

- | #   | Author(s)  | Title   |
|-----|--|---|
| 165 | D. Arnold and R.S. Falk                          | Continuous Dependence on the Elastic Coefficients for a Class of Anisotropic Materials  |
| 166 | I.J. Bakelman                                    | The Boundary Value Problems for Non-linear Elliptic Equation and the Maximum Principle for Euler-Lagrange Equations             |
| 167 | Ingo Müller                                      | Gases and Rubbers   |
| 168 | Ingo Müller                                      | Pseudoelasticity in Shape Memory Alloys - an Extreme Case of Thermoelasticity   |
| 169 | Luis Magalhães                                   | Persistence and Smoothness of Hyperbolic Invariant Manifolds for Functional Differential Equations                              |
| 170 | A. Demianian and M. Vogelius                     | Homogenization Limits of the Equations of Elasticity in Thin Domains  |
| 171 | H.C. Simpson and S.-J. Spector                   | On Hadamard Stability in Finite Elasticity  |
| 172 | J.L. Vazquez and C. Yavar                        | Isolated Singularities of the Solutions of the Schrödinger Equation with a Radial Potential                                     |
| 173 | G. Dal Maso and U. Mosco                         | Wiener's Criterion and $\Gamma$ -Convergence  |
| 174 | John H. Maddocks                                 | Stability and Folds   |
| 175 | R. Hardt and D. Kinderlehrer                     | Existence and Partial Regularity of Static Liquid Crystal Configurations  |
| 176 | M. Nerukar                                       | Construction of Smooth Ergodic Cocycles for Systems with Fast Periodic Approximations   |
| 177 | J.L. Ericksen                                    | Stable Equilibrium Configurations of Elastic Crystals   |
| 178 | Patricio Aviles                                  | Local Behavior of Solutions of Some Elliptic Equations  |
| 179 | S.-N. Chow and R. Lauterbach                     | A Bifurcation Theorem for Critical Points of Variational Problems   |
| 180 | R. Pego  | Phase Transitions: Stability and Admissibility in One Dimensional Nonlinear Viscoelasticity                                     |
| 181 | Mariano Giaquinta                                | Quadratic Functions and Partial Regularity  |
| 182 | J. Bone  | Fully Discrete Galerkin Methods for the Korteweg De Vries Equation  |
| 183 | J. Maddocks and J. Keller                        | Mechanics of Robes  |
| 184 | F. Bernis  | Qualitative Properties for some nonlinear higher order Nonlinear Higher Order Parabolic Equations and Asymptotic Rates for some |
| 185 | F. Bernis  | Game Forms with Minimal Strategy Spaces   |
| 186 | S. Reichstein and S. Reiter                      | Game Forms with Minimal Strategy Spaces   |
| 188 | J. Rubinstein and R. Mauri                       | Dispersion and Convexity in Periodic Media  |
| 189 | M.H. Fleming and P.E. Souganidis                 | Asymptotic Series and the Method of Vanishing Viscosity   |
| 190 | H. Beirão da Veiga                               | Existence and Asymptotic Behavior for Strong Solutions of Navier-Stokes Equations in the Whole Space                            |
| 191 | L.A. Caffarelli, J.L. Vazquez, and N.I. Wolanski | Lipschitz Continuity of Solutions and Interfaces of the N-Dimensional Porous Medium Equation                                    |
| 192 | R. Johnson                                       | m-Functions and Floquet Exponents for Linear Differential Systems   |
| 193 | F.V. Atkinson and L.A. Peletier                  | Ground States and Dirichlet Problems for $-\Delta = F(U)$ in $\mathbb{R}^N$   |
| 194 | G. Dal Maso, U. Mosco                            | The Wiener Modulus of a Radial Measure  |
| 195 | H. A. Levine and H.F. Weinberger                 | Inequalities between Dirichlet and Neumann Eigenvalues  |
| 196 | J. Rubinstein                                    | On the Macroscopic Description of Slow Viscous Flow Past a Dirichlet Problem  |
| 197 | G. Dal Maso and U. Mosco                         | Wiener Criteria and Energy Decay for Relaxed Dirichlet Problems   |
| 198 | V. Ofikar and P. Walfman                         | On the Monge-Ampere Equation Arising in the Reflect Mapping Problem   |
| 199 | M. Chipot, D. Kinderlehrer and L. Caffarelli     | Some Smoothness Properties of Linear Laminae  |
| 200 | Y. Giga and R. Kohn                              | Characterizing Blow-up Using Similarity Variables   |
| 201 | P. Cannarsa and H. M. Soner                      | On the Singularities of the Viscosity Solutions to Hamilton-Jacobi-Bellman Equations  |
| 202 | Andrew Majda                                     | Nonlinear Geometric Optics for Hyperbolic Systems of Conservation Laws  |
| 203 | G. Buttazzo, G. Dal Maso and U. Mosco            | A Derivation Theorem for Capacities with Respect to a Radon Measure   |
| 204 | S. Cowin, M. Mehrabadi                           | On the Identification of Material Symmetry for Anisotropic Elastic Materials  |
| 205 | R.W.R. Darling                                   | Constructing Nonhomeomorphic Stochastic Flows.  |
| 206 | M. Chipot  | On the Reynolds Lubrication Equation  |
| 207 | R.V. Kohn and G.W. Milfong                       | On Bounding the Effective Conductivity of Anisotropic Composites  |
| 208 | I.J. Bakelman                                    | Notes Concerning the Torsion of Hardening Rods and its N-Dimensional Generalizations  |
| 209 | I.J. Bakelman                                    | The Boundary Value Problems for Non-Linear Elliptic Equation  |
| 210 | Guanglu Gong & Mingping Qian                     | On the Large Deviation Functions of Markov Cha  |
| 211 | Arie Leizarowitz                                 | Control Problems with Random and Progressively Known Target   |
| 212 | R.W.R. Darling                                   | Ergodicity of a Measure-Valued Markov Chain Induced by Random Transformations   |
| 213 | G. Gong, M. Qian & Zhongxin Zhao                 | Killed Diffusions and its Conditioning  |
| 214 | Arie Leizarowitz                                 | Controlling Diffusion Processes on Infinite Horizon with the Overtaking Criterion   |
| 215 | Milford Beatty                                   | The Poisson Function of Finite Elasticity   |
| 216 | David Terman                                     | Traveling Wave Solutions Arising From a Combustion Model  |
| 217 | Yuh-Jia Lee                                      | Sharp Inequalities and Regularity of Heat Semigroup on Infinite Dimensional Spaces  |
| 218 | D. Stroock                                       | Lecture Notes   |
| 219 | Claudio Canuto                                   | Spectral Methods and Maximum Principle  |
| 220 | Thomas O'Brien                                   | A Two Parameter Family of Pension Contribution Functions and Stochastic Optimization  |
| 221 | Takeyuki Hida                                    | Analysis of Brownian Functionals  |
| 222 | Leonid Murwicz                                   | On Informational Decentralization and Efficiency of Resource Allocation Mechanisms  |
| 223 | E.B. Fabes and D.W. Stroock                      | A New Proof of Moser's Parabolic Harnack Inequality via the Old Ideas of Nash   |
| 224 | Minoru Murata                                    | Structure of Positive Solution to $(-\Delta + V)u = 0$ in $\mathbb{R}^n$  |
| 225 | Paul Dupuis                                      | Large Deviations Analysis of Reflected Diffusion  |
| 226 | F. Bernis  | Constrained Stochastic Approximation Algorithms in Convex Sets  |
| 227 | S. Orey and S. Peifan                            | Existence Results for Doubly Nonlinear Higher Order Parabolic Equations on Unbounded Domains.                                   |
| 228 | R. Gulliver and S. Hildebrandt                   | Boundary Configurations Spanning Continua of Minimal Surfaces.  |
| 229 | J. Baxter, G. Dal Maso & U. Mosco                | Stopping Times and $\Gamma$ -Convergence.   |
| 230 | Julio Boulliet                                   | Self-Similar Solutions, Having Jumps and Intervals of Constancy of a Diffusion-heat Conduction Equation                         |
| 231 | R. Hardt, D. Kinderlehrer & F.-H. Lin            | A Remark About the Stability of Smooth Equilibrium Configurations of Static Liquid Crystal                                      |
| 232 | M. Chipot and M. Luskin                          | The Compressible Reynolds Lubrication Equation  |
| 233 | J.H. Maddocks                                    | A Model for Disclinations in Nematic Liquid Crystal   |
| 234 | C. Foles, G.R. Sell and R. Temam                 | Inertial Manifolds for Nonlinear Evolutionary Equations   |
| 235 | P.-L. Chow                                       | Expectation Functionals Associated with Some Stochastic Evolution.  |
| 236 | Gulisepe Buttazzo                                | Reinforcement by a Thin Layer with Oscillating Thickness  |
| 237 | W.H. Fleming, S.-J. Sheu and H.M. Soner          | On Existence of the Dominant Eigenfunction and its Application to the Large Deviation Properties of Ergodic Markov Process      |
| 238 | R. Jensen and P.E. Souganidis                    | A Regularity Result for Viscosity Solution of Hamilton-Jacobi Equations in one Space Dimension                                  |
| 239 | B. Boczor-Karakli, J.L. Bone and D.L. Cohen      | Interaction of Shallow-Water Waves and Bottom Topography  |
| 240 | F. Colonius and W. Killemann                     | Infinite Time Optimal Control and Periodic Dirichlet Problems   |

- Recent IMA Preprints (continued)
- | #   | Author(s)  | Title  | #   | Author(s)   | Title   |
|-----|--|--|-----|---|---|
| 241 | Harry Kesten,  | Scaling Relations for 2D-Percolation   | 277 | Cockburn, B.  | The Quasi-Monotone Schemes for Scalar Conservation Laws Part I  |
| 242 | A. Leizarowitz,  | Infinite Horizon Optimization for Markov Process with Finite States Spaces                                 | 278 | M. Chipot,  | T. Sideris, On the Abelian Higgs Model  |
| 243 | Louis H.-Y. Chen,  | The Rate of Convergence in A Central Limit Theorem for Dependent Random Variables with Arbitrary Index Set | 279 | C. Fofas, B. Nicolaenko, G.R.Sell, R. Temam,              | Inertial Manifolds for the Kuramoto-Sivashinsky Equation and an Estimate of their Lowest Dimension                      |
| 244 | G. Kafilanur,  | Stochastic Differential Equations in Duals of Nuclear Spaces with some Applications                        | 280 | R. Duran,   | On the Approximation of Miscible Displacement in Porous Media by Method of Characteristics Combined with a Mixed Method |
| 245 | Tzuu-Shuh Chiang, Yunshong Chow and Yuh-Jia Lee,           | Evaluation of Certain Functional Integrals   | 281 | H. Aizang, Zhang Bo,                                      | The Convergence for Nodal Expansion Method Random Distributions   |
| 246 | L. Karp and M. Pinsky,                                     | The First Eigenvalue of a Small Geodesic Ball in a Riemannian Manifold                                     | 282 | V. Twersky,   | Dispersive Bulk Parameters for Coherent Propagation in Correl. Systems  |
| 247 | Chi-Sing Man,  | Towards An Acoustoelastic Theory for Measurement of Residual Stress  | 283 | W.Th.F. den Hollander,                                    | Mixing Properties for Random Walk in Random Scenery   |
| 248 | Andreas Stoll,   | Invariance Principles for Brownian Intersection  | 284 | H.R. Jauslin,   | Non-differentiable Potentials for Nonequilibrium Steady State Systems   |
| 249 | R.W.R. Darling,  | Rate of Growth of the Coalescent Set in a Coalescing Stochastic Flow                                       | 285 | K. Meyer, G.R.Sell,                                       | Homoclinic Orbits and Bernoulli Bundles in Almost Per. Systems  |
| 250 | R. Cohen, R. Hardt, D. Kinderlehrer, S.-Y. Lin, M. Luskin, | Minimum Energy for Liquid Crystals: Computational Results  | 286 | J. Douglas, Jr., Y. Yuan,                                 | Finite Difference Methods for the Transient Behavior of a Semiconductor Device  |
| 251 | Suzanne M. Lenhart,  | Viscosity Solutions for Weakly Coupled Systems of First Order PDEs   | 287 | Li Kaitai, Yan Ningning,                                  | The Extrapolation for Boundary Finite Elements  |
| 252 | M. Cranston, E. Fabes, Z. Zhao,                            | Condition Gauge and Potential Theory for the Schrödinger Operator  | 288 | R. Durrett, R.H. Schonmann,                               | Stochastic Growth Models  |
| 253 | H. Brezis, J.M. Coron, E.H. Lieb,                          | Harmonic Maps with Defects   | 289 | David Kinderlehrer,                                       | Remarks about Equilibrium Configurations of Crystals  |
| 254 | A. Corverhili,   | Flows of Stochastic Dynamical Systems: Nontriviality of the Lyapunov Spectrum                              | 290 | D.G. Aronson, J.L. Vazquez,                               | Eventual Co-regularity and Concavity for Flow in One-Dimensional Porous Media   |
| 255 | A. Carverhill,   | Conditioning A 'Lifted' Stochastic System in a Product Case  | 291 | L.R. Scott, J.M. Boyle, B. Bagheri,                       | Distributed Data Structures for Scientific Computation  |
| 256 | R.J. Williams,   | Local Time and Excursions of Reflected Brownian Motion   | 292 | J. Douglas, Jr., P.J. Poes Leme, T. Arbogast, T. Schmitt, | Simulation of Flow in Naturally Fractured Petroleum Reservoirs  |
| 257 | H. Föllmer, S. Orey,                                       | Large Deviations for the Empirical Field of a Gibbs Measure  | 293 | D.G. Aronson, L.A. Caffarelli,                            | Optimal Regularity for One-Dimensional Porous Medium Flow   |
| 258 | A. Leizarowitz,  | Characterization of Optimal Trajectories On an Infinite Horizon  | 294 | Haim Brezis,  | Liquid Crystals and Energy Estimates for $S^2$ -valued Maps   |
| 259 | Y.Giga, T. Miyakawa, H. Osada,                             | Two Dimensional Navier Stokes Flow with Measures As Initial Vorticity                                      |     |   |   |
| 260 | M. Chipot, V. Olliker,                                     | Sur Une Propriété Des Fonctions Propres De L'Operateur De Laplace Beltrami                                 |     |   |   |
| 261 | V. Perez-Abreu,  | Decompositions of Semimartingales On Duals of Countably Nuclear Spaces                                     |     |   |   |
| 262 | J.M. Ball,   | Does Rank-One Convexity Imply Quasiconvexity   |     |   |   |
| 263 | B. Cockburn,   | The Quasi-Monotone Schemes for Scalar Conservation Laws. Part I  |     |   |   |
| 264 | K.A. Pericak-Spector,                                      | On Radially Symmetric Simple Waves in Elasticity   |     |   |   |
| 265 | P.N. Shivakumar, Chi-Sing Man, Simon W. Rabbkin,           | Modelling of the Heart and Pericardium at End-Diastole   |     |   |   |
| 266 | Jose-Luis Menaldi,   | Probabilistic View of Estimates for Finite Difference Methods  |     |   |   |
| 267 | Robert Hardt, Harold Rosenberg,                            | Open Book Structures and Unicity of Minimal Submanifolds   |     |   |   |
| 268 | Bernardo Cockburn,   | The Quasi-Monotone Schemes for Scalar Conservation Laws  |     |   |   |
| 269 | H.R. Jauslin, W. Zimmerman, Jr.,                           | Dynamics of a Model for an Ac Josephson Effect in Superfluid   |     |   |   |
| 270 | A.K. Kapila,   | Introductory Lecture on Reacting Flows   |     |   |   |
| 271 | J.C. Taylor,   | Do Minimal Solutions of Heat Equations Characterize Diffusions?  |     |   |   |
| 272 | J.C. Taylor,   | The Minimal Eigenfunctions Characterize the Ornstein-Uhlenbeck Process                                     |     |   |   |
| 273 | Chi-Sing Man, Quan-Xin Sun,                                | On the Significance of Normal Stress Effects in the Flow of Glaciers                                       |     |   |   |
| 274 | Omar Hijab,  | On Partially Observed Control of Markov Processes  |     |   |   |
| 275 | Lawrence Gray,   | The Behavior of Processes with Statistical Mechanical Properties   |     |   |   |
| 276 | R. Hardt, D. Kinderlehrer, M. Luskin,                      | Remarks About the Mathematical Theory of Liquid Crystals   |     |   |   |

STOPPING TIMES AND  $\Gamma$  CONVERGENCE

BY

J. BAXTER, G. DAL MASO, U. MOSCO

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# **Stopping Times and $\Gamma$ -Convergence<sup>(1)</sup>**

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### Summary

The equation  $\partial u/\partial t = \Delta u - \mu u$  represents diffusion with killing. The strength of the killing is described by the measure  $\mu$ , which is not assumed to be finite or even  $\sigma$ -finite (to illustrate the effect of infinite values for  $\mu$ , it may be noted that the diffusion is completely absorbed on any set  $A$  such that  $\mu(B) = \infty$  for every nonpolar subset  $B$  of  $A$ ). In order to give a rigorous mathematical meaning to this general diffusion with killing, one may interpret the solution  $u$  as arising from a variational problem, via the resolvent, or one may construct a semigroup probabilistically, using a multiplicative functional. Both constructions are carried out here, and applied to the study of the diffusion equation, as well as to the study of the related Dirichlet problem for the equation  $\Delta u - \mu u = 0$ . The class of diffusions studied here is closed with respect to limits when the domain is allowed to vary. Two appropriate forms of convergence are considered, the first being  $\mathcal{V}$ -convergence of the measures  $\mu$ , which is defined in terms of the variational problem, and the second being stable convergence in distribution of the multiplicative functionals associated with the measures  $\mu$ . These two forms of convergence are shown to be equivalent.

1. Let  $D$  be an open set in  $\mathbb{R}^d$ ,  $d \geq 0$ . Let  $\mathbf{M}_0$  be the class of nonnegative measures, not necessarily  $\sigma$ -finite, which do not charge polar sets. For each  $\mu$  in  $\mathbf{M}_0$ , we wish to consider two problems:

Problem 1 To find the solution  $u$  of the  $\mu$ -Dirichlet problem on  $D$  with data  $g$  on  $\partial D$ , that is:

$$(1.1) \quad -\Delta u + \mu u = 0 \text{ on } D,$$

$$(1.2) \quad u = g \text{ on } \partial D.$$

For brevity, we will say that a solution of (1.1) is  $\mu$ -harmonic on  $D$ . (This usage is not related to the notion of  $h$ -harmonic functions, as defined for example in [12], VIII.1.)

Problem 2 To find the solution  $v: (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$  of the  $\mu$ -diffusion equation with initial data equal to some measure  $\nu$  on  $\mathbb{R}^d$ , that is:

$$(1.3) \quad \partial v / \partial t = \Delta v - \mu v \text{ on } (0, \infty) \times \mathbb{R}^d,$$

$$(1.4) \quad \lim_{t \downarrow 0} v(t, \cdot) = \nu \text{ in distribution sense.}$$

We could generalize problem 2 to the case that  $v$  satisfies a boundary condition

$$(1.5) \quad v(t, \cdot) = 0 \text{ on } D^c \text{ for all } t,$$

but this problem is really included in the previous form of problem 2 for an appropriate choice of  $\mu$ , as we shall see.

Naturally, it is necessary to give a precise meaning to equations (1.1) and (1.3) when  $\mu$  is a general measure. This was done for (1.1) in [8] and [9], by defining  $u$  as the solution of a variational problem (see section 2). The measure  $\mu$  in this case represents a penalization on the solution. It was shown in [8] that the general form of problem 1 provides an appropriate framework for studying limits of solutions of Dirichlet problems in varying domains with "holes" (cf. [7],[16],[17],[18],[19]). Equation (1.3) can also be interpreted variationally, in terms of the resolvent family associated with  $-\Delta + \mu$ . In the present paper we will consider this formulation, and at the same time develop a probabilistic interpretation for both problems 1 and 2. The solution of (1.3) will be defined (section 4) using an appropriate multiplicative functional  $M(\mu)$  associated with  $\mu$  for each  $\mu$  in  $\mathbf{M}_0$ . A multiplicative functional is a special type of randomized stopping time (see section 3). We will show (section 6) that the usual probabilistic method can be applied to solve problem 1, that is, the analogue of the Feynman-Kac formula which is used when  $\mu$  has a finite density  $h$  with respect to Lebesgue on  $\mathbb{R}^d$  (see equation (6.22)). We will also (Theorem 6.2 and Lemma 6.4) give two criteria for the Dirichlet regularity of a point for a  $\mu$ -harmonic function.

