\|_{Y_q} \leq CM.$$

By Poincaré–Sobolev inequality (Proposition A.2),

$$\left\|p_{1,m} - [p_{1,m}]_{B_r^\Omega}\right\|_{L^{3/2}(B_r^\Omega)} \leq Ar^{3-3/q}\|\nabla p_{1,m}\|_{L^q(B_r^\Omega)}.$$

Taking into account the integral over time, we get

$$D_r(p_{1,m}) \leq C_M r^{\frac{9}{2}-\frac{15}{2q}} \quad \forall m \in \mathbb{N}. \tag{5.31}$$

As $r \to 0$, this term contributes insignificantly to the left hand side of (5.30). Note that the positive power of r is due to the fact that the force terms f_m are bounded in a subcritical space $L^q_{t,x}$. This is not the case for $D_r(p_{2,m})$. The force term $-u_m \cdot \nabla u_m$ of w_m is only bounded in a supercritical space, for example $L^{5/4}_{t,x}$. Nevertheless, an upper bound for $p_{2,m}$ in $L^{3/2}(Q_1)$ can be obtained as follows. First, replace $p_{2,m}$ by $p_{2,m} - [p_{2,m}]_{B_1}$. (Note that pressures are determined up to a quantity only depending on t.) Then $[p_{2,m}]_{B_1} = 0$. Thanks to Poincaré–Sobolev inequality and Proposition 5.5,

$$\|p_{2,m}\|_{L^{3/2}(Q_1)} \leq C\|\nabla p_{2,m}\|_{L^{3/2}_t L^{9/8}_x} \leq C_M \quad \forall m \in \mathbb{N}. \tag{5.32}$$

Consider the case $x_0 \in \Omega$.

There is no need to distinguish Ω from $\mathbb{R}^3$ because (u_m, p_m) and (u,p) are interior suitable weak solutions in the parabolic cylinders Q_r in $\mathbb{R}^3 \times (0,\infty)$ as long as r is sufficiently small (see Proposition 5.16). The interior regularity of the pressure is already treated in [76, Lem. 2.1]. We explain it here for a self-contained proof. Suppose by contradiction that $z_0 = (x_0, 5)$ is a regular point of u. We assume that u is bounded in Q_1 for simplicity. Recall that (w, p_2) satisfies the equation

$$\partial_t w - \Delta w + \nabla p_2 = -u \cdot \nabla u$$

in sense of distribution in Q_1. Taking the divergence of both sides, we get $\Delta p_2 = -\partial_i\partial_j(u_iu_j)$. Let $\chi : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function supported in B_1 and equal to 1 in $B_{1/2}$. Then p_2 is decomposed at $p_2 = p_3 + h$ where

$$p_3 = -R_iR_j(u_iu_j\chi),$$

R_1, R_2, R_3 are Riesz operators in $\mathbb{R}^3$, and h is a harmonic function on $B_{1/2}$. We see that p_3 is the pressure caused by local convection; and h is caused by the boundary and convection elsewhere. The pressure $p_{2,m}$ is decomposed likewise. Then (5.30) implies

$$C_r(u_m) + D_r(p_{1,m}) + D_r(p_{3,m}) + D_r(h_m) \geq A\varepsilon \quad \forall\, m \in \mathbb{N},\ \forall\, r \in (0,\delta). \tag{5.33}$$

Because (u_m) converges strongly in $L^3(Q_1)$, the sequence $(p_{3,m})$ converges strongly in $L^{3/2}(Q_1)$. Thus,

$$\lim_{m\to\infty} D_r(p_{3,m}) = D_r(p_3) \quad \forall\, r \in (0,\delta). \tag{5.34}$$

By the boundedness of Riesz operators in $L^{3/2}$,

$$\|p_{3,m}\|_{L^{3/2}(Q_1)} \leq \|u_m\|^2_{L^3(Q_1)} \leq C_M \quad \forall\, m \in \mathbb{N}. \tag{5.35}$$

By (5.32) and (5.35), (h_m) is bounded in $L^{3/2}(Q_1)$. It follows from the theory of harmonic functions that for all $r < 1/4$,

$$\big\|h_m - [h_m]_{B_r}\big\|_{L^{3/2}(B_r)} \leq Ar^3\|h_m\|_{L^{3/2}(B_{2r})} \leq C_Mr^3. \tag{5.36}$$

This implies

$$D_r(h_m) \leq C_Mr^3 \quad \forall\, m \in \mathbb{N},\ \forall\, r \in (0,\delta). \tag{5.37}$$

For small r, by (5.31) and (5.37) the terms involving $p_{1,m}$ and h_m can be dropped from the left hand side of (5.33). Then with $m \to \infty$, we get

$$C_r(u) + D_r(p_3) \geq A\varepsilon \quad \forall\, r \in (0,\delta). \tag{5.38}$$

So far, we have not used the assumption by contradiction that u is bounded in Q_1. This leads to $C_r(u) \leq Cr^3$. Although p_3 is not necessary bounded (since the Riesz operators are not defined from L^∞ to L^∞), it is L^α-integrable with respect to the spatial variables for any $\alpha < \infty$. In particular, $p_3 \in L^{3/2}_tL^6_x(\mathbb{R}^3 \times (4,5))$. Then $D_r(p_3) \leq Cr^{1/4}$. This

contradicts (5.38).

Consider the case $x_0 \in \partial\Omega$.

For simplicity, assume $x_0 = 0$. Suppose by contradiction that $z_0 = (x_0, 5) = (0, 5)$ is a regular point of u. We can assume u is bounded in $Q_2^+ = B_2^+ \times (1, 5)$ without loss of generality. Let $\tilde{B}$ be a smooth domain such that $B_1^+ \subset \tilde{B} \subset B_2^+$, and let Γ be the flat portion of the boundary of $\tilde{B}$ on $\partial\Omega$. We decompose $w = u^{(1)} + u^{(2)}$ and $p_2 = \pi^{(1)} + \pi^{(2)}$ such that

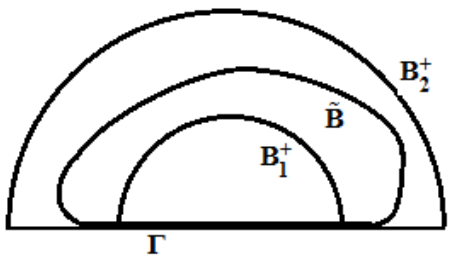

Figure 5.2: The smoothed half-ball $\tilde{B}$.

$$\begin{cases} \partial_t u^{(1)} - \Delta u^{(1)} + \nabla \pi^{(1)} &= -u \cdot \nabla u \quad \text{in } \tilde{B} \times (0, 5), \\ \text{div } u^{(1)} &= 0, \\ u^{(1)}|_{\partial\tilde{B}} &= 0, \\ u^{(1)}(0) &= 0. \end{cases} \tag{5.39}$$

$$\begin{cases} \partial_t u^{(2)} - \Delta u^{(2)} + \nabla \pi^{(2)} &= 0 \quad \text{in } \tilde{B} \times (0, 5), \\ \text{div } u^{(2)} &= 0, \\ u^{(2)}|_{\Gamma} &= 0, \\ u^{(2)}(0) &= 0. \end{cases} \tag{5.40}$$

The functions w_m and $p_{2,m}$ are decomposed likewise. Then (5.30) implies

$$C_r(u_m) + D_r(p_{1,m}) + D_r(\pi_m^{(1)}) + D_r(\pi_m^{(2)}) \geq A\varepsilon \quad \forall\, m \in \mathbb{N},\ \forall\, r \in (0, \delta). \tag{5.41}$$

Put $\tilde{Q} = \tilde{B} \times (0, 5)$. By the compactness theorem (Proposition 5.7), $\nabla u_m \to \nabla u$ in $L^{15/8}(\tilde{Q})$ and $u_m \to u$ in $L^1(\tilde{Q})$. Because (u_m) is bounded energy norm on $\tilde{Q}$ (Corollary 5.6), it is bounded in $L_t^8 L_x^{12/5}(\tilde{Q})$. By Lemma A.5, $u_m \to u$ in $L_t^{15/2} L_x^{9/4}(\tilde{Q})$. Thus,

$u_m \cdot \nabla u_m \to u \cdot \nabla u$ in $L_t^{3/2} L_x^{45/44}(\tilde{Q})$. Replace $\pi_m^{(1)}$ by $\pi_m^{(1)} - [\pi_m^{(1)}]_{\tilde{B}}$. By Solonnikov's coercive estimates (Proposition 3.3),

$$\left\|\nabla \pi_m^{(1)} - \nabla \pi^{(1)}\right\|_{L_t^{3/2} L_x^{45/44}(\tilde{Q})} \leq A \|u_m \cdot \nabla u_m - u \cdot \nabla u\|_{L_t^{3/2} L_x^{45/44}(\tilde{Q})}$$

which goes to 0 as $m \to \infty$. By Poincaré–Sobolev inequality,

$$\left\|\pi_m^{(1)} - \pi^{(1)}\right\|_{L_{t,x}^{3/2} L(\tilde{Q})} \leq \left\|\nabla \pi_m^{(1)} - \nabla \pi^{(1)}\right\|_{L_t^{3/2} L_x^{45/44}(\tilde{Q})} \to 0 \quad \text{as } m \to \infty.$$

Hence,

$$\lim_{m \to \infty} D_r(\pi_m^{(1)}) = D_r(\pi^{(1)}) \quad \forall\, r \in (0, \delta). \tag{5.42}$$

To estimate $\pi^{(2)}$, we need Lemma 5.22 due to Seregin. Some estimates of $\pi_m^{(2)}$ can be found through Solonnikov's coercive estimates of $\pi_m^{(1)}$. Indeed, the force term $-u_m \cdot \nabla u_m$ of the system (5.39) is bounded in $L_x^{3/2} L_x^{9/8}(\tilde{Q})$ (by Proposition 5.5).

$$\|u_m \cdot \nabla u_m\|_{L_t^{3/2} L_x^{9/8}(\tilde{Q})} \leq C_M \quad \forall\, m \in \mathbb{N}.$$

By coercive estimates, the sequences $\partial_t u_m^{(1)}$, $u_m^{(1)}$, $\nabla u_m^{(1)}$, $\nabla^2 u_m^{(1)}$, $\nabla \pi_m^{(1)}$ are bounded in $L_x^{3/2} L_x^{9/8}(\tilde{Q})$. Hence, $(u_m^{(2)}, \pi_m^{(2)}) = (w_m, p_{2,m}) - (u_m^{(1)}, \pi_m^{(1)})$ satisfies

$$\left\|\partial_t u_m^{(2)}\right\|_X, \left\|u_m^{(2)}\right\|_X, \left\|\nabla u_m^{(2)}\right\|_X, \left\|\nabla^2 u_m^{(2)}\right\|_X, \left\|\nabla \pi_m^{(2)}\right\|_X \leq C_M$$

where $X = L_x^{3/2} L_x^{9/8}(\tilde{Q})$. Replace $\pi_m^{(2)}$ by $\pi_m^{(2)} - [\pi_m^{(2)}]_{\tilde{B}}$. Then $\|\pi_m^{(2)}\| \leq C_M$ by Poincaré–Sobolev inequality. Therefore, we can apply Lemma 5.22 for $\alpha = 3/2$, $\beta = 9/8$, $\gamma = 2$, and get $\left\|\nabla \pi_m^{(2)}\right\|_{L_t^{3/2} L_x^2(Q_{1/2}^+)} \leq C_M$ for all $m \in \mathbb{N}$. By Poincaré–Sobolev inequality,

$$\left\|\pi_m^{(2)} - [\pi_m^{(2)}]_{B_r^+}\right\|_{L^{3/2}(Q_r^+)} \leq r^{3/2} \left\|\nabla \pi_m^{(2)}\right\|_{L_t^{3/2} L_x^2(Q_r^+)} \leq C_M r^{3/2}.$$

Hence,

$$D_r(\pi_m^{(2)}) \leq r^{1/4} C_M \quad \forall\, m \in \mathbb{N},\ \forall\, r \in (0, \delta). \tag{5.43}$$

For small r, by (5.31) and (5.43) the terms involving $p_{1,m}$ and $\pi_m^{(2)}$ can be dropped from the left hand side of (5.41). Then with $m \to \infty$, we get

$$C_r(u) + D_r(\pi^{(1)}) \geq A\varepsilon \quad \forall\, r \in (0, \delta). \tag{5.44}$$

So far, we have not used the assumption by contradiction that u is bounded in Q_2^+. This leads to $C_r(u) \le Cr^3$. Let $\eta = \eta(t)$ be a smooth function equal to 0 on [0,1] and equal to 1 on $[2, \infty)$. Put $u^{(3)} = u^{(1)}\eta$ and $\pi^{(3)} = \pi^{(1)}\eta$. Then $(u^{(3)}, \pi^{(3)})$ solves the system

$$\begin{cases} \partial_t u^{(3)} - \Delta u^{(3)} + \nabla \pi^{(3)} &= -\eta u \cdot \nabla u + u^{(1)}\eta' \qquad \text{in } \tilde{Q} = \tilde{B} \times (0,5), \\ \text{div } u^{(3)} &= 0, \\ u^{(3)}|_{\partial \tilde{B}} &= 0, \end{cases} \tag{5.45}$$

The new force term is $g = g_1 + g_2$ where $g_1 = -\eta u \cdot \nabla u$ and $g_2 = u^{(1)}\eta'$. Because u is bounded in Q_2^+ and $\nabla u \in L^2(\tilde{Q})$, $g_1 \in L^2(\tilde{Q})$. As mentioned earlier, $\partial_t u^{(1)}$ and $\nabla^2 u^{(1)}$ belong to $L_t^{3/2} L_x^{9/8}(\tilde{Q})$. By the Sobolev embedding argument mentioned in Remark 3.4, $\nabla u^{(1)} \in L_t^3 L_x^{9/8}(\tilde{Q})$. Then by the embedding $W_0^{1,9/8}(\tilde{B}) \hookrightarrow L^{9/5}(\tilde{B})$, we get $u^{(1)} \in L_t^3 L_x^{9/5}(\tilde{Q})$. Hence, $g = g_1 + g_2 \in L_t^2 L_x^{9/5}(\tilde{Q})$. By Solonnikov's coercive estimates, $\nabla \pi^{(3)} \in L_t^2 L_x^{9/5}(\tilde{Q})$. By Poincaré–Sobolev inequality,

$$D_r(\pi^{(1)}) = D_r(\pi^{(3)}) \le r^{1/2} \left\| \nabla \pi^{(3)} \right\|_{L_t^2 L_x^{9/5}(Q_r^+)} \le Cr^{1/2} \quad \forall\, r \in (0, \delta).$$

This contradicts (5.44). □

Lemma 5.22. *[81, Prop. 2] Denote $Q_r^+ = B_r^+ \times (5 - r^2, 5)$, and let Γ be the flat portion of ∂B_1^+. Let $1 < \alpha, \beta \le 2$, $\beta < \gamma < \infty$. Suppose (u, p) is a solution to the system*

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= 0 \qquad \text{in } Q_1^+, \\ \text{div } u &= 0, \\ u|_\Gamma &= 0, \end{cases}$$

with regularity properties $\partial_t u$, u, ∇u, $\nabla^2 u$, p, $\nabla p \in X = L_t^\alpha L_x^\beta(Q_1^+)$. Then these six functions also belong to $Y = L_t^\alpha L_x^\gamma(Q_{1/2}^+)$. Their norms in Y are bounded by $C_{\alpha,\beta,\gamma}(\|u\|_X + \|\nabla u\|_X + \|p\|_X)$.

Chapter 6

Minimal blowup data for potential Navier–Stokes singularities

6.1 Formulation of the problem

Consider the Navier–Stokes system with initial condition in a critical space X and force term in a critical space Y. We refer to the pair (u_0, f) as the *data* of $(\mathrm{NSE})_\Omega$. The product space $X \times Y$ equipped with norm $\|(u_0, f)\|_{X\times Y} = \|u_0\|_X + \|f\|_Y$ is a critical space of the data with respect to the scaling transformation (1.1). It is known as a rule of thumb that if the data is sufficiently small in a critical space then $(\mathrm{NSE})_\Omega$ has a global mild solution. The idea that the smallness of a scale-invariant quantity implies global regularity traces back to the pioneering work of Leray in 1934 [48, Para. 21], in which he proves global regularity given the smallness of $\|u_0\|_{L^\infty}\|u_0\|_{L^2}^2$ or $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}$. Since then, the local and global existence of mild solutions have been studied in a number of critical spaces, for example

- $u_0 \in L^3$, $f = 0$ in [35],
- $u_0 \in L^3$, $t^\alpha f \in L_t^\infty L_x^p$, $1 < p < 3$ in [47, Thm. 7.5],
- $u_0 = \dot{H}^{1/2}$, $f \in L_t^2 \dot{H}_x^{-1/2}$ in [5, Thm. 5.6], [47, Thm. 7.4],

- $u_0 \in M^{p,\lambda}$ (Morrey spaces) in [36], [103],
- $u_0 \in \dot{B}^{-1+3/p}_{p,\infty}$ (homegeneous Besov spaces) in [13], [72],
- $u_0 \in BMO^{-1}$ in [38], [47, Ch. 9].

The common tool is Picard's contraction mapping principle (Lemma 1.3). The task is to select critical spaces for u_0, f, and u (the mild solution) so that the contraction mapping principle is applicable. In this light, Lemarié-Rieusset gives quite general choices for (X, Y) such as $X = E$, $Y = L^1_t E$ where E satisfies several reasonably weak premises. Among admissible choices for E are L^3, $L^{3,\infty}$ and critical Morrey spaces [47, p. 163-175], [46, Ch. 1, §2], [46, Ch. 15, §1].

Denote by $\rho^{\Omega}_{\max}$ the supremum of all $\rho > 0$ such that $(\mathrm{NSE})_\Omega$ is globally well-posed for every (u_0, f) with $\|(u_0, f)\|_{X\times Y} < \rho$. For $\Omega = \mathbb{R}^3$ and $Y = \{0\}$, whether $\rho^{\Omega}_{\max}$ is finite is essentially the millennium problem of fluid dynamics [18]. The global well-posedness in any other domain, with or without boundaries, is still not known. However, we are interested in the hypothetical situation when $\rho^{\Omega}_{\max}$ is finite. In particular, we consider the following question:

> (Q1) If $\rho^{\Omega}_{\max}$ is finite, does there exist a data $(u_0, f) \in X\times Y$ with $\|(u_0, f)\| = \rho^{\Omega}_{\max}$, such that the solution u of the system $(\mathrm{NSE})_\Omega$ blows up in finite time?

We call such data a *minimal blowup-generating data*, or simply *minimal blowup data*. This question is already addressed in several settings of the initial conditions (with zero force). Affirmative answers are established for $X = \dot{H}^{1/2}(\mathbb{R}^3)$ in [76], for $X = L^3(\mathbb{R}^3)$ in [32] and [21], for $X = \dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$, $3 < p$, $q < \infty$ in [22]. The global existence of mild solutions for small $u_0 \in \dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ is known, but the local existence for large u_0 is not known. The difficulty lies in the fact that one cannot make function in L^∞_t small by reducing the time interval. Albritton and Barker [3] reformulate question (Q1) for global weak Besov solutions with initial data in suitable subspaces of $\dot{B}^{-1}_{\infty,\infty}$, including $\dot{B}^{-1+3/p}_{p,\infty}(\mathbb{R}^3)$ with $3 < p < \infty$, and force term of the form

$$f = \operatorname{div} F, \quad t^\alpha F \in L^\infty_t L^q_x, \quad \text{for } 3 < q < \infty. \tag{6.1}$$

Using a treatment similar to [76, 32], they also obtain a positive answer.