It should be noted that, when measures  $\mu$  which are not Radon are used, that two distinct measures  $\mu_1$  and  $\mu_2$  can induce the same variational solutions for problems 1 and 2. Thus we define (section 2) two measures  $\mu_1$  and  $\mu_2$  to be equivalent if this is the case. Because of the variational formulation of these problems, we may express the equivalence of  $\mu_1$  and  $\mu_2$  more briefly by requiring that

$$(1.6) \quad \int u^2 d\mu_1 = \int u^2 d\mu_2,$$

for all functions  $u$  in  $H^1(\mathbb{R}^d)$ .

We will show (Lemma 4.2) that  $\mu_1$  and  $\mu_2$  are equivalent if and only if  $\mu_1(V) = \mu_2(V)$  for every finely open set  $V$  in  $\mathbb{R}^d$ .

The study of limits of solutions of problem 1 in varying domains [8] referred to earlier, was carried out using the notion of  $\mathcal{X}$ -convergence of measures (see section 2). In particular, it was shown in [8] that the space of measures is compact with respect to  $\mathcal{X}$ -convergence, and that finite measures with smooth densities are dense in  $\mathbf{M}_0$  with respect to  $\mathcal{X}$ -convergence. A similar analysis can be carried out for problem 2, using the resolvent family, and we shall show this in section 4. At the same time, we will develop the connection between  $\mathcal{X}$ -convergence of measures and stable convergence of stopping times. This latter convergence was applied in [4] and [5] to study the limits of diffusions in varying regions with holes, using, in particular, the fact that the space of stopping times is compact with respect to stable convergence [3]. In the present paper we will show (Theorem 3.2) that the space of multiplicative functionals is compact with respect to stable convergence. Furthermore, we show (Theorem 4.3) that a sequence  $\mu_n$   $\mathcal{X}$ -converges to  $\mu$  if and only if the associated sequence of multiplicative functionals  $M(\mu_n)$  converges stably to  $M(\mu)$ . We thus obtain a probabilistic interpretation of  $\mathcal{X}$ -convergence of measures. Whereas in the analytical theory of  $\mathcal{X}$ -convergence, the convergence is defined in terms of functionals on  $L^2$ -spaces, the probabilistic notion is expressed as a weak convergence for associated probability measures on an appropriate sample space.

In section 3 we develop some general facts concerning stable convergence (Lemmas 3.3, 3.4, Theorem 3.1). As one application, in Theorem 3.3 we show the relation between stable convergence of multiplicative functionals and strong resolvent convergence for the associated semigroups. This enables us to show the correspondence between  $\mathcal{X}$ -convergence and stable convergence, and to prove a criterion

for  $\chi$ -convergence (Theorem 4.3, Corollary).

2. In this section we shall approach problems 1 and 2 of section 1 analytically, from the variational standpoint. The results we state are for the most part proved in [8] and [9], and we shall follow the terminology of those papers. We will give a precise meaning to the weak inhomogeneous boundary value problem (2.1),(2.2), and thence to the resolvent operator. The resolvent operator would provide an indirect route to the definition of diffusion with a general killing measure, i.e. to problem 2, but we will use a more direct probabilistic approach in sections 3 and 4 to accomplish the same task. After proving some convergence and regularity results for solutions of (2.1), (2.2), and in particular for resolvents, we introduce the idea of  $\chi$ -convergence of measures (Definition 2.7). As we show in section 4, this form of convergence is entirely parallel to the stable convergence defined probabilistically in section 3, so we will develop many of our later convergence results in terms of stable convergence. The link between the two forms of convergence is made through the convergence of the resolvent operator.

Before proceeding we note some terminology. A measure will mean as usual a countably additive set function, taking values in  $[0, \infty]$ , and so not necessarily finite. In particular a measure is nonnegative unless explicitly stated to be a signed measure. We may at times refer to a measure as nonnegative for emphasis. A set of classical capacity zero is a polar set, and a property that is true except on a polar set will be said to hold quasi everywhere or q.e. . We will often consider functions in  $H^1(D)$  where  $D$  is an open set in  $\mathbb{R}^d$ . Such functions are given quasi everywhere. More precisely, for any function  $v \in H^1(D)$  the limit

$$\lim_{r \downarrow 0} \int_{B_r(x)} v(y) dy / m(B_r(x))$$

exists and is finite for quasi every  $x$  in  $D$ . Here  $B_r(x)$  denotes the open ball with center  $x$  and radius  $r$ , and  $m(B)$  denotes the Lebesgue measure of  $B$ , for any Borel set  $B$ . We will adopt the following convention concerning the pointwise values of a function  $v$  in  $H^1(D)$ : for every  $x \in D$  we will always require that

$$\liminf_{r \downarrow 0} \int_{B_r(x)} v(y) dy / m(B_r(x)) \leq v(x) \leq \limsup_{r \downarrow 0} \int_{B_r(x)} v(y) dy / m(B_r(x))$$

With this convention the pointwise value  $v(x)$  is determined quasi everywhere in  $D$ , and the function  $v$  is quasi continuous in  $D$ , i. e. for any  $\epsilon > 0$  there exists an open set  $U$  of capacity less than  $\epsilon$  such that the

restriction of  $v$  to  $D-U$  is continuous.

We now consider an inhomogenous version of problem 1. Let  $D$  be an open set in  $\mathbb{R}^d$ ,  $\mu$  a member of the class  $\mathcal{M}_0$  defined in section 1, so that  $\mu$  is a measure that does not charge polar sets but may be infinite on nonpolar sets. Let  $f \in L^2(D)$ ,  $g \in H^1(D)$ . ( $L^2(D)$  denotes the measurable functions on  $D$  which are square-integrable with respect to Lebesgue measure on  $D$ .) We consider a solution  $u$  of

$$(2.1) \quad -\Delta u + \mu u = f \text{ in } D$$

$$(2.2) \quad u = g \quad \text{on } \partial D.$$

In order to interpret these equations rigorously, we make the following definitions:

**Definition 2.1** A function  $u \in H^1_{loc}(D) \cap L^2_{loc}(D, \mu)$  will be called a local weak solution of (2.1) if

$$(2.3) \quad \int_D \nabla u \cdot \nabla v \, dx + \int_D u v \, d\mu = \int_D f v \, dx$$

for every  $v \in H^1(D) \cap L^2(D, \mu)$  with support  $v$  compact in  $D$ . When  $f=0$  we will also say that  $u$  is  $\mu$ -harmonic on  $D$ . A local weak solution  $u$  of (2.1) will be called a weak solution of the boundary problem (2.1), (2.2) if

$$(2.4) \quad u - g \in H^1_0(D).$$

(Of course, (2.4) implies that  $u \in H^1(D)$ .)

We note that unless  $\mu$  is a Radon measure (that is,  $\mu(K) < \infty$  for every compact set  $K$ ), the weak solutions just defined are not solutions in the distribution sense in  $D$  (see [8], Remark 3.9). However if  $\mu \in \mathcal{M}_0$  is Radon, it is proved in [8] (Proposition 3.8) that  $u$  is a local weak solution of (2.1), according to Definition 2.1, if and only if

$$(2.5) \quad u \in H^1_{loc}(D) \cap L^2_{loc}(D, \mu)$$

and  $u$  is a solution of the equation (2.1) in the sense of distributions, that is

$$(2.6) \quad \int_D \nabla u \cdot \nabla \phi \, dx + \int_D u \phi \, d\mu = \int_D f \phi \, dx \text{ for every } \phi \in C^\infty_0(D).$$

The solutions of (2.1), (2.2) can be characterized in variational terms as follows

**Proposition 2.1** Let  $D$  be any open set in  $\mathbb{R}^d$ . Let  $f \in L^2(D)$  be given and let  $g \in H^1(D)$  be given such that there exists some  $w \in H^1(D) \cap L^2(D, \mu)$  with  $w - g \in H^1_0(D)$ . Then  $u$  is a weak solution of (2.1), (2.2) if and only if  $u$  is the (unique) minimum point of the functional

$$(2.7) \quad F(v) = \int_D |\nabla v|^2 dx + \int_D v^2 d\mu - 2 \int_D f v dx$$

on the set  $\{v: v \in H^1(D), v|_D \in H^1_0(D)\}$ . Moreover,  $u \in H^1(D) \cap L^2(D, \mu)$  and condition (2.3) holds for every  $v \in H^1_0(D) \cap L^2(D, \mu)$ . Furthermore, if  $D$  is bounded, such a solution  $u$  exists for arbitrary  $\mu \in \mathcal{M}_0$ . If  $D$  is unbounded,  $u$  exists for every  $\mu \in \mathcal{M}_0$ , such that  $\mu \geq \lambda m$ , where  $m$  denotes Lebesgue measure in  $\mathbb{R}^d$  and  $\lambda$  is any positive constant.

For  $D$  bounded, the proof can be found in [9], Theorem 2.4 and Proposition 2.5. The same proof can be adapted to the case  $D$  unbounded.

Note that in (2.7) the integral  $\int_D v^2 d\mu$  is well defined, because  $v \in H^1(D)$  can be specified up to sets of capacity zero and these sets have  $\mu$  measure zero.

Let us introduce a special class of measures  $\mu \in \mathcal{M}_0$ , corresponding to homogeneous Dirichlet conditions on Borel sets of  $\mathbb{R}^d$ .

**Definition 2.2** For any Borel set  $E$ , let  $\infty_E$  denote the measure which is  $+\infty$  on all nonpolar Borel subsets of  $E$ , and 0 on every Borel subset of  $E^c$  and on every polar set.

The boundary problem (2.1), (2.2), with  $g=0$  on  $\partial D$ , can be formulated as an equation of the form (2.1) in  $\mathbb{R}^d$ , provided we replace the measure  $\mu$  with the measure  $\mu + \infty_E$ , with  $E = D^c$ .

**Proposition 2.2** Let  $D$  be an open set in  $\mathbb{R}^d$ ,  $f \in L^2(D)$ ,  $\mu \in \mathcal{M}_0$ . Then  $u$  is a weak solution of the boundary problem

$$(2.8) \quad -\Delta u + \mu u = f \text{ in } D,$$

$$(2.9) \quad u = 0 \text{ on } \partial D,$$

(in particular,  $u \in H^1_0(D)$ ), if and only if  $u = U|_D$ , where  $U$  is a weak solution of the equation

$$(2.10) \quad -\Delta U + (\mu + \infty_E)U = f \text{ in } \mathbb{R}^d, \text{ with } E = D^c.$$

**Proof** Let us first recall that  $V \in H^1(\mathbb{R}^d)$  and  $V=0$  q.e. on  $E = D^c$  implies that  $V|_D \in H^1_0(D)$ , see e.g. J. Deny [11], L. Hedberg [15].

Let  $u$  be a solution of (2.8), (2.9). By Proposition 2.1,  $u \in H^1_0(D) \cap L^2(D, \mu)$  and (2.3) holds for every  $v \in H^1_0(D) \cap L^2(D, \mu)$ . Let  $U = u$  in  $D$ ,  $U = 0$  in  $E = D^c$ . Then  $U \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, \mu + \infty_E)$ . Now let  $V \in H^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, \mu + \infty_E)$  with compact support; since  $V=0$  q.e. on  $E$ , we have  $V|_D \in H^1_0(D)$ , and therefore

$\int_{\mathbb{R}^d} \nabla U \nabla V dx + \int_{\mathbb{R}^d} UV d(\mu + \infty_E) = \int_{\mathbb{R}^d} fV dx$ , and hence  $U$  is a weak solution of (2.10).

Now let  $U$  be a solution of (2.10) and let  $u = U|_D$ . Since  $U \in L^2(\mathbb{R}^d, \mu + \infty_E)$  we have  $U = 0$  q.e. on  $E$ , thus  $u \in H^1_0(D)$  and  $u$  is a weak solution of (2.8), so the proposition is proved.

In view of Proposition 2.1, with every  $\mu \in \mathbf{M}_0$  we associate the family of resolvent operators

$$R^D_{\lambda}(\mu) = (-\Delta + \lambda m)^{-1}, \quad D \text{ open } \subset \mathbb{R}^d, \quad \lambda \in \mathbb{R}^+, \text{ as in the following}$$

**Definition 2.3** Let  $D$  be bounded open and  $\lambda \geq 0$  or  $D$  unbounded open and  $\lambda > 0$ . Then the operator

$$R^D_{\lambda}(\mu): L^2(D) \rightarrow L^2(D)$$

is defined to be the mapping that associates with every  $f \in L^2(D)$  the (unique) weak solution  $u \in H^1_0(D) \cap L^2(D, \mu) \subset L^2(D)$  of the problem:

$$\begin{aligned} -\Delta u + (\mu + \lambda m)u &= f \text{ in } D \\ u &= 0 \text{ on } \partial D, \end{aligned}$$

where  $m$  denotes the Lebesgue measure on  $\mathbb{R}^d$ .

By Proposition 2.1, the linear operators  $R^D_{\lambda}(\mu)$  are well defined and continuous with

$$(2.11) \quad \|R^D_{\lambda}(\mu)\| \leq (\lambda + \lambda_1(\mu, D))^{-1},$$

where

$$\lambda_1(\mu, D) = \inf \left[ \left( \int_D |\nabla v|^2 dx + \int_D v^2 d\mu \right) / \int_D v^2 dx \right]$$

(note that  $\lambda_1(\mu, D) > 0$  if  $D$  is bounded).

This is easily proved by taking  $v = u$  in (2.3), where  $\mu$  has been replaced by  $\mu + \lambda m$ , giving

$$\int_D |\nabla u|^2 dx + \int_D u^2 d\mu + \lambda \int_D u^2 dx \leq \|f\|_{L^2(D)} \|u\|_{L^2(D)},$$

hence

$$[\lambda + \lambda_1(\mu, D)] \int_D u^2 dx \leq \|f\|_{L^2(D)} \|u\|_{L^2(D)},$$

which implies (2.11).

Let us also remark that the range of  $R^D_{\lambda}(\mu)$  is dense in  $H^1_0(D) \cap L^2(D, \mu)$  with respect to the norm

$$\left[ \int_D |\nabla u|^2 dx + \int_D u^2 d\mu + \int_D u^2 dx \right]^{1/2}.$$

In fact, if not there exists  $v \in H_0^1(D) \cap L^2(D, \mu)$ ,  $v \neq 0$ , such that

$$\int_D \nabla u \cdot \nabla v dx + \int_D uv d\mu + \lambda \int_D uv dx = 0$$

for every  $u = R_\lambda^D(\mu)f$  and every  $f \in L^2(D)$ . Taking Definition 2.3 into account,

this implies  $\int_D f v dx = 0$  for every  $f \in L^2(D)$ , giving a contradiction.

For every  $f \in L^2(\mathbb{R}^d)$  and every open subset  $D$  of  $\mathbb{R}^d$  we now define  $R_\lambda^D(\mu)f$  to be  $R_\lambda^D(\mu)$  applied to the restriction of  $f$  to  $D$ . We will then define  $R_\lambda^D(\mu)f$  to be zero outside  $D$ , so that  $R_\lambda^D(\mu)f$  is defined on all of  $\mathbb{R}^d$  when convenient.

The following comparison principle holds, for local weak solutions of equation (2.1) (see [9], Theorem 2.10):

**Proposition 2.3** Let  $\mu_1, \mu_2 \in \mathbf{M}_0$  and let  $u_1, u_2$  be local weak solutions of the equations

$$\begin{aligned} -\Delta u_1 + \mu_1 u_1 &= f_1 \text{ in } D, \\ -\Delta u_2 + \mu_2 u_2 &= f_2 \text{ in } D, \end{aligned}$$

where  $D$  is an open set in  $\mathbb{R}^d$ . If  $\mu_1 \leq \mu_2$  as measures on  $D$ ,  $0 \leq f_2 \leq f_1$  on  $D$ , and  $0 \leq u_2 \leq u_1$  on  $\partial D$ , then  $0 \leq u_2 \leq u_1$  quasi everywhere in  $D$ .

We recall that for  $u, v \in H_{loc}^1(D)$  we say that  $u \leq v$  on  $\partial D$  if and only if  $(v-u) \wedge 0 \in H_0^1(D)$ .

**Corollary** Let  $D_1$  and  $D_2$  be open sets in  $\mathbb{R}^d$ ,  $\mu \in \mathbf{M}_0$ ,  $f \in L_2(D)$ ,  $f \geq 0$ . Then  $R_{\lambda_1}^{D_1}(\mu)f \leq R_{\lambda_2}^{D_2}(\mu)f$  quasi everywhere on  $\mathbb{R}^d$ , provided  $\lambda \geq 0$  and  $D$  is bounded or  $\lambda > 0$  and  $D$  is unbounded.

Resolvents on unbounded regions can be approximated by resolvents on bounded domains, according to the following

**Lemma 2.1** Let  $D_n, D$  be open sets in  $\mathbb{R}^d$ , with  $D_n \uparrow D$ . Let  $\mu \in \mathbf{M}_0$ ,  $\lambda > 0$ ,  $f \in L_2(\mathbb{R}^d)$ . Then  $R_{\lambda}^{D_n}(\mu)f \rightarrow R_{\lambda}^D(\mu)f$  in  $L^2(\mathbb{R}^d)$  as  $n \rightarrow \infty$ .

**Proof** By (2.11), the operators  $R_{\lambda}^{D_n}(\mu)$  are uniformly bounded in  $n$  from  $L^2(\mathbb{R}^d)$  into  $L^2(\mathbb{R}^d)$ . Therefore, it suffices to prove the lemma for every  $f$  in a dense subset of  $L^2(\mathbb{R}^d)$ . We assume that  $f$  has compact support in some  $D_{n_0}$ . We may also assume that  $f \geq 0$  in  $D$ . Let  $u_n = R_{\lambda}^{D_n}(\mu)f$  and  $u = R_{\lambda}^D(\mu)f$ . Since  $f \geq 0$ ,  $u_n$  converges monotonically downward to a limit  $w$  in  $D$ .

Clearly  $u_n \in H^1_0(D_n)$  for each  $n \geq n_0$  and hence by (2.7)

$$\begin{aligned} \int_{D_n} |\nabla u_{n_0}|^2 dx + \int_{D_n} u_{n_0}^{2d(\mu+\lambda m)-2} \int_{D_n} f u_{n_0} dx &\geq \\ &\geq \int_{D_n} |\nabla u_n|^2 dx + \int_{D_n} u_n^{2d(\mu+\lambda m)-2} \int_{D_n} f u_n dx. \end{aligned}$$

$$\begin{aligned} \text{Hence } \int_D |\nabla u_{n_0}|^2 dx + \int_D u_{n_0}^{2d(\mu+\lambda m)+2} \int_{D_n} f u_n dx &\geq \\ &\geq \int_D |\nabla u_n|^2 dx + \lambda \int_D u_n^2 dx. \end{aligned}$$

Thus  $\|u_n\|_{H^1(D)}$  is bounded and hence, since  $u_n \uparrow w$ ,  $u_n \rightarrow w$  weakly in  $H^1(D)$ .

Let  $v \in H^1(D) \cap L^2(D, \mu)$  and let the support of  $v$  be compact in  $D$ . Let us suppose  $v \geq 0$ . Let  $n_1$  be such that support  $v \subset D_{n_1}$ . Then, by (2.3) for each  $n \geq \max(n_0, n_1)$ ,

$$\int_D \nabla u_n \cdot \nabla v dx + \int_D u_n v d(\mu + \lambda m) = \int_D f v dx.$$

By weak convergence in the first term and monotone convergence in the second term, we obtain

$$\int_D \nabla w \cdot \nabla v dx + \int_D w v d(\mu + \lambda m) = \int_D f v dx.$$

Since  $w$  is a weak limit of elements in  $H^1_0(D)$ , then  $w \in H^1_0(D)$ . Since  $w \leq u$ ,  $w \in L^2(D, \mu)$ . Hence, by the uniqueness part of Proposition 2.1,  $u = w$ . This proves Lemma 2.1.

**Remark 2.1** It follows in particular from Lemma 2.1 that if  $f \geq 0$  a.e. in  $R^d$  and if the functions  $R^D_{n, \lambda}(\mu) f \in H^1(R^d)$  are pointwise defined in  $R^d$  according to the convention mentioned above, then  $R^D_{n, \lambda}(\mu) f \uparrow R^D_{\lambda}(\mu) f$  q.e. in  $R^d$ .

With each  $\mu \in \mathbf{M}_0$  and each open set  $D$  in  $R^d$  we associate the following functional  $F^D_{\mu}(v)$  defined on  $L^2_{loc}(D)$  by setting:

$$(2.12) \quad F^D_{\mu}(v) = \int_D |\nabla v|^2 dx + \int_D v^2 d\mu \text{ if } v \in H^1_0(D),$$

$$(2.13) \quad F^D_{\mu}(v) = +\infty \text{ if } v \in L^2(D), \text{ but } v \text{ not in } H^1_0(D).$$

If  $D = R^d$ , we denote the corresponding functional by  $F_{\mu}$ .

Since  $\mu$  does not charge polar sets, the functional  $F^D_{\mu}$  is lower semicontinuous in  $L^2(D)$ .

By Proposition 2.1, knowledge of  $F_{\mu}^D$  is sufficient to determine the solution of the  $\mu$ -Dirichlet problem. Clearly two different measures  $\mu_1, \mu_2$  may give rise to the same functional. This leads to

**Definition 2.4** Two measures  $\mu_1, \mu_2 \in \mathbf{M}_0$  are equivalent, in which case we write  $\mu_1 \sim \mu_2$ , if  $F_{\mu_1}^D(v) = F_{\mu_2}^D(v)$  for every open set  $D \subset \mathbb{R}^d$ , and every  $v \in L^2(D)$ .

Obviously,  $\mu_1 \sim \mu_2$  if and only if

$$(2.14) \quad \int_{\mathbb{R}^d} v^2 d\mu_1 = \int_{\mathbb{R}^d} v^2 d\mu_2 \text{ for every } v \in H^1(\mathbb{R}^d).$$

We will see in Lemma 4.2 that two measures are equivalent if and only if they agree on all finely open sets.

We need the following result from [8] (Lemma 4.5):

**Proposition 2.4** For every  $\mu \in \mathbf{M}_0$  there exists a nonnegative Radon measure  $\nu$ , with  $\nu \in H^{-1}(\mathbb{R}^d)$ , and a nonnegative Borel function  $q: \mathbb{R}^d \rightarrow [0, \infty]$ , such that  $\mu \sim q\nu$  in the sense of Definition 2.4.

We recall that a Radon measure  $\nu$  on  $\mathbb{R}^d$  belongs to the space  $H^{-1}(\mathbb{R}^d)$  (the dual space of  $H^1(\mathbb{R}^d)$ ) if there exists a constant  $c > 0$  such that

$$(2.15) \quad \left| \int_{\mathbb{R}^d} \varphi d\nu \right| \leq c \|\varphi\|_{H^1(\mathbb{R}^d)}$$

for every  $\varphi \in C_0^\infty(\mathbb{R}^d)$ . We also recall that if  $\nu$  is a nonnegative Radon measure on  $\mathbb{R}^d$  which belongs to  $H^{-1}(\mathbb{R}^d)$ , then  $\nu \in \mathbf{M}_0$  and (2.15) holds for every  $\varphi \in H^1(\mathbb{R}^d)$ .

We will denote by  $\mathbf{M}_1$  the space of measures of the form  $q\nu$ , with  $\nu \in H^{-1}(\mathbb{R}^d)$ , and  $q$  a nonnegative Borel function from  $\mathbb{R}^d$  to  $[0, \infty]$ .

Proposition 2.4 can now be expressed by saying that for each  $\mu \in \mathbf{M}_0$  there exists  $\mu_1 \in \mathbf{M}_1$  with  $\mu_1 \sim \mu$ .

For later work we will need the classical Newtonian potential operator and its analogues on  $\mathbb{R}^d$ ,  $d \geq 2$ . Thus we make the following

**Definition 2.5** Let  $\mu$  be a measure on  $\mathbb{R}^d$ ,  $d \geq 2$ . Define the operators  $\text{Pot}^+, \text{Pot}^-$  by

$$(2.16) \quad \text{Pot}^+ \mu(x) = \int k^+(|x-y|) \mu(dy), \quad \text{Pot}^- \mu(x) = \int k^-(|x-y|) \mu(dy),$$

where  $k(r) = c_d r^{d-2}$  if  $d \geq 3$ , and  $k(r) = -c_d \log r$  if  $d = 2$ . Let  $a_d$  denote the area of the unit sphere in  $\mathbb{R}^d$ . Then  $c_d = 2/(d-2)a_d$  for  $d \geq 3$ ,  $c_d = 2/a_d$  for  $d = 2$ .

$\text{Pot}^+ \mu, \text{Pot}^- \mu$  are defined everywhere on  $\mathbb{R}^d$  in the finite or  $+\infty$  sense, for all measures  $\mu$ . Of course  $\text{Pot}^- \mu = 0$  if  $d \geq 3$ .

For any  $\mu$  such that  $\text{Pot}^- \mu$  is finite on  $\mathbb{R}^d$ , we define the classical potential  $\text{Pot} \mu$  of  $\mu$  by  $\text{Pot} \mu = \text{Pot}^+ \mu - \text{Pot}^- \mu$ . In particular  $\text{Pot} \mu$  is defined for any finite measure when  $d \geq 3$ , and  $\text{Pot} \mu$  is defined for any finite measure with compact support when  $d=2$ . In these cases  $\text{Pot} \mu$  is finite except on a polar set.

Let  $\mathbf{M}_2$  denote the space of finite measures  $\mu$  on  $\mathbb{R}^d$  such that  $\text{Pot}^+ \mu$  is bounded and continuous on  $\mathbb{R}^d$  and  $\text{Pot}^- \mu$  is finite on  $\mathbb{R}^d$ . It is well known that  $\text{Pot} \mu$  is continuous if and only if  $\text{Pot}^- \mu$  is finite on  $\mathbb{R}^d$  and

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in B} \int_{\{|x-y| < \varepsilon\}} k^+(|x-y|) \mu(dy) = 0$$

for every bounded set  $B$  in  $\mathbb{R}^d$  (see, for instance, [1]). It follows easily that if  $\mu \in \mathbf{M}_2$  and  $\nu$  is a measure with  $0 \leq \nu \leq \mu$ , then  $\nu \in \mathbf{M}_2$ .

We may strengthen Proposition 2.4 somewhat:

**Proposition 2.5** For every  $\mu \in \mathbf{M}_0$  there exists a measure  $\psi \in \mathbf{M}_2$ , and a nonnegative Borel function  $h: \mathbb{R}^d \rightarrow [0, \infty]$ , such that  $\mu \sim h\psi$  in the sense of Definition 2.4.

**Proof** By Proposition 2.4 there exists a nonnegative Radon measure  $\nu \in H^{-1}(\mathbb{R}^d)$ , and a nonnegative Borel function  $q: \mathbb{R}^d \rightarrow [0, \infty]$ , such that  $\mu \sim q\nu$ . Replacing  $\nu$  by  $q_1\nu$ , where  $q_1$  is an appropriate small positive Borel function, we may assume that  $\nu$  is finite and  $\text{Pot}^- \nu$  is finite on  $\mathbb{R}^d$ . Since  $\nu$  is finite,  $\text{Pot}^+ \nu$  is finite except on at most a set of capacity zero. Thus  $\text{Pot} \nu < \infty$ ,  $\nu$ -a.e. It follows from [12], I.V.9, that there exists a sequence  $\nu_n \in \mathbf{M}_2$  with  $\nu = \sum_{n=1}^{\infty} \nu_n$ . By choosing positive constants  $c_n$  sufficiently small, we have  $\psi = \sum_{n=1}^{\infty} c_n \nu_n$  such that  $\psi \in \mathbf{M}_2$ . Since  $\nu \ll \psi$ , the proposition follows.

Measures in  $\mathbf{M}_2$  will play an important role in what follows. The next proposition gives one useful property of these measures.

**Proposition 2.6** Let  $\mu \in \mathbf{M}_2$ , let  $D$  be open in  $\mathbb{R}^d$ ,  $u \in H^1_{loc}(D)$ ,  $u$   $\mu$ -harmonic on  $D$ . Then there exists  $w$ , continuous on  $D$ , such that  $u=w$  quasi everywhere.

**Proof** Let  $x \in D$ . Let  $B$  be an open ball containing  $x$  with compact closure in  $D$ , with diameter less than 1. Let  $w^\pm$  denote the solution of the  $\mu$ -Dirichlet problem on  $B$  with boundary data  $u^\pm$ . Let  $v^\pm$  denote the solution of the ordinary Dirichlet problem on  $B$  with boundary data  $u^\pm$ .  $w^\pm, v^\pm$  exist by Proposition 2.1. Clearly  $w^\pm - v^\pm \in H^1_0(B)$ . Hence  $w^\pm = v^\pm$  on  $\partial B$ , so by Proposition 2.3  $w^\pm \leq v^\pm$  q.e. on  $B$ . By uniqueness for the  $\mu$ -Dirichlet problem (Proposition 2.1),  $u = w^+ - w^-$  on  $B$ . Hence  $|u| \leq v^+ + v^-$

q.e. on  $B$ , and hence  $u$  is locally bounded on  $D$

Since  $\Delta u = \mu u$  in distribution sense in  $B$  (see [8], Proposition 3.8), we have

$$u = -\text{Pot}^+(u^+ |_{B\mu}) + \text{Pot}^+(u^- |_{B\mu}) + h$$

where  $h$  is harmonic in  $B$ . Since  $u^+$  and  $u^-$  are bounded in  $B$ , the measures  $u^+ |_{B\mu}$  and  $u^- |_{B\mu}$  belong to  $\mathbf{M}_2$ , therefore the corresponding potentials are continuous, hence  $u$  is continuous on  $B$ .

**Proposition 2.7** Let  $D$  be a bounded open set in  $\mathbb{R}^d$ ,  $\mu_n, \mu \in \mathbf{M}_0$  with  $\mu_n \uparrow \mu$  as  $n \rightarrow \infty$ ,  $g \in H^1(D) \cap L^2(D, \mu)$ ,  $g \geq 0$ ,  $f \in L^2(D)$ ,  $f \geq 0$ . Let  $u_n, u$  denote the solutions, in the sense of Definition 2.1, of

$$-\Delta u + \mu_n u = f \text{ in } D, u_n = g \text{ on } \partial D, \text{ and}$$

$$-\Delta u + \mu u = f \text{ in } D, u = g \text{ on } \partial D.$$

Then  $u_n \downarrow u$  quasi everywhere as  $n \rightarrow \infty$ , and  $\|u_n - u\|_{H^1(D)} \rightarrow 0$  as  $n \rightarrow \infty$ .

**Proof** By Proposition 2.3,  $u_n \downarrow$  and  $u_n \geq u \geq 0$  for all  $n$ . Let  $w = \lim_{n \rightarrow \infty} u_n$ . By (2.12), (2.13), and Proposition 2.1, for every  $n \geq m$  we have

$$(2.17) \quad F_{\mu}^D(u) - 2 \int_D f u dx \geq F_{\mu_n}^D(u) - 2 \int_D f u dx \geq F_{\mu_n}^D(u_n) - 2 \int_D f u_n dx \\ \geq F_{\mu_m}^D(u_n) - 2 \int_D f u_n dx.$$

$$\text{Hence } \int_D |\nabla u|^2 dx + \int_D u^2 d\mu + 2 \int_D f u dx \geq \int_D |\nabla u_n|^2 dx.$$

Thus  $\|u_n\|_{H^1(D)}$  is bounded, so  $u_n \rightarrow w$  weakly in  $H^1(D)$ , and also  $u_n - g \rightarrow w - g$  weakly in  $H^1(D)$ . Since  $u_n - g \in H_0^1(D)$ ,  $w - g \in H_0^1(D)$ .

By the lower semicontinuity of  $F_{\mu_m}^D$  we obtain from (2.17)

$$F_{\mu_m}^D(w) - 2 \int_D f w dx \leq \liminf_{n \rightarrow \infty} [F_{\mu_m}^D(u_n) - 2 \int_D f u_n dx] \leq \\ \leq \liminf_{n \rightarrow \infty} [F_{\mu_n}^D(u_n) - 2 \int_D f u_n dx] \leq \limsup_{n \rightarrow \infty} [F_{\mu_n}^D(u_n) - 2 \int_D f u_n dx] \leq \\ \leq F_{\mu}^D(u) - 2 \int_D f u dx.$$

By taking the limit as  $m \rightarrow \infty$  we obtain

$$(2.18) \quad F_{\mu}^D(w) - 2 \int_D f w dx \leq F_{\mu}^D(u) - 2 \int_D f u dx.$$

By Proposition 2.1  $u$  is the unique minimum point of the functional

$$F_{\mu}^D(v) - 2 \int_D f v dx$$

on the set  $\{v: v \in H^1(D), v - g \in H_0^1(D)\}$ . Since  $w - g \in H_0^1(D)$ , from (2.18) we obtain  $w = u$  and

$$(2.19) \quad F_{\mu}^D(u) = \lim_{n \rightarrow \infty} F_{\mu_n}^D(u_n).$$

Since  $\mu_n \uparrow \mu$  and  $u_n \downarrow u$  we have

$$\int_D u^2 d\mu \leq \lim_{n \rightarrow \infty} \int_D u_n^2 d\mu_n.$$

Therefore from (2.19) we obtain

$$(2.20) \quad \int_D |\nabla u|^2 dx = \lim_{n \rightarrow \infty} \int_D |\nabla u_n|^2 dx$$

Since  $(u_n)$  converges to  $u$  weakly in  $H^1(D)$  and  $u_n - g \in H_0^1(D)$ , (2.20) implies that  $(u_n)$  converges to  $u$  strongly in  $H^1(D)$ , so Proposition 2.7 is proved.

We now introduce a variational convergence for sequences  $(\mu_n)$  in  $\mathbf{M}_0$ , using the notion of  $\Gamma$ -convergence for functionals in the calculus of variations (see [10] and also [2], where such convergence is called epi-convergence).  $\Gamma$ -convergence in turn is based on the abstract Kuratowski convergence, which can be formulated in an arbitrary metric space  $X$  as follows:

**Definition 2.6** Let  $(F_n)$  be a sequence of functions from  $X$  into  $\bar{\mathbf{R}}$ , and let  $F$  be a function from  $X$  into  $\bar{\mathbf{R}}$ . We say that  $(F_n)$   $\Gamma$ -converges to  $F$  in  $X$  if the following conditions are satisfied:

(a) for every  $u \in X$  and for every sequence  $(u_n)$  converging to  $u$  in  $X$

$$F(u) \leq \liminf_{n \rightarrow \infty} F_n(u_n);$$

(b) for every  $u \in X$  there exists a sequence  $(u_n)$  converging to  $u$  in  $X$  such that

$$F(u) \geq \limsup_{n \rightarrow \infty} F_n(u_n).$$

We now define the  $\mathcal{X}$ -convergence of a sequence  $(\mu_n)$  in  $\mathbf{M}_0$  in terms of the  $\Gamma$ -convergence of the corresponding functionals  $F_{\mu_n}$  defined on the space  $L^2(\mathbf{R}^d)$  by the equations (2.12) and (2.13) with  $D = \mathbf{R}^d$ .

**Definition 2.7** We say that a sequence  $(\mu_n)$  in  $\mathbf{M}_0$   $\mathcal{X}$ -converges to the measure  $\mu \in \mathbf{M}_0$  if the sequence of functionals  $(F_{\mu_n})$   $\Gamma$ -converges to the functional  $F_\mu$  in  $L^2(\mathbb{R}^d)$  as in Definition 2.5.

The following proposition shows that our definition of  $\mathcal{X}$ -convergence is equivalent to Definition 4.8 of [8].

**Proposition 2.8** Let  $(\mu_n)$  be a sequence in  $\mathbf{M}_0$  and let  $\mu \in \mathbf{M}_0$ . The following conditions are equivalent:

- (a)  $(\mu_n)$   $\mathcal{X}$ -converges to  $\mu$ ;
- (b)  $(F_{\mu_n}^D)$   $\Gamma$ -converges to  $F_\mu^D$  in  $L^2(D)$  for every open set  $D$  in  $\mathbb{R}^d$ ;
- (c)  $(F_{\mu_n}^D)$   $\Gamma$ -converges to  $F_\mu^D$  in  $L^2(D)$  for every bounded open set  $D$  in  $\mathbb{R}^d$ .

**Proof** For every open set  $D$  in  $\mathbb{R}^d$  we consider the following functionals on  $L^2(D)$ :

$$(2.21) \quad F_+^D(u) = \inf\{\limsup_{n \rightarrow \infty} F_{\mu_n}^D(u_n) : u_n \rightarrow u \text{ in } L^2(D)\},$$

$$(2.22) \quad F_-^D(u) = \inf\{\liminf_{n \rightarrow \infty} F_{\mu_n}^D(u_n) : u_n \rightarrow u \text{ in } L^2(D)\}.$$

If  $D = \mathbb{R}^d$ , we denote the corresponding functionals by  $F_+$  and  $F_-$ .

It is easy to see (by a diagonal argument) that the infima in (2.21) and (2.22) are achieved by suitable sequences and that  $F_+^D$  and  $F_-^D$  are lower semicontinuous on  $L^2(D)$  (see [10], Proposition 1.8). Moreover,  $F_\mu^D$  is the  $\Gamma$ -limit in  $L^2(D)$  of the sequence  $(F_{\mu_n}^D)$  if and only if

$$F_+^D = F_-^D \text{ and both equal } F_\mu^D \text{ on } L^2(D).$$

Since  $F_-^D \leq F_+^D$  on  $L^2(D)$  and  $F_-^D(u) = F_+^D(u) = F_\mu^D(u)$  if  $u$  is not in  $H_0^1(D)$ , it follows that the  $\Gamma$ -convergence of  $F_{\mu_n}^D$  to  $F_\mu^D$  is equivalent to the inequalities

$$F_+^D \leq F_\mu^D \leq F_-^D \text{ on } H_0^1(D).$$

Let us prove that (a) implies (b). Assume (a), which is equivalent to  $F_\mu = F_- = F_+$  on  $L^2(\mathbb{R}^d)$ .

Let  $D$  be open in  $\mathbb{R}^d$ . Let us prove that

$$(2.23) \quad F_\mu^D \leq F_-^D \text{ on } H_0^1(D).$$

Let  $u \in H_0^1(D)$  with  $F_-^D(u) < \infty$ . By (2.22) there exists a sequence  $(u_n)$  converging to  $u$  in  $L^2(D)$  such that

$$F_{-}^D(u) = \liminf_{n \rightarrow \infty} F_{\mu_n}^D(u_n).$$

We may assume  $u_n \in H_0^1(D)$  for every  $n$ . If we extend  $u_n$  to  $\mathbb{R}^d$  by putting  $u_n = 0$  outside  $D$ , we have that  $u_n \in H^1(\mathbb{R}^d)$  and  $F_{\mu_n}^D(u_n) = F_{\mu_n}(u_n)$ , therefore

$$F_{-}^D(u) = \liminf_{n \rightarrow \infty} F_{\mu_n}(u_n) \geq F_{-}(u) = F_{\mu}(u)$$

and (2.23) is proved.

Let us prove that

$$(2.24) \quad F_{+}^D \leq F_{\mu}^D \text{ on } H_0^1(D).$$

Let  $u \in H_0^1(D)$  with compact support in  $d$  and such that  $F_{\mu}^D(u) < \infty$ . We extend  $u$  to  $\mathbb{R}^d$  by putting  $u = 0$  outside  $D$ , so that  $u \in H^1(\mathbb{R}^d)$  and  $F_{+}(u) = F_{\mu}(u) = F_{\mu}^D(u)$ .

By (2.21) there exists a sequence  $(u_n)$  converging to  $u$  in  $L^2(\mathbb{R}^d)$  such that

$$F_{\mu}^D(u) = F_{+}(u) = \limsup_{n \rightarrow \infty} F_{\mu_n}(u_n) < \infty.$$

Therefore  $u_n \in H^1(\mathbb{R}^d)$  for  $n$  large enough and

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^d} |\nabla u_n|^2 dx \leq \limsup_{n \rightarrow \infty} F_{\mu_n}(u_n) < \infty,$$

so that  $(u_n)$  converges to  $u$  weakly in  $H^1(\mathbb{R}^d)$ .

Let  $\varphi \in C_0^\infty(D)$  with  $0 \leq \varphi \leq 1$  and  $\varphi = 1$  on the support of  $u$ . Then  $\varphi u_n \in H_0^1(D)$  and, by Rellich's theorem, the sequence  $(\varphi u_n)$  converges to  $u$  in  $L^2(D)$ , hence

$$\begin{aligned} F_{+}^D(u) &\leq \limsup_{n \rightarrow \infty} F_{\mu_n}^D(\varphi u_n) = \\ &= \limsup_{n \rightarrow \infty} \left\{ \int_D [\varphi^2 |\nabla u_n|^2 + 2\varphi u_n \nabla \varphi \nabla u_n + u_n^2 |\nabla \varphi|^2] dx + \int_D \varphi^2 u_n^2 d\mu_n \right\} \leq \\ &\leq \limsup_{n \rightarrow \infty} F_{\mu_n}(u_n) + \int_D [2\varphi u \nabla \varphi \nabla u + u^2 |\nabla \varphi|^2] dx = F_{\mu}(u). \end{aligned}$$

Therefore (2.24) is proved when  $u$  has compact support on  $D$ . In the general case  $u \in H_0^1(D)$ , there exists a sequence  $(v_n)$  of functions in  $H_0^1(D)$  with compact support in  $D$  which converges to  $u$  strongly in  $H_0^1(D)$  and such that the sequence  $(v_n^2)$  is increasing and converges to  $u^2$  pointwise quasi everywhere on  $D$ . By the monotone convergence theorem and by the lower semicontinuity of  $F_{+}^D$ , we obtain

$$F_{+}^D(u) \leq \liminf_{n \rightarrow \infty} F_{+}^D(v_n) \leq \lim_{n \rightarrow \infty} F_{\mu}^D(v_n) = F_{\mu}^D(u),$$

which proves (2.24).