Physical boundaries are known to complicate the regularity theory. For example, the proof the ε-regularity criteria near the boundary requires a different decomposition of the pressure [80, Lem. 7.2]. The same decomposition is needed to prove the persistence of singularities of sw-solutions (Proposition 5.21), which is a key step to construct a minimal blowup data in the half-space (under certain assumptions). Our introduction of the right hand side enables us to overcome stability issues when passing from one domain to another. In particular, one can obtain a blowup solution on the half-space from a blowup solution on the whole space via cutting off and Bogovskiĭ correction.

The most natural critical space for the force term is perhaps $L^{5/3}_{t,x}$. However, this space does not sustain the persistence of singularities. Indeed, consider the Navier–Stokes system in the whole space with $u_0 = 0$ and $f \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R})$ supported in $Q_1(0) = B_1(0) \times (1,2)$. Suppose the mild solution (u,p) has finite-time singularity at $z_0 = (0,2)$. For $\lambda > 0$, put

$$\begin{cases} f_\lambda(x,t) &= \lambda^3 f\left(\lambda x, \lambda^2\left(t + \dfrac{2}{\lambda^2} - 2\right)\right), \\ u_\lambda(x,t) &= \lambda u\left(\lambda x, \lambda^2\left(t + \dfrac{2}{\lambda^2} - 2\right)\right), \\ p_\lambda(x,t) &= \lambda^2 p\left(\lambda x, \lambda^2\left(t + \dfrac{2}{\lambda^2} - 2\right)\right). \end{cases} \tag{6.2}$$

Then (u_λ, p_λ) solves the Navier–Stokes system with force term f_λ and has singularity at z_0. Thus, (f_λ) is a family of blowup-generating forces with the same norm in $L^{5/3}_{t,x}$. We have

$$\|f_\lambda\|_{L^\alpha_{t,x}} = \lambda^{3-5/\alpha}\|f\|_{L^\alpha_{t,x}}$$

which tends to zero as $\lambda \to \infty$ for any $\alpha > 5/3$. This implies f_λ converges weakly to 0 in $L^{5/3}(\mathbb{R}^3 \times \mathbb{R})$. We see that the limit corresponds to the zero solution, which has no singularities. The failure to preserve singularities in the limit is due to the fact that the $L^{5/3}_{t,x}$-norm is invariant not only under the scaling but also under time-shift transformation $f(x,t) \to f(x,t+\tau)$. The two are coupled in (6.2). One way to exclude the time-shift symmetry is to consider time-weighted critical spaces, for example

$$Y_q = \{f : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^3 : \ t^{q_*} f \in L^q(\mathbb{R}^3 \times (0,\infty))\}$$

with

$$\|f\|_{Y_q} = \|t^{q_*} f\|_{L^q_{t,x}}, \qquad q_* = \frac{3}{2} - \frac{5}{2q}.$$

It can be seen from (6.1) that Albritton and Barker [3] choose a type of time-weighted critical space for f. Next, we discuss the range for q. If the force term f is divergence-free, the solution to the Stokes equations is exactly the solution to the heat equation, which is

$$\int_0^t \int_{\mathbb{R}^3} \Gamma(x-y,t-s) f(y,s) dyds.$$

For this function to be well-defined, it is reasonable to require $f \in L^1_{\text{loc}}$. By Hölder inequality,

$$\int_0^1 \int_{B_1} |f(y,s)| dyds \leq t^{1/q'} \left(\int_0^1 s^{q_* q'} \right)^{1/q'} \left(\int_0^1 \int_{B_1} |s^{q_*} f|^q dyds \right)^{1/q},$$

which is finite if and only if $q_* q' < 1$. This leads to $q < 3$. On the other hand, to be able to apply ε-regularity criteria, we need that, away form $t = 0$, f is locally in $L^\alpha_{t,x}$ for some $\alpha > 5/2$. As discussed in Section 5.4, the exponent $5/2$ cannot be lowered without losing the local L^∞-estimate for the solution. Therefore, $(\frac{5}{2}, 3)$ is quite a natural range for q.

In the sequel, we will use the following notations. For $\Omega = \mathbb{R}^3$ or $\mathbb{R}^3_+$ and $5/2 < q < 3$, put

$$Y^\Omega_q = \{ f : \Omega \times (0,\infty) \to \mathbb{R}^3 : \ t^{q_*} f \in L^q(\Omega \times (0,\infty)) \}.$$

Denote by $\rho^\Omega_{\max}$ the supremum of all $\rho > 0$ such that $(\text{NSE})_\Omega$ with $u_0 = 0$ has a global mild solution for every $f \in Y^\Omega_q$ with $\|f\|_{Y^\Omega_q} < \rho$. For simplicity, let us denote these objects by (NSE), Y_q and $\rho_{\max}$ for $\Omega = \mathbb{R}^3$, and $(\text{NSE})_+$, Y^+_q and $\rho^+_{\max}$ for $\Omega = \mathbb{R}^3_+$.

6.2 Bogovskiĭ localization

To localize a divergence-free field u on $\mathbb{R}^n$, we multiply it by a smooth cutoff function χ and add a correction term ϕ such that $v = u\chi + \phi$ is divergence free. The correction term satisfies

$$\text{div } \phi = -u \cdot \nabla \chi. \tag{6.3}$$

Consider a more general problem

$$\text{div } \phi = g. \tag{6.4}$$

If ϕ is not subjected to any boundary conditions, (6.3) is simple to solve: for $g \in C_0^\infty(\mathbb{R}^n)$, we choose $\phi = \nabla\psi$. Function ψ solves the Poisson equation $\Delta\psi = g$. The only solution that decays at infinity is $\psi = \Phi * g$, where Φ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$. Then a solution to (6.3) is

$$\phi = \nabla\Phi * g. \tag{6.5}$$

Because $\nabla^2\Phi$ is a Calderón–Zygmund kernel, we have the *a priori* estimates $\|\nabla\phi\|_{L^p(\mathbb{R}^n)} \leq C_{n,p}\|g\|_{L^p(\mathbb{R}^n)}$ for all $p \in (1,\infty)$. Now go back to Equation 6.3. Suppose χ is supported in the ball of radius 2 and equal to one on the ball of radius 1. Then $\nabla\chi$ is supported in a spherical shell which we denote by $S_{1,2}$. Then $g = -u \cdot \nabla\phi$ is supported in $S_{1,2}$. It is natural to ask the following question: *for $g \in C_0^\infty(S_{1,2})$, does there exist $\phi \in C_0^\infty(S_{1,2})$ such that div $\phi = g$ and $\|\nabla\phi\|_{L^p(\mathbb{R}^n)} \leq C_{n,p}\|g\|_{L^p(\mathbb{R}^n)}$ for all $p \in (1,\infty)$?*

The choice at (6.5) fails to have a compact support. In fact, the gradient form $\phi = \nabla\rho$ does not work because it would lead to an overdetermined system

$$\begin{cases} \Delta\rho &= g. \\ (\nabla\rho)|_{\partial S_{1,2}} &= 0. \end{cases}$$

Bogovskiĭ [8, 9] gives an affirmative answer provided that

$$[g]_{B_{1,2}} = \int_{S_{1,2}} g(x)dx = 0. \tag{6.6}$$

This is simply a compatibility condition on g. It is automatically satisfied by the right hand side of (6.3) because u is divergence free. In star-shaped domains, Bogovskiĭ constructs an explicit formula for ϕ. To treat a domain of more general shapes, he partitions it into star-shaped domains. A map $B_0 : g \mapsto \phi$ obtained by this construction is called *Bogovskiĭ's operator*. The properties of B_0 can be summarized as follows. Denote $\tilde{C}_0^\infty(\Omega) = \{g \in C_0^\infty(\Omega) : \ [g]_\Omega = 0\}$.

Proposition 6.1. *[20, Thm. III.3.3] Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded and locally Lipschitz domain. There exists a linear map $B_0 : \tilde{C}_0^\infty(\Omega) \to C_0^\infty(\Omega)$ such that* $\mathrm{div} B_0 g = g$ *and*

$$\left\|\nabla^{m+1}B_0 g\right\|_{L^p} \leq C_{n,m,p,\Omega}\|\nabla^m g\|_{L^p} \quad \forall\ m \geq 0,\ 1 < p < \infty. \tag{6.7}$$

Remark 6.2. By density arguments, one can extend B_0 to a bounded linear map from $\tilde{W}_0^{m,p}(\Omega)$ to $W_0^{m,p}(\Omega)$, where $\tilde{W}_0^{m,p}(\Omega)$ is the subspace of $W_0^{m,p}(\Omega)$ consisting of all functions with zero average.

In general, the moduli of continuity of a Bogovskiĭ's operator depend not only on the domain itself but also on how it is partitioned into star-shaped domains. For certain family of shapes, however, these constants can be chosen to be independent of the domain. For example, let $\{\Omega_R\}_{R>0}$ be the dilations of Ω

$$\Omega_R = R\,\Omega := \{y = Rx : \ x \in \Omega\}.$$

The problem

$$\begin{cases} \operatorname{div} \ \phi = g & \text{in} \quad \Omega_R \\ \phi = 0 & \text{on} \quad \partial\Omega_R \end{cases}$$

has a solution $\phi(x) = R^{-1}B_0 g^{(R)}(x)$ where $g^{(R)}(x) = g(Rx)$ and B_0 is a Bogovskiĭ's operator of Ω. Then

$$B_0^R g(x) = R(B_0 g^{(R)})\left(\frac{x}{R}\right) \tag{6.8}$$

is a Bogovskiĭ's operator of Ω_R. By changing variables $x \mapsto Rx$ or $x \mapsto R^{-1}x$ in evaluating integrals, we see that the moduli of continuity C_{n,m,p,Ω_R} do not depend on R, only on the original domain Ω.

To localize the Navier–Stokes equations, we will be working with spherical shells $\Omega_R = S_{R,2R}$. The moduli of continuity do not depend on R. Note that function $g = -u\cdot\nabla\chi$ depends also on time parameter. Thanks to the linearity of B_0 and the estimates (6.7), we have

$$\left\|\nabla^{m+1}\partial_t\phi\right\|_{L^p(\Omega_R)} \le C_{n,m,p,\Omega}\|\nabla^m\partial_t g\|_{L^p(\Omega_R)}$$

for $m \ge 0$ and $1 < p < \infty$, provided that the right hand side is well-defined and finite.

6.3 Transition from the whole space to half-space

Recall that

$$\begin{aligned} Y_q &= \{f : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^3 : \ t^{q_*} f \in L^q(\mathbb{R}^3 \times (0,\infty))\}, \\ Y_q^+ &= \{f : \mathbb{R}^3_+ \times (0,\infty) \to \mathbb{R}^3 : \ t^{q_*} f \in L^q(\mathbb{R}^3_+ \times (0,\infty))\}. \end{aligned}$$

and

$$
(\mathrm{NSE}) : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= f, \\ \operatorname{div} u &= 0, \\ u(0) &= 0. \end{cases}
$$

$$
(\mathrm{NSE})_+ : \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p &= f, \\ \operatorname{div} u &= 0, \\ u|_{\partial \mathbb{R}^3_+} &= 0, \\ u(0) &= 0. \end{cases}
$$

The main statement in this section is the following.

Proposition 6.3. *Suppose that for some $f \in Y_q$, the mild solution to (NSE) with force f blows up. Then there exists a sequence (f_n) in Y_q^+ with $\|f_n\|_{Y_q^+} \to \|f\|_{Y_q}$ such that the mild solution to (NSE)$_+$ with force f_n also blows up.*

The strategy of proof is to cutoff the blowup solution of (NSE) by a smooth function χ_R supported in B_{2R} and equal to 1 in B_R, taking into account Bogovskiĭ correction, to obtain a blowup solution to (NSE)$_+$. The right hand side of (NSE)$_+$ is then a perturbation of f. The error terms can be estimated thanks to the regularity of u, ∇u, $\partial_t u$, $\nabla^2 u$ and ∇p when $|x|$ is large. The boundedness of u is given in Proposition 5.17 as an application of ε-regularity criteria. The integrability of ∇u, $\partial_t u$, $\nabla^2 u$ and ∇p is given in Proposition 5.18 as a consequence of regularity bootstrapping of parabolic equations.

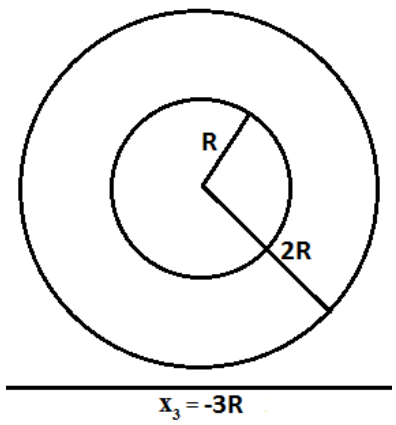

Figure 6.1: Localization of blowup solution on the whole space.

Proof of Proposition 6.3. Let (u,p) be a mild solution to (NSE) with force $f \in Y_q$. Assume that u blows up at time $t=2$ and has a singular point at $z_0=(0,2)$. Let $\rho:\mathbb{R}^3\to\mathbb{R}$ be a smooth function supported in B_2 and equal to 1 in B_1. For each $R>0$, put $\chi_R(x)=\rho(\frac{x}{R})$. Then

$$\|\nabla^m\chi_R\|_{L^r}\le C_m R^{\frac{3}{r}-m}\quad \forall\ m\ge 0,\ 1\le r\le\infty.$$

We localize u and p in space by setting $\tilde u_R=u\chi_R+\phi_R$ and $\tilde p_R=p\chi_R$, where ϕ_R satisfies

$$\begin{cases}\operatorname{div}\phi_R &= -u\cdot\nabla\chi_R,\\ \operatorname{supp}\phi_R &\subset S_{R,2R}.\end{cases}$$

Let B_0 be a Bogovskiĭ's operator of $S_{1,2}$ and B_0^R be the Bogovskiĭ's operator of $S_{R,2R}$ given by (6.8). As discussed in Section 6.2, the moduli of continuity of B_0^R do not depend on R. Functions $\tilde u_R$ and $\tilde p_R$ satisfy a differential equation of the form

$$\partial_t\tilde u_R-\Delta\tilde u_R+\tilde u_R\cdot\nabla\tilde u_R+\nabla\tilde p_R=f\chi_R+g_R,$$

where g_R is a residue term supported in $S_{R,2R}\times[0,\infty)$. Suppose g_R is negligible as $R\to\infty$ in the following sense:

$$\lim_{R\to\infty}\|t^{q_*}g_R\|_{L^q(\mathbb{R}^3\times(0,2))}=0. \tag{6.9}$$

Put $f_R=f\chi_R+g_RI_{(0,2)}(t)$. Then $f_R\to f$ in Y_q as $R\to\infty$. Put

$$\begin{cases}\bar u_R(x,t) &= \tilde u_R(x',x_3-3R,t),\\ \bar p_R(x,t) &= \tilde p_R(x',x_3-3R,t),\\ \bar f_R(x,t) &= f_R(x',x_3-3R,t).\end{cases}$$

Then $(\bar u_R,\bar p_R)$ solves the problem $(\mathrm{NSE})_+$ with force $\bar f_R$. Function $\bar u_R$ has a singularity at $((0,0,3R),2)$. Moreover,

$$\|\bar f_R\|_{Y_q^+}=\|f_R\|_{Y_q}\to\|f\|_{Y_q}\quad\text{as}\quad R\to\infty.$$

Therefore, we only need to show (6.9). By simple calculations, we obtain an expression for g_R as follows. (To simplify the notations, we drop subscript R.)

$$\begin{aligned} g &= -\underbrace{\chi(1-\chi)u\nabla u}_{\{1\}}+\underbrace{\partial_t\phi}_{\{2\}}-\underbrace{\nabla u\nabla\chi}_{\{3\}}+\underbrace{uu\chi\nabla\chi}_{\{4\}}+\underbrace{\phi u\nabla\chi}_{\{5\}}\\ &+ \underbrace{p\nabla\chi}_{\{6\}}+\underbrace{\phi\chi\nabla u}_{\{7\}}+\underbrace{u\chi\nabla\phi}_{\{8\}}-\underbrace{u\Delta\chi}_{\{9\}}-\underbrace{\Delta\phi}_{\{10\}}+\underbrace{\phi\nabla\phi}_{\{11\}}.\end{aligned}$$

Each term is estimated on the time intervals $(0,1)$ and $(1,2)$ differently. On $\mathscr{O}_1 = \mathbb{R}^3 \times (0,1)$, critical norms of u are finite. On $\mathscr{O}_2 = \mathbb{R}^3 \backslash B_{R_0} \times (1,2)$ with $R_0 > 0$ given by Proposition 5.17 and Proposition 5.18, $u \in L^\infty$, $\nabla u \in L^{10/3}$, and $\partial_t u$, $\nabla^2 u$, $\nabla p \in L^q$. In the following, we only consider $R > R_0$.

Next, we observe that by Proposition 6.1, for any $1 < r < \infty$

$$\|\nabla \phi\|_{L^r(S_{R,2R})} \leq C_r \|u \nabla \chi\|_{L^r(S_{R,2R})} \leq C_r R^{-1} \|u\|_{L^r(S_{R,2R})}. \tag{6.10}$$

By Poincaré inequality,

$$\|\phi\|_{L^r(S_{R,2R})} \leq C_r R \|\nabla \phi\|_{L^r(S_{R,2R})} \leq C_r \|u\|_{L^r(S_{R,2R})}. \tag{6.11}$$

Estimate (6.11) indicates that for estimation purposes ϕ plays essentially the same role as u. In particular, terms $\{5\}$, $\{7\}$, $\{11\}$ can be treated similarly to terms $\{4\}$, $\{1\}$, $\{8\}$ respectively.

- *Term* $\{1\}$.

By Corollary 4.4, $\|t^{q_*} u \nabla u\|_{L^q(\mathscr{O}_1)} < \infty$. Thus,

$$\|t^{q_*} u \nabla u\|_{L^q(S_{R,2R} \times (0,1))} \to 0 \quad \text{as} \quad R \to \infty.$$

Put $r = 10q/(10-3q) \in (10,30)$. Since $u = v + w \in L^3(\mathscr{O}_2)$ and $u \in L^\infty(\mathscr{O}_2)$, we have $u \in L^r(\mathscr{O}_2)$. By Hölder inequality, $u \nabla u \in L^q(\mathscr{O}_2)$. Therefore,

$$\|u \nabla u\|_{L^q(S_{R,2R} \times (1,2))} \to 0 \quad \text{as} \quad R \to \infty.$$

- *Term* $\{2\}$.

By the coercive estimate (Proposition 3.5), $t^{q_*} \partial_t u \in L^q(\mathscr{O}_1)$. We have $\partial_t \phi = \partial_t B_0(-u \nabla \chi) = B_0(-\partial_t u \nabla \chi)$. By Proposition 6.1,

$$\|\nabla \partial_t \phi\|_{L^q(S_{R,2R})} \leq A \|\partial_t u \nabla \chi\|_{L^q(S_{R,2R})} \leq A R^{-1} \|\partial_t u\|_{L^q(S_{R,2R})}.$$

By Poincaré inequality (Proposition A.2), $\|\partial_t \phi\|_{L^q(S_{R,2R})} \leq AR \|\nabla \partial_t \phi\|_{L^q(S_{R,2R})}$. Thus,

$$\|\partial_t \phi\|_{L^q(S_{R,2R})} \leq A \|\partial_t u\|_{L^q(S_{R,2R})}.$$

This leads to

$$\begin{aligned} \|t^{q_*} \partial_t \phi\|_{L^q(S_{R,2R} \times (0,1))} &\leq A \|t^{q_*} \partial_t u\|_{L^q(S_{R,2R} \times (0,1))}, \\ \|\partial_t \phi\|_{L^q(S_{R,2R} \times (1,2))} &\leq A \|\partial_t u\|_{L^q(S_{R,2R} \times (1,2))}, \end{aligned}$$

which tend to zero as $R \to \infty$. Therefore, $\|\partial_t \phi\|_{L^q(S_{R,2R}\times(0,2))} \to 0$ as $R \to \infty$.

- *Term* $\{3\}$.

By Remark 4.5 and Remark 3.6, $t^{q_*}\nabla u \in L_t^{2q}L_x^q(\mathscr{O}_1)$. Thus,

$$\|t^{q_*}\nabla u \nabla\chi\|_{L^q(\mathscr{O}_1)} \le AR^{-1}\|t^{q_*}\nabla u\|_{L_t^{2q}L_x^q(\mathscr{O}_1)} \to 0 \quad \text{as} \quad R \to \infty.$$

Put $r = 10q/(10-3q) \in (10, 30)$. Then $1/r + 3/10 = 1/q$. By Hölder inequality,

$$\|\nabla u \nabla\chi\|_{L^q(S_{R,2R})} \le \|\nabla u\|_{L^{10/3}(S_{R,2R})}\|\nabla\chi\|_{L^r} \le AR^{3/r-1}\|\nabla u\|_{L^{10/3}(S_{R,2R})}.$$

Thus,

$$\|\nabla u \nabla\chi\|_{L^q(S_{R,2R}\times(1,2))} \le AR^{3/r-1}\|\nabla u\|_{L^{10/3}(S_{R,2R}\times(1,2))}$$

which tends to zero as $R \to \infty$.

- *Term* $\{4\}$.

Put $\bar{q} = \frac{1}{2} - \frac{3}{4q} > 0$. Note that $2\bar{q} < q_*$ since $q \in (5/2, 3)$. By Part (iii) of Proposition 4.2, $t^{\bar{q}}u \in L_t^\infty L_x^{2q}(\mathscr{O}_1)$. Then

$$t^{q_*}uu = t^{q_*-2\bar{q}}(\underbrace{t^{\bar{q}}u}_{L_t^\infty L_x^{2q}})(\underbrace{t^{\bar{q}}u}_{L_t^\infty L_x^{2q}}) \in L^q(\mathscr{O}_1).$$

Thus, $\|t^{q_*}uu\chi\nabla\chi\|_{L^q(\mathscr{O}_1)} \le AR^{-1}\|t^{q_*}uu\|_{L^q(\mathscr{O}_1)}$ which converges to zero as $R \to \infty$. On the other hand, since $u \in L^3 \cap L^\infty(\mathscr{O}_2)$, we have $u \in L^{2q}(\mathscr{O}_2)$ by interpolation. Then $\|uu\chi\nabla\chi\|_{L^q(\mathscr{O}_2)} \le AR^{-1}\|u\|^2_{L^{2q}(\mathscr{O}_2)}$ which tends to zero as $R \to \infty$.

- *Term* $\{5\}$.

Thanks to (6.11), this term is treated similarly to term $\{4\}$.

- *Term* $\{6\}$.

For each R, we replace p by $p - [p]_{S_{R,2R}}$ before estimating term $\{6\}$. By Poincaré inequality and the coercive estimate (Proposition 3.5),

$$\left\|t^{q_*}(p - [p]_{S_{R,2R}})\nabla\chi\right\|_{L^q(\mathscr{O}_1)} \le AR^{-1}\|t^{q_*}\nabla p\|_{L^q(\mathscr{O}_1)} \to 0 \quad \text{as} \quad R \to \infty.$$

By Poincaré inequality,

$$\left\|(p-[p]_{S_{R,2R}})\nabla\chi\right\|_{L^q(\mathscr{O}_2)} \leq AR^{-1}\|\nabla p\|_{L^q(\mathscr{O}_2)} \to 0 \quad \text{as} \quad R\to\infty.$$

- *Term* $\{7\}$.

Thanks to (6.11), this term is treated similarly to term $\{1\}$.

- *Term* $\{8\}$.

Denote $\bar{q}$ as in the estimate of term $\{4\}$. By (6.10),

$$\|t^{q_*}u\chi\nabla\phi\|_{L^q(S_{R,2R})} \leq A\left\|t^{\bar{q}}u\right\|_{L^{2q}(S_{R,2R})}\left\|t^{\bar{q}}\nabla\phi\right\|_{L^{2q}(S_{R,2R})} \leq AR^{-1}\left\|t^{\bar{q}}u\right\|^2_{L^{2q}(S_{R,2R})}.$$

Thus,

$$\|t^{q_*}u\chi\nabla\phi\|_{L^q(S_{R,2R}\times(0,1))} \leq AR^{-1}\left\|t^{\bar{q}}u\right\|^2_{L^\infty_t L^{2q}_x(\mathscr{O}_1)} \to 0 \quad \text{as} \quad R\to\infty.$$

On the other hand, since $u\in L^3\cap L^\infty(\mathscr{O}_2)$, we have $u\in L^{2q}(\mathscr{O}_2)$ by interpolation. Then

$$\|u\chi\nabla\phi\|_{L^q(S_{R,2R}\times(1,2))} \leq C_q\|u\|_{L^{2q}(\mathscr{O}_2)}\|\nabla\phi\|_{L^{2q}(S_{R,2R}\times(1,2))} \leq C_qR^{-1}\|u\|^2_{L^{2q}(\mathscr{O}_2)},$$

which tends to zero as $R\to\infty$.

- *Term* $\{9\}$.

By Remark 4.5 and Proposition 3.5, $u\in L^\infty_t L^q_x(\mathscr{O}_1)$. Then

$$\left\|t^{q_*}u\nabla^2\chi\right\|_{L^q(\mathscr{O}_1)} \leq \|u\|_{L^\infty_t L^q_x(\mathscr{O}_1)}\left\|\nabla^2\chi\right\|_{L^\infty} \leq AR^{-2}\|u\|_{L^\infty_t L^q_x(\mathscr{O}_1)} \to 0 \quad \text{as} \quad R\to\infty.$$

Put $r=3q/(3-q)$. By Hölder inequality,

$$\left\|u\nabla^2\chi\right\|_{L^q(\mathscr{O}_2)} \leq \|u\|_{L^3(\mathscr{O}_2)}\left\|\nabla^2\chi\right\|_{L^r} \leq C_qR^{\frac{3}{q}-3}\|u\|_{L^3(\mathscr{O}_2)} \to 0 \quad \text{as} \quad R\to\infty.$$

- *Term* $\{10\}$.

By Proposition 6.1, for any $1<r<\infty$

$$\left\|\nabla^2\phi\right\|_{L^r(S_{R,2R})} \leq C_r\|\nabla(u\nabla\chi)\|_{L^r(S_{R,2R})} \leq C_r\|\nabla u\nabla\chi\|_{L^r(S_{R,2R})} + C_r\left\|u\nabla^2\chi\right\|_{L^r(S_{R,2R})}.$$

Hence, the estimating of term $\{10\}$ can be split into estimating terms $\{3\}$ and $\{9\}$. These terms have already been treated.

- *Term* $\{11\}$.

Thanks to (6.11), this term is treated similarly to term $\{8\}$. □

Corollary 6.4. $\rho^+_{\max} \le \rho_{\max}$.

Proof. If $\rho_{\max} = \infty$, the inequality is true. Suppose $\rho_{\max} < \infty$. For each $\varepsilon > 0$, let $f_\varepsilon \in Y_q$ be a blowup data of (NSE) with $\|f_\varepsilon\|_{Y_q} < \rho_{\max} + \varepsilon$. By Proposition 6.3, there exists a blowup data $g_\varepsilon \in Y_q^+$ of $(\text{NSE})_+$ such that $\|g_\varepsilon\|_{Y_q^+} < \|f_\varepsilon\|_{Y_q} + \varepsilon < \rho_{\max} + 2\varepsilon$. Thus, $\rho^+_{\max} < \rho_{\max} + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\rho^+_{\max} \le \rho_{\max}$. □

6.4 Transition from the half-space to whole space

Proposition 6.5. *We have the following statements.*

(i) If $\rho_{\max} < \infty$ then there exists a minimal blowup data $f \in Y_q$ for (NSE).

(ii) If $\rho^+_{\max} < \rho_{\max}$ then there exists a minimal blowup data $f \in Y_q^+$ for $(\text{NSE})_+$.

Proof. For the proof of Part (i), we follow the procedure given in [76] and [32].
(i) Let $f_k \in Y_q$ be a sequence of blowup-generating data such that $\|f_k\|_{Y_q} \to \rho_{\max}$ as $k \to \infty$. Let (u_k, p_k) be the corresponding mild solutions. By Proposition 4.6, (u_k, p_k) is an sw-solution on its maximal time-interval of existence. By Proposition 5.20, u_k has a singular point at the blowup time, say $z_k = (x_k, t_k)$. By the scaling symmetry

$$\begin{aligned} f_k &\to t_k^{3/2} f_k(t_k^{1/2} x, t_k t), \\ u_k &\to t_k^{1/2} u_k(t_k^{1/2} x, t_k t), \\ p_k &\to t_k p_k(t_k^{1/2} x, t_k t), \end{aligned}$$

we can assume $t_k = 1$ for all k. That is, blowup solutions u_k have singularities at the same time. The whole space is also invariant under translation (in arbitrary directions). By the translation symmetry

$$\begin{aligned} f_k &\to f_k(x - x_k, t), \\ u_k &\to u_k(x - x_k, t), \\ p_k &\to p_k(x - x_k, t), \end{aligned}$$

we can assume $x_k = 0$ for all k. In other words, by scaling and translation we can assume that solutions u_k are singular at the same point $z_0 = (0, 1)$. The norm of f_k remains

unchanged in these transformations. We can localize the time interval by replacing f_k with $\tilde{f}_k(x,t) = f_k(x,t)\chi_{[0,1]}(t)$ if necessary. Note that after this replacement, (f_k) is still a minimizing sequence because

$$\rho_{\max} \leq \|\tilde{f}_k\|_{Y_q} \leq \|f_k\|_{Y_q}.$$

Up to a subsequence, f_k converges weakly to some $f \in Y_q$. By the compactness theorem (Proposition 5.7), up to a subsequence u_k converges to some u, and p_k converges to some p in the sense described therein. Moreover, (u,p) is an sw-solution to (NSE) with force f. By the persistence of singularity (Proposition 5.21), z_0 is a singular point of u. By the weak-strong uniqueness (Proposition 5.9), u coincides the mild solution $\tilde{u}$ of (NSE) with force f as long as $\tilde{u}$ remains in the critical space $L^5_{t,x}$. We see that $\tilde{u}$ cannot remain in $L^5_{t,x}$ at time $t=1$ because u has a singular point at this time. Hence, $\tilde{u}$ must blowup either before or at time $t=1$. Then $\|f\|_{Y_q} \geq \rho_{\max}$ by the definition of $\rho_{\max}$. By the weak convergence $f_k \rightharpoonup f$,

$$\|f\|_{Y_q} \leq \liminf_{k\to\infty} \|f_k\|_{Y_q} = \rho_{\max}.$$

This implies $\|f\|_{Y_q} = \rho_{\max}$. In other words, f is a minimal blowup data for (NSE).

(ii) We repeat as much as possible the proof of Part (i), which is for the whole space, for the half-space. Let $f_k \in Y_q^+$ be a sequence of blowup-generating data such that $\|f_k\|_{Y_q^+} \to \rho^+_{\max}$ as $k \to \infty$. Let (u_k, p_k) be the corresponding mild solutions. By Proposition 4.6, (u_k, p_k) is an sw-solution on its maximal time-interval of existence. By Proposition 5.20, u_k has a singular point at the blowup time, say $z_k = (x_k, t_k)$. By the scaling symmetry

$$\begin{aligned} f_k &\to t_k^{3/2} f_k(t_k^{1/2}x, t_k t), \\ u_k &\to t_k^{1/2} u_k(t_k^{1/2}x, t_k t), \\ p_k &\to t_k p_k(t_k^{1/2}x, t_k t), \end{aligned}$$

we can assume $t_k = 1$ for all k. That is, blowup solutions u_k have singularities at the same time. The half-space is only invariant under horizontal translation, i.e. in directions $a = (a_1, a_2, 0)$. Write $x_k = \bar{x}_k + (0,0,d_k)$ where $d_k \geq 0$ and $\bar{x}_k$ has zero third component. By the horizontal translation symmetry

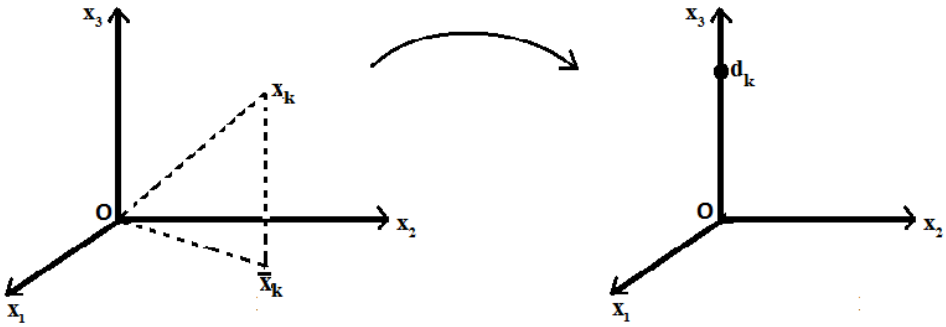

Figure 6.2: Horizontal shifting of singular points.

$$\begin{aligned} f_k &\to f_k(x-\bar{x}_k,t), \\ u_k &\to u_k(x-\bar{x}_k,t), \\ p_k &\to p_k(x-\bar{x}_k,t), \end{aligned}$$

we can assume that $x_k = (0,0,d_k)$ for all k. Replacing d_k by a subsequence if necessary, we see that there are three scenarios:

- $d_k \to d > 0$,
- $d_k \to 0$,
- $d_k \to \infty$.

In the first and second cases, the sequence of singular points has an accumulation point $z_0 \in \overline{\mathbb{R}^3_+}$. By the persistence of singularity (Proposition 5.21), z_0 is a singular point of the limit function u. As discussed in Section 5.5, the persistence of singularity in the first case is proved in [76]. In the second case, it is proved by using a different technique of decomposing the pressure near the boundary due to Seregin [80]. In these two cases, a minimal blowup data for $(\text{NSE})_+$ exists by the same arguments as in Part (i).

Now consider the third case. We show that if this case happens then $\rho^+_{\max} \geq \rho_{\max}$, which would be a contraction. The idea is to shift the half-space by vertical vector $(0,0,-d_k)$ to bring singular points to $z_0 = (0,1)$. The limit solution is expected to be a sw-solution in the whole space which is singular at z_0. Put

$$g_k(x,t) = \begin{cases} f_k(x', x_3 + d_k, t) & \text{if} \quad x_3 \geq -d_k, \\ 0 & \text{if} \quad x_3 < -d_k. \end{cases}$$

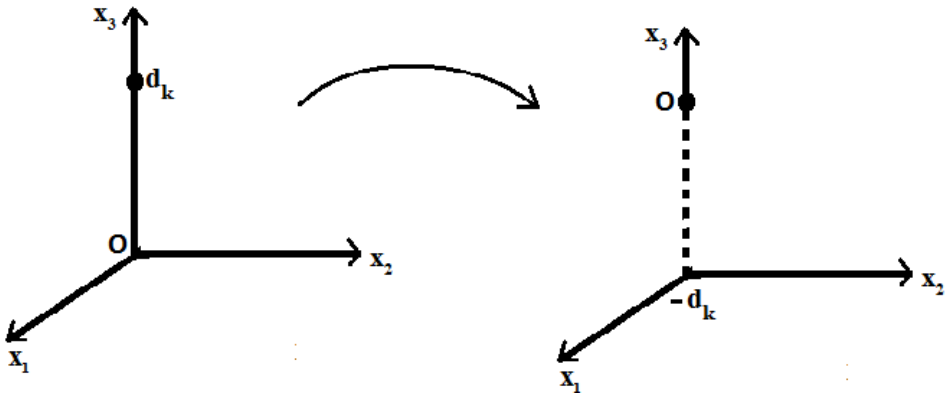

Figure 6.3: Vertical shifting of singular points.

$$v_k(x,t) = \begin{cases} u_k(x', x_3 + d_k, t) & \text{if} \quad x_3 \geq -d_k, \\ 0 & \text{if} \quad x_3 < -d_k. \end{cases}$$

$$\pi_k(x,t) = \begin{cases} p_k(x', x_3 + d_k, t) & \text{if} \quad x_3 \geq -d_k, \\ 0 & \text{if} \quad x_3 < -d_k. \end{cases}$$

Then (v_k, p_k) satisfies the Navier–Stokes system

$$\begin{cases} \partial_t v_k - \Delta v_k + v_k \cdot \nabla v_k + \nabla \pi_k &= g_k, \\ \text{div } v_k &= 0, \\ v_k(\cdot, 0) &= 0. \end{cases}$$

in $\mathscr{O}_k = \mathbb{R}^2 \times (-d_k, \infty) \times (0,1)$. The compactness theorem (Proposition 5.7) can be adapted to this situation. Up to a subsequence, g_k converges weakly to some $g \in Y_q$. The limit function satisfies all properties of an sw-solution because these properties are local. Thus, g generates a blowup mild solution for (NSE). Then

$$\rho_{\max} \leq \|g\|_{Y_q} \leq \liminf_{k\to\infty} \|g_k\|_{Y_q} = \liminf_{k\to\infty} \|f_k\|_{Y_q^+} = \rho_{\max}^+.$$

This is a contradiction. □

Chapter 7

A quantitative criterion for global strong solutions

7.1 Formulation of the problem

We consider the initial value problem for the mollified Navier–Stokes equations in the three-dimensional space

$$(\mathrm{NSE})_\varepsilon : \begin{cases} \partial_t u - \Delta u + (u * \eta_\varepsilon) \cdot \nabla u + \nabla p = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^3, \end{cases}$$

where $(\eta_\varepsilon)_{\varepsilon>0}$ is a mollifier in $\mathbb{R}^3$, e.g. $\eta_\varepsilon(x) = \varepsilon^{-3}\eta(x/\varepsilon)$ where

$$\eta(x) = \begin{cases} A \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Denote by (NSE) the exact Navier–Stokes system. Assume that the initial condition u_0 belongs to $L^2_\sigma \cap L^\infty$. A natural class of solutions in this setting is mild solutions (called *strong solutions* by Leray), defined in the space $L^\infty_{t,x}$.