Condition (b) follows now from (2.23) and (2.24).

The implication (b) $\Rightarrow$ (c) is trivial.

Let us prove that (c) implies (a). Assume (c). We prove that

$$(2.25) \quad F_\mu \leq F_- \text{ on } H^1(\mathbb{R}^d).$$

Let  $u \in H^1(\mathbb{R}^d)$  with  $F_-(u) < \infty$ . By (2.22) there exists a sequence  $(u_n)$  converging to  $u$  in  $L^2(\mathbb{R}^d)$  such that

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^d} |\nabla u_n|^2 dx \leq \liminf_{n \rightarrow \infty} F_{\mu_n}(u_n) = F_-(u) < +\infty.$$

We may assume that  $u_n \in H^1(\mathbb{R}^d)$  and that  $(u_n)$  converges to  $u$  weakly in  $H^1(\mathbb{R}^d)$ .

Let  $\psi$  be a nonincreasing function on  $\mathbb{R}_+$  of class  $C^\infty$  such that  $0 \leq \psi \leq 1$ ,  $\psi(t) = 1$  for  $0 \leq t \leq 1$ , and  $\psi(t) = 0$  for  $t \geq 2$ . For every  $k \in \mathbb{N}$ , let  $\phi_k \in C^\infty_0(\mathbb{R}^d)$  be defined by

$$(2.26) \quad \phi_k(x) = \psi(|x|/k).$$

Let us fix  $k \in \mathbb{N}$  and a bounded open set  $D$  containing the support of  $\phi_k$ , so that  $(F^D_{\mu_n})$   $\Gamma$ -converges to  $F^D_\mu$  in  $L^2(D)$ . Then  $(\phi_k u_n)$  converges to  $\phi_k u$  in  $L^2(D)$  as  $n \rightarrow \infty$ , hence

$$\begin{aligned} F_\mu(\phi_k u) &= F^D_\mu(\phi_k u) \leq \liminf_{n \rightarrow \infty} F^D_{\mu_n}(\phi_k u_n) = \\ &= \liminf_{n \rightarrow \infty} \left\{ \int_D [\phi_k^2 |\nabla u_n|^2 + 2\phi_k u_n \nabla \phi_k \nabla u_n + u_n^2 |\nabla \phi_k|^2] dx + \right. \\ &\quad \left. \int_D \phi_k^2 u_n^2 d\mu_n \right\} \leq \liminf_{n \rightarrow \infty} F_{\mu_n}(u_n) + \int_{\mathbb{R}^d} [2\phi_k u \nabla \phi_k \nabla u + u^2 |\nabla \phi_k|^2] dx. \end{aligned}$$

Since  $(\nabla \phi_k)$  converges to 0 uniformly on  $\mathbb{R}^d$  and  $(\phi_k u)$  converges to  $u$  in  $H^1(\mathbb{R}^d)$ , by the lower semicontinuity of  $F_\mu$  we obtain

$$F_\mu(u) \leq \liminf_{k \rightarrow \infty} F_\mu(\phi_k u) \leq \liminf_{n \rightarrow \infty} F_{\mu_n}(u_n) = F_-(u),$$

which proves (2.25).

Let us prove that

$$(2.27) \quad F_+ \leq F_\mu \text{ on } H^1(\mathbb{R}^d).$$

Let  $u \in H^1(\mathbb{R}^d)$  with  $F_\mu(u) < \infty$  and let  $(\phi_k)$  be the sequence in  $C^\infty_0(\mathbb{R}^d)$  defined in the previous step of the proof.

Let us fix  $k \in \mathbb{N}$  and a bounded open set  $D$  containing the support of  $\phi_k$  so that  $(F^D_{\mu_n})$   $\Gamma$ -converges to  $F^D_\mu$  in  $L^2(D)$ . Since  $\phi_k u \in H^1_0(\mathbb{R}^d)$ , there exists a sequence  $(u_n)$  converging to  $\phi_k u$  in  $L^2(D)$  such that

$$F_\mu(\varphi_k u) = F_\mu^D(\varphi_k u) = \lim_{n \rightarrow \infty} F_{\mu_n}^D(u_n) < \infty.$$

Then  $u_n \in H^1_0(D)$  for  $n$  large enough. If we extend  $u_n$  to  $\mathbb{R}^d$  by putting  $u_n = 0$  outside  $D$  we obtain that  $u_n \in H^1(\mathbb{R}^d)$  and  $(u_n)$  converges to  $\varphi_k u$  in  $L^2(\mathbb{R}^d)$ , hence

$$F_+(\varphi_k u) \leq \limsup_{n \rightarrow \infty} F_{\mu_n}(u_n) = \lim_{n \rightarrow \infty} F_{\mu_n}^D(u_n) = F_\mu(\varphi_k u).$$

Since  $(\varphi_k u)$  converges to  $u$  strongly in  $H^1(\mathbb{R}^d)$  and  $(\varphi_k^2 u^2)$  is increasing and converges to  $u^2$  quasi everywhere in  $\mathbb{R}^d$ , by the monotone convergence theorem and by the lower semicontinuity of  $F_+$  we obtain

$$F_+(u) \leq \liminf_{k \rightarrow \infty} F_+(\varphi_k u) = \lim_{k \rightarrow \infty} F_\mu(\varphi_k u) = F_\mu(u),$$

which proves (2.27).

Condition (a) now follows from (2.25) and (2.27).

Let us mention some general properties of  $\mathcal{X}$ -convergence as established in [8].

$\mathcal{X}$ -convergence in  $\mathbf{M}_0$  is metrizable ([8], Prop. 4.9), and  $\mathbf{M}_0$  is compact under  $\mathcal{X}$  ([8], Theorem 4.14).

The  $\mathcal{X}$ -convergence of a sequence  $\mu_n$  to  $\mu$  in  $\mathbf{M}_0$  can be characterized in variational terms ([8], Proposition 4.10), as convergence for every bounded open set  $D \subset \mathbb{R}^d$  and every  $f \in L^2(D)$  of the minimum values  $m_n$  to  $m$ , where

$$m_n \equiv \min_{v \in H^1_0(D)} \left[ F_{\mu_n}^D(v) + \int_D f v dx \right]$$

$$m \equiv \min_{v \in H^1_0(D)} \left[ F_\mu^D(v) + \int_D f v dx \right].$$

However, more conveniently for our present purposes,  $\Gamma$ -convergence (hence  $\mathcal{X}$ -convergence) can be characterized in terms of strong convergence of the resolvents in  $L^2(D)$ , as we shall see in the next proposition.

With each  $\mu \in \mathbf{M}_0$ , each open set  $D$  in  $\mathbb{R}^d$ , and each  $\lambda > 0$  we associate the Moreau-Yosida approximation  $(F_\mu^D)_\lambda$  of  $F_\mu^D$ , defined for every  $f \in L^2(D)$  by

$$(2.28) \quad (F_\mu^D)_\lambda(f) \equiv \min_{v \in H^1_0(D)} \left[ F_\mu^D(v) + \lambda \int_D (v-f)^2 v dx \right].$$

**Proposition 2.9** Let  $(\mu_n)$  be a sequence in  $\mathbf{M}_0$ , let  $\mu \in \mathbf{M}_0$ , let  $D$  be open in  $\mathbb{R}^d$ , and let  $\lambda > 0$ . The following conditions are equivalent:

- (a) the functionals  $F_{\mu_n}^D$   $\Gamma$ -converge to  $F_{\mu}^D$  in  $L^2(D)$  as  $n \rightarrow \infty$ ;  
 (b) the resolvent operators  $R_{\lambda}^D(\mu_n)$  converge strongly to  $R_{\lambda}^D(\mu)$  in  $L^2(D)$  as  $n \rightarrow \infty$ ;  
 (c) the Moreau-Yosida approximations  $(F_{\mu_n}^D)_{\lambda}$  converge pointwise to  $(F_{\mu}^D)_{\lambda}$  in  $L^2(D)$  as  $n \rightarrow \infty$ .

To prove Proposition 2.9 we need the following

**Lemma 2.2** Suppose that

$$(2.29) \quad F_{\mu}^D(u) \leq \liminf_{n \rightarrow \infty} F_{\mu_n}^D(u_n)$$

for every  $u \in L^2(D)$  and every sequence  $(u_n)$  converging to  $u$  strongly in  $L^2(D)$ . Then (2.29) holds for every  $u \in L^2(D)$  and every sequence  $(u_n)$  converging to  $u$  weakly in  $L^2(D)$ .

**Proof** Let  $u \in L^2(D)$  and let  $(u_n)$  be a sequence converging to  $u$  weakly in  $L^2(D)$  such that

$$\liminf_{n \rightarrow \infty} F_{\mu_n}^D(u_n) < \infty.$$

Then  $u_n \in H_0^1(D)$  for infinitely many  $n$  and

$$\liminf_{n \rightarrow \infty} \int_D |\nabla u_n|^2 dx < \infty,$$

so we may assume that  $(u_n)$  converges to  $u$  weakly in  $H_0^1(D)$ . Let  $(\phi_k)$  be the sequence in  $C_0^{\infty}(R^d)$  defined by (2.26). Let us fix  $k \in \mathbb{N}$ . Then  $\phi_k u_n \in H_0^1(D)$  and  $(\phi_k u_n)$  converges to  $\phi_k u$  weakly in  $H_0^1(D)$  and, by Rellich's theorem, strongly in  $L^2(D)$ . Therefore, by the hypothesis,

$$\begin{aligned} F_{\mu}^D(\phi_k u) &\leq \liminf_{n \rightarrow \infty} F_{\mu_n}^D(\phi_k u_n) = \\ &= \liminf_{n \rightarrow \infty} \left\{ \int_D [\phi_k^2 |\nabla u_n|^2 + 2\phi_k u_n \nabla \phi_k \nabla u_n + u_n^2 |\nabla \phi_k|^2] dx + \right. \\ &\quad \left. \int_D \phi_k^2 u_n^2 d\mu_n \right\} \leq \liminf_{n \rightarrow \infty} F_{\mu_n}^D(u_n) + \int_D [2\phi_k u \nabla \phi_k \nabla u + u^2 |\nabla \phi_k|^2] dx. \end{aligned}$$

In the last inequality we have used the fact that, by Rellich's theorem,  $(u_n)$  converges to  $u$  in  $L^2(D_k)$  where  $D_k = \{x : x \in D, \nabla \phi_k(x) \neq 0\}$ . Since  $(\nabla \phi_k)$  converges to 0 uniformly on  $R^d$  and  $(\nabla \phi_k u)$  converges to  $u$  in  $H_0^1(D)$ , we then obtain

$$F_{\mu}^D(u) \leq \liminf_{n \rightarrow \infty} F_{\mu}^D(\phi_k u) \leq \liminf_{n \rightarrow \infty} F_{\mu_n}^D(u_n),$$

and the lemma follows.

We will now proceed with the

**Proof of Proposition 2.9** By Lemma 2.2 the equivalence of (a), (b), and (c) could be obtained as a consequence of an abstract result ([2], Theorem 3.26 with  $X=L^2(D)$ ). However we prefer to give a more direct proof here.

Let  $F_n = F_{\mu_n}^D$ ,  $F = F_{\mu}^D$  and, correspondingly,  $(F_n)_\lambda = (F_{\mu_n}^D)_\lambda$ ,  $(F)_\lambda = (F_{\mu}^D)_\lambda$ .  
(a) $\Rightarrow$ (b). Assume (a). Let  $f \in L^2(D)$ ,  $u_n = \lambda R_{\lambda}^D(\mu_n)f$ , and  $u = \lambda R_{\lambda}^D(\mu)f$ . We have to prove that  $(u_n)$  converges to  $u$  strongly in  $L^2(D)$ .

By the definition of  $\Gamma$ -convergence there exists a sequence  $(v_n)$  converging to  $u$  in  $L^2(D)$  such that

$$F(u) = \lim_{n \rightarrow \infty} F_n(v_n).$$

By the minimum property of  $u_n$  (Proposition 2.1) we have

$$F_n(u_n) + \lambda \int_D (u_n - f)^2 dx \leq F_n(v_n) + \lambda \int_D (v_n - f)^2 dx,$$

hence

$$(2.30) \quad \limsup_{n \rightarrow \infty} [F_n(u_n) + \lambda \int_D (u_n - f)^2 dx] \leq \\ \leq \lim_{n \rightarrow \infty} [F_n(v_n) + \lambda \int_D (v_n - f)^2 dx] = F(u) + \lambda \int_D (u - f)^2 dx.$$

Since

$$F_n(u_n) + \lambda \int_D (u_n - f)^2 dx \geq \int_D |\nabla u_n|^2 dx + (\lambda/2) \int_D u_n^2 dx - \lambda \int_D f^2 dx,$$

the sequence  $(u_n)$  is bounded in  $H^1_0(D)$ , hence it contains a subsequence which converges weakly in  $H^1_0(D)$  to a function  $w \in H^1_0(D)$ . We will relabel this subsequence as  $(u_n)$  again. By Lemma 2.1, we have

$$(2.31) \quad F(w) \leq \liminf_{n \rightarrow \infty} F_n(u_n).$$

Moreover,

$$(2.32) \quad \int_D (w - f)^2 dx \leq \liminf_{n \rightarrow \infty} \int_D (u_n - f)^2 dx.$$

Hence, by using (2.30), we obtain

$$F(w) + \lambda \int_D (w - f)^2 dx \leq \liminf_{n \rightarrow \infty} [F_n(u_n) + \lambda \int_D (u_n - f)^2 dx] \leq \\ \leq \limsup_{n \rightarrow \infty} [F_n(u_n) + \lambda \int_D (u_n - f)^2 dx] \leq F(u) + \lambda \int_D (u - f)^2 dx.$$

Since  $u$  is the unique minimum point of problem (2.28) (see

Proposition 2.1), we obtain  $w=u$  and

$$(2.33) \quad F(u) + \lambda \int_D (u-f)^2 dx = \lim_{n \rightarrow \infty} [F_n(u_n) + \lambda \int_D (u_n-f)^2 dx].$$

From (2.31), (2.32), (2.33) and from the equality  $u=w$  it follows that

$$\int_D (u-f)^2 dx = \lim_{n \rightarrow \infty} \int_D (u_n-f)^2 dx,$$

and hence that  $(u_n)$  converges to  $u=w$  strongly in  $L^2(D)$ . The limit being independent of the subsequence, the whole original sequence  $(u_n)$  converges to  $u$  in  $L^2(D)$  and (b) is proved.

(b) $\Rightarrow$ (c) Assume (b). Let  $f \in L^2(D)$ ,  $u_n = \lambda R_\lambda^D(\mu_n)f$ , and  $u = \lambda R_\lambda^D(\mu)f$ . By taking the test function  $u=u_n$  in the equation satisfied by  $u_n$ , we obtain

$$F_n(u_n) + \lambda \int_D u_n^2 dx = \lambda \int_D f u_n dx.$$

Since  $u_n$  is the minimum point of problem (2.28) we have

$$(F_n)_\lambda f = F_n(u_n) + \lambda \int_D (u_n-f)^2 dx = \lambda \int_D f(f-u_n) dx.$$

In the same sense we obtain

$$(F)_\lambda f = \lambda \int_D f(f-u) dx.$$

By (b), the sequence  $(u_n)$  converges to  $u$  in  $L^2(D)$ , hence  $(F_n)_\lambda f$  converges to  $(F)_\lambda f$  as  $n \rightarrow \infty$ , and (c) is proved.

(c) $\Rightarrow$ (a) Assume (c). We set

$$G_n(v) = F_n(v) + \lambda \int_D v^2 dx,$$

$$G(v) = F(v) + \lambda \int_D v^2 dx.$$

By condition (c) we have

$$(2.34) \quad \min_{v \in H_0^1(D)} [G(v) + \int_D f v dx] = \lim_{n \rightarrow \infty} \min_{v \in H_0^1(D)} [G_n(v) + \int_D f v dx]$$

for every  $f \in L^2(D)$ .

Let us consider the following functionals on  $L^2(D)$ :

$$F_+(u) = \inf \{ \limsup_{n \rightarrow \infty} F_n(u_n) : u_n \rightarrow u \text{ in } L^2(D) \},$$

$$F_-(u) = \inf \{ \liminf_{n \rightarrow \infty} F_n(u_n) : u_n \rightarrow u \text{ in } L^2(D) \}.$$

In order to prove that  $(F_n)$   $\Gamma$ -converges to  $F$  in  $L^2(D)$ , it is enough to show that

$$(2.35) \quad F \leq F_- \text{ on } L^2(D),$$

$$(2.36) \quad F_+ \leq F \text{ on } H^1_0(D) \cap L^2(D, \mu).$$

Let us prove (2.35). Let  $u \in L^2(D)$ . Since  $G$  is convex and lower semicontinuous on  $L^2(D)$  we have (see for example [3], Proposition 4.1)

$$(2.37) \quad G(u) = \sup_{f \in L^2(D)} \inf_{v \in L^2(D)} [G(v) + \int_D f(v-u) dx].$$

If  $(u_n)$  converges to  $u$  in  $L^2(D)$ , by (2.34) we have for every  $f \in L^2(D)$

$$\begin{aligned} & \min_{v \in H^1_0(D)} [G(v) + \int_D f(v-u) dx] = \\ & = \lim_{n \rightarrow \infty} \min_{v \in H^1_0(D)} [G_n(v) + \int_D f(v-u_n) dx] \leq \liminf_{n \rightarrow \infty} G_n(u_n). \end{aligned}$$

By taking the supremum with respect to  $f \in L^2(D)$  in the previous inequality and by using (2.37) we obtain

$$G(u) \leq \liminf_{n \rightarrow \infty} G_n(u_n).$$

Hence

$$F(u) \leq \liminf_{n \rightarrow \infty} F_n(u_n).$$

Since  $(u_n)$  is an arbitrary sequence converging to  $u$  in  $L^2(D)$  we obtain  $F(u) \leq F_-(u)$  and (2.35) is proved.

Let us prove (2.36). Let  $f \in L^2(D)$ ,  $u_n = -(1/2)R^D_{\lambda}(\mu_n)f$ , and  $u = -(1/2)R^D_{\lambda}(\mu)f$ . By Proposition 2.1 we have

$$\begin{aligned} G(u) + \int_D f u dx &= \min_{v \in H^1_0(D)} [G(v) + \int_D f v dx], \\ G_n(u_n) + \int_D f u_n dx &= \min_{v \in H^1_0(D)} [G_n(v) + \int_D f v dx]. \end{aligned}$$

Therefore by (2.34) we have

$$(2.38) \quad \lim_{n \rightarrow \infty} [G_n(u_n) + \int_D f u_n dx] = G(u) + \int_D f u dx < \infty.$$

Since

$$G_n(u_n) + \int_D f u_n dx \geq \int_D |\nabla u_n|^2 dx + (\lambda/2) \int_D u_n^2 dx - (1/2\lambda) \int_D f^2 dx,$$

the sequence  $(u_n)$  is bounded in  $H^1_0(D)$ , hence it contains a

subsequence, which by relabelling we may still denote by  $(u_n)$ , which converges weakly in  $H^1_0(D)$  to a function  $w \in H^1_0(D)$ . By the inequality  $F \leq F_-$  and Lemma 2.2 we have

$$(2.39) \quad F(w) \leq \liminf_{n \rightarrow \infty} F_n(u_n).$$

Since

$$(2.40) \quad \int_D w^2 dx \leq \liminf_{n \rightarrow \infty} \int_D u_n^2 dx,$$

we have

$$\begin{aligned} G(w) + \int_D f w dx &\leq \lim_{n \rightarrow \infty} [G_n(u_n) + \int_D f u_n dx] = G(u) + \int_D f u dx = \\ &= \min_{v \in H^1_0(D)} [G_n(v) + \int_D f v dx]. \end{aligned}$$

By the uniqueness of the minimum point we have  $w = u$ , hence  $(u_n)$  converges to  $u$  weakly in  $H^1_0(D)$ . By (2.38), (2.39), (2.40) we have

$$\int_D u^2 dx = \lim_{n \rightarrow \infty} \int_D u_n^2 dx,$$

and hence  $(u_n)$  converges to  $u$  strongly in  $L^2(D)$ . It follows from this argument that the entire original sequence  $(u_n)$  must converge strongly to  $u$  in  $L^2(D)$  and we obtain easily from (2.38) that

$$F_+(u) \leq F(u)$$

for every  $u \in H^1_0(D) \cap L^2(D, \mu)$  of the form  $u = R^D_\lambda(\mu)f$  with  $f \in L^2(D)$ .