In the pioneering work [48], Leray shows that (NSE) is locally well-posed. Specifically, if $T \leq A\|u_0\|^{-2}_{L^\infty_x}$ for some absolute constant A then (NSE) has a unique mild solution in $X_T = L^\infty(\mathbb{R}^3 \times (0, T))$ [48, Para. 19]. Moreover, $u \in C([0, T], L^\infty_x)$ (see e.g. [95, Lecture 61] and [71, Thm. 3.2]). The global well-posedness of (NSE) is still not

known. Heuristically, solution u_ε of $(\text{NSE})_\varepsilon$ is more regular than solution u of (NSE) because the mollified nonlinear term $(u_\varepsilon * \eta_\varepsilon) \cdot \nabla u_\varepsilon$ is dominated by the linear term Δu_ε. In fact, $(\text{NSE})_\varepsilon$ is globally well-posed for each $\varepsilon > 0$ (see [48, Para. 26] and [71, Thm. 4.2]). Moreover, u_ε solves $(\text{NSE})_\varepsilon$ in classical sense and satisfies the energy identity

$$\|u_\varepsilon(t)\|_{L^2_x}^2 + 2\int_0^t \|\nabla u_\varepsilon(s)\|_{L^2_x}^2 \, ds = \|u_0\|_{L^2}^2 \quad \forall\, t,\, \varepsilon > 0.$$

This uniform bound allows one to take the limit as $\varepsilon \to 0$ to obtain a global weak solution. However, the uniqueness of Leray's weak solutions is not known due to possible lack of regularity of this class of solutions. Since the Stokes operator on the whole space is equal to the Laplacian, by (1.3) mild solutions u and u_ε are given implicitly by

$$u(t) = \Gamma(t) * u_0 + \int_0^t K'(t-s) * (u(s) \otimes u(s)) ds, \tag{7.1}$$

$$u_\varepsilon(t) = \Gamma(t) * u_0 + \int_0^t K'(t-s) * ((u_\varepsilon(s) * \eta_\varepsilon) \otimes u_\varepsilon(s)) ds, \tag{7.2}$$

where $K' = \nabla \mathbb{P} \Gamma$. Leray's construction of weak solutions is purely qualitative (using limit, compactness, etc). Without a sufficiently strong *a priori* estimate, many regularity properties are lost in the limit $\varepsilon \to 0$. All *a priori* estimates that have been known so far can trace back to the energy estimate, which is not strong enough to preserve the boundedness of solutions in the limit process (see [98, Sec. 3.4]). In this light, it is reasonable to find a quantitative assumption on u_ε that can retain sufficient regularity in the limit process. Specifically, we are interested in the question:

> (Q2) For $M > 0$, how large ε can we take so that the following is true: "If u_ε is bounded in $\mathbb{R}^3 \times (0, \infty)$ by M then (NSE) has a global strong solution u which is bounded in $\mathbb{R}^3 \times (0, \infty)$ by $2M$"?

This question is addressed by Li [50] from a numerical perspective. Considering a discretized Navier–Stokes system on a polyhedron, he introduces a hypothetical relation between the mesh size and the size of the corresponding numerical solution which guarantees the global existence of the exact solutions. Li essentially suggests that $\varepsilon \lesssim \exp(-M^{225})$. We formulate this problem for the continuous setting $(\text{NSE})_\varepsilon$, in which ε, the resolution of the approximation, is analogous to the mesh size. Although question (Q2) can be formulated for domains with boundaries, system $(\text{NSE})_\varepsilon$ seems to

be a more natural model to study because of the scaling symmetry and the absence of boundaries. This setting already contains some key difficulties. Like the exact Navier–Stokes system, $(\text{NSE})_\varepsilon$ has scaling symmetry:

$$\begin{aligned} u(x,t) &\to u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \\ p(x,t) &\to p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \\ \varepsilon &\to \varepsilon_\lambda = \lambda^{-1}\varepsilon. \end{aligned}$$

We are interested in obtaining a bound for ε that has the same scaling as ε. The bound $\|u_\varepsilon\|_{L^\infty} \le M$ gives the Navier–Stokes problem a natural length scale, which is M^{-1}. The "resolution" ε also has dimension length, so the ideal bound for ε would be $\varepsilon \lesssim M^{-1}$. Our goal is to investigate the following conjecture.

> Let $M > 0$ and $u_0 \in L^2 \cap L^\infty$. There exists a constant $C > 0$ independent of M, possibly dependent on $\|u_0\|_{L^3}$, $\|u_0\|_{L^2}^2 \|u_0\|_{L^\infty}$ or other scaling-invariant quantities involving u_0, such that the following is true: Suppose for some $0 < \varepsilon \le CM^{-1}$, the solution u_ε of $(\text{NSE})_\varepsilon$ is bounded by M. Then (NSE) has a global strong solution bounded by $2M$.

We obtain a partial result that $\varepsilon \le F(M)$ where $F(M)$ has the same scaling as ε decays as $\exp(-M^3)$ as $M \to \infty$, improving the results of Li [50]. One strategy to improve the decay of $F(M)$ is to choose a different approximate Navier–Stokes system which may result in a decay better than exponential. In fact, proving a decay $F(M) \sim M^{-\alpha}$ for some $\alpha > 0$ would already be significant. The key difficulty is that the "naive" estimates at global scale by Grönwall inequality seems not sufficiently strong. A treatment at local scale is perhaps needed to access the strength of the local regularity theory.

7.2 Estimate of $u_\varepsilon * \eta_\varepsilon - u_\varepsilon$

By Proposition 2.4 and Proposition 2.2, K' is a smooth function in $\mathbb{R}^3 \times (0,\infty)$ satisfying

$$K'(\lambda x, \lambda^2 t) = \lambda^{-4} K'(x,t) \quad \forall\, \lambda > 0 \tag{7.3}$$

$$\left\|K'(t)\right\|_{L^q_x} \le C(q) t^{\frac{3}{2q}-2} \quad \forall\, 1 \le q \le \infty, \tag{7.4}$$

By simple calculation, one can check that

$$\left|K'(-z,\tau) - K'(a-z,\tau)\right| \le \frac{A}{\left(|z|^2+\tau\right)^{5/2}} \tag{7.5}$$

for all a, $z \in \mathbb{R}^3$ with $|a| = 1$ and $\tau \ge 1$ (see [95, Lectures 59, 60]). For each $\alpha > 0$, denote $\log_+\alpha = \max\{\log\alpha, 0\}$. Put $X_T = L^\infty(\mathbb{R}^3 \times (0,T))$.

Proposition 7.1. *Let T, $\varepsilon > 0$. Suppose* $(\mathrm{NSE})_\varepsilon$ *has solution $u_\varepsilon \in X_T$. Then*

$$\|u_\varepsilon(t) * \eta_\varepsilon - u_\varepsilon(t)\|_{L^\infty_x} \le \frac{A\varepsilon\|u_0\|_{L^\infty}}{\sqrt{t}} + A\varepsilon\left(1 + \log_+\frac{t}{\varepsilon^2}\right)\|u_\varepsilon\|^2_{X_T} \quad \forall\, t \in (0,T).$$

Proof. The proof follows from [95, Lecture 60]. We have

$$\begin{aligned} u_\varepsilon(t) * \eta_\varepsilon - u_\varepsilon(t) &= \int_{\mathbb{R}^3} \left[u_\varepsilon(x-y,t) - u_\varepsilon(x,t)\right]\eta_\varepsilon(y)dy \\ &\overset{z=y/\varepsilon}{=\!=\!=} \int_{\mathbb{R}^3} \left[u_\varepsilon(x-\varepsilon z,t) - u_\varepsilon(x,t)\right]\eta(z)dz \\ &= \int_{B_1} \left[u_\varepsilon(x-\varepsilon z,t) - u_\varepsilon(x,t)\right]\eta(z)dz. \end{aligned}$$

Thus,

$$\begin{aligned} |u_\varepsilon(t) * \eta_\varepsilon - u_\varepsilon(t)| &\le \sup_{|z|\le 1}|u_\varepsilon(x-\varepsilon z,t) - u_\varepsilon(x,t)| \int_{B_1}\eta(z)dz \\ &= \sup_{|z|\le 1}|u_\varepsilon(x-\varepsilon z,t) - u_\varepsilon(x,t)|. \end{aligned} \tag{7.6}$$

By (7.2),

$$\begin{aligned} u_\varepsilon(x-\varepsilon z,t) &= \int_{\mathbb{R}^3}\Gamma(x-\varepsilon z-y,t)u_0(y)dy \\ &+ \int_0^t\int_{\mathbb{R}^3} K'(x-\varepsilon z-y,t-s)[(u_\varepsilon(y,s)*\eta_\varepsilon)\otimes u_\varepsilon(y,s)]dyds, \end{aligned}$$

$$u_\varepsilon(x,t) = \int_{\mathbb{R}^3}\Gamma(x-y,t)u_0(y)dy + \int_0^t\int_{\mathbb{R}^3} K'(x-y,t-s)[(u_\varepsilon(y,s)*\eta_\varepsilon)\otimes u_\varepsilon(y,s)]dyds.$$

Put

$$\{1\} = \int_{\mathbb{R}^3}\left[\Gamma(x-\varepsilon z-y,t) - \Gamma(x-y,t)\right]u_0(y)dy,$$

$$\{2\} = \int_0^t \int_{\mathbb{R}^3} [K'(x-\varepsilon z-y,t-s) - K'(x-y,t-s)][(u_\varepsilon(y,s) * \eta_\varepsilon) \otimes u_\varepsilon(y,s)]dyds.$$

Then

$$|u_\varepsilon(x-\varepsilon z,t) - u_\varepsilon(x,t)| \leq |\{1\}| + |\{2\}|. \tag{7.7}$$

For $|z| \leq 1$, we have

$$\begin{aligned}
|\{1\}| &\leq \|u_0\|_{L^\infty} \int_{\mathbb{R}^3} |\Gamma(x-\varepsilon z-y,t) - \Gamma(x-y,t)|dy \\
&\leq \|u_0\|_{L^\infty} \int_{\mathbb{R}^3} |\varepsilon z| \int_0^1 |\nabla\Gamma(x-s\varepsilon z-y,t)|dsdy \\
&\leq \varepsilon\|u_0\|_{L^\infty} \int_{\mathbb{R}^3} \frac{1}{2t(4\pi t)^{3/2}} \int_0^1 |x-s\varepsilon z-y| \exp\left(-\frac{|x-s\varepsilon z-y|^2}{4t}\right)dsdy \\
&= A\varepsilon t^{-5/2}\|u_0\|_{L^\infty} \int_0^1 \int_{\mathbb{R}^3} |x-s\varepsilon z-y| \exp\left(-\frac{|x-s\varepsilon z-y|^2}{4t}\right)dyds \\
&= A\varepsilon t^{-5/2}\|u_0\|_{L^\infty} \int_0^1 \int_{\mathbb{R}^3} t^2|y'| \exp\left(-|y'|^2\right)dyds,
\end{aligned}$$

where

$$y' = \frac{x-s\varepsilon z-y}{2\sqrt{t}}.$$

Thus,

$$|\{1\}| \leq \frac{A\varepsilon\|u_0\|_{L^\infty}}{\sqrt{t}}. \tag{7.8}$$

We have

$$\begin{aligned}
|\{2\}| &\leq \int_0^t \int_{\mathbb{R}^3} |K'(x-\varepsilon z-y,t-s) - K'(x-y,t-s)|\|u_\varepsilon(s) * \eta_\varepsilon\|_{L^\infty_x}\|u_\varepsilon(s)\|_{L^\infty_x}dyds \\
&\leq \|u_\varepsilon\|^2_{X_T} \underbrace{\int_0^t \int_{\mathbb{R}^3} |K'(x-\varepsilon z-y,t-s) - K'(x-y,t-s)|dyds.}_{I} \qquad (7.9)
\end{aligned}$$

Changing variables $y \mapsto x-y$ and $s \mapsto t-s$, we get

$$I = \int_0^t \int_{\mathbb{R}^3} |K'(y-\varepsilon z,s) - K'(y,s)|dyds.$$

For $z=0$, $I=0$. Consider the case $0<|z|\leq 1$. Put $\lambda=\varepsilon|z|$. By (7.3),

$$I=\lambda^{-4}\int\limits_0^t\int\limits_{\mathbb{R}^3}\left|K'\left(\frac{y}{\lambda}-\frac{\varepsilon z}{\lambda},\frac{s}{\lambda^2}\right)-K'\left(\frac{y}{\lambda},\frac{s}{\lambda^2}\right)\right|dyds.$$

Changing variables $y\mapsto\frac{y}{\lambda}$ and $s\mapsto\frac{s}{\lambda^2}$, we get

$$I=\lambda\int\limits_0^{\frac{t}{\lambda^2}}\int\limits_{\mathbb{R}^3}\left|K'\left(y-a,s\right)-K'\left(y,s\right)\right|dyds$$

where $a=\frac{\varepsilon z}{\lambda}$. If $t\leq\lambda^2$ then

$$\begin{aligned}
I &\leq \lambda\int\limits_0^1\int\limits_{\mathbb{R}^3}\left|K'\left(y-a,s\right)-K'\left(y,s\right)\right|dyds\\
&\leq \lambda\int\limits_0^1\int\limits_{\mathbb{R}^3}\left|K'\left(y-a,s\right)\right|dyds+\lambda\int\limits_0^1\int\limits_{\mathbb{R}^3}\left|K'\left(y,s\right)\right|dyds\\
&= 2\lambda\int\limits_0^1\left\|K'(s)\right\|_{L^1_x}ds\\
&\overset{(7.4)}{\leq} A\lambda\int\limits_0^1\frac{1}{\sqrt{s}}ds\\
&= A\lambda.
\end{aligned}$$

If $t>\lambda^2$ then

$$\begin{aligned}
I &\leq \lambda\int\limits_0^1\int\limits_{\mathbb{R}^3}\left|K'\left(y-a,s\right)-K'\left(y,s\right)\right|dyds+\lambda\int\limits_1^{\frac{t}{\lambda^2}}\int\limits_{\mathbb{R}^3}\left|K'\left(y-a,s\right)-K'\left(y,s\right)\right|dyds\\
&\leq A\lambda+\lambda\int\limits_1^{\frac{t}{\lambda^2}}\int\limits_{\mathbb{R}^3}\left|K'\left(y-a,s\right)-K'\left(y,s\right)\right|dyds\\
&\overset{(7.5)}{\leq} A\lambda+\lambda\int\limits_1^{\frac{t}{\lambda^2}}\int\limits_{\mathbb{R}^3}\frac{A}{\left(|y|^2+s\right)^{5/2}}dyds.
\end{aligned}$$

Changing variable $y\mapsto y\sqrt{s}$, we get

$$I\leq A\lambda+\lambda\int_1^{\frac{t}{\lambda^2}}\frac{A}{s}ds=A\lambda+A\lambda\log\frac{t}{\lambda^2}.$$

Combining two cases, we get

$$I \le A \max\left\{\lambda,\ \lambda + \lambda \log \frac{t}{\lambda^2}\right\}.$$

Because $0 < \lambda \le \varepsilon$,

$$I \le A \max\left\{\varepsilon,\ \sup_{\lambda \in (0,\varepsilon)} f(\lambda)\right\},$$

where

$$f(\lambda) = \lambda + \lambda \log \frac{t}{\lambda^2}.$$

λ	0		$\sqrt{\frac{t}{e}}$		$\sqrt{t}$
$f'(\lambda)$		$+$	0	$-$	
$f(\lambda)$		$\nearrow$	$A\sqrt{t}$	$\searrow$	

If $\varepsilon > \sqrt{\frac{t}{e}}$ then $\sup_{\lambda \in (0,\varepsilon)} f(\lambda) = A\sqrt{t} < A\varepsilon$.
If $\varepsilon \le \sqrt{\frac{t}{e}}$ then $\sup_{\lambda \in (0,\varepsilon)} f(\lambda) = f(\varepsilon) = \varepsilon + \varepsilon \log \frac{t}{\varepsilon^2}$.
Combining two cases, we get

$$I \le A\varepsilon \left(1 + \log_+ \frac{t}{\varepsilon^2}\right).$$

Substitute this estimate into (7.9),

$$|\{2\}| \le A\varepsilon \left(1 + \log_+ \frac{t}{\varepsilon^2}\right) \|u_\varepsilon\|_{X_T}^2. \tag{7.10}$$

Substitute (7.8) and (7.10) into (7.7),

$$|u_\varepsilon(x - \varepsilon z, t) - u_\varepsilon(x,t)| \le \frac{A\varepsilon \|u_0\|_{L^\infty}}{\sqrt{t}} + A\varepsilon \left(1 + \log_+ \frac{t}{\varepsilon^2}\right) \|u_\varepsilon\|_{X_T}^2.$$

From (7.6), we obtain

$$\|u_\varepsilon(t) * \eta_\varepsilon - u_\varepsilon(t)\|_{L_x^\infty} \le \frac{A\varepsilon \|u_0\|_{L^\infty}}{\sqrt{t}} + A\varepsilon \left(1 + \log_+ \frac{t}{\varepsilon^2}\right) \|u_\varepsilon\|_{X_T}^2.$$

□

7.3 Estimate of $u_\varepsilon - u$

Proposition 7.2. *Let M, T, $\varepsilon > 0$ and $u_0 \in L^2 \cap L^\infty$, $||u_0||_{L^\infty} \leq M$. Suppose (NSE) has a mild solution u on a maximal interval $(0, T_*)$. Suppose* $(\mathrm{NSE})_\varepsilon$ *has a mild solution u_ε satisfying $||u_\varepsilon||_{X_T} \leq M$. Then*

$$\begin{aligned} \|u_\varepsilon(t) - u(t)\|_{L^\infty_x} &\leq A\left(\varepsilon M^2 + \varepsilon^2 M^3 + \varepsilon M^3 \sqrt{T}\left(1 + \log_+ \frac{T}{\varepsilon^2}\right)\right) \\ &\times \exp\left(AT^{3/2}\left(M^3 + \sup_{s\in(0,t)} \|u(s)\|^3_{L^\infty_x}\right)\right) \end{aligned}$$

for all $0 < t < \min\{T, T_*\}$.