Since the range of  $R^D_\lambda(\mu)$  is dense in  $H^1_0(D) \cap L^2(D, \mu)$ , for every  $u \in H^1_0(D) \cap L^2(D, \mu)$  there exists a sequence  $(f_n)$  in  $L^2(D)$  such that the functions  $v_n = R^D_\lambda(\mu)f_n$  converge to  $u$  in  $H^1_0(D)$  and in  $L^2(D)$ . By the lower semicontinuity of  $F_+$ , we obtain

$$F_+(u) \leq \liminf_{n \rightarrow \infty} F_+(v_n) \leq \lim_{n \rightarrow \infty} F(v_n) = F(u)$$

and inequality (2.36) is proved.

Condition (a) follows now from (2.35) and (2.36), and Proposition 2.9 is proved.

We conclude the present section by stating the following theorem which follows immediately from Propositions 2.8 and 2.9.

**Theorem 2.1** Let  $(\mu_n)$  be a sequence in  $\mathbf{M}_0$ , let  $\mu \in \mathbf{M}_0$ , and let  $\lambda > 0$ . The following conditions are equivalent:

- (a)  $(\mu_n)$   $\mathfrak{X}$ -converges to  $\mu$ ;
- (b) the resolvent operators  $R_\lambda(\mu_n)$  converge to  $R_\lambda(\mu)$  strongly in  $L^2(\mathbb{R}^d)$  as  $n \rightarrow \infty$ ;
- (c) the resolvent operators  $R_\lambda^D(\mu_n)$  converge to  $R_\lambda^D(\mu)$  strongly in  $L^2(D)$  as  $n \rightarrow \infty$  for every open set  $D$  in  $\mathbb{R}^d$ ;
- (d) the resolvent operators  $R_\lambda^D(\mu_n)$  converge to  $R_\lambda^D(\mu)$  strongly in  $L^2(D)$  as  $n \rightarrow \infty$  for every bounded open set  $D$  in  $\mathbb{R}^d$ .

This result will be convenient for connecting  $\mathfrak{X}$ -convergence with the probabilistic notion of stable convergence to be defined in the next section.

3. We shall use the concept of a randomized stopping time for Brownian motion. Some notations and results will be taken from [4], section 2.

Let  $C = \mathcal{C}([0, \infty), \mathbb{R}^d)$ , the space of continuous  $\mathbb{R}^d$ -valued functions of nonnegative time, endowed with the topology of uniform convergence on compact time sets.  $C$  is the sample space for standard Brownian motion  $(B_t)$ , where  $B_t: C \rightarrow \mathbb{R}^d$  is the projection map defined by  $B_t(\omega) = \omega(t)$ ,  $\omega$  denoting a typical point or "sample path" in  $C$ . The relevant  $\sigma$ -algebras on  $C$  are  $\mathcal{F}_t = \sigma(B_s: 0 \leq s \leq t)$ , and  $\mathcal{G}_t = \mathcal{F}_{t+}$ . We let  $\mathcal{G} = \mathcal{G}_\infty = \mathcal{F}_\infty = \sigma(B_s: 0 \leq s < \infty)$ .

A **stopping time**  $\tau$  with respect to the fields  $(\mathcal{G}_t)$  is defined as usual to be a map  $\tau: C \rightarrow [0, \infty]$ , such that  $\{\tau \leq t\}$  is in  $\mathcal{G}_t$  for all  $t$ ,  $0 \leq t < \infty$ . A **randomized stopping time**  $T$  is defined to be a map  $T: C \times [0, 1] \rightarrow [0, \infty]$ , such that  $T$  is a stopping time with respect to the  $\sigma$ -algebras  $(\mathcal{G}_t \times \mathcal{B}_1)$ , where  $\mathcal{B}_1$  denotes the Borel sets on  $[0, 1]$ . We shall require  $T(\omega, \cdot)$  to be nondecreasing and left continuous on  $[0, 1]$ , with  $T(\omega, 0) = 0$ , for every  $\omega$  in  $C$ . When convenient we shall regard an ordinary stopping time  $\tau$  also as a randomized one, by setting  $\tau(\omega, t) = \tau(\omega)$  for all  $t$  in  $[0, 1]$ . If  $T$  is a randomized stopping time, then  $T(\cdot, a)$  is an ordinary stopping time, for each  $a$  in  $[0, 1]$ .

A randomized stopping time  $T$  can be expressed by an equivalent object, the **stopping time measure**  $F$  induced by  $T$ .  $F$  is the map  $F: C \times \mathcal{B} \rightarrow [0, 1]$ , where  $\mathcal{B}$  = the Borel sets of  $[0, \infty]$ , defined by

$$(3.1) \quad F(\omega, [0, t]) = \sup \{ a: T(\omega, a) \leq t \}$$

and the condition that  $F(\omega, \cdot)$  be a measure on  $\mathcal{B}$ .

We shall often write  $F(\cdot, (t, \infty))$  as  $F((t, \infty))$  or as  $F_t$ . If  $P$  denotes a probability measure on  $(C, \mathcal{G})$ , and  $m_1$  denotes Lebesgue measure on  $[0, 1]$ , then  $F_t$  is a version of the conditional expectation of  $\chi_{\{T > t\}}$  using the probability  $P \times m_1$ , with respect to the  $\sigma$ -algebra  $\mathcal{G} \times \{\emptyset, [0, 1]\}$ .  $F(\omega, \cdot)$  is thus a version of the conditional distribution of  $T$  given the entire path  $\omega$ .

We can recover  $T$  from  $F$  by

$$(3.2) \quad T(\omega, a) = \inf \{ t: F(\omega, [0, t]) \geq a \}.$$

Furthermore, given any map  $F: C \times \mathcal{B} \rightarrow [0, 1]$  such that

$$(3.3) \quad F(\omega, \cdot) \text{ is a probability for each } \omega \text{ in } C,$$

we can define  $T$  by (3.2).  $T$  will be a randomized stopping time, provided

that

$$(3.4) \quad F(\cdot, [0, t]) \text{ is } \mathcal{G}_t\text{-measurable for each } t.$$

Any  $F : C \times \mathcal{B} \rightarrow [0, 1]$  such that (3.3) and (3.4) hold will be called a stopping time measure. We see that there is a complete correspondence between the notions of stopping time measure and randomized stopping time.

There is another useful representation for a randomized stopping time  $T$ . Let  $P$  be a probability on  $(C, \mathcal{G})$ . Consider the map  $(\omega, a) \rightarrow (\omega, T(\omega, a))$  from  $C \times [0, 1]$  to  $C \times [0, \infty]$ . The distribution of this map on  $(C \times [0, \infty], \mathcal{G} \times \mathcal{B})$  with respect to  $P \times m_1$  on  $(C \times [0, 1], \mathcal{G} \times \mathcal{B}_1)$  will be called the **stopped process** corresponding to  $T$  and  $P$ . If  $F$  denotes the stopping time measure corresponding to  $T$ , and  $Q$  denotes the stopped process corresponding to  $T$  and  $P$ , we see easily that for any bounded  $\mathcal{G} \times \mathcal{B}$  measurable function  $Z$  on  $C \times [0, \infty]$ ,

$$(3.5) \quad \int Z dQ = \int \left[ \int Z(\omega, t) F(\omega, dt) \right] P(d\omega).$$

We shall often denote the stopped process  $Q$  by  $P \times F$ .

It can be shown ([3],[4]) that any probability  $Q$  on  $(C \times [0, \infty], \mathcal{G} \times \mathcal{B})$  is a stopped process  $P \times F$  for some  $P$  and some  $F$ , if and only if: for every function  $f : [0, \infty] \rightarrow \mathbb{R}$  continuous and having support in  $[0, t]$ , and every  $Y : C \rightarrow \mathbb{R}$  of form  $Y(\omega) = g_1(\omega(t_1)) \dots g_k(\omega(t_k))$ , where  $g_i$  is continuous and has compact support on  $\mathbb{R}^d$ ,  $0 \leq t_i < \infty$ ,  $i = 1, \dots, k$ , the following holds:

$$(3.6) \quad \int Y f dQ = \int E(Y | \mathcal{G}_t) f dQ.$$

Here  $E(Y | \mathcal{G}_t)$  means the conditional expectation using the marginal of  $Q$  on  $(C, \mathcal{G})$ , and we regard  $Y$  and  $E(Y | \mathcal{G}_t)$  as functions both on  $C$  and on  $C \times [0, \infty]$  in the obvious way, and regard  $f$  similarly as a function on  $[0, \infty]$  and on  $C \times [0, \infty]$ .

It is easy to show, by a monotone class argument, that if (3.6) holds for the choices of  $Y$  and  $f$  just described, then it also holds for any  $Y$  in  $L^1(C, \mathcal{G}, P)$  and any  $f$  bounded measurable on  $[0, \infty]$  with  $f=0$  on  $(t, \infty)$ .

Till now we have discussed arbitrary probabilities  $P$  on  $(C, \mathcal{G})$ . Since our interest is in Brownian motion, we now consider  $P^\nu$ , the usual Wiener measure on  $(C, \mathcal{G})$  with initial probability distribution  $\nu$  on  $\mathbb{R}^d$ , that is

$$(3.7) \quad P^\nu(B_0 \in A) = \nu(A)$$

for any Borel set  $A$  in  $\mathbb{R}^d$ . We will refer to such a probability measure  $P = P^\nu$  as a **Brownian** probability on  $C$ .

The usefulness of characterizing a stopped process by (3.6) with the

original limited choices of  $Y$  and  $f$ , is that, for Brownian probabilities, this relation is preserved under weak limits. Hence we have the fact that (cf. [3] Lemma 2.9, [4] Lemma 2.2):

**Lemma 3.1** A weak limit of stopped processes  $P_n \times F_n$  is again a stopped process if (i)  $P_n = P$  for all  $n$ , or if (ii)  $P_n$  is Brownian for all  $n$ .

Since  $[0, \infty]$  is compact, a sequence  $Q_n = P_n \times F_n$  will be tight if and only if the sequence  $P_n$  is tight. Here by tight we mean as usual that for any  $\epsilon > 0$  there exists a compact set  $A$  in the sample space such that  $Q_n(A^c) < \epsilon$  for all  $n$ . From a tight sequence of measures a weakly convergent subsequence can be selected. Thus we have a convenient compactness principle available for stopped Brownian processes, and hence for randomized stopping times.

We denote the usual heat semigroup by  $P_t$ , where  $P_t$  acts both on measures and functions as a Markov operator, so that the distribution of  $B_t$  under  $P^\nu$  is  $\nu P_t$ , and  $P_t h(x) = E^x[h \circ B_t]$  for any bounded Borel  $h$  on  $\mathbb{R}^d$ .

**Definition 3.1** A sequence  $T_n$  of randomized stopping times will be said to converge **stably** to a limit  $T$ , **with respect to a probability measure**  $P$  on  $(C, \mathcal{B})$ , if  $T_n$  has stopping measure  $F_n$ ,  $T$  has stopping measure  $F$ , and  $P \times F_n$  converges weakly to  $P \times F$ . If  $P^\nu \times F_n$  converges weakly to  $P^\nu \times F$ , for one (and hence for all) probability measures  $\nu$  such that  $\nu \ll m$ ,  $m \ll \nu$ , where  $m$  denotes Lebesgue measure on  $\mathbb{R}^d$ , then we will simply say that  $T_n$  converges **stably** to  $T$ . Since the stopping time measures  $F_n, F$ , characterize  $T_n, T$ , we will also speak of  $F_n$  as converging stably to  $F$ .

The fact that  $P^\nu \times F_n \rightarrow P^\nu \times F$  weakly for one  $\nu$  with  $\nu \ll m$ ,  $m \ll \nu$  implies that  $P^\lambda \times F_n \rightarrow P^\lambda \times F$  for every  $\lambda$  with  $\lambda \ll m$  is clear from the following lemma, and the fact that  $P^\lambda \ll P^\nu$ :

**Lemma 3.2** Let  $F_n, F$  be stopping time measures, and let  $P$  be a probability measure on  $(C, \mathcal{B})$  such that  $P \times F_n$  converges to  $P \times F$  weakly. Then for every  $Y$  in  $L^1(C, \mathcal{B}, P)$ , and every continuous  $f$  on  $[0, \infty]$ ,

$$(3.8) \quad \int Y(\omega) \int f(t) F_n(\omega, dt) P(d\omega) \rightarrow \int Y(\omega) \int f(t) F(\omega, dt) P(d\omega).$$

Lemma 3.2 follows at once from the triangle inequality on  $L^1$ , and is a special case of Lemma 3.3 below.

A statement will be said to hold **almost surely** (a.s.) on  $C$  if it

holds  $P^x$ -a.e. for every  $x$  in  $R^d$ . If  $T_n \rightarrow T$  stably, as defined above, we see easily that although  $T$  is not uniquely determined by  $T_n$ ,  $T \circ \theta_t$  is uniquely determined almost surely for every  $t > 0$ , where  $\theta_t$  denotes the usual shift operator.

We now prove some convergence facts for later use.

**Lemma 3.3** Let  $F_n, F$  be stopping time measures, and let  $P$  be a probability measure on  $(C, \mathcal{G})$  such that  $P \times F_n \rightarrow P \times F$  weakly. Let  $\mathcal{L}$  denote the space of all functions  $Z: C \times [0, \infty] \rightarrow R$ , such that  $Z(\omega, \cdot)$  is continuous on  $[0, \infty]$  for  $P$ -a.e.  $\omega$  in  $C$ , such that  $Z_t = Z(\cdot, t)$  is  $\mathcal{G}$ -measurable for all  $t$ , and such that

$$(3.9) \quad \int \sup\{|Z_t|: 0 \leq t \leq \infty\} dP < \infty.$$

Then for every  $Z$  in  $\mathcal{L}$ , as  $n \rightarrow \infty$ ,

$$(3.10) \quad \int Z d(P \times F_n) \rightarrow \int Z d(P \times F).$$

**Proof** When  $Z$  is of the form  $Z(\omega, t) = Y(\omega)f(t)$ , where  $Y$  is bounded and continuous on  $C$  and  $f$  is continuous on  $[0, \infty]$ , (3.10) is true by the definition of weak convergence. Continuous functions  $Y$  on  $C$  are dense in  $L^1(C, \mathcal{G}, P)$ , so, as noted in Lemma 3.2, (3.10) remains true for  $Z(\omega, t) = Y(\omega)f(t)$ , where  $Y$  is in  $L^1(C, \mathcal{G}, P)$  and  $f$  is continuous on  $[0, \infty]$ . Hence (3.10) is true when  $Z$  is a finite linear combination of functions of this type.

$$\text{For any } Z \text{ in } \mathcal{L}, \text{ define } \|Z\| = \int \sup\{|Z_t|: 0 \leq t \leq \infty\} dP.$$

It is easy to see that the collection  $\mathcal{H}$  of  $Z$  in  $\mathcal{L}$  for which (3.10) is true is closed with respect to the semi-norm just defined.

Let  $Z$  be in  $\mathcal{L}$ ,  $Z$  bounded. Let  $\epsilon > 0$  be given. We can choose an integer  $k > 0$  and a partition  $0 = t_0 < t_1 < \dots < t_k = \infty$  such that if  $W: C \times [0, \infty] \rightarrow R$  is defined by  $W(\omega, t) = [kZ(\omega, t)]/k$  for all  $t = t_0, t_1, \dots, t_{k-1}$ ,  $W(\omega, \cdot)$  linear on each interval  $[t_{j-1}, t_j]$ ,  $j = 1, \dots, k-1$ ,  $W(\omega, \cdot)$  constant on  $[t_{k-1}, t_k]$ , then  $\|W - Z\| < \epsilon$ .

$W(\omega, \cdot)$  is determined by the values  $W(\omega, t_j)$ ,  $j = 0, \dots, k-1$ , and, since  $Z$  is bounded, there are only finitely many possible choices for these values.  $W$  is thus a finite linear combination of product functions of the sort described earlier. Thus  $W$  is in  $\mathcal{H}$ , and hence we have shown that any bounded function  $Z$  in  $\mathcal{L}$  is in  $\mathcal{H}$ .

Any  $Z$  in  $\mathcal{L}$  can be approximated in the semi-norm sense by its bounded truncations, so  $\mathcal{H} = \mathcal{L}$ , and Lemma 3.3 is proved.

We may write equation (3.10) more conveniently as

$$(3.10)' \quad E[Z_{T_n}] \rightarrow [Z_T].$$

Here we follow a **convention** which we will at times use later, of employing the same letter  $E$  for expectation on the randomized space  $C \times [0,1]$ , that is, with respect to  $P \times m_1$ , as we employ for expectation on  $C$  with respect to  $P$ .

**Corollary to Lemma 3.3** Let  $T_n, T$  be randomized stopping times,  $P$  a probability measure on  $(C, \mathcal{B})$ .

(i) Let  $\sigma$  be a  $\mathcal{G}_t$ -stopping time. If  $T_n \rightarrow T$  stably with respect to  $P$  then  $T_n \wedge \sigma \rightarrow T \wedge \sigma$  stably with respect to  $P$ .

(ii) If there exists  $\tau_j$  a sequence of  $\mathcal{G}_t$ -stopping times such that  $\tau_j \uparrow \infty$  and  $T_n \wedge \tau_j \rightarrow T \wedge \tau_j$  stably with respect to  $P$  for each  $j$ , then  $T_n \rightarrow T$  stably with respect to  $P$  for each  $j$ .

**Proof (i)** Given  $f \in \mathcal{C}([0, \infty))$ ,  $Y$  bounded measurable, let  $Z_t(\omega) = f(t \wedge \sigma(\omega))Y(\omega)$ . Since  $Z_{T_n} = f(T_n \wedge \sigma(\omega))Y(\omega)$ , and  $Z_T = f(T \wedge \sigma(\omega))Y(\omega)$ , the result follows at once from (3.10)'.

(ii) Given  $f \in \mathcal{C}([0, \infty))$ ,  $Y$  bounded measurable, for every  $\epsilon > 0$  there exists  $j$  such that  $E[\sup_{t \geq \tau_j} |f(t) - f(\tau_j)|] \leq \epsilon$ . Then

$$|E[f(T_n)Y] - E[f(T_n \wedge \tau_j)Y]| \leq \epsilon \|Y\|_\infty \text{ and } |E[f(T)Y] - E[f(T \wedge \tau_j)Y]| \leq \epsilon \|Y\|_\infty.$$

Since  $E[f(T_n \wedge \tau_j)Y] \rightarrow E[f(T \wedge \tau_j)Y]$ , the corollary follows.

**Lemma 3.4** Let  $F_n, F$  be stopping time measures, and let  $P$  be a probability measure on  $(C, \mathcal{B})$  such that  $P \times F_n \rightarrow P \times F$  weakly. Let  $Z: C \times [0, \infty) \rightarrow R$  be given. We will write  $Z(\cdot, t) = Z_t$  where convenient. Suppose  $Z$  is measurable from  $\mathcal{G} \times \mathcal{B}$  to  $\mathcal{B}'$ , where  $\mathcal{B}$  denotes the Borel sets of  $[0, \infty)$  and  $\mathcal{B}'$  denotes the Borel sets of  $R$ .

(i) Let  $W: C \times [0, \infty) \rightarrow R$  be defined by

$$(3.12) \quad W(\omega, t) = \sup\{ \inf\{Z(\omega, s) : s \in U\} \}$$

where the supremum is over all open subsets  $U$  of  $[0, \infty)$  containing  $t$ . That is,  $W(\omega, t) = \liminf_{s \rightarrow t} Z(\omega, s)$ . (One sees easily that  $W(\omega, \cdot)$  is lower semicontinuous.  $W(\omega, \cdot)$  is the lower semicontinuous regularization of  $Z(\omega, \cdot)$ .)

Let

$$(3.13) \quad D(\omega) = \{t : W(\omega, t) < Z(\omega, t)\}.$$

Suppose that for  $P$ -a.e.  $\omega$ ,  $F(\omega, D(\omega)) = 0$ . Suppose also that  $Z$  is

bounded below. Then

$$(3.14) \quad \liminf_{n \rightarrow \infty} \int Z d(P \times F_n) \geq \int Z d(P \times F).$$

(ii) Let

$$(3.15) \quad L(\omega) = \{t: Z(\omega, \cdot) \text{ is discontinuous at } t\}.$$

Suppose that for P-a.e.  $\omega$ ,  $F(\omega, L(\omega))=0$ . Suppose also that Z is bounded. Then

$$(3.16) \quad \lim_{n \rightarrow \infty} \int Z d(P \times F_n) = \int Z d(P \times F).$$

**Proof** Let  $a$  be in  $\mathbb{R}$ . Let  $H = Z^{-1}((a, \infty))$ . That is,  $H = \{(\omega, t): Z(\omega, t) > a\}$ .

Let  $f_j$  be a sequence of functions in  $\mathcal{C}([0, \infty], \mathbb{R})$ ,  $0 \leq f_j \leq 1$ , such that for any open set  $U$  contained in  $[0, \infty]$ ,  $\chi_U = \sup\{f_j: \text{support } f_j \subset U\}$ .

For any set  $K \subset \mathbb{C} \times [0, \infty]$ , let  $K_\omega = \{t: (\omega, t) \in K\}$ . If  $K \in \mathcal{G} \times \mathcal{B}$ , and  $G$  is any stopping time measure,  $G(\omega, K_\omega)$  is a  $\mathcal{G}$ -measurable function of  $\omega$ , by the monotone class theorem.

Let  $G = \sum_{n=1}^{\infty} F_n / 2^n$ . Let  $V_j = \{\omega: G(\omega, \text{support } f_j - H_\omega) = 0\}$ .  $V_j$  is  $\mathcal{G}$ -measurable, by the previous remark.

Let  $J_\omega = \{j: \omega \in V_j\}$ . Let  $Y_k(\omega, t) = \sup\{f_j(t): j \leq k \text{ and } j \in J_\omega\}$ .  $Y_k(\omega, t)$  is the supremum of  $\chi_{V_j}(\omega) f_j(t)$ ,  $1 \leq j \leq k$ , so  $Y_k$  is  $\mathcal{G} \times \mathcal{B}$ -measurable.

For each  $k$ , by Lemma 3.2,

$$(3.17) \quad \lim_{n \rightarrow \infty} \int Y_k d(P \times F_n) = \int Y_k d(P \times F).$$

For each  $k$ ,

$$\begin{aligned} \{(\omega, t): Y_k(\omega, t) > \chi_H(\omega, t)\} &\subset \{(\omega, t): \exists j \in J_\omega \text{ with } f_j(t) > 0, \chi_H(\omega, t) = 0\} \\ &\subset \{(\omega, t): \exists j \text{ with } \omega \in V_j \text{ and } t \in (\text{support } f_j - H_\omega)\}. \end{aligned}$$

$$\text{Let } N_k = \{(\omega, t): Y_k(\omega, t) > \chi_H(\omega, t)\}, M_j = \{(\omega, t): \omega \in V_j \text{ and } t \in (\text{support } f_j - H_\omega)\}.$$

We have shown that  $N_k \subset \bigcup_{j=1}^{\infty} M_j$ . Thus  $P \times F_n(N_k) \leq \sum_{j=1}^{\infty} P \times F_n(M_j)$ .

$$P \times F_n(M_j) = \int F_n(\omega, (M_j)_\omega) P(d\omega) = \int_{V_j} F_n(\omega, \text{support } f_j - H_\omega) P(d\omega) = 0.$$

Thus  $P \times F_n(N_k) = 0$  for every  $k$  and every  $n$ . Hence  $Y_k \leq \chi_H$ ,  $P \times F_n$ -a.e., for every  $n, k$ . Thus, by (3.17), for every  $k$ ,

$$(3.18) \quad \liminf_{n \rightarrow \infty} P \times F_n(H) \geq \int Y_k d(P \times F).$$

$Y_k$  increases to a limit  $Y$ , where  $Y(\omega, t) = \sup\{f_j(t): j \in J_\omega\}$ . Clearly

$$(3.19) \quad \liminf_{n \rightarrow \infty} P \times F_n(H) \geq \int Y d(P \times F).$$

For fixed  $\omega$  in  $C$ , let  $t$  be in  $D^C(\omega) \cap H_\omega$ . Then  $W(\omega, t) \geq Z(\omega, t) > a$ . Hence there exists an open set  $U$  in  $[0, \infty]$  with  $t$  in  $U$  such that  $Z(\omega, s) > a$  for all  $s$  in  $U$ . Thus  $U \subset H_\omega$ . For any  $j$  such that  $\text{support } f_j \subset U$ , clearly  $j \in J_\omega$ . Hence  $Y(\omega, t) \geq \sup\{f_j(t) : \text{support } f_j \subset U\}$ , so  $Y(\omega, t) = 1$ .

Let  $S = \{(\omega, t) : Y(\omega, t) < \chi_H(\omega, t)\}$ . We have shown that  $S_\omega \subset D(\omega)$ . Hence

$P \times F(S) = \int F(\omega, S_\omega) P(d\omega) = 0$ , since  $F(\omega, D(\omega)) = 0$  except on a  $P$ -null set. Thus  $Y \geq \chi_H$ ,  $P \times F$ -a.e., and so

$$(3.20) \quad \liminf_{n \rightarrow \infty} P \times F_n(H) \geq P \times F(H).$$

Let  $\psi_n, \psi$  denote the distributions of  $Z$  with respect to  $P \times F_n$  and  $P \times F$ , respectively. Assume  $Z$  is bounded below by  $b$ . Then

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int Z d(P \times F_n) &= \liminf_{n \rightarrow \infty} (b + \int_{(b, \infty)} \psi_n((a, \infty)) da) \\ &\geq b + \int_{(0, \infty)} \psi((a, \infty)) da = \int Z d(P \times F). \end{aligned}$$

This proves part (i) of Lemma 3.4. Part (ii) follows from part (i) applied to  $Z$  and to  $-Z$ , so Lemma 3.4 is proved.

**Corollary** When the function  $Z$  of Lemma 3.4 is such that  $Z$  is bounded below(above) and  $Z(\omega, \cdot)$  is l.s.c.(u.s.c.)  $P$ -a.e., then

$$(3.21) \quad \liminf_{n \rightarrow \infty} (\limsup_{n \rightarrow \infty}) \int Z d(P \times F_n) \geq (\leq) \int Z d(P \times F).$$

**Theorem 3.1** Let  $T_n$  and  $T$  be randomized stopping times, with corresponding stopping time measures  $F_n$  and  $F$ , respectively. Let  $\nu$  be a probability measure on  $\mathbb{R}^d$ , and  $P = P^\nu$ , the Wiener measure with initial distribution  $\nu$ . Suppose  $P \times F_n \rightarrow P \times F$  weakly. Let  $K$  be a closed set in  $\mathbb{R}^d$ , and let  $\tau$  denote the first entrance time of  $K$ . Suppose that  $P \times m_1(T = \tau) = 0$ ,

that is, that  $\int F(\omega, \{\tau(\omega)\}) P(d\omega) = 0$ . Let  $Y \in L^1(C, \mathcal{G}, P)$  with  $Y = 0$  on  $\{\tau = \infty\}$ . Define signed measures  $\psi_n$  and  $\psi$  by the equations

$$(3.20) \quad \int h d\psi_n = \int h \circ B_{\tau_n} F_n((\tau, \infty)) Y dP, \quad \int h d\psi = \int h \circ B_\tau F((\tau, \infty)) Y dP,$$

for every bounded Borel function  $h$  on  $\mathbb{R}^d$ .

Then  $\psi_n \rightarrow \psi$  in total variation norm as  $n \rightarrow \infty$ .

**Proof** Clearly  $\|\psi_n\| \leq \|Y\|_1$ ,  $\|\psi\| \leq \|Y\|_1$ . Thus the collection of  $Y$ 's for which the theorem is true is closed in  $L^1(C, \mathcal{G}, P)$ . Hence it is enough to prove the theorem when  $Y = 0$  on  $\{\tau > \zeta\}$ , where  $\zeta =$  the first entrance time of

the complement of some large ball. Thus we may assume that  $K$  contains the complement of some ball, and hence that  $\tau < \infty$  almost surely.

Since  $\int_{\{\tau=0\}} F(\{0\}) dP = 0$ , by the corollary to lemma 3.4 we have  $\limsup_{n \rightarrow \infty} \int F_n(\{0\}) dP = 0$ . Clearly  $\|\psi_n - \psi\| \leq \int |F_n((\tau, \infty)) - F((\tau, \infty))| |Y| dP$ , so if  $Y=0$  on  $\{\tau > 0\}$  we see easily that Theorem 3.1 holds. Thus we may, by decomposing  $Y$  into two parts, assume that  $Y=0$  on  $\{\tau=0\}$ . Then, discarding part of  $\nu$ , we may assume  $P^\nu(\tau=0) = 0$  also. Thus we assume  $\tau > 0$ ,  $P$ -a.e..

Let  $W_k$  be open sets in  $R^d$ ,  $W_k \downarrow K$ . Let  $\tau_k$  denote the first entrance time of  $W_k$ . Then  $\tau_k < \tau$  on  $\{\tau > 0\}$ , and  $\tau_k \uparrow \tau$  everywhere. Standard arguments show that  $\mathcal{G}$  is generated by the two  $\sigma$ -algebras  $\mathcal{K}_\tau$  and  $\theta_\tau^{-1}(\mathcal{G})$ , where  $\mathcal{K}_\tau = \sigma(\tau, B_t \wedge \tau: 0 \leq t < \infty)$ . Using again the fact that the set of  $Y$ 's for which the theorem is true is closed in  $L^1$ -norm, we see that it is enough to prove Theorem 3.1 when  $Y$  is of the form  $U(V \circ \theta_\tau)$ , where  $U$  is bounded and  $\mathcal{G}_{\tau_k}$ -measurable for some  $k$ , and  $V$  is bounded and  $\mathcal{G}$ -measurable. We assume  $|U| \leq 1$ ,  $|V| \leq 1$  without loss of generality, and consider fixed  $U, V, k$ . Let  $M$  denote the collection of bounded Borel functions  $h$  on  $R^d$ , having sup norm  $\leq 1$ , and having support in a fixed compact subset  $S$  of  $R^d$ . In order to prove Theorem 3.1 it is clearly sufficient to prove that

$$(3.21) \quad \int h \circ B_\tau F_n((\tau, \infty)) Y dP \rightarrow \int h \circ B_\tau F((\tau, \infty)) Y dP$$

as  $n \rightarrow \infty$ , uniformly over all  $h$  in  $M$ .

Suppose not. We will show a contradiction. There exists some  $\delta > 0$ , and a sequence  $h_{n_i}$  in  $M$ , such that

$$(3.22) \quad \left| \int h_{n_i} \circ B_\tau F_{n_i}((\tau, \infty)) Y dP - \int h_{n_i} \circ B_\tau F((\tau, \infty)) Y dP \right| > \delta \text{ for all } i.$$

Passing to a subsequence and relabelling, we may assume that  $h_n$  converges to a limit  $h$  in the weak\*-topology on  $L^\infty(R^d, \mathcal{B}''; \lambda)$ , where  $\mathcal{B}''$  denotes the Borel sets in  $R^d$ , and  $\lambda$  denotes the distribution of  $B_\tau$  with respect to  $P^\mu$ , where  $\mu$  is chosen so that  $m \ll \mu$ . We note that if  $\lambda^x$  denotes the distribution of  $B_\tau$  with respect to  $P^x$ , then  $\lambda^x \ll \lambda$  for every  $x$  in  $K^c$ .

Let  $\epsilon > 0$  be given. Choose  $j \geq k$  such that  $P^{x_{n_j}}(\tau_j \leq T \leq \tau) < \epsilon/2$ , that is,

$\int F([\tau_j, \tau]) dP < \epsilon/2$ . By the Corollary to Lemma 3.4, there exists  $n_0$  such that for every  $n \geq n_0$ ,

$$(3.23) \quad \int F([\tau_j, \tau]) dP < \epsilon/2.$$

Then, for  $n \geq n_0$ ,

$$(3.24) \quad \left| \int h_n \circ B_{\tau} F_n((\tau, \infty)) Y dP - \int h_n \circ B_{\tau} F_n((\tau_j, \infty)) Y dP \right| < \epsilon/2.$$

$$\begin{aligned} & \left| \int h_n \circ B_{\tau} F_n((\tau_j, \infty)) Y dP - \int h \circ B_{\tau} F_n((\tau_j, \infty)) Y dP \right| = \\ & \left| \int (h_n - h) \circ B_{\tau} F_n((\tau_j, \infty)) U(V \circ \theta_{\tau}) dP \right| = \\ & \left| \int E[(h_n - h) \circ B_{\tau} F_n((\tau_j, \infty)) U(V \circ \theta_{\tau}) | \mathfrak{G}_{\tau}] dP \right| = \\ & \left| \int U F_n((\tau_j, \infty)) ((h_n - h)g) \circ B_{\tau} dP \right|, \text{ where } g(x) = E^*[V], \text{ so that } g \text{ is Borel on } R^d \\ & \text{and } |g| \leq 1 \text{ everywhere on } R^d. \text{ Thus} \end{aligned}$$

$$\begin{aligned} & \left| \int h_n \circ B_{\tau} F_n((\tau_j, \infty)) Y dP - \int h \circ B_{\tau} F_n((\tau_j, \infty)) Y dP \right| = \\ & \left| \int E[U F_n((\tau_j, \infty)) ((h_n - h)g) \circ B_{\tau} | \mathfrak{G}_{\tau_j}] dP \right| = \\ & \left| \int U F_n((\tau_j, \infty)) E^{B_{\tau_j}} [((h_n - h)g) \circ B_{\tau}] dP \right| \leq \int |E^{B_{\tau_j}} [((h_n - h)g) \circ B_{\tau}]| dP. \end{aligned}$$

For each  $x$  in  $K^C$ ,  $h_n - h \rightarrow 0$  in the weak\*-topology on  $L^1(R^d, \mathfrak{B}''', \lambda^*)$ .

Thus  $E^{B_{\tau_j}} [((h_n - h)g) \circ B_{\tau}] \rightarrow 0$ , P-a.e.. This shows that

$$(3.25) \quad \left| \int h_n \circ B_{\tau} F_n((\tau_j, \infty)) Y dP - \int h \circ B_{\tau} F_n((\tau_j, \infty)) Y dP \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By (3.23), for every  $n \geq n_0$ ,

$$(3.26) \quad \left| \int h \circ B_{\tau} F_n((\tau_j, \infty)) Y dP - \int h \circ B_{\tau} F_n((\tau, \infty)) Y dP \right| < \epsilon/2.$$

By (3.24), (3.25), and (3.26),

$$(3.27) \quad \left| \int h_n \circ B_{\tau} F_n((\tau, \infty)) Y dP - \int h \circ B_{\tau} F_n((\tau, \infty)) Y dP \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By the same argument (applied to the constant sequence (F)),

$$(3.28) \quad \left| \int h_n \circ B_{\tau} F((\tau, \infty)) Y dP - \int h \circ B_{\tau} F((\tau, \infty)) Y dP \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By lemma 3.4(ii), with  $Z(\omega, t) = h \circ B_{\tau}(\omega)(\omega) Y(\omega) \chi_{(\tau(\omega), \infty)}(t)$ , we have

$$(3.29) \quad \left| \int h \circ B_{\tau} F_n((\tau, \infty)) Y dP - \int h \circ B_{\tau} F((\tau, \infty)) Y dP \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(3.27), (3.28), and (3.29) contradict (3.22), so theorem 3.1 is proved.

**Remark 3.1** The conclusion of Theorem 3.1 is also true if  $\tau$  is any stopping time of form  $\sigma+u$ , where  $\sigma$  is any stopping time and  $u>0$ . This result was proved in [4] for the case  $\sigma=0$ . The proof is a simpler version of the one just given. It seems possible that this result could be proved in a more general form.

Now that we have obtained some properties of weak convergence of general randomized stopping times, we turn to the more restricted class of randomized stopping times that we shall actually work with, namely the multiplicative functionals.

**Definition 3.2** A stopping time measure  $M$  will be called a multiplicative functional if for every  $s \geq 0$ ,  $t \geq 0$ ,

$$(3.30) \quad M_{t+s} = M_t(M_s \circ \theta_t) \text{ almost surely.}$$

We recall that almost surely means  $P^x$ -a.e. for every  $x$  in  $R^d$ , and that  $\theta_t$  is the usual shift operator, so that  $(\theta_t \omega)(s) = \omega(s+t)$ .

We shall assume, unless otherwise stated, that any multiplicative functional  $M$  also has the following continuity property: for every  $t > 0$ , and every sequence  $\varepsilon_k$  of positive real numbers with  $\varepsilon_k \downarrow 0$ ,

$$(3.31) \quad \lim_{k \rightarrow \infty} M_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} = M_t \text{ almost surely.}$$

If  $\tau$  is a first hitting time or a first entrance time, and  $M$  is the associated multiplicative functional, then  $M_t = \chi_{(t, \infty]} \circ \tau$  gives the stopping time measure associated with  $\tau$ , and  $M$  satisfies (3.30).  $M$  also satisfies (3.31) if  $\tau$  is a first hitting time but not if  $\tau$  is a first entrance time. For a general treatment of multiplicative functionals see [6].

Associated with any multiplicative functional  $M$  there is a sub-Markov semigroup  $Q_t(M)$ , defined by

$$(3.32) \quad (Q_t(M) h)(x) = E^x[h \circ B_t M_t], \text{ for every bounded Borel function } h.$$

As usual, we also define  $\nu Q_t(M)$  for any finite measure  $\nu$  by the equation  $\int h d(\nu Q_t(M)) = \int Q_t(M) h d\nu$ .

Although we will refer to  $Q_t(M)$  as a semigroup, it should be stressed that  $Q_0(M)$  will not in general be the identity operator.

If  $P_t$  is the usual heat semigroup associated with Brownian motion, clearly  $P_t = Q_t(1)$ . Also, one shows easily from 3.31 that

$$(3.33) \quad Q_t(M) h = \lim_{\varepsilon \downarrow 0} P_\varepsilon Q_{t-\varepsilon}(M) h$$

Furthermore, if  $h \geq 0$ , the limit in (3.33) is decreasing. It follows that for any  $h \geq 0$  the function  $Q_t(M) h$  is upper semicontinuous.

The following fact is sometimes useful:

**Lemma 3.5** Let  $M$  be a stopping time measure having the multiplicative property (3.30). For any  $t > 0$ ,  $M(\{t\}) = 0$  almost surely.

**Proof** Let  $\lambda$  be a probability on  $\mathbb{R}^d$ ,  $m \ll \lambda$ . There exists  $s$ ,  $0 < s < t$ , such that  $\int M(\{s\}) dP^\lambda = 0$ . Then  $M(\{s\}) = 0$ ,  $P^\lambda$ -a.e., and hence  $M(\{s\}) = 0$ ,  $P^\nu$ -a.e., for any probability  $\nu \ll m$ . For any  $x$  in  $\mathbb{R}^d$ ,  $M(\{t\}) = M_{t-s}(M(\{s\})) \circ \theta_{t-s}$ ,  $P^x$ -a.e.. Since  $P^x \theta_{t-s}^{-1} = P^{\lambda^x}$ , where  $\lambda^x \ll m$ , we see that  $(M(\{s\})) \circ \theta_{t-s} = 0$ ,  $P^x$ -a.e., proving Lemma 3.5.

The semigroup  $Q_t(M)$  characterizes  $M$  uniquely. More generally:

**Lemma 3.6** Let  $M$  and  $N$  be any two stopping time measures having the multiplicative property (3.30). Define the sub-Markov semigroups  $Q_t(M)$  and  $Q_t(N)$  as in (3.32). If  $Q_t(M) = Q_t(N)$  for all  $t$  then  $M = N$  almost surely.

**Proof.** Fix  $t \geq 0$ . Let  $s = t/k$  for some  $k > 0$ , and let  $t_j = js$ ,  $j = 0, 1, \dots, k$ .

$M_t = \prod_{j=0}^{k-1} (M_s \circ \theta_{t_j})$ , almost surely. Let  $f_0, \dots, f_k$  be bounded Borel functions on  $\mathbb{R}^d$ . A straightforward induction shows that for any  $x$  in  $\mathbb{R}^d$ ,  $E^x[M_t f_0 \circ B_{t_0} \dots f_k \circ B_{t_k}] = f_0 Q_s(M)(f_1 Q_s(M) \dots (Q_s(M) f_k) \dots)(x)$ .

Thus  $E^x[M_t | B_{t_0}, \dots, B_{t_k}] = E^x[N_t | B_{t_0}, \dots, B_{t_k}]$ . It follows that  $M_t = N_t$ ,  $P^x$ -a.e.. By right continuity,  $M_t = N_t$  for all  $t$ ,  $P^x$ -a.e., so lemma 3.6 is proved.

We now wish to consider in what sense a limit of multiplicative functionals is again a multiplicative functional. The definition of convergence we wish to use is that given in Definition 3.1, stable convergence of stopping time measures. As noted after Definition 3.1, this definition of convergence does not specify the limit uniquely. However, we now show:

**Theorem 3.2** Let  $F^n, F$  be stopping time measures such that  $F^n \rightarrow F$  stably and such that for every  $n$  and every  $s \geq 0, t \geq 0$

$$(3.34) \quad F^n_{t+s} = F^n_t(F^n_s \circ \theta_t), \quad P^\lambda\text{-a.e.}, \quad \text{where } \lambda \text{ is a fixed probability}$$

measure with  $m \ll \lambda$ ,  $\lambda \ll m$ . There exists a multiplicative functional  $M$  (satisfying (3.30) and (3.31)) such that  $F^n \rightarrow M$  stably.  $M$  is unique a.s. .