Proof. Let $0 < T' < \min\{T, T^*\}$. Define a bilinear form $B : X_{T'} \times X_{T'} \to X_{T'}$,

$$B(v, w)(x, t) = \int_0^t K'(t - s) * (v(s) \otimes w(s))ds.$$

Then

$$\begin{aligned} \|B(v, w)\|_{L^\infty_x} &\leq \int_0^t \left\|K'(t - s) * (v(s) \otimes w(s))\right\|_{L^\infty_x} ds \\ &\leq \int_0^t \left\|K'(t - s)\right\|_{L^1_x} \|v(s) \otimes w(s)\|_{L^\infty_x} ds \\ &\overset{(7.4)}{\leq} \int_0^t \frac{A}{\sqrt{t - s}} \|v(s)\|_{L^\infty_x} \|w(s)\|_{L^\infty_x} ds. \end{aligned} \tag{7.11}$$

We know that $u, u_\varepsilon \in X_{T'}$ and

$$\begin{aligned} u_\varepsilon(t) &= \Gamma(t) * u_0 + B(u_\varepsilon * \eta_\varepsilon, u_\varepsilon), \\ u(t) &= \Gamma(t) * u_0 + B(u, u). \end{aligned}$$

Then

$$\begin{aligned} \|u_\varepsilon(t) - u(t)\|_{L^\infty_x} &= \|B(u_\varepsilon * \eta_\varepsilon, u_\varepsilon) - B(u, u)\|_{L^\infty_x} \\ &= \|B(u_\varepsilon * \eta_\varepsilon - u_\varepsilon, u_\varepsilon) + B(u_\varepsilon - u, u_\varepsilon) + B(u, u_\varepsilon - u)\|_{L^\infty_x} \end{aligned}$$

$$\leq \underbrace{\|B(u_\varepsilon * \eta_\varepsilon - u_\varepsilon, u_\varepsilon)\|_{L^\infty_x}}_{\{1\}} + \underbrace{\|B(u_\varepsilon - u, u_\varepsilon)\|_{L^\infty_x}}_{\{2\}} + \underbrace{\|B(u, u_\varepsilon - u)\|_{L^\infty_x}}_{\{3\}}. \tag{7.12}$$

Thanks to Proposition 7.1, for $t \in (0, T')$ we have

$$\begin{aligned} \{1\} &\overset{(7.11)}{\leq} \int_0^t \frac{A}{\sqrt{t - s}} \|u_\varepsilon(s) * \eta_\varepsilon - u_\varepsilon(s)\|_{L^\infty_x} \|u_\varepsilon(s)\|_{L^\infty_x} ds \\ &\leq \int_0^t \frac{A}{\sqrt{t - s}} \left[\frac{A\varepsilon \|u_0\|_{L^\infty}}{\sqrt{s}} + A\varepsilon\left(1 + \log_+ \frac{s}{\varepsilon^2}\right) \|u_\varepsilon\|^2_{X_{T'}}\right] \|u_\varepsilon(s)\|_{L^\infty_x} ds. \end{aligned}$$

Because $\|u_0\|_{L^\infty}, \|u_\varepsilon\|_{X_T} \le M$,

$$\begin{aligned}
\{1\} &\le \int_0^t \frac{A}{\sqrt{t-s}}\left[\frac{A\varepsilon M}{\sqrt{s}} + A\varepsilon M^2\left(1+\log_+\frac{s}{\varepsilon^2}\right)\right] M ds \\
&= A\varepsilon M^2 \underbrace{\int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} ds}_{\{4\}} + A\varepsilon M^3 \underbrace{\int_0^t \frac{1}{\sqrt{t-s}}\left(1+\log_+\frac{s}{\varepsilon^2}\right) ds}_{\{5\}}. \qquad (7.13)
\end{aligned}$$

We have

$$\begin{aligned}
\{4\} &= \int_0^{t/2} \frac{1}{\sqrt{t-s}\sqrt{s}} ds + \int_{t/2}^t \frac{1}{\sqrt{t-s}\sqrt{s}} ds \\
&\le \sqrt{\frac{2}{t}} \int_0^{t/2} \frac{1}{\sqrt{s}} ds + \sqrt{\frac{2}{t}} \int_{t/2}^t \frac{1}{\sqrt{t-s}} ds \\
&= \sqrt{\frac{2}{t}}\left(2\sqrt{\frac{t}{2}}\right) + \sqrt{\frac{2}{t}}\left(2\sqrt{\frac{t}{2}}\right) \\
&= 4. \qquad (7.14)
\end{aligned}$$

If $t \le \varepsilon^2$ then

$$\{5\} = \int_0^t \frac{1}{\sqrt{t-s}} ds = 2\sqrt{t} \le 2\varepsilon.$$

If $t \ge \varepsilon^2$ then

$$\begin{aligned}
\{5\} &= \int_0^{\varepsilon^2} \frac{1}{\sqrt{t-s}} ds + \int_{\varepsilon^2}^t \frac{1}{\sqrt{t-s}}\left(1+\log\frac{s}{\varepsilon^2}\right) ds \\
&\le \int_0^{\varepsilon^2} \frac{1}{\sqrt{\varepsilon^2-s}} ds + \left(1+\log\frac{t}{\varepsilon^2}\right)\int_{\varepsilon^2}^t \frac{1}{\sqrt{t-s}} ds \\
&= 2\varepsilon + 2\left(1+\log\frac{t}{\varepsilon^2}\right)\sqrt{t-\varepsilon^2} \\
&\le 2\varepsilon + 2\sqrt{T}\left(1+\log\frac{T}{\varepsilon^2}\right).
\end{aligned}$$

Combining two cases, we get

$$\{5\} \le A\varepsilon + A\sqrt{T}\left(1+\log_+\frac{T}{\varepsilon^2}\right) \quad \forall\, t \in (0, T'). \qquad (7.15)$$

Substitute (7.14) and (7.15) into (7.13),

$$\{1\} \le A\varepsilon M^2 + A\varepsilon^2 M^3 + A\varepsilon M^3\sqrt{T}\left(1+\log_+\frac{T}{\varepsilon^2}\right). \qquad (7.16)$$

□

We have

$$\begin{aligned}
\{2\} &\overset{(7.11)}{\leq} \int_0^t \frac{A}{\sqrt{t-s}} \|u_\varepsilon(s) - u(s)\|_{L_x^\infty} \|u_\varepsilon(s)\|_{L_x^\infty} ds \\
&\leq AM \int_0^t \frac{1}{\sqrt{t-s}} \|u_\varepsilon(s) - u(s)\|_{L_x^\infty} ds. \qquad (7.17)
\end{aligned}$$

$$\begin{aligned}
\{3\} &\overset{(7.11)}{\leq} \int_0^t \frac{A}{\sqrt{t-s}} \|u(s)\|_{L_x^\infty} \|u_\varepsilon(s) - u(s)\|_{L_x^\infty} ds \\
&\leq A \sup_{s\in(0,t)} \|u(s)\|_{L_x^\infty} \int_0^t \frac{1}{\sqrt{t-s}} \|u_\varepsilon(s) - u(s)\|_{L_x^\infty} ds. \qquad (7.18)
\end{aligned}$$

Substituting (7.16), (7.17), (7.18) into (7.12), we get

$$\begin{aligned}
\|u_\varepsilon(t) - u(t)\|_{L_x^\infty} &\leq A\varepsilon M^2 + A\varepsilon^2 M^3 + A\varepsilon M^3 \sqrt{T} \left(1 + \log_+ \frac{T}{\varepsilon^2}\right) \\
&+ A \left(M + \sup_{s\in(0,t)} \|u(s)\|_{L_x^\infty}\right) \int_0^t \frac{1}{\sqrt{t-s}} \|u_\varepsilon(s) - u(s)\|_{L_x^\infty} ds.
\end{aligned}$$

Put

$$\begin{aligned}
\alpha &= \varepsilon M^2 + \varepsilon^2 M^3 + \varepsilon M^3 \sqrt{T} \left(1 + \log_+ \frac{T}{\varepsilon^2}\right), \\
\beta &= M + \sup_{s\in(0,t)} \|u(s)\|_{L_x^\infty}, \\
\varphi(s) &= \|u_\varepsilon(s) - u(s)\|_{L_x^\infty}.
\end{aligned}$$

Then

$$\begin{aligned}
\varphi(t) &\leq A\alpha + A\beta(t) \int_0^t \frac{1}{\sqrt{t-s}} \varphi(s) ds \\
&\overset{\text{Holder}}{\leq} A\alpha + A\beta(t) \left[\int_0^t \frac{1}{(t-s)^{3/4}} ds\right]^{\frac{2}{3}} \left(\int_0^t \varphi(s)^3 ds\right)^{\frac{1}{3}} \\
&= A\alpha + A\beta(t) t^{1/6} \left(\int_0^t \varphi(s)^3 ds\right)^{\frac{1}{3}} \\
&\leq A\alpha + A\beta(t) T^{1/6} \left(\int_0^t \varphi(s)^3 ds\right)^{\frac{1}{3}}.
\end{aligned}$$

Raise both sides to the third power,

$$\varphi(t)^3 \leq A\alpha^3 + A\beta(t)^3 \sqrt{T} \int_0^t \varphi(s)^3 ds.$$

Note that β is an increasing function. For each $\tau \in (0, T')$,

$$\varphi(t)^3 \leq A\alpha^3 + A\beta(\tau)^3 \sqrt{T} \int_0^t \varphi(s)^3 ds \quad \forall\, t \in (0, \tau).$$

By the same method of showing that the map $t \in (0, T^*) \mapsto u(t) \in L^\infty_x$ is continuous, we can show that the map $t \in (0, T) \mapsto u_\varepsilon(t) \in L^\infty_x$ is also continuous. Thus, φ is a bounded continuous function on $(0, T')$. By Grönwall inequality,

$$\varphi(t)^3 \leq A\alpha^3 \exp\left(A\beta(\tau)^3\sqrt{T}t\right) \quad \forall t \in (0, \tau).$$

Letting $t \to \tau^-$, we get

$$\varphi(\tau)^3 \leq A\alpha^3 \exp\left(A\beta(\tau)^3\sqrt{T}\tau\right) \leq A\alpha^3 \exp\left(AT^{3/2}\beta(\tau)^3\right).$$

Since τ is chosen arbitrarily in $(0, T')$,

$$\begin{aligned}
\varphi(t) &\leq A\alpha \exp\left(AT^{3/2}\beta(t)^3\right) \\
&\leq A\left(\varepsilon M^2 + \varepsilon^2 M^3 + \varepsilon M^3\sqrt{T}\left(1 + \log_+\frac{T}{\varepsilon^2}\right)\right) \\
&\times \exp\left(AT^{3/2}\left(M^3 + \sup_{s\in(0,t)} \|u(s)\|^3_{L^\infty_x}\right)\right) \quad \forall\, t \in (0, T').
\end{aligned}$$

7.4 Global existence of u indicated by suitable size of u_ε

Proposition 7.3. *Let M, $T > 0$ and $u_0 \in L^2 \cap L^\infty$, $\|u_0\|_{L^\infty} \leq M$. Suppose there exists $\varepsilon > 0$ such that*

$$A\left(\varepsilon M + \varepsilon^2 M^2 + \varepsilon M^2\sqrt{T}\left(1 + \log_+\frac{T}{\varepsilon^2}\right)\right)\exp\left(AT^{3/2}M^3\right) \leq 1, \tag{7.19}$$

and that $(\mathrm{NSE})_\varepsilon$ *has mild solution u_ε satisfying $\|u_\varepsilon\|_{X_T} \leq M$. Then (NSE) has a mild solution u satisfying $\|u\|_{X_T} \leq 2M$.*

Remark 7.4. For fixed $M, T \in (0, \infty)$, Condition (7.19) is satisfied if ε is sufficiently small. To respect the scaling symmetry, we need to make sure that two quantities to be compared have the same dimension. Condition (7.19) can be written as

$$A\left[\varepsilon M + (\varepsilon M)^2 + (\varepsilon M)(\sqrt{T}M)\left(1 + 2\log_+\frac{\sqrt{T}M}{\varepsilon M}\right)\right]\exp\left[A(\sqrt{T}M)^3\right] \leq 1.$$

Put $C_1 = \varepsilon M$ and $C_2 = \sqrt{T}M$. These are dimensionless quantities. Condition (7.19) becomes

$$A\left[C_1 + C_1^2 + C_1C_2\left(1 + 2\log_+\frac{C_2}{C_1}\right)\right]\exp\left(AC_2^3\right) \leq 1. \tag{7.20}$$

Applying the inequality $\log_+ x \le 2(\sqrt{x}+1)$, we have

$$\begin{aligned}\text{LHS}(7.20) &\le A\left[C_1 + C_1^2 + C_1C_2\left(5 + 4\sqrt{\frac{C_2}{C_1}}\right)\right]\exp\left(AC_2^3\right)\\ &\le A\sqrt{C_1}\left(\sqrt{C_1} + \sqrt{C_1}^3 + \sqrt{C_1}C_2 + \sqrt{C_2}^3\right)\exp\left(AC_2^3\right).\end{aligned}$$

Provided that $C_1 \le C_2$,

$$\text{LHS}(7.20) \le A\sqrt{C_1}\left(\sqrt{C_2} + \sqrt{C_2}^3\right)\exp\left(AC_2^3\right) \le A\sqrt{C_1}\exp\left(AC_2^3\right).$$

Then (7.20) is satisfied if $C_1 \le \min\left\{A\exp(-AC_2^3), C_2\right\}$. Thus, (7.19) is satisfied if

$$\varepsilon \le \frac{1}{M}\min\left\{A\exp(-AC_2^3), C_2\right\}.$$

Proof of Proposition 7.3. Suppose that (NSE)has a mild solution u on an some interval $[0, T']$. By [48, p. 223], it has a mild solution on $[0, T' + \tau']$ with

$$\tau' = \frac{A}{\|u(T')\|_{L_x^\infty}^2},$$

$$\sup_{t\in[T',T'+\tau']}\|u(t)\|_{L_x^\infty} \le 16\left\|u(T')\right\|_{L_x^\infty}.$$

For each $\delta \in (0, T)$, we show that (NSE) has a mild solution on $[0, T-\delta]$. Set

$$\tau = \frac{A}{M^2}.$$

Divide the interval $[0, T-\delta]$ into subintervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N]$, each having length $\le \tau$. We prove by induction on $k \in \{0, 1, \ldots, N\}$ that (NSE) has a mild solution u on $[t_0, t_k]$ satisfying $\sup_{t\in[t_0,t_k]}\|u(t)\|_{L_x^\infty} \le 2M$.

The claim is true for $k = 0$. Suppose it is true for some $k \in \{0, 1, \ldots, N-1\}$. Because

$$\tau = \frac{A}{M^2} \le \frac{A}{\|u(t_k)\|_{L_x^\infty}^2},$$

(NSE) has a mild solution on $[t_k, t_{k+1}]$. Moreover,

$$\sup_{t\in[t_k,t_{k+1}]}\|u(t)\|_{L_x^\infty} \le 16\|u(t_k)\|_{L_x^\infty} \le 32M.$$

Thus,

$$\sup_{t\in[0,t_{k+1}]}\|u(t)\|_{L_x^\infty} \le 32M.$$

By Proposition 7.2, for any $t \in (t_0, t_{k+1})$,

$$\begin{aligned}
\|u_\varepsilon(t) - u(t)\|_{L^\infty_x} &\leq A\left(\varepsilon M^2 + \varepsilon^2 M^3 + \varepsilon M^3 \sqrt{T}\left(1 + \log_+ \frac{T}{\varepsilon^2}\right)\right) \\
&\times \exp\left(AT^{3/2}\left(M^3 + \sup_{s\in(0,t)} \|u(s)\|^3_{L^\infty_x}\right)\right) \\
&\leq MA\left(\varepsilon M + \varepsilon^2 M^2 + \varepsilon M^2 \sqrt{T}\left(1 + \log_+ \frac{T}{\varepsilon^2}\right)\right) \\
&\times \exp\left(AT^{3/2}\left(M^3 + 32^3 M^3\right)\right) \\
&= MA\left(\varepsilon M + \varepsilon^2 M^2 + \varepsilon M^2 \sqrt{T}\left(1 + \log_+ \frac{T}{\varepsilon^2}\right)\right) \exp\left(AT^{3/2} M^3\right) \\
&\leq M.
\end{aligned}$$

Thus,

$$\|u(t)\|_{L^\infty_x} \leq \|u_\varepsilon(t) - u(t)\|_{L^\infty_x} + \|u_\varepsilon(t)\|_{L^\infty_x} \leq M + M = 2M \quad \forall\, t \in (t_0, t_{k+1}).$$

Because $\|u(t)\|_{L^\infty_x} = \|u_0\|_{L^\infty} \leq M$ and $\|u(t_{k+1})\|_{L^\infty_x} = \lim_{t \to t_{k+1}^-} \|u(t)\|_{L^\infty_x} \leq M$, we get

$$\|u(t)\|_{L^\infty_x} \leq 2M \quad \forall\, t \in [t_0, t_{k+1}].$$

We have showed that (NSE) has a mild solution on $[0, T - \delta]$ satisfying

$$\|u(t)\|_{L^\infty_x} \leq 2M \quad \forall\, t \in [0, T - \delta].$$

Because δ is arbitrarily small, u exists on $[0, T)$ and $\|u(t)\|_{L^\infty_x} \leq 2M$ for all $t \in [0, T)$. □

By Littlewood–Paley theory, T. Tao shows that the mild solution u of (NSE) satisfies an *a priori* estimate called *bounded total speed* [96, Prop. 9.1]:

$$\int_0^T \|u(t)\|_{L^\infty_x} dt \leq A\left(\|u_0\|_{L^2} T^{1/4} + \|u_0\|^2_{L^2}\right) \quad \forall\, T \in (0, T_*). \tag{7.21}$$

If u blows up at finite time T_*, we have

$$\|u(t)\|_{L^\infty_x} \geq \frac{A}{\sqrt{T_* - t}} \quad \forall\, t \in (0, T_*).$$

This is called by Leray the *first characterization of irregularities* [48, p. 224]. Applying this inequality to (7.21), we get

$$A\sqrt{T_*} = \int_0^{T_*} \frac{A}{\sqrt{T_* - t}} dt \leq \int_0^{T_*} \|u(t)\|_{L^\infty_x} dt \leq A\left(\|u_0\|_{L^2} T_*^{1/4} + \|u_0\|^2_{L^2}\right).$$

We obtain an upper bound for T_*

$$T_* \leq A \|u_0\|_{L^2}^4 . \tag{7.22}$$

In other words, if the mild solution to (NSE) blows up at a finite time, it must blow up before the time $T_0 = A \|u_0\|_{L^2}^4$. This conclusion is consistent with the structure of Leray's weak solutions [48, Para. 34], which says that all weak solutions become regular after a fixed time $T \sim \|u_0\|_{L^2}^4$. We obtain the following consequence of Proposition 7.3.

Proposition 7.5. *Let $M > 0$ and $u_0 \in L^2 \cap L^\infty$ be such that $\|u_0\|_{L^\infty} \leq M$. Suppose there exists $\varepsilon > 0$ such that*

$$\varepsilon \leq \min \left\{ \frac{A}{M} \exp(-A {T_0}^{3/2} M^3), \sqrt{T_0} \right\} \tag{7.23}$$

and that $(\mathrm{NSE})_\varepsilon$ *has mild solution u_ε with $\|u_\varepsilon\|_{X_{T_0}} \leq M$. Then (NSE) has a global mild solution u. Moreover, $\|u\|_{X_{T_0}} \leq 2M$.*

Proof. By Proposition 7.3 and Remark 7.4, Condition (7.23) guarantees that (NSE) has a mild solution u on the time interval $(0, T_0)$ and $\|u\|_{X_{T_0}} \leq 2M$. Then u does not blow up before T_0. Thus, it never blows up at any finite time. We conclude that (NSE) has a global solution. □

7.5 Another approximate Navier–Stokes system

One can also approximate the Navier–Stokes system in the frequency domain. For $0 < \kappa < \infty$, let $\mathbb{P}_{\leq \kappa}$ be a Fourier multiplier that preserves all frequencies ξ with magnitude $\leq \kappa$ and suppresses all with magnitude $\geq 2\kappa$. One can take, for example, $\widehat{\mathbb{P}_{\leq \kappa} f}(\xi) = \phi(\xi/\kappa)\hat{f}(\xi)$ where $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function supported in the ball of radius 2 and equal to one in the ball of radius 1. Consider the approximate system

$$(\mathrm{NSE})_{\leq \kappa} : \begin{cases} \partial_t u - \Delta u + \mathbb{P}_{\leq \kappa}(u \cdot \nabla u) + \nabla p = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in} \quad \mathbb{R}^3 \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{in} \quad \mathbb{R}^3. \end{cases}$$

We see that κ gives the problem a natural length scale κ^{-1}, which plays the role as resolution of the approximation. One can fix the length scale by setting $\kappa = 1$. Suppose the

initial condition belongs to L^2 and is compactly supported in the Fourier domain. Then $(\mathrm{NSE})_{\leq 1}$ is globally well-posed in the setting $L^\infty_t \dot{H}^s_x$ for any $s \geq 1/2$ (Proposition 7.7). Here $\dot{H}^s$ denotes the homogeneous Sobolev space with exponent s (see Appendix A.4). The critical space in this scale is $\dot{H}^{1/2}$. Let v be the mild solution of $(\mathrm{NSE})_{\leq 1}$

$$v(t) = \Gamma(t) * u_0 + \int_0^t \mathbb{P}_{\leq 1}\left[K'(t-s)\right] * (v(s) \otimes v(s))ds.$$

A variant of Conjecture 1.2 for system $(\mathrm{NSE})_{\leq 1}$ is the following.