**Proof** We wish to prove first that for every  $s \geq 0, t > 0$ ,

$$(3.35) \quad F_{t+s} = F_t(F_s \circ \theta_t), \quad P^\lambda\text{-a.e.}$$

By right continuity it is enough to prove (3.35) when  $s$  is such that

$\int F(\{t+s\})dP^\lambda=0$  and  $\int F(\{s\})dP^\lambda=0$ . (3.35) is true if for every  $Y$  in  $L^1(C, \mathfrak{G}, P^\lambda)$ ,  $\int YF_{t+s}dP^\lambda = \int YF_t(F_s \circ \theta_t)dP^\lambda$ , or by the usual density argument, if this equality holds for every  $Y=W(V \circ \theta_t)$ , where  $W$  is bounded and  $\mathfrak{G}_t$ -measurable, and  $V$  is bounded and  $\mathfrak{G}$ -measurable.

We have, using Lemma 3.4(ii),

$$\begin{aligned} \int YF_{t+s}dP^\lambda &= \lim_{n \rightarrow \infty} \int YF_{t+s}^n dP^\lambda = \lim_{n \rightarrow \infty} \int W(V \circ \theta_t)F_t^n(F_s^n \circ \theta_t)dP^\lambda = \\ &= \lim_{n \rightarrow \infty} \int W F_t^n E^{B_t}[VF_s^n]dP^\lambda = \lim_{n \rightarrow \infty} \int E^X[VF_s^n]\psi^n(dx), \text{ where } \psi^n \text{ is the} \\ & \text{signed measure on } R^d \text{ defined by} \\ (3.36) \quad & \int h d\psi^n = E^\lambda[h \circ B_t W F_t^n] \text{ for any } h \text{ bounded Borel on } R^d. \end{aligned}$$

Let  $\psi$  be the signed measure on  $R^d$  defined by  $\int h d\psi = E^\lambda[h \circ B_t W F_t]$ . By Remark 3.1 (the case proved in [4]),

$$(3.37) \quad \|\psi^n - \psi\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Clearly we may assume  $\lambda$  is such that we can write  $\psi = g d\lambda$ , where  $g$  is a bounded Borel function on  $R^d$ . Thus  $\int E^X[VF_s^n]\psi(dx) = E^\lambda[g \circ B_0 VF_s^n]$ . It then follows from Lemma 3.4(ii) that

$$(3.38) \quad \int E^X[VF_s^n]\psi(dx) \rightarrow \int E^X[VF_s]\psi(dx) \text{ as } n \rightarrow \infty.$$

$\int E^X[VF_s]\psi(dx) = E^\lambda[W F_t E^{B_t}[VF_s]] = E^\lambda[YF_t(F_s \circ \theta_t)]$ . Thus, by (3.37) and

$$(3.38), \quad \int YF_{t+s}dP^\lambda = \int YF_t(F_s \circ \theta_t)dP^\lambda, \text{ and (3.35) is proved.}$$

Having established (3.35), we finish the proof of Theorem 4.2 by showing that for any stopping time measure  $F$ , such that (3.35) holds for a fixed probability  $\lambda$  with  $m \ll \lambda$ , we can find a multiplicative functional  $M$  such that (3.30) and (3.31) hold, and such that  $M=F$ ,  $P^\nu$ -a.e., for any  $\nu \ll m$ .

For any  $\nu \ll m$ , and any  $t > 0$ ,

$$(3.39) \quad \int F(\{t\})dP^\nu = 0.$$

This is proved in the same way as Lemma 3.5, using (3.35) instead of (3.30).

As another consequence of (3.35), we see easily that for any  $t > 0$ , and any sequence of positive real numbers  $\epsilon_k$  with  $\epsilon_k \downarrow 0$ , we have

(3.40)  $F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}$  is almost surely decreasing in  $k$ .

In particular  $F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}$  has a limit as  $k \rightarrow \infty$ , almost surely.

Let  $Y$  denote the limit of  $F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}$  as  $k \rightarrow \infty$ . By (3.35),  $Y \geq F_t$ ,  $P^\lambda$ -a.e..

On the other hand, if  $\nu \ll m$ , we have  $\int Y dP^\nu = \lim_{k \rightarrow \infty} E^\nu [F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}] =$

$\lim_{k \rightarrow \infty} E^{\nu_k} [F_{t-\varepsilon_k}]$ , where  $\nu_k = \nu P_{\varepsilon_k}$ . Since  $\|\nu_k - \nu\| \rightarrow 0$  as  $k \rightarrow \infty$ ,  $\int Y dP^\nu =$

$E^\nu [F_{t-}] = \int F_t dP^\nu$  by (3.39). This proves that if  $\nu \ll m$ ,  $t > 0$ ,

(3.41)  $\lim_{k \rightarrow \infty} F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} = F_t$ ,  $P^\nu$ -a.e..

We now define  $N_t(\omega)$ , for  $t > 0$  and  $\omega$  in  $C$ , by

(3.42)  $N_t(\omega) = \inf \{F_{t-r} \circ \theta_r : r \text{ rational}, 0 < r < t\}$ .

It follows easily that for every  $t > 0$ , for every  $s \geq 0$ ,

(3.43)  $N_{t+s} \leq N_t$  everywhere.

For any  $\varepsilon_k$  real positive,  $\varepsilon_k \downarrow 0$ , we find from (3.40) that for  $t > 0$ ,

(3.44)  $N_t = \lim_{k \rightarrow \infty} F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}$  almost surely, and hence that for any

$\nu \ll m$ ,

(3.45)  $N_t = F_t$ ,  $P^\nu$ -a.e.

From (3.44) and (3.35), for  $t > 0$ ,  $s \geq 0$ ,

(3.46)  $N_{t+s} = N_t F_s \circ \theta_t$ , almost surely.

We define  $M$ , a stopping time measure, by

(3.47)  $M_t = N_{t+}$  for  $0 \leq t < \infty$ ,  $M_\infty = 1$ .

For  $t > 0$ ,  $N_{t+s+1/k} = N_t F_{s+1/k} \circ \theta_t$  almost surely, by (3.46). Thus  $M_{t+s} = N_t F_s \circ \theta_t$  almost surely, using the right continuity of  $M$  and  $F$ . Hence  $M_{t+s} = N_{t+s}$  almost surely, by (3.46) again. This shows that for  $0 < t < \infty$ ,

(3.48)  $M_t = N_t$  almost surely.

Fix  $u > 0$ , consider  $t$ ,  $0 < t < u$ , and let  $s = u - t$ . Applying (3.46),

$N_u = N_t F_{u-t} \circ \theta_t$ , almost surely. Letting  $t \downarrow 0$  through a sequence gives

$N_u = M_0 N_u$  almost surely, by (3.47) and (3.44). Hence, by (3.47),  $M_r = M_0 M_r$

almost surely for all  $r \geq 0$ . This proves that (3.30) holds for our  $M$ , when  $t = 0$ .

When  $t > 0$ ,  $M_{t+s} = N_{t+s} = N_t F_s \circ \theta_t$  almost surely by (3.48) and (3.46), and  $F_s \circ \theta_t = N_s \circ \theta_t = M_s \circ \theta_t$  by (3.45) and (3.48), so (3.30) holds in all cases.

$M_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} = N_{t-\varepsilon_k} \circ \theta_{\varepsilon_k} = F_{t-\varepsilon_k} \circ \theta_{\varepsilon_k}$  almost surely by (3.48) and (3.45), so (3.31) holds by (3.44) and (3.48). This proves Theorem 4.2.

Although we will not need this fact in the present paper, we note that a multiplicative functional satisfying (3.30) and (3.31) has the strong Markov property:

**Lemma 3.7** Let  $M$  satisfy (3.30) and (3.31), and let  $\tau$  be any  $\mathcal{G}_t$ -stopping time. For any  $s \geq 0$ ,

$$(3.49) \quad M_{\tau+s} = M_\tau(M_s \circ \theta_\tau) \text{ almost surely on } \{\tau < \infty\}.$$

The proof is omitted.

**Definition 3.3** Let  $M$  be a multiplicative functional,  $Q_t(M)$  the corresponding sub-Markov semigroup. The resolvent  $R_\alpha(M)$ ,  $\alpha > 0$ , associated with  $M$  is defined by

$$(3.50) \quad R_\alpha(M) = \int_0^\infty e^{-\alpha t} Q_t(M) dt, \text{ i.e. } R_\alpha(M)h(x) = E^x \left[ \int_0^\infty e^{-\alpha t} M_t h \circ B_t dt \right].$$

We note that the usual resolvent equation argument shows that if  $M$  and  $N$  are multiplicative functionals and  $R_\alpha(M) = R_\alpha(N)$  for one  $\alpha > 0$  then  $R_\beta(M) = R_\beta(N)$  for every  $\beta > 0$ .

We may consider  $Q_t(M)$  and  $R_t(M)$  as defined initially for  $h$  bounded Borel, and then extend to  $h$  in  $L^p(\mathbb{R}^d, m)$ , since  $Q_t(M)$  is a contraction for  $1 \leq p \leq \infty$ .

**Lemma 3.8** Let  $M_n$  and  $M$  be multiplicative functionals. The following statements are equivalent:

- (i)  $M_n \rightarrow M$  stably as  $n \rightarrow \infty$ ;
- (ii)  $R_\alpha(M_n) \rightarrow R_\alpha(M)$  weakly on  $L^2(\mathbb{R}^d, m)$  for each  $\alpha > 0$  as  $n \rightarrow \infty$ .

**Proof** Assume (i). Let  $\lambda$  be a probability measure with  $m \ll \lambda$ ,  $\lambda \ll m$ . By lemma 3.5 and Remark 3.1, for any  $t > 0$ ,

$$(3.51) \quad \|\lambda Q_t(M_n) - \lambda Q_t(M)\| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where  $Q_t(M)$  is defined by (3.32).

Hence for any bounded Borel functions  $h$  and  $g$  with compact support,

$$(3.52) \quad \int h(Q_t(M_n) - Q_t(M))g dm \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since  $Q_t(M_n)$  and  $Q_t(M)$  are contractions on  $L^2(\mathbb{R}^d, m)$  for all  $t$ , (ii) follows.

Conversely, assume (ii). Let  $N$  be a stable limit point of  $M_n$ . Then  $R_\alpha(N) = R_\alpha(M)$  on  $L^2(\mathbb{R}^d, m)$ .  $Q_t(N)$  and  $Q_t(M)$  are clearly strongly right continuous on  $L^2(\mathbb{R}^d, m)$  as functions of  $t$ .  $Q_t(N) = Q_t(M)$  on  $L^2(\mathbb{R}^d, m)$  for all

a.e.  $t$  and hence for all  $t$  since as functions of  $t$  they have the same Laplace transforms. Hence, by (3.33),  $Q_t(N)=Q_t(M)$  for all  $t$ , as sub-Markov operators. Thus  $N=M$  by Lemma 3.6, so (i) holds, and Lemma 3.8 is proved.

**Definition 3.4** A multiplicative functional  $M$  will be called symmetric if the corresponding operators  $Q_t(M)$  are symmetric on  $L^2(R^d, m)$ .

**Lemma 3.9** Let  $M_n$  be a sequence of symmetric multiplicative functionals, and let  $M$  be a multiplicative functional such that  $M_n \rightarrow M$  stably. Then  $M$  is symmetric.

**Proof** Let  $h$  and  $g$  be bounded Borel functions on  $R^d$  with compact support. As noted in (3.52), if  $t > 0$  then  $\int h Q_t(M_n) g dm \rightarrow \int h Q_t(M) g dm$  and  $\int g Q_t(M_n) h dm \rightarrow \int g Q_t(M) h dm$ . It follows at once that  $Q_t(M)$  is symmetric on  $L^2(R^d, m)$ , so Lemma 3.9 is proved.

It seems to be open whether nonsymmetric multiplicative functionals (satisfying (3.30) and (3.31)) exist. In any case we have:

**Theorem 3.3** Let  $M_n$  and  $M$  be symmetric multiplicative functionals. The following statements are equivalent:

- (i)  $M_n \rightarrow M$  stably as  $n \rightarrow \infty$ ;
- (ii)  $R_\alpha(M_n) \rightarrow R_\alpha(M)$  strongly on  $L^2(R^d, m)$  for each  $\alpha > 0$  as  $n \rightarrow \infty$ .

**Proof** If (i) holds, we know by the proof of Lemma 3.8 that for  $t > 0$   $Q_{2t}(M_n) \rightarrow Q_{2t}(M)$  weakly, hence  $Q_t(M_n) \rightarrow Q_t(M)$  strongly, and (ii) holds. The converse follows from Lemma 3.8, so Theorem 3.3 is proved.

**4.** Let  $\mu$  be a measure on  $R^d$ ,  $d \geq 2$ ,  $\mu \geq 0$ , not necessarily finite. Define the operators  $Pot^+$ ,  $Pot^-$  as in section 2 by

$$(4.1) \quad Pot^+ \mu(x) = \int k^+(|x-y|) \mu(dy), \quad Pot^- \mu(x) = \int k^-(|x-y|) \mu(dy),$$

where  $k(r) = c_d r^{d-2}$  if  $d \geq 3$ , and  $k(r) = -c_d \log r$  if  $d=2$ . Let  $a_d$  denote the area of the unit sphere in  $R^d$ . Then  $c_d = 2/(d-2)a_d$  for  $d \geq 3$  and  $c_d = 2/a_d$  for  $d=2$ .

As in section 2,  $\mathbf{M}_2$  will denote the space of finite measures  $\mu$  on  $R^d$  such that  $Pot^+ \mu$  is bounded and continuous on  $R^d$  and  $Pot^- \mu$  is finite on  $R^d$ .

For any measure  $\mu$  with  $Pot^+ \mu$  bounded, in particular for  $\mu$  in  $\mathbf{M}_2$ , there exists a continuous additive functional  $J(\mu) \geq 0$  such that  $Pot \mu \bullet B_t + J_t$  is a martingale, a particular case of the Doob-Meyer decomposition theorem for supermartingales (cf. [6], [14]).  $J_t$  is

$\mathcal{G}_t$ -measurable for each  $t$ , is continuous as a function of  $t$ , and is additive in the sense that  $J_{t+s} = J_t + J_s \circ \theta_t$  a.s. for all  $t, s \geq 0$ . By [6], 5.2.1, we may in fact take  $J(\mu)$  to be perfect, meaning that almost surely,  $J_{t+s}(\mu) = J_t(\mu) + J_s(\mu) \circ \theta_t$  for all  $t$  and all  $s$ . For  $d \geq 3$   $J_t$  has a finite limit  $J_\infty$  such that  $E^x[J_\infty(\mu)] = \text{Pot } \mu(x)$  for all  $x$  in  $R^d$ . For all  $d$ , using the symmetry of Pot, we see from the fact that  $\text{Pot } \mu \circ B_t + J_t$  is a martingale that for any probability measure  $\lambda$  with compact support, and any finite stopping times  $\tau_1 \leq \tau_2$   $P$ -a.e., if  $\lambda_{\tau_1}$  denotes the distribution of  $B_{\tau_1}$  with respect to  $P^\lambda$ , and  $\text{Pot } \lambda_{\tau_1} \geq \text{Pot } \lambda_{\tau_2}$  on  $R^d$ , then

$$(4.2) \quad \int (\text{Pot } \lambda_{\tau_1} - \text{Pot } \lambda_{\tau_2}) d\mu = E^\lambda [J_{\tau_2}(\mu) - J_{\tau_1}(\mu)].$$

We note that the condition  $\text{Pot } \lambda_{\tau_2} \leq \text{Pot } \lambda_{\tau_1}$  is automatically true when  $d \geq 3$ , so that  $\tau_1, \tau_2$  can be any finite stopping times in this case. In the  $d=2$  case we will have  $\text{Pot } \lambda_{\tau_2} \leq \text{Pot } \lambda_{\tau_1}$  when  $\tau_2$  is bounded, or when  $\tau_2 \leq$  the first exit time of a bounded region. Also, by Fubini's theorem we can extend (4.2) to randomized stopping times.

By [6], 6.3.1 and 4.2.13, for any measure  $\mu$  in  $\mathbf{M}_2$ , and any bounded Borel function  $q$ , if we let  $\nu = q d\mu$ , then

$$(4.3) \quad J_t(\nu) = \int_{[0,t]} q \circ B_s dJ_s(\mu).$$

**Lemma 4.1** Let  $K_t(i): C \rightarrow [0, \infty]$  be such that  $K_t(i)$  is  $\mathcal{G}_t$ -measurable for each  $t$  and  $i=1,2$ . Suppose that for every  $x$  in  $R^d$  and  $i=1,2$  the following conditions hold  $P^x$ -a.e.:

- (1)  $K_t(i)$  is nondecreasing and left continuous,
- (2)  $K_t(i)$  is continuous on the interval where  $K_t(i)$  is finite,
- (3)  $K_0(i) = 0$ .

Finally, suppose that for every finite stopping time  $\tau$ , for every  $x$  in  $R^d$

$$(4.4) \quad E^x[K_\tau(1)] = E^x[K_\tau(2)].$$

Then for every  $x$  in  $R^d$ ,  $P^x$ -a.e.  $K_t(1) = K_t(2)$ .

**Proof** Fix  $a, 0 \leq a < \infty$ . Let  $\tau_a = \inf\{t: K_t(1) \geq a\}$ .  $\{\tau_a < s\}$  is the union of  $\{K_r(1) \geq a\}$  over all  $r$  rational,  $r < s$ , so  $\tau_a$  is a  $\mathcal{G}_t$ -stopping time.

If  $t < \tau_a$ , clearly  $K_t(1) < a$ . Hence, if  $0 < \tau_a < \infty$  then for every  $x$ ,  $P^x$ -a.e.  $K_{\tau_a}(1) \leq a$  by left continuity. If  $\tau_a = 0$  the same is clearly true. Thus for

every  $x$ ,  $P^x$ -a.e.  $K_t(1) \leq a$  for all finite  $t \leq \tau_a$ .

Fix  $x$  in  $R^d$  and a positive number  $n$ . Let  $\tau = \tau_a \wedge n$ . Then  $K_\tau(1) \leq a$ ,  $P^x$ -a.e.. Hence  $E^x[K_\tau(1)] < \infty$ , so  $E^x[K_\tau(2)] < \infty$ .

Let  $Y_t = K_{t \wedge \tau}(1) - K_{t \wedge \tau}(2)$ . For any  $\mathcal{G}_t$ -stopping time  $\sigma$ ,  $E^x[Y_\sigma] = E^x[K_{\sigma \wedge \tau}(1)] - E^x[K_{\sigma \wedge \tau}(2)] = 0 = E^x[Y_0]$ . It follows easily that  $(Y_t, \mathcal{G}_t)$  is a martingale.

Since  $Y_t$  is continuous and has bounded variation,  $Y_t$  must be constant,  $P^x$ -a.e.. Hence  $Y_t = 0$  for all  $t$ ,  $P^x$ -a.e.. That is,  $K_{t \wedge \tau}(1) = K_{t \wedge \tau}(2)$  for all  $t$ ,  $P^x$ -a.e.. Thus for every  $n$ ,  $P^x$ -a.e., if  $t \leq n$  and  $K_t(1) < a$ , then  $K_t(1) = K_t(2)$ . Since this is true for all finite  $a$ , we have  $P^x$ -a.e. that  $K_t(1) \geq K_t(2)$  for every  $t$ . Lemma 4.1 then follows by symmetry.

We recall the space  $\mathcal{M}_1$  defined in section 2:  $\mathcal{M}_1$  is the space of measures  $\mu$  which can be expressed in the form  $\mu = q\nu$ , where  $\nu$  is in  $\mathcal{M}_2$  and  $q: R^d \rightarrow [0, \infty]$  is Borel.  $\mathcal{M}_1 \subset \mathcal{M}_0$ . As noted in section 2, for every  $\mu \in \mathcal{M}_0$  there exists  $\mu' \in \mathcal{M}_1$  with  $\mu \sim \mu'$ , in the sense of Definition 2.4.