Conjecture 7.6. *Let $u_0 \in L^2$ with* supp $\hat{u}_0 \subset \{\xi : |\xi| \leq 2\}$. *For any $s > 1/2$ there exists $\delta = \delta(s, \|u_0\|_{\dot{H}^{1/2}})$ such that if $\|v(t)\|_{\dot{H}^s} \leq \delta$ for all $t > 0$ then* (NSE) *has a global solution u with $\|u\|_{L^\infty_t \dot{H}^s_x} \leq 2\delta$.*

Denote $K'_{\leq 1}(t) = \mathbb{P}_{\leq 1}[K'(t)]$. We have an estimate for the m'th derivatives with respect to the spatial coordinates.

$$\left\|\nabla^m K'_{\leq 1}(t)\right\|_{L^p_x} = \left\|\mathbb{P}_{\leq 1}[\nabla^m K'(t)]\right\|_{L^p_x} \leq A\left\|\nabla^m K'(t)\right\|_{L^p_x} \leq \frac{C(m,p)}{t^{2+\frac{m}{2}-\frac{3}{2p}}}$$

for all $1 \leq p \leq \infty$ and $m = 0, 1, 2, \ldots$ Many properties of the Navier–Stokes equations are still true for $(\mathrm{NSE})_{\leq 1}$, for example the local-in-time existence, blowup criterion, regularity, and energy identity.

Proposition 7.7. *We have the following statements.*

(a) For $1 \leq p \leq \infty$ and $m = 0, 1, 2, \ldots$

$$\left\|\nabla^m K'_{\leq 1}(t)\right\|_{L^p_x} \leq \min\left\{\frac{C_{m,p}}{t^{2+\frac{m}{2}-\frac{3}{2p}}}, C_{m,p}\right\} \quad \forall\, t > 0.$$

(b) $(\mathrm{NSE})_{\leq 1}$ has a global solution v with

$$\|v\|_{L^\infty_{t,x}} \leq \|u_0\|_{L^\infty} + A\,\|u_0\|^2_{L^2}.$$

(c) For each $2 < p < 3$, there exists $C(p) > 0$ such that if $\|u_0\|_{L^p} < \varepsilon \leq C(p)$ then $\|u\|_{L^\infty_t L^p_x} \leq 2\varepsilon$.

Proof. (a) The Fourier transform of $K'_{\leq 1}(t)$ is supported in $\{|\xi| \leq 2\}$. By Bernstein's lemma [5, Lem. 2.1],

$$\left\|\nabla^m K'_{\leq 1}(t)\right\|_{L^p_x} \leq C(m,p)\left\|K'_{\leq 1}(t)\right\|_{L^p_x}.$$

It suffices to show that

$$\left\|K'_{\leq 1}(t)\right\|_{L^p_x} \leq C(p). \tag{7.24}$$

By Proposition 2.2,

$$\begin{aligned}
\left|\widehat{K'_{\leq 1}(t)}(\xi)\right| &= \left|\phi(\xi)\widehat{K'(t)}(\xi)\right| \\
&= A|\phi(\xi)|\left||\xi|(\mathbb{I}_3 - \xi\otimes\xi)e^{-t|\xi|^2}\right| \\
&\leq A|\phi(\xi)||\xi||\mathbb{I}_3 - \xi\otimes\xi| \\
&\leq A\chi_{|\xi|\leq 2}.
\end{aligned}$$

Then

$$\left\|\widehat{K'_{\leq 1}(t)}\right\|_{L^1_\xi} \leq A\int\limits_{|\xi|\leq 2} d\xi = A$$

and

$$\left\|K'_{\leq 1}(t)\right\|_{L^\infty_x} \leq A\left\|\widehat{K'_{\leq 1}(t)}\right\|_{L^1_\xi} = A.$$

Using the interpolation inequality of Lebesgue spaces, to prove (7.24) for all $1 \leq p \leq \infty$ we only need to prove it for $p = 1$. That is to show $\left\|K'_{\leq 1}(t)\right\|_{L^1_x} \leq A$. By the inequality $\left\|K'_{\leq 1}(t)\right\|_{L^1_x} \leq \frac{A}{\sqrt{t}}$, it suffices to show that

$$\left\|K'_{\leq 1}(t) - K'_{\leq 1}(1)\right\|_{L^1_x} \leq A \quad \forall\, 0 < t \leq 1.$$

We have

$$\begin{aligned}
K'_{\leq 1}(t) - K'_{\leq 1}(1) &= A\int\limits_{\mathbb{R}^3}\left(\widehat{K'_{\leq 1}(t)}(\xi) - \widehat{K'_{\leq 1}(1)}(\xi)\right)e^{ix\cdot\xi}d\xi \\
&= A\int\limits_{\mathbb{R}^3}\phi(\xi)\left(e^{-t|\xi|^2} - e^{-|\xi|^2}\right)i\xi\left(\mathbb{I}_3 - \frac{\xi\otimes\xi}{|\xi|^2}\right)e^{ix\cdot\xi}d\xi.
\end{aligned}$$

Denote

$$\phi_0(\xi) = \frac{1 - e^{-|\xi|^2}}{|\xi|^2}.$$

This is a smooth function in $\mathbb{R}^3$ with bounded derivatives. Then

$$\begin{aligned}
K'_{\leq 1}(t) - K'_{\leq 1}(1) &= A\int\limits_{\mathbb{R}^3}\phi(\xi)\left(|\xi|^2\phi_0(\xi) - t|\xi|^2\phi_0(\sqrt{t}\xi)\right)i\xi\left(\mathbb{I}_3 - \frac{\xi\otimes\xi}{|\xi|^2}\right)e^{ix\cdot\xi}d\xi \\
&= A\int\limits_{\mathbb{R}^3}\phi(\xi)\left(\phi_0(\xi) - t\phi_0(\sqrt{t}\xi)\right)i\xi\left(|\xi|^2\mathbb{I}_3 - \xi\otimes\xi\right)e^{ix\cdot\xi}d\xi.
\end{aligned}$$

By the integration by parts,

$$K'_{\leq 1}(t) - K'_{\leq 1}(1) =$$
$$\frac{A}{(1+|x|^2)^2} \int\limits_{\mathbb{R}^3} (Id - \Delta_\xi)^2 \left[\phi(\xi)\left(\phi_0(\xi) - t\phi_0(\sqrt{t}\xi)\right) i\xi \left(|\xi|^2 \mathbb{I}_3 - \xi \otimes \xi\right)\right] e^{ix\cdot\xi} d\xi.$$

Thus,

$$\left\|K'_{\leq 1}(t) - K'_{\leq 1}(1)\right\|_{L^1_x}$$
$$\leq \int\limits_{\mathbb{R}^3} \frac{A}{(1+|x|^2)^2} dx \int\limits_{\mathbb{R}^3} \left|(Id - \Delta_\xi)^2 \left[\phi(\xi)\left(\phi_0(\xi) - t\phi_0(\sqrt{t}\xi)\right) \xi \left(|\xi|^2 \mathbb{I}_3 - \xi \otimes \xi\right)\right]\right| d\xi.$$

The integrand in the second integral is continuous,and is bounded by $A(1+t)^3 I_{|\xi|\leq 2}$. Because $t \leq 1$, the integral is bounded by an absolute constant A. Thus, $\|\{2\}\|_{L^1_x} \leq A$. We conclude that

$$\left\|K'_{\leq 1}(t) - K'_{\leq 1}(1)\right\|_{L^1_x} \leq A.$$

(b) We have

$$v(t) = \Gamma(t) * u_0 + \int_0^t K'_{\leq 1}(t-s) * (v(s) \otimes v(s)) ds.$$

Take the L^∞_x-norm of both sides

$$\begin{aligned}
\|v(t)\|_{L^\infty_x} &\leq \|\Gamma(t) * u_0\|_{L^\infty} + \int_0^t \left\|K'_{\leq 1}(t-s) * (v(s) \otimes v(s))\right\|_{L^\infty_x} ds \\
&\leq \|u_0\|_{L^\infty} + \int_0^t \left\|K'_{\leq 1}(t-s)\right\|_{L^\infty_x} \|u(s) \otimes u(s)\|_{L^1_x} ds \\
&\leq \|u_0\|_{L^\infty} + A \|u_0\|^2_{L^2} \int_0^t \left\|K'_{\leq 1}(t-s)\right\|_{L^\infty_x} ds \\
&\leq \|u_0\|_{L^\infty} + A \|u_0\|^2_{L^2} \int_0^t \left\|K'_{\leq 1}(s)\right\|_{L^\infty_x} ds. \qquad (7.25)
\end{aligned}$$

Denote $t \wedge 1 = \min\{t, 1\}$ and $t \vee 1 = \max\{t, 1\}$. Because

$$\left\|K'_{\leq 1}(s)\right\|_{L^\infty_x} \leq \min\left\{A, \frac{A}{s^2}\right\},$$

$$\begin{aligned}
\int_0^t \left\|K'_{\leq 1}(s)\right\|_{L^\infty_x} ds &= \int_0^{t\wedge 1} \left\|K'_{\leq 1}(s)\right\|_{L^\infty_x} ds + \int_1^{t\vee 1} \left\|K'_{\leq 1}(s)\right\|_{L^\infty_x} ds \\
&\leq A(t \wedge 1) + \int_1^{t\vee 1} \frac{A}{s^2} ds \\
&= A + A\left(1 - \frac{1}{t \vee 1}\right) \\
&\leq A.
\end{aligned}$$

Substituting this estimate into (7.25), we get $\|v\|_{L^\infty_{t,x}} \le \|u_0\|_{L^\infty} + A\|u_0\|^2_{L^2}$.

(c) We have

$$\begin{aligned}
\|v(t)\|_{L^p_x} &\le \|u_0\|_{L^p} + \int_0^t \left\|K'_{\le 1}(t-s)\right\|_{L^1_x} \|v(s)\otimes v(s)\|_{L^p_x} ds \\
&\le \|u_0\|_{L^p} + \int_0^t \frac{A}{\sqrt{t-s}} \|v(s)\|^2_{L^{2p}_x} ds \\
&\le \|u_0\|_{L^p} + \int_0^t \frac{A}{\sqrt{t-s}} \|v(s)\|^{\frac{2}{p}}_{L^2_x} \|v(s)\|^{2-\frac{2}{p}}_{L^\infty_x} ds \\
&\le \|u_0\|_{L^p} + A\sqrt{t}\,\|u_0\|^{\frac{2}{p}}_{L^2} \|v\|^{2-\frac{2}{p}}_{L^\infty_{t,x}} . \qquad (7.26)
\end{aligned}$$

We show that the map $t \in (0,\infty) \mapsto v(t) \in L^p_x(\mathbb{R}^3)$ is continuous. The proof is exactly the same as for the Navier–Stokes equations. For t, $t' > 0$,

$$\begin{aligned}
v(t') - v(t) &= \underbrace{(\Gamma(t') - \Gamma(t)) * u_0}_{\{1\}} + \underbrace{\int_0^t \left[K'_{\le 1}(t'-s) - K'_{\le 1}(t-s)\right] * (v(s)\otimes v(s)) ds}_{\{2\}} \\
&+ \underbrace{\int_t^{t'} K'_{\le 1}(t'-s) * (v(s)\otimes v(s)) ds}_{\{3\}} .
\end{aligned}$$

We show that $\{1\}, \{2\}, \{3\} \to 0$ in L^p_x as $t' \to t$.

$$\|\{1\}\|_{L^p_x} \le \left\|\Gamma(t') - \Gamma(t)\right\|_{L^1_x} \|u_0\|_{L^p}.$$

Recall that

$$\Gamma(x,s) = \frac{A}{s^{3/2}} \exp\left(-\frac{|x|^2}{4s}\right).$$

For $t' \in \left(\frac{t}{2}, \frac{3t}{2}\right)$,

$$\Gamma(x,t') \le \frac{A}{(t/2)^{3/2}} \exp\left(-\frac{|x|^2}{4(3t/2)}\right) = A\Gamma\left(x, \frac{3t}{2}\right).$$

Because $\Gamma\left(\frac{3t}{2}\right) \in L^1_x$, by the Dominated Convergence Theorem $\|\Gamma(t') - \Gamma(t)\|_{L^1_x} \to 0$ as $t' \to t$. Thus, $\|\{1\}\|_{L^p_x} \to 0$ as $t' \to t$.

$$\begin{aligned}
\|\{2\}\|_{L^p_x} &\le \int_0^t \left\|K'_{\le 1}(t'-s) - K'_{\le 1}(t-s)\right\|_{L^1_x} * \|u(s)\otimes u(s)\|_{L^p_x} ds \\
&\le \underbrace{\left(\int_0^t \left\|K'_{\le 1}(t'-s) - K'_{\le 1}(t-s)\right\|_{L^1_x} ds\right)}_{\{4\}} \|u_0\|^{\frac{2}{p}}_{L^2} \|u\|^{2-\frac{2}{p}}_{L^\infty_{t,x}} .
\end{aligned}$$

We have

$$\begin{aligned}
\{4\} &= \int_0^t \int_{\mathbb{R}^3} \left|K'_{\leq 1}(x, t'-s) - K'_{\leq 1}(x, t-s)\right| dxds \\
&= \int_0^t \int_{\mathbb{R}^3} \left|K'_{\leq 1}(x, s+t'-t) - K'_{\leq 1}(x, s)\right| dxds \\
&= \left\|K'_{\leq 1}(\cdot + (0, t'-t)) - K'_{\leq 1}(\cdot)\right\|_{L^1(\mathbb{R}^3 \times (0,t))}.
\end{aligned}$$

This quantity will converge to 0 as $t' \to t$ if $K'_{\leq 1} \in L^1(\mathbb{R}^3 \times (0,T))$ for all $T > 0$. It is true because

$$\int_0^T \int_{\mathbb{R}^3} \left|K'_{\leq 1}(x,s)\right| dxds \leq \int_0^T \frac{A}{\sqrt{s}} ds \leq A\sqrt{T} < \infty.$$

Therefore, $\|\{2\}\|_{L^p_x} \to 0$ as $t' \to t$.

$$\begin{aligned}
\|\{3\}\|_{L^p_x} &\leq \left| \int_t^{t'} \left\|K'_{\leq 1}(t'-s)\right\|_{L^1_x} \|v(s) \otimes v(s)\|_{L^p_x} ds \right| \\
&\leq \left| \int_t^{t'} \frac{A}{\sqrt{t'-s}} ds \right| \|u_0\|_{L^2}^{\frac{2}{p}} \|u\|_{L^\infty_{t,x}}^{2-\frac{2}{p}} \\
&\leq A\sqrt{|t'-t|}\, \|u_0\|_{L^2}^{\frac{2}{p}} \|v\|_{L^\infty_{t,x}}^{2-\frac{2}{p}},
\end{aligned}$$

which goes to 0 as $t' \to t$. We have showed that the map $t \in (0,\infty) \mapsto v(t) \in L^p_x(\mathbb{R}^3)$ is continuous.

Denote $f(s) = \|v(s)\|_{L^p_x}$ for all $s > 0$. Then f is continuous on $(0,\infty)$. Suppose $\|u_0\|_{L^p} < \varepsilon \leq C(p)$ where $C(p) > 0$ is a constant to be determined. We have

$$\begin{aligned}
f(t) &= \|v(t)\|_{L^p_x} \leq \|u_0\|_{L^p} + \int_0^t \left\|K'_{\leq 1}(t-s)\right\|_{L^{p'}_x} \|v(s) \otimes v(s)\|_{L^{p/2}_x} ds \\
&\leq \|u_0\|_{L^p} + A \int_0^t \left\|K'_{\leq 1}(t-s)\right\|_{L^{p'}_x} f(s)^2 ds \\
&\leq \|u_0\|_{L^p} + A \left(\int_0^t \left\|K'_{\leq 1}(s)\right\|_{L^{p'}_x} ds \right) \sup_{s \in (0,t)} f(s)^2. \qquad (7.27)
\end{aligned}$$

By Part (a),

$$\left\|K'_{\leq 1}(s)\right\|_{L^{p'}_x} \leq \min\left\{ C_1(p),\ \frac{C_1(p)}{s^{\frac{1}{2}+\frac{3}{2p}}} \right\}.$$

Then

$$\begin{aligned}
\int_0^t \left\|K'_{\leq 1}(s)\right\|_{L^{p'}_x} ds &= \int_0^{t\wedge 1} \left\|K'_{\leq 1}(s)\right\|_{L^{p'}_x} ds + \int_1^{t\vee 1} \left\|K'_{\leq 1}(s)\right\|_{L^{p'}_x} ds \\
&\leq \int_0^{t\wedge 1} C_1(p) ds + \int_1^{t\vee 1} \frac{C_1(p)}{s^{\frac{1}{2}+\frac{3}{2p}}} ds \\
&\leq C_2(p).
\end{aligned}$$

With this estimate, (7.27) implies

$$f(t) \leq \|u_0\|_{L^p} + C_2(p) \sup_{s\in(0,t)} f(s)^2. \tag{7.28}$$

By (7.26), there exists $T_1 > 0$ such that $f(s) \leq 2\varepsilon$ for all $s \in (0, T_1)$. Let $T = \sup\{s > 0 : f(s) \leq 2\varepsilon\}$. Suppose $T < \infty$. From (7.28) we get

$$f(T) \leq \|u_0\|_{L^p} + C_2(p)(2\varepsilon)^2 \leq \varepsilon(1 + 4\varepsilon C_2(p)) \leq \varepsilon(1 + 4C(p)C_2(p)).$$

If we choose

$$C(p) = \frac{1}{8C_2(p)}$$

then $f(T) < 2\varepsilon$. By the continuity of f, there exists $\delta > 0$ such that $f(T + \delta) \leq 2\varepsilon$. This contradicts the definition of T. $\square$

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Appendix A

Standard tools in PDE

A.1 Some inequalities

The following inequality is an interpolation version of the well-known Gagliardo-Nirenberg inequality.

Proposition A.1 (Gagliardo-Nirenberg inequality)**.** *[97, Prop. A.3] Let* $1 \le p \le q \le \infty$*,* $k \ge 1$*,* $0 \le \theta \le 1$ *be such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{k\theta}{n}.$$

Then

$$\|u\|_{L^q} \le C_{p,q,k,n} \|u\|_{L^p}^{1-\theta} \left\|\nabla^k u\right\|_{L^p}^{\theta} \quad \forall\ u \in W^{k,p}(\mathbb{R}^n).$$

By an approximation procedure, one can obtain a variation of Gagliardo-Nirenberg inequality for appropriately regular domains $\Omega \subset \mathbb{R}^n$ (not necessarily bounded):

$$\|u\|_{L^{p^*}(\Omega)} \le C_{p,n,\Omega} \|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega)$$

where $p^* = np/(n-p)$ (see e.g. [17, p. 279], [2, Lem. 5.19]).

Let Ω be a bounded connected open subset in $\mathbb{R}^n$ with Lipschitz boundary. Denote by $[u]_\Omega$ the average value of u over Ω. Then for $1 \le p < n$ and $1 \le q \le p^*$,

$$\|u - [u]_\Omega\|_{L^q} \le C_1(n,p,q,\Omega) \|\nabla u\|_{L^p} \quad \forall\ u \in W^{1,p}(\Omega), \tag{A.1}$$

$$\|u\|_{L^q} \le C_2(n,p,q,\Omega) \|\nabla u\|_{L^p} \quad \forall\ u \in W_0^{1,p}(\Omega). \tag{A.2}$$

These are known as Poincaré–Sobolev inequalities [102, Rem. 1.3.5], [29, Lec. 4]. Inequalities (A.1) and (A.2) also hold for any $1 \le p = q \le \infty$, which are known as

Poincaré–Wirtinger inequalities [17, p. 280, 290]. If Ω is a convex domain and $p = q$, one can choose

$$C_1 = A \operatorname{diam}(\Omega)$$

(See e.g. [94, Thm. 2].) For $p < q \leq p^*$, the Poincaré constants may depend non-trivially on the volume of Ω. This can be seen by a scaling argument: let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective linear map. By changing variables in integration, one notices that the constants C_i, $i = 1, 2$, are rescaled as

$$C_i(n, p, q, T(\Omega)) \sim |\det T|^{\frac{1}{q}-\frac{1}{p}} \|T\| \, C_i(n, p, q, \Omega).$$

It T acts as a squeezing operator in one direction, e.g. $T(x_1, x_2, \ldots, x_n) = (\varepsilon x_1, x_2, \ldots, x_n)$, leaving $\operatorname{diam}(\Omega)$ unchanged then $C_i(\Omega_\varepsilon) \sim \varepsilon^{1/q-1/p} \to \infty$ as $\varepsilon \to 0$. In fact, for Ω convex and $p \leq q < p^*$ the authors in [66] derive a constant

$$C_1 = C(n, p, q) \frac{\operatorname{diam}(\Omega)^{1+n\left(1+\frac{1}{q}-\frac{1}{p}\right)}}{\operatorname{vol}(\Omega)}.$$

See also [42]. Nevertheless, for families of domains with similar shapes, including balls, half-balls, cubics, regular tetrahedra,... the Poincaré constant depends only on $\operatorname{diam}(\Omega)$. In particular, we have

Proposition A.2 (Poincaré–Sobolev inequality)**.** *Suppose Ω be a bounded and connected domain with unit diameter. Let $\{\Omega_r\}_{r>0}$ be dilations Ω , i.e. $\Omega_r = \{y = rx : \ x \in \Omega\}$. Then for $1 \leq p < n$ and $1 \leq q \leq p^* = np/(n-p)$,*

$$\left\|u - [u]_{\Omega_r}\right\|_{L^q(\Omega_r)} \leq C(n, p, \Omega) r^{n\left(\frac{1}{q}-\frac{1}{p^*}\right)} \|\nabla u\|_{L^p(\Omega_r)} \quad \forall \ u \in W^{1,p}(\Omega_r).$$

$$\|u\|_{L^q(\Omega_r)} \leq C(n, p, \Omega) r^{n\left(\frac{1}{q}-\frac{1}{p^*}\right)} \|\nabla u\|_{L^p(\Omega_r)} \quad \forall \ u \in W_0^{1,p}(\Omega_r).$$

A stronger version of Young's inequality for convolution is the so-called Young–O'Neil inequality, which applies for functions in weak Lebesgue spaces and Lorentz spaces. We use it to verify regularity of the Stokes equations with force term in a mixed-norm Lebesgue space (Remark 3.2). A list of several familiar inequalities on Lebesgue spaces generalized to weak Lebesgue spaces can be found in [62].

Proposition A.3 (Young–O'Neil inequality)**.** *[7, Thm. 7.6], [28, Thm. 1.4.25] Let $1 < p, q, r < \infty$ be such that $1 + 1/p = 1/q + 1/r$. Then for any $f \in L^{q,\infty}(\mathbb{R}^n)$ and $g \in L^r(\mathbb{R}^n)$,*

$$\|f * g\|_{L^p} \leq C_{p,q,r} \|f\|_{L^{q,\infty}} \|g\|_{L^r}$$

A.2 Compactness theorem for Banach spaces

Aubin-Lions lemma is an abstract compactness theorem for Banach spaces. It is particularly useful in the theory of weak solutions of the Navier–Stokes equations (see Section 5.2). The formulation of the lemma is as follows.

Let X_0, X, X_1 be Banach spaces such that

- $X_0 \subset X \subset X_1$ where each injection is continuous,
- the injection $X_0 \to X$ is compact.

Let $0 < T < \infty$ and $1 < \alpha_0,\ \alpha_1 < \infty$. Put

$$\mathscr{Y} = \left\{ u \in L^{\alpha_0}((0,T), X_0) : \quad u' = \frac{du}{dt} \in L^{\alpha_1}((0,T), X_1) \right\}.$$

It is a Banach space with norm $\|u\|_{\mathscr{Y}} = \|u\|_{L^{\alpha_0}((0,T),X_0)} + \|u'\|_{L^{\alpha_1}((0,T),X_1)}$.

Proposition A.4 (Aubin-Lions lemma)**.** *[99, p. 271], [85]*
The injection $\mathscr{Y} \to L^{\alpha_0}((0,T), X)$ is compact.

Note that the original formulation by Aubin (1963) requires X_0 and X_1 to be reflexive. It turns out that this condition can be dropped (see [85]).

A.3 Mixed-norm Lebesgue spaces

The notion of mixed-norm Lebesgue spaces was introduced by Benedek and Panzone [6]. For $\vec{p} = (p_1, p_2, \ldots, p_n)$ and $\mathscr{O} = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n) \subset \mathbb{R}^n$, a function $v : \mathscr{O} \to \mathbb{R}$ is said to belong to $L^{\vec{p}}(\mathscr{O})$ if

$$\|v\|_{L^{\vec{p}}} = \left\| \ldots \left\| \|v\|_{L^{p_1}_{x_1}} \right\|_{L^{p_2}_{x_2}} \ldots \right\|_{L^{p_n}_{x_n}} < \infty.$$

Some basic properties of $L^{\vec{p}}$ are studied in [6], including the completeness, reflexivity and characterization of dual space. In particular, if $1 < p_j < \infty$ for all $1 \le j \le n$ (which we simply denote as $1 < \vec{p} < \infty$), then $L^{\vec{p}}$ is reflexive.

Lemma A.5. *Let $D_1 \subset \mathbb{R}^k$ and $D_2 \subset \mathbb{R}^m$ be bounded sets. Put $D = D_1 \times D_2$. Consider a bounded sequence (v_n) in a mixed-norm Lebesgue space $L^a_x L^b_y(D)$ for some $1 < a, b \le \infty$. Suppose that (v_n) converges to v in $L^1(D)$. Then $v_n \to v$ in $L^\alpha_x L^\beta_y(D)$ for all $1 \le \alpha < a$ and $1 \le \beta < b$.*

Proof. For each $\theta \in (0,1)$, put

$$\frac{1}{\gamma_\theta} = \frac{\theta}{1} + \frac{1-\theta}{a}, \qquad \frac{1}{\kappa_\theta} = \frac{\theta}{1} + \frac{1-\theta}{b}.$$

By Hölder inequality,

$$\|v_n - v\|_{L_x^{\gamma_\theta} L_y^{\kappa_\theta}} \leq \|v_n - v\|^{\theta}_{L_x^1 L_y^1} \|v_n - v\|^{1-\theta}_{L_x^a L_y^b} \to 0.$$

Because $\gamma_\theta \to a^-$ and $\kappa_\theta \to b^-$ as $\theta \to 1^-$, there exists $\theta > 0$ so that $1 \leq \alpha < \gamma_\theta$ and $1 \leq \beta < \kappa_\theta$. By the boundedness of D and Hölder inequality, $L_x^{\gamma_\theta} L_y^{\kappa_\theta}(D)$ is continuously embedded into $L_x^\alpha L_y^\beta(D)$. Therefore, $v_n \to v$ in $L_x^\alpha L_y^\beta(D)$. □

A.4 Homogeneous Sobolev spaces

Let s be a real number. The homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$, or simply denoted by $\dot{H}^s$, is defined as the space of all tempered distributions u over $\mathbb{R}^3$ such that $\hat{u} \in L^1_{\text{loc}}(\mathbb{R}^3)$ and

$$\|u\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}} < \infty.$$

As a consequence of Hölder inequality, the following inequality, so called "interpolation inequality", for homogeneous Sobolev spaces is satisfied:

$$\|u\|_{\dot{H}^s} \leq \|u\|^{1-\theta}_{\dot{H}^{s_0}} \|u\|^{\theta}_{\dot{H}^{s_1}},$$

where $s_0 < s < s_1$ and $s = (1-\theta)s_0 + \theta s_1$. If $s < \frac{3}{2}$ then $\dot{H}^s(\mathbb{R}^3)$ is a Hilbert space with the inner product

$$(f,g)_{\dot{H}^s} = \int_{R^3} |\xi|^{2s} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi.$$

Denote by $\mathscr{S}$ the Schwartz space. If $|s| < \frac{3}{2}$ then the bilinear functional $B : \mathscr{S} \times \mathscr{S} \to \mathbb{C}$,

$$B(\phi_1, \phi_2) = \int_{\mathbb{R}^3} \phi_1(x)\phi_2(x) dx$$

can be extended to a continuous bilinear functional on $\dot{H}^{-s} \times \dot{H}^s$. Moreover, if L is a continuous linear functional on $\dot{H}^s$ then there exists a unique tempered distribution u in $\dot{H}^{-s}$ such that $\|L\|_{(\dot{H}^s)'} = \|u\|_{\dot{H}^{-s}}$ and

$$L\phi = B(u, \phi) \quad \forall \phi \in \dot{H}^s.$$

For this reason, the space $\dot{H}^{-s}$ can be considered as the dual space of $\dot{H}^s$.

Proposition A.6. *[5, Thm. 1.38]We have the following embedding properties.*

(a) If $0 \le s < \frac{3}{2}$ then $\dot{H}^s(\mathbb{R}^3)$ is continuously embedded in $L^{\frac{6}{3-2s}}(\mathbb{R}^3)$.

(b) If $1 < p \le 2$ then $L^p(\mathbb{R}^3)$ is continuously embedded in $\dot{H}^{\frac{3}{2}-\frac{3}{p}}(\mathbb{R}^3)$.

A.5 Difference quotients

Proposition A.7. *Let $n \ge 2$ and $1 \le i \le n$. Consider a domain $\mathscr{O} \subset \mathbb{R}^n$ such that for some $\delta > 0$,*

$$\begin{aligned} \mathscr{O} + he_i \subset \mathscr{O} \quad & \forall h \in (0,\delta) \\ \text{or} \quad \mathscr{O} - he_i \subset \mathscr{O} \quad & \forall h \in (0,\delta). \end{aligned} \tag{A.3}$$

Denote

$$D_i^h v(x) = \frac{v(x+he_i) - v(x)}{h} \tag{A.4}$$

where $e_1, e_2, \dots, e_n$ are the standard basis vectors of $\mathbb{R}^n$.

(i) Suppose v and $\partial_i v$ belong to $L^p(\mathscr{O})$ for some $1 \le p < \infty$. Then $\left\|D_i^h v\right\|_{L^p(\mathscr{O})} \le \|\partial_i v\|_{L^p(\mathscr{O})}$.

(ii) Suppose $v \in L^1_{\text{loc}}(\mathscr{O})$ for some $1 < p < \infty$, and that there exists a constant $C > 0$ such that $\left\|D_i^h v\right\|_{L^p(\mathscr{O})} \le C$ for all $h \in (0,\delta)$. Then the weak derivative $\partial_i v$ exists and $\|\partial_i v\|_{L^p(\mathscr{O})} \le C$.

Such relations between the difference quotients and weak derivatives are already addressed in many texts (see e.g. [27, Sec. 7.11], [17, p. 292]). Typically, $D_i^h v$ and $\partial_i v$ are estimated only over a subset $\mathscr{O}'$ away from the boundary $\partial\mathscr{O}$. It turns out that when $\mathscr{O}$ satisfies (A.3), which is the case for half-space, quadrant, octant, orthant, infinite strip, semi-infinite strip,... the estimates hold for $\mathscr{O}' = \mathscr{O}$.

Proof. Let $(\eta_\varepsilon)_{\varepsilon>0}$ be an approximate of identity (or mollifiers) in $\mathbb{R}^n$. It suffices to only consider the case $\mathscr{O} + he_i \subset \mathscr{O}$ for all $h \in (0,\delta)$.
(i) Consider the case where v is smooth. Then

$$\begin{aligned} \int_{\mathscr{O}} |D_i^h v(x)|^p dx &= \frac{1}{h^p}\int_{\mathscr{O}} |v(x+he_i) - v(x)|^p dx = \int_{\mathscr{O}} \left|\int_0^1 \partial_i v(x+the_i)dt\right|^p dx \\ &\overset{\text{Hölder}}{\le} \int_{\mathscr{O}}\int_0^1 |\partial_i v(x+the_i)|^p dt dx \overset{\text{Fubini}}{=\!=} \int_0^1\int_{\mathscr{O}} |\partial_i v(x+the_i)|^p dx dt \\ &\le \int_0^1\int_{\mathscr{O}} |\partial_i v(y)|^p dy dt = \|\partial_i v\|^p_{L^p(\mathscr{O})}. \end{aligned}$$

For general $v \in L^p(\mathscr{O})$, approximate v by smooth functions $v_\varepsilon = \tilde{v} * \eta_\varepsilon$ where $\tilde{v}$ is the extension of v to $\mathbb{R}^n$ by zero. We have $\left\|D_i^h v_\varepsilon\right\|_{L^p(\mathscr{O})} \le \|\partial_i v_\varepsilon\|_{L^p(\mathscr{O})}$. Letting $\varepsilon \to 0$, we get $\left\|D_i^h v\right\|_{L^p(\mathscr{O})} \le \|\partial_i v\|_{L^p(\mathscr{O})}$.

(ii) Because $L^p(\mathscr{O})$ is a reflexive space, there exists a sequence $h_k \downarrow 0$ such that $(D_i^{h_k} v)$ weakly converges to some $w \in L^p(\mathscr{O})$. For $\phi \in \mathscr{D}(\mathscr{O})$ and $0 < h_k < \operatorname{dist}(\operatorname{supp} \phi,\ \partial\mathscr{O})$,

$$\begin{aligned}\int_{\mathscr{O}} \phi D_i^{h_k} v dx &= \frac{1}{h_k}\left(\int_{\mathscr{O}} \phi(x) v(x+h_k e_i) dx - \int_{\mathscr{O}} \phi(x) v(x) dx\right) \\ &= \frac{1}{h_k}\left(\int_{\mathscr{O}} \phi(y-h_k e_i) v(y) dy - \int_{\mathscr{O}} \phi(y) v(y) dy\right) = -\int_{\mathscr{O}} v D_i^{-h_k} \phi dx.\end{aligned}$$

Note that $D_i^{h_k}\phi \to \partial_i \phi$ pointwise and $|v D_i^{h_k}\phi| \le |v| \|\partial_i \phi\|_{L^\infty}$, which is an L^1-function. By Lebesgue's Dominated Convergence Theorem,

$$\int_{\mathscr{O}} \phi w dx = -\int_{\mathscr{O}} v \partial_i \phi dx.$$

Thus, w is the weak derivative $\partial_i v$. Moreover, $\|w\|_{L^p} \le \liminf_{k\to\infty} \left\|D_i^{h_k} v\right\|_{L^p} \le C$. □

The following lemma is a variation Proposition A.7 for mixed-norm Lebesgue space.

Proposition A.8. *Let $n \ge 2$, $1 \le i \le n$ and $1 < \vec{p} < \infty$. Consider a domain $\mathscr{O} = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n) \subset \mathbb{R}^n$ with $b_i = \infty$.*

(i) Suppose v and $\partial_i v$ belong to $L^{\vec{p}}(\mathscr{O})$. Then $\left\|D_i^h v\right\|_{L^{\vec{p}}(\mathscr{O})} \le \|\partial_i v\|_{L^{\vec{p}}(\mathscr{O})}$.

(ii) Suppose $v \in L^1_{\mathrm{loc}}(\mathscr{O})$ and that there exist $\delta, C > 0$ such that $\left\|D_i^h v\right\|_{L^{\vec{p}}(\mathscr{O})} \le C$ for all $h \in (0, \delta)$. Then the weak derivative $\partial_i v$ exists and $\|\partial_i v\|_{L^{\vec{p}}(\mathscr{O})} \le C$.

Proof. Without loss of generality, we assume $i = 1$. The proofs are essentially the same as in Proposition A.7.

$$\begin{aligned}\int_{a_1}^{\infty} |D_1^h v(x)|^{p_1} dx_1 &= \frac{1}{h^{p_1}} \int_{a_1}^{\infty} |v(x+he_1) - v(x)|^{p_1} dx_1 \\ &\le \int_0^1 \int_{a_1}^{\infty} |\partial_1 v(y)|^{p_1} dy_1 dt = \|\partial_1 v\|_{L^{p_1}_{y_1}}^{p_1}.\end{aligned}$$

Thus, $\left\|D_1^h v\right\|_{L^{p_1}_{x_1}} \le \|\partial_1 v\|_{L^{p_1}_{x_1}}$. By integrating both sides over the rest of the variables, we get $\left\|D_1^h v\right\|_{L^{\vec{p}}(\mathscr{O})} \le \|\partial_1 v\|_{L^{\vec{p}}(\mathscr{O})}$. Thanks to the reflexivity of $L^{\vec{p}}(\mathscr{O})$, the proof of Part (ii) is the same as in Part (ii) of Proposition A.7. □

A.6 Anisotropic Sobolev embeddings

Consider a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. Suppose that for some $1 \le p \le \infty$ and $r \in \{0, 1, 2, \ldots\}$, u and all its generalized derivatives up to order r belong to $L^p(\Omega)$. Sobolev embedding theorems (see e.g. [2, Thm. 5.4]) describe the values of q for which $u \in L^q(\Omega)$. These are called *isotropic Sobolev embeddings* because the coordinates x_1, x_2, ..., x_n play

the same role. There are situations when it is necessary to view coordinates as having different scalings. A typical example is the heat system

$$\begin{cases} \partial_t u - \Delta u = f & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ u(\cdot, 0) = u_0 \end{cases}$$

which has scaling symmetry

$$\begin{aligned} u(x,t) &\to u(\lambda x, \lambda^2 t), \\ f(x,t) &\to \lambda^2 f(\lambda x, \lambda^2 t), \\ u_0(x) &\to u_0(\lambda x). \end{aligned}$$

In this regard, the time variable is "worth" two spatial variables. Suppose $f \in L^p_{t,x}$ and the solution belongs to the regularity class $\partial_t u$, $\nabla^2 u \in L^p_{t,x}$. It is natural to ask for what values of q does u belong to $L^q_{t,x}$. The question can be formulated more generally as follows.

> Let $r_1, r_2, \ldots, r_n$ be nonnegative integers and $1 \leq p_1, p_2, \ldots, p_n < \infty$. Suppose $u \in L^p$ and $\partial^{r_i}_{x_i} u \in L^{p_i}$ for all $1 \leq i \leq n$. For what values of $q_1, q_2, \ldots, q_n$ do we have $u \in L^{q_1}_{x_1} L^{q_2}_{x_2} ... L^{q_n}_{x_n}$?

This is answered by *anisotropic Sobolev embeddings*, which were first studied by Nikol'skiĭ, Solonnikov, Besov, Golovkin and others in 1950s and 1960s. In the following, we state an embedding theorem for W and H classes due to Nikol'skiĭ [69]. Let Ω be a domain in $\mathbb{R}^n$. For $r = 0, 1, 2, \ldots$ and $1 \leq p \leq \infty$, the Sobolev space with respect to the i'th direction is defined as

$$W^r_{x_i p}(\Omega) = \left\{ u \in L^p(\Omega) : \ \partial^l_{x_i} u \in L^p(\Omega) \ \forall \, 1 \leq l \leq r \right\}$$

which is a Banach space with norm

$$\|u\|_{W^r_{x_i p}(\Omega)} = \sum_{l=0}^{r} \left\| \partial^l_{x_i} u \right\|_{L^p(\Omega)}.$$

By the interpolation of intermediate derivatives (see e.g. [2, Thm. 4.14]), this norm is equivalent to $||u||'_{W^r_{x_i p}} = ||u||_{L^p(\Omega)} + ||\partial^r_{x_i} u||_{L^p(\Omega)}$. For $r > 0$, not necessarily integral,

write $r = \bar{r} + \alpha$ with $\bar{r} = 0, 1, 2, \dots$ and $0 < \alpha \le 1$. Given a function u, for $0 < \alpha < 1$ we denote

$$M_{x_i p}^{(r)} = \sup_{\substack{0<|t|<\delta \\ \delta>0}} \frac{||\partial_{x_i}^{\bar{r}} u(x+te_i) - \partial_{x_i}^{\bar{r}} u(x)||_{L^p(\Omega_\delta^i)}}{|t|^\alpha}$$

where e_i is the i'th standard unit vector in $\mathbb{R}^n$ and $\Omega_\delta^i = \{x \in \Omega : \, x + te_i \in \Omega \text{ for all } t \in (-\delta, \delta)\}$. For $\alpha = 1$, denote

$$M_{x_i p}^{(r)} = \sup_{\substack{0<|t|<\delta \\ \delta>0}} \frac{||\partial_{x_i}^{\bar{r}} u(x+te_i) - 2\partial_{x_i}^{\bar{r}} u(x) + \partial_{x_i}^{\bar{r}} u(x-te_i)||_{L^p(\Omega_\delta^i)}}{|t|}.$$

Define the space $H_{x_i p}^r$ as

$$H_{x_i p}^r(\Omega) = \left\{ u \in W_{x_i p}^{\bar{r}}(\Omega) : \; M_{x_i p}^{(r)} < \infty \right\}$$

which is a Banach space with norm $\|u\|_{H_{x_i p}^r(\Omega)} = \|u\|_{W_{x_i p}^{\bar{r}}(\Omega)} + M_{x_i p}^{(r)}$. We say that a vector $\vec{a} = (a_1, \dots, a_n)$ is nonnegative integral if each a_i is a nonnegative integer, and use the notation $1 < \vec{a} < \infty$ if $1 < a_i < \infty$ for all $i = 1, 2, ..., n$. Other notations (e.g. $\vec{a} > 0$, $\vec{a} > \vec{b}, \dots$) are understood in similar manner (i.e. componentwise). Define anisotropic spaces of W and H classes as follows.

$$W_{\vec{p}}^{\vec{r}}(\Omega) = \bigcap_{i=1}^n W_{x_i p_i}^{r_i}(\Omega), \quad \|u\|_{W_{\vec{p}}^{\vec{r}}(\Omega)} = \sum_{i=1}^n \|u\|_{W_{x_i p_i}^{r_i}(\Omega)}.$$

$$H_{\vec{p}}^{\vec{r}}(\Omega) = \bigcap_{i=1}^n H_{x_i p_i}^{r_i}(\Omega), \quad \|u\|_{H_{\vec{p}}^{\vec{r}}(\Omega)} = \sum_{i=1}^n \|u\|_{H_{x_i p_i}^{r_i}(\Omega)}.$$

Although the definition of $W_{\vec{p}}^{\vec{r}}$ involves only unmixed partial derivatives, the integrability of certain mixed derivatives can be deduced. Good guesses can be seen through scaling arguments: take $\Omega = \mathbb{R}^2$, $\vec{r} = (3, 2)$ and $\vec{p} = (q, q)$ for example. Then $W_{q,q}^{3,2} = \{u \in L^q : \; \partial_{tt} u, \, \partial_{xxx} u \in L^q\}$. Let us consider a scaled version of u, namely $u_\lambda(x,t) = u(\lambda^2 x, \lambda^3 t)$. The quantities u, $\partial_{tt} u$, $\partial_{xxx} u$ are scaled by λ^0, λ^6, λ^6 respectively. Any mixed derivative scaled by λ^k with $0 \le k \le 6$, such as $\partial_{tx} u$, can possibly be bounded by $\max\{\|u\|_{L^q}, \, \|\partial_{tt} u\|_{L^q}, \, \|\partial_{xxx} u\|_{L^q}\}$ in L^q-norm. One cannot expect the same for $\partial_{txx} u$, which is scaled by λ^7. In fact, it is shown in [56] and [89, Thm. 2'] that if $u \in W_q^{\vec{r}} := W_{(q,\dots,q)}^{\vec{r}}$ then $\partial^{\vec{s}} u \in L^q$ for all $\vec{s} = (s_1, \, s_2, \dots, s_n)$ satisfying

$$\sum_{i=1}^n \frac{s_i}{r_i} \le 1. \tag{A.5}$$

Consequently, the space $W_q^m := W_{q,\dots,q}^{m,\dots,m}$ coincides the usual isotropic Sobolev space $W^{m,p}$. If Condition (A.5) is not satisfied, $\partial^{\vec{s}}u$ may not belong to L^q (see e.g. [74, Sec. 5] for a counterexample).

Likewise, certain mixed derivatives of a function in $H_\infty^{\vec{r}} := H_{\infty,\dots,\infty}^{\vec{r}}$ are Hölder continuous: take $\Omega = \mathbb{R}^n \times \mathbb{R}$ and $u = u(x,t)$ for example. It is shown in [44, Lem. 3.1] that if u is Hölder continuous with respect to t with exponent α, and $\partial_x u$ is Hölder continuous with respect to x with exponent β, then $\partial_x u$ is Hölder continuous with respect to t with exponent $\alpha\beta/(1+\beta)$. See also [89, Thm. 2].

Using the idea of Parts (i) and (ii) of Proposition A.7, one can check that $W_{\vec{p}}^{\vec{r}} = H_{\vec{p}}^{\vec{r}}$ if $\vec{r}$ is positive integral and $1 < \vec{p} < \infty$. The two spaces are not the same when $\vec{p} = \infty$ (see [70, p. 271-273] for a counterexample). The following embedding theorem is due to Nikol'skiĭ.

Proposition A.9 (Anisotropic Sobolev embedding for class H)**.** *[68, Sec. 6.9] Let $\Omega \subset \mathbb{R}^n$ be a domain with cone property, i.e. there exists a fixed finite cone $\mathscr{C}$ such that if its vertex is placed at any point in $\bar{\Omega}$, the cone itself can be swung so that it lies in $\bar{\Omega}$. Let $\vec{r} > 0$ and $1 \le \vec{p} \le \vec{q} \le \infty$. Suppose*

$$\tau_i = 1 - \sum_{l=1}^n \left(\frac{1}{p_l} - \frac{1}{q_i}\right)\frac{1}{r_l} > 0, \qquad \kappa_i = 1 - \sum_{l=1}^n \left(\frac{1}{p_l} - \frac{1}{p_i}\right)\frac{1}{r_l} > 0$$

for all $i = 1, 2, \dots, n$. Put $s_i = r_i\tau_i/\kappa_i$. Then we have a continuous embedding $H_{\vec{p}}^{\vec{r}}(\Omega) \hookrightarrow H_{\vec{q}}^{\vec{s}}(\Omega)$.

Strictly speaking, [68, Sec. 6.9] addresses the case $\Omega = \mathbb{R}^n$, and [69, Thm. 6, 7, 8] the case Ω equals the whole space or a rectangular parallelepiped. Since any domain with cone property can be expressed as a union of translations of finitely many parallelepipeds [2, Thm. 4.8], the embedding theorem extends to this case as well. We now state a useful consequence of Proposition A.9 in regard to regularity of the heat equation. For $\Omega \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$, and a function u defined on $Q = \Omega \times I$, let us denote

$$\begin{aligned} \langle\!\langle u \rangle\!\rangle_{q,Q}^{(j)} &= \sum_{2\alpha+\beta=j} \left\| \partial_t^\alpha \nabla_x^\beta u \right\|_{L^q(Q)}, \\ \langle u \rangle_Q^{(\lambda)} &= \operatorname*{ess\,sup}_{\substack{x,x'\in\Omega \\ t,t'\in I}} \frac{|u(x',t') - u(x,t)|}{(|x'-x|^2 + |t'-t|)^{\lambda/2}}. \end{aligned}$$

Proposition A.10 (Parabolic Sobolev embedding)**.** *[44, Lem. 3.1], [102, Thm. 1.4.1], [75, Appendix D3, D4] Let Ω be a domain in $\mathbb{R}^n$ with cone property, $T \in (0,\infty)$ and*

$\Omega_T = \Omega \times (0,T)$. *Consider a function* $u \in W_p^{2l,l}(\Omega_T) := W_{q,\dots,q,q}^{2l,\dots,2l,l}(\Omega_T)$. *For nonnegative integers* r *and* s, *and* $0 < \delta \le \min\{d, \sqrt{T}\}$ *where* d *is the altitude of the cone, we have the following statements.*

(i) If $p \ge q$ *and* $2l - m - (1/q - 1/p)(n+2) \ge 0$ *with* $m = 2r + s$ *then*

$$\|\partial_t^r \nabla_x^s u\|_{L^p(\Omega_T)} \le C\delta^{2l-m-\left(\frac{1}{q}-\frac{1}{p}\right)(n+2)} \langle\!\langle u \rangle\!\rangle_{q,\Omega_T}^{(2l)} + C\delta^{-m-\left(\frac{1}{q}-\frac{1}{p}\right)(n+2)} \|u\|_{L^q(\Omega_T)},$$

(ii) If $p \ge q$ *and* $0 \le \lambda < 2l - m - (n+2)/q$ *then*

$$\langle \partial_t^r \nabla_x^s u \rangle_{\Omega_T}^{(\lambda)} \le C\delta^{2l-m-\frac{n+2}{q}-\lambda} \langle\!\langle u \rangle\!\rangle_{q,\Omega_T}^{(2l)} + C\delta^{-m-\frac{n+2}{q}-\lambda} \|u\|_{L^q(\Omega_T)}.$$

Moreover, the inequality still holds for $\lambda = 2l - m - (n+2)/q$ *if it is not an integer.*

A.7 Singular integrals

An important tool in harmonic analysis and PDE is the Calderón–Zygmund's theory of singular integrals, an interest of which is the boundedness of operators $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y) dy.$$

Under certain conditions on K such that T possesses a so-called Calderón–Zygmund (C-Z) decomposition, T is called a C-Z operator, K the associated C-Z kernel. The two terms may be used interchangeably when the context is clear. If the kernel is of the form $K(x-y)$, then T is said to be of convolution type. There are many available definitions in literature for C-Z kernels. However, we will follow the definition in [28, Sec. 5.4]: $K = K(x)$ is said to be a C-Z kernel if it satisfies

(a) Size condition:

$$\sup_{R>0} \int_{R \le |x| \le 2R} |K(x)| dx = A_1 < \infty, \tag{A.6}$$

(b) Smoothness condition:

$$\sup_{y \ne 0} \int_{|x| \ge 2|y|} |K(x-y) - K(x)| dx = A_2 < \infty, \tag{A.7}$$

(c) Cancellation condition:

$$\sup_{0<R_1<R_2<\infty} \left| \int_{R_1<|x|<R_2} K(x) dx \right| = A_3 < \infty. \tag{A.8}$$

A sufficient condition of (A.6) is $|K(x)| \le C|x|^{-n}$. A sufficient condition of (A.7) is

$$K \in C^1(\mathbb{R}^n \backslash \{0\}) \quad \text{and} \quad |\nabla K(x)| \le C|x|^{-n-1}.$$

If the size condition (A.6) and the boundedness of Fourier transform $\hat{K} \in L^\infty(\mathbb{R}^n)$ hold, then (A.8) also holds [28, Prop. 5.4.4]. In many applications, K is odd with respect to at least one coordinate (as in case of the Hilbert transform and Riesz transform). In such a case, (A.8) is satisfied with $A_3 = 0$.

Proposition A.11 (Calderón–Zygmund)**.** *[28, Thm. 5.4.1] If a function K satisfies (A.6), (A.7), (A.8) then $Tf = K * f$ defines a bounded linear operator from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. Moreover,*

$$\|T\|_{L^p \to L^p} \le C_n \max\left\{p, \frac{1}{p-1}\right\}(A_1 + A_2 + A_3).$$

Remark A.12. The boundedness of T from $L^p(\mathbb{R}^n)$ to itself also holds if K satisfies the smoothness condition and $\hat{K} \in L^\infty(\mathbb{R}^n)$ [75, Thm. B3]. Note that the size condition is not required.

Another framework to study singular integrals is the Littlewood–Paley (L-P) theory. We know that the kernel of an operator of convolution type acts as a multiplier in the Fourier frequency domain. If an operator $Tf = K * f$ is bounded from $L^p(\mathbb{R}^n)$ to itself, then $\hat{K}$ is called a Fourier multiplier, or *multiplier* for short. As an abuse of terminology, the operator T itself is sometimes called Fourier multiplier to avoid reference to the kernel. L-P theory provides sufficient conditions for a bounded function to be Fourier multiplier. Certain versions of L-P theorem can be proved by Calderón–Zygmund theorem and vice versa. In many cases, L-P decomposition in the frequency domain is able to extract subtle information about the kernel that is not easily seen in the physical domain, for example the rate of decay of the Stokes kernel (see e.g. [46, Prop. 11.1]). Well-known multiplier theorems include those of Mikhlin (1957), Hörmander (1960), and Lizorkin (1963). Their results have been improved over time and are sometimes stated quite differently from their original versions. See e.g. [93] for a history of multiplier theorems.

Proposition A.13 (Hörmander-Mikhlin multiplier theorem)**.** *[28, Thm. 6.2.7] Let $m(\xi)$ be a complex-valued bounded function on $\mathbb{R}^n \backslash \{0\}$ such that*

$$\left(\int_{R \le |\xi| \le 2R} |\partial^\alpha m(\xi)|^2 d\xi\right)^{1/2} \le A R^{\frac{n}{2} - |\alpha|}$$

for all $R > 0$ and multi-indices $|\alpha| \leq \left[\frac{n}{2}\right] + 1$. Then m is a Fourier multiplier with

$$\|m\|_{L^p \to L^p} \leq C_n \max\left\{p, \frac{1}{p-1}\right\}(A + \|m\|_{L^\infty}) \quad \forall\ 1 < p < \infty.$$

Proposition A.14 (Lizorkin multiplier theorem)**.** *[28, Cor. 6.2.5], [57, Thm. 5] Let $m(\xi)$ be a complex-valued bounded function on $\mathbb{R}^n \backslash \{0\}$ such that for any distinct $j_1, j_2, \ldots, j_k \in \{1, 2, \ldots, n\}$ we have*

$$|\partial_{j_1} \ldots \partial_{j_k} m(\xi)| \leq \frac{A}{|\xi_{j_1}| \ldots |\xi_{j_k}|} \quad \forall\ \xi \neq 0.$$

Then m is a Fourier multiplier with

$$\|m\|_{L^p \to L^p} \leq C_n \max\left\{p, \frac{1}{p-1}\right\}^{6n}(A + \|m\|_{L^\infty}) \quad \forall\ 1 < p < \infty.$$

Marcinkiewicz interpolation theorem is a useful tool to study the boundedness of linear operators from one Lebesgue space $L^p(X, \mu)$ to another $L^q(Y, \nu)$. There exist many Marcinkiewicz-type theorems: the original one is for L^p spaces (Marcinkiewicz 1939), the formulation for weak L^p spaces, which coincide Lorentz spaces $L^{p,\infty}$, due to Zygmund (1956), the formulation for Lorentz spaces $L^{p,r}$ due to Calderón (1966). See e.g. [59] for a history of the development of this theorem. The following statement is essentially due to Zygmund (1956).

Proposition A.15 (Marcinkiewicz interpolation theorem)**.** *[7, Thm. 4.13] Suppose $1 \leq p_1 < p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$ with $q_1 \neq q_2$. Let $0 < \theta < 1$ and define p and q such that*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Let T be simultaneously a bounded operator from $L^{p_1}(X, \mu)$ to $L^{q_1}(Y, \nu)$, and from $L^{p_2}(X, \mu)$ to $L^{q_2}(Y, \nu)$. Then T is a bounded operator from $L^p(X, \mu)$ to $L^q(Y, \nu)$ with

$$\|T\|_{L^p \to L^q} \leq \frac{C_{p_1,q_1,p_2,q_2}}{\theta(1-\theta)} \max\left\{\|T\|_{L^{p_1} \to L^{q_1,\infty}}, \|T\|_{L^{p_2} \to L^{q_2,\infty}}\right\}.$$

Marcinkiewicz interpolation theorem can be used to derive an important result in the theory of singular integrals known as Hardy-Littlewood-Sobolev theorem for fractional integration. We call it *fractional inequality* for short.

Proposition A.16 (Fractional inequality)**.** *[7, Thm. 4.18] If $1 < p < q < \infty$ and $0 < \kappa < n$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{\kappa}{n}$ then for any $f \in L^p(\mathbb{R}^n)$, the function*

$$I_\kappa f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\kappa}} dy$$

is well-defined almost everywhere and lies in $L^q(\mathbb{R}^n)$. Moreover,

$$\|I_\kappa f\|_{L^q(\mathbb{R}^n)} \leq C(p, q, n)\|f\|_{L^p(\mathbb{R}^n)} \quad \forall\ f \in L^p(\mathbb{R}^n).$$