**Lemma 4.2** Let  $D$  be an open set. Let  $\mu_1$  and  $\mu_2$  be in  $\mathcal{M}_0$ . The following statements are equivalent:

- (i)  $\mu_1 \sim \mu_2(D)$ ;
- (ii) For every bounded open set  $G \subset D$ , every stopping time  $\tau \leq$  the first exit time of  $G$ , and every probability measure  $\lambda$  on  $G$ ,

$$(4.5) \quad \int (\text{Pot } \lambda - \text{Pot } \lambda_\tau) d\mu = \int (\text{Pot } \lambda - \text{Pot } \lambda_\tau) d\mu_2;$$

- (iii) For every bounded open set  $G \subset D$ , every probability measure  $\lambda$  on  $G$  with  $\lambda \ll m$ , if  $V$  is a finely open subset of  $G$ , and  $h$  is the reduction of  $\text{Pot } \lambda$  onto  $V^c$  then

$$(4.6) \quad \int (\text{Pot } \lambda - h) d\mu_1 = \int (\text{Pot } \lambda - h) d\mu_2;$$

- (iv)  $\mu_1(V) = \mu_2(V)$  for all finely open subsets  $V$  of  $D$ .

**Proof** (i)  $\Rightarrow$  (ii):  $(\text{Pot } \lambda - \text{Pot } \lambda_\tau) \wedge k = \text{Pot } \lambda \wedge (\text{Pot } \lambda_\tau + k) - \text{Pot } \lambda_\tau$  is nonnegative and bounded, and so is easily seen to be in  $H_0^1(G)$ . Thus

$$\int (\text{Pot } \lambda - \text{Pot } \lambda_\tau) \wedge k d\mu_1 = \int (\text{Pot } \lambda - \text{Pot } \lambda_\tau) \wedge k d\mu_2 \text{ for all } k, \text{ and we may let } k \rightarrow \infty \text{ to prove (ii).}$$

- (ii)  $\Rightarrow$  (iii):  $h = \text{Pot } \lambda_\tau$ , where  $\tau =$  the first exit time of  $V$ .

- (iii)  $\Rightarrow$  (iv): Let  $G$  be a bounded open subset of  $D$ ,  $\lambda$  a probability measure on

$\mathbb{R}^d$ , with  $\lambda \ll m$ ,  $\text{Pot } \lambda$  bounded on  $G$ , and  $\lambda(G^c) = 0$ . Let  $V$  be a finely open subset of  $G$ . Let  $h$  be the reduced function of  $\text{Pot } \lambda$  on  $V^c$ . Let  $\alpha > 0$  be fixed. Let  $W = \{\text{Pot } \lambda > \alpha + h\}$ . Let  $g$  be the reduced function of  $\text{Pot } \lambda$  on  $W^c$ .  $W$

is finely open, so  $\int (\text{Pot } \lambda - g) d\mu_1 = \int (\text{Pot } \lambda - g) d\mu_2$ . If  $\int (\text{Pot } \lambda - g) d\mu_1 = \infty$  then  $\mu_1(W) = \mu_2(W) = \infty$ , so  $\int (g - h) d\mu_1 \geq \int_W \alpha d\mu_1 = \infty$ , and

$\int (g - h) d\mu_1 = \int (g - h) d\mu_2$ . If  $\int (\text{Pot } \lambda - g) d\mu_1 < \infty$ , then, subtracting from (4.6),

we obtain  $\int (g - h) d\mu_1 = \int (g - h) d\mu_2$  again. Since  $g = \text{Pot } \lambda \wedge (\alpha + h)$ , we have

shown that  $\int (\text{Pot } \lambda \wedge (\alpha + h) - h) d\mu_1 = \int (\text{Pot } \lambda \wedge (\alpha + h) - h) d\mu_2$ , or

$\int (\text{Pot } \lambda - h) \wedge \alpha d\mu_1 = \int (\text{Pot } \lambda - h) \wedge \alpha d\mu_2$ . By the proof of I.XI.10 in [12] we can choose our probability  $\lambda$  (with  $\text{Pot } \lambda$  even continuous), such that for any finely open subset  $A$  of  $G$ , if  $f$  is the reduced function of  $\text{Pot } \lambda$  on  $A^c$  then  $A = \{\text{Pot } \lambda > h\}$ , up to a polar set. Let  $U = \{\text{Pot } \lambda > h\}$ .  $U = V$  up to a polar set.  $k((\text{Pot } \lambda - h) \wedge (1/k)) = (k(\text{Pot } \lambda - h)) \wedge 1 \upharpoonright_U$ , so  $\mu_1(U) = \mu_2(U)$ . This proves (iv).

(iv)  $\Rightarrow$  (i): As mentioned earlier, if  $G$  is a bounded open set, any function  $f$  in  $H^1(G)$  has the property that for any  $\epsilon > 0$  there is a set  $B_\epsilon$  of capacity (relative to  $G$ , say) less than  $\epsilon$ , such that the restriction of  $f$  to  $B_\epsilon^c$  is continuous. We may enlarge  $B_\epsilon$  to make it fine closed without changing its capacity.  $f$  is finely continuous at each point of the complement of the closure of  $B_\epsilon$ , a finely open set. It follows that  $f$  on  $G$  is fine continuous at each point of a finely open set in  $G$  whose complement is polar. It then follows that the inverse image under  $f$  of any open set differs from a finely open set by a polar set. Thus

$\int f^2 d\mu_1 = \int_{[0, \infty)} \mu_1(\{f^2 > t\}) dt = \int_{[0, \infty)} \mu_2(\{f^2 > t\}) dt = \int f^2 d\mu_2$ . This proves

Lemma 4.2.

**Remark 4.1** The proof that (iii)  $\Rightarrow$  (iv) shows that following fact: let  $\mu_1, \mu_2 \in \mathbf{M}_0$  and let  $U$  be a finely open set, such that for any probability  $\lambda$  on  $\mathbb{R}^d$  with  $\lambda \ll m$  and  $\text{Pot } \lambda$  bounded, and any finely open subset  $V$  of  $U$ , we

have  $\int (\text{Pot } \lambda - h) d\mu_1 = \int (\text{Pot } \lambda - h) d\mu_2$ , where  $h$  denotes the reduction of  $\text{Pot } \lambda$  onto  $V^c$ . Then  $\mu_1 = \mu_2$  on all finely open subsets of  $U$ .

We shall now show how to associate an additive functional with any measure in  $\mathbf{M}_1$ . Given  $\mu \in \mathbf{M}_1$ , let  $\mu$  be expressed as  $q d\nu$ , where  $\nu$  is in  $\mathbf{M}_2$  and

$q: \mathbb{R}^d \rightarrow [0, \infty]$  is Borel. Let  $K(\mu)$  be defined by

$$(4.7) \quad K_t(\mu) = \int_{[0, t]} q \circ B_S dJ_S(\nu).$$

We see that  $K_t(\mu): C \rightarrow [0, \infty]$  is  $\mathcal{G}_t$ -measurable for each  $t$ , and for every  $x \in \mathbb{R}^d$ ,  $P^x$ -a.e.:  $K_t(\mu)$  is nondecreasing and left continuous,  $K_t(\mu)$  is continuous on the interval where  $K_t(\mu)$  is finite, and  $K_0(\mu) = 0$ . Furthermore,  $K_t(\mu)$  is additive, and also: for every  $t > 0$ , for any sequence  $t_k > 0$  of real numbers such that  $t_k \rightarrow 0$ , we have for every  $x \in \mathbb{R}^d$ ,  $P^x$ -a.e. that

$$(4.8) \quad K_t(\mu) = \lim_{k \rightarrow \infty} K_{t-t_k}(\mu) \circ \theta_{t_k}.$$

We will regard two functionals as being the same if for every  $x \in \mathbb{R}^d$ ,  $P^x$ -a.e. the functionals agree for all time. In this sense we have the following lemma:

**Lemma 4.3** The functional  $K(\mu)$  defined by (4.7) is independent of the representation  $\mu = q d\nu$  chosen for  $\mu$ . Furthermore, if  $\mu_1 \sim \mu_2$  then  $K(\mu_1) = K(\mu_2)$ . Conversely, if  $K(\mu_1) = K(\mu_2)$  then  $\mu_1 \sim \mu_2$ .

**Proof** Let  $\mu = q_1 d\nu_1 = q_2 d\nu_2$ . Let  $\lambda$  be any probability measure on  $\mathbb{R}^d$ ,  $\tau$  any finite stopping time such that  $\text{Pot } \lambda_{\tau} \leq \text{Pot } \lambda$ . For all  $n$ , by (4.2) and (4.3)

$$(4.9) \quad \int (\text{Pot } \lambda - \text{Pot } \lambda_{\tau})(q_i \wedge n) d\nu = E^{\lambda} \left[ \int_{[0, \tau]} (q_i \wedge n) \circ B_S dJ_S(\nu_i) \right].$$

Letting  $n \rightarrow \infty$ , we see that

$$(4.10) \quad \int (\text{Pot } \lambda - \text{Pot } \lambda_{\tau}) d\mu = E^{\lambda} \left[ \int_{[0, \tau]} q_i \circ B_S dJ_S(\nu_i) \right].$$

For any finite stopping time  $\tau$  whatsoever, it then follows that

$$E^{\lambda} \left[ \int_{[0, \tau]} q_1 \circ B_S dJ_S(\nu_1) \right] = E^{\lambda} \left[ \int_{[0, \tau]} q_2 \circ B_S dJ_S(\nu_2) \right].$$

By Lemma 4.1,

$\int_{[0, t]} q_i \circ B_S dJ_S(\nu_i)$ ,  $i=1,2$  defines the same functional, which is now unambiguously denoted by  $K(\mu)$ . By (4.10)

$$(4.11) \quad \int (\text{Pot } \lambda - \text{Pot } \lambda_{\tau}) d\mu = E^{\lambda} \left[ \int K_{\tau}(\mu) \right] \text{ for any stopping time } \tau \text{ such that } \text{Pot } \lambda_{\tau} \leq \text{Pot } \lambda.$$

Suppose now  $\mu_1 \sim \mu_2$ . Let  $x \in \mathbb{R}^d$ , and let  $\tau$  be any finite stopping time. Let  $G$  be any bounded open set, and let  $\sigma$  be the first exit time of  $G$ . Then by (4.11) and lemma 4.2

$$(4.12) \quad \int (\text{Pot } \lambda - \text{Pot } \lambda_{\tau \wedge \sigma}) d\mu_i = E^{\lambda} [K_{\tau \wedge \sigma}(\mu_i)] \text{ for } i=1,2, \text{ where } \lambda = \delta_x.$$

Letting  $G$  expand to  $\infty$ , we see that  $E^X[K_\tau(\mu_1)] = E^X[K_\tau(\mu_2)]$ . Thus Lemma 4.1 applies again to show that  $K(\mu_1) = K(\mu_2)$ .

Finally, let  $\mu_1, \mu_2$  be in  $\mathfrak{M}_1$ , such that  $K(\mu_1) = K(\mu_2)$ . Let  $G$  be a bounded open set,  $\lambda$  a probability on  $G$ , and  $\tau$  a stopping time  $\leq$  the first exit time

of  $G$ . By (4.11),  $\int (\text{Pot } \lambda - \text{Pot } \lambda_\tau) d\mu_1 = \int (\text{Pot } \lambda - \text{Pot } \lambda_\tau) d\mu_2$  and thus  $\mu_1 \sim \mu_2$  by Lemma 4.2. This proves Lemma 4.3.

**Lemma 4.4** For every  $\mu \in \mathfrak{M}_1$ ,  $K_{t+}(\mu)$  defines an additive functional.

**Proof** For every  $s > 0$ ,  $t \geq 0$ , for every  $\varepsilon$ ,  $0 < \varepsilon < s$ , for every  $x \in \mathbb{R}^d$ ,

$$(4.13) \quad K_{t+s}(\mu) = K_{t+\varepsilon}(\mu) + K_{s-\varepsilon}(\mu) \circ \theta_\varepsilon \circ \theta_t, \quad P^x\text{-a.e.}$$

Letting  $\varepsilon = \varepsilon_k \downarrow 0$ , by (4.8),

$$(4.14) \quad K_{t+s}(\mu) = K_{t+}(\mu) + K_s(\mu) \circ \theta_t, \quad P^x\text{-a.e.}$$

Letting  $s \downarrow 0 \geq 0$  proves lemma 4.4.

**Remarks:**

(a) Clearly, for every  $x \in \mathbb{R}^d$ , since  $K_{0+}(\mu) = K_{0+}(\mu) + K_{0+}(\mu)$ ,  $P^x$ -a.e., either  $K_{0+}(\mu) = \infty$ ,  $P^x$ -a.e., or  $K_{0+}(\mu) = 0$ ,  $P^x$ -a.e.. (We recall that  $\mathfrak{G}_0$  is trivial a.s..)

(b) Let  $t > 0$ ,  $t_k > 0$ ,  $t_k \downarrow 0$ . For large  $k$

$$(4.15) \quad K_{t+}(\mu) = K_{t_k+}(\mu) + K_{(t-t_k)+}(\mu) \circ \theta_{t_k}, \quad P^x\text{-a.e.}$$

As  $k \rightarrow \infty$ , clearly  $K_{(t-t_k)+}(\mu) \circ \theta_{t_k}$  increases to a limit,  $P^x$ -a.e.. Thus we have

$$(4.16) \quad K_{t+}(\mu) = K_{0+}(\mu) + \lim_{k \rightarrow \infty} K_{(t-t_k)+}(\mu) \circ \theta_{t_k}, \quad P^x\text{-a.e.}$$

But  $\lim_{k \rightarrow \infty} K_{(t-t_k)+}(\mu) \circ \theta_{t_k} \geq \lim_{k \rightarrow \infty} K_{(t-t_k)}(\mu) \circ \theta_{t_k} = K_t(\mu) \geq K_{0+}(\mu)$ . So whether  $K_{0+}(\mu) = 0$  or  $K_{0+}(\mu) = \infty$ , we have

$$(4.17) \quad K_{t+}(\mu) = \lim_{k \rightarrow \infty} K_{(t-t_k)+}(\mu) \circ \theta_{t_k}, \quad P^x\text{-a.e.}$$

Now we can define the multiplicative functional  $M(\mu)$  associated with  $\mu \in \mathfrak{M}_1$  by

$$(4.18) \quad M_t(\mu) = \exp(-K_{t+}(\mu)).$$

Clearly (3.30) and (3.31) hold, because of Lemma 4.4 and equation (4.17).

Lemma 4.3 gives at once:

**Theorem 4.1** Let  $\mu_1$  and  $\mu_2$  be in  $\mathfrak{M}_1$ . Then  $\mu_1 \sim \mu_2$  if and only if  $M(\mu_1) = M(\mu_2)$ .

Because of theorem 4.1, we can define  $M(\mu)$  for  $\mu \in \mathfrak{M}_0$  by  $M(\mu) = M(\nu)$ ,

where  $\nu \in \mathbf{M}_1$  and  $\nu \sim \mu$ .

We note that  $M(\mu)$  has been defined in terms of  $\mu$  by a probabilistic construction. As in section 3, we can then define the resolvent  $R_\lambda(M(\mu))$  corresponding to the semigroup associated with  $M(\mu)$ . At the same time, we can consider the resolvent  $R_\lambda(\mu)$  defined by the variational problem discussed in section 2. We now prove:

**Theorem 4.2** Let  $\mu$  be in  $\mathbf{M}_0$ . Let  $T$  be the randomized stopping time corresponding to  $M(\mu)$ . Then for any  $\lambda > 0$ ,  $R_\lambda(\mu) = R_\lambda(M(\mu))$ .

**Proof** As usual we may take  $\mu \in \mathbf{M}_1$ . Let  $\mu$  be expressed as  $\mu = qd\nu$ , where  $\nu \in \mathbf{M}_2$  and  $q: R^d \rightarrow [0, \infty]$  is Borel. Let  $\mu_n = (q \wedge n)d\nu$ . Then  $\mu_n \uparrow \mu$ , so  $R_\lambda(\mu_n) \rightarrow R_\lambda(\mu)$  strongly, by Theorem 2.1. Also  $M_t(\mu_n) = \exp(-K_t(\mu_n))$  decreases pointwise to  $\exp(-K_t(\mu)) = M_{t-}(\mu)$ , so as a random measure  $M(\mu_n)$  converges weakly to  $M(\mu)$  on  $[0, \infty]$ , pointwise for each  $\omega$ . Thus  $P^\nu \times M(\mu_n) \rightarrow P^\nu \times M(\mu)$  weakly for any probability measure  $\nu$ , so in particular  $M(\mu_n) \rightarrow M(\mu)$ . Thus  $R(M(\mu_n)) \rightarrow R(M(\mu))$  strongly, by Theorem 3.3. Hence it is enough to show  $R_\lambda(\mu) = R_\lambda(M(\mu))$  for  $\mu \in \mathbf{M}_2$ .

Accordingly, let  $\mu$  be a measure in  $\mathbf{M}_2$ . We must show that  $R_\lambda(\mu) = R_\lambda(M(\mu))$ . Let  $\nu_k = \mu P_{1/k}$ , where  $P_t$  is the usual Brownian motion semigroup. The usual arguments concerning energy show that for any  $x \in R^d$ ,

$$(4.19) \quad E^x[(J_t(\nu_k) - J_t(\mu))^2] \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Let  $\tilde{k}_j$  be any subsequence. We can find a subsequence  $k_j$  of  $\tilde{k}_j$ , such that

$$(4.20) \quad J_t(\nu_{k_j}) \rightarrow J_t(\mu) \text{ pointwise } P^x\text{-a.e., for all rational } t.$$

Thus as a measure  $M(\nu_{k_j})$  converges weakly to  $M(\mu)$  on  $[0, \infty]$ , for  $P^x$ -a.e.  $\omega$ . Hence  $P^x \times M(\nu_{k_j})$  converges weakly to  $M(\mu)$ . Since  $\tilde{k}_j$  was any subsequence of the original sequence, we see that

$$(4.21) \quad P^x \times M(\nu_k) \rightarrow P^x \times M(\mu) \text{ weakly, for all } x \in R^d.$$

In particular we have shown that  $M(\nu_k) \rightarrow M(\mu)$ , so  $R_\lambda(M(\nu_k)) \rightarrow R_\lambda(M(\mu))$  strongly. By [8], Proposition 4.12,  $\nu_k$   $\mathcal{V}$ -converges to  $\mu$ , so by theorem 2.1,  $R_\lambda(\nu_k) \rightarrow R_\lambda(\mu)$  strongly. Thus it is sufficient to consider  $\mu$  in  $\mathbf{M}_2$  such that  $\mu$  has a  $C^\infty$  density with respect to Lebesgue measure. In this case both  $R_\lambda(\mu)$  and  $R_\lambda(M(\mu))$  are defined by the same classical differential equation, so Theorem 4.2 is proved.

**Corollary** For any open set  $D$ , let  $\sigma$  be the last exit time of  $D$ . Let  $M^D(\mu)$

be the multiplicative functional corresponding to  $T(\mu) \wedge \sigma$ . Then for any  $\lambda > 0$ ,  $R_\lambda^D(\mu) = R_\lambda(M^D(\mu))$ .

**Proof** The same argument used in the proof of Theorem 4.2 can be used again. Or by Proposition 2.2 and the method of example 4.1 below we can just apply Theorem 4.2 to the measure  $\mu^{+\infty}_E$ , where  $E = D^C$ .

**Theorem 4.3** A sequence  $\mu_n \in \mathbf{M}_0$   $\mathcal{Y}$ -converges if and only if the corresponding sequence  $M(\mu_n)$  converges stably.

**Proof** By Theorem 2.1,  $\mu_n$   $\mathcal{Y}$ -converges if and only if the resolvents  $R_\lambda(\mu_n)$  converge strongly.  $M(\mu_n)$  converges if and only if  $R_\lambda(M(\mu_n))$  converges strongly by Theorem 3.3. Since  $R_\lambda(\mu_n) = R_\lambda(M(\mu_n))$  by Theorem 4.2, the result is proved.

**Remark 4.2** Since  $\mathcal{Y}$ -convergence and stable convergence are now linked, we see that the Corollary to Lemma 3.3 gives a probabilistic proof of Theorem 2.1.

**Example 4.1** Let  $K$  be any Borel set in  $R^d$ ,  $d \geq 3$ . Let  $\lambda$  be a probability,  $\lambda \ll m$ ,  $m \ll \lambda$ . Let  $\tau$  be the first hitting time of  $K$ , and let  $\nu$  denote the distribution of  $B_\tau$  on  $\{\tau < \infty\}$  with respect to  $P^\lambda$ , i.e.  $\nu$  is the swept measure of  $\lambda$  on  $K$ .

**Then:**  $\infty\nu = \infty_K$ , and  $M(\infty\nu)$  is the stopping time measure for  $\tau$ , where  $\infty_K$  is given by Definition 2.2, and  $\infty\nu(A) = \infty$  if  $\nu(A) > 0$ ,  $\infty\nu(A) = 0$  otherwise.

**Proof** Clearly it does not change  $\infty\nu$  if we replace  $\lambda$  by any probability which is mutually absolutely continuous with respect to  $\lambda$ . Thus without loss of generality we assume that  $\lambda$  has a bounded density with respect to  $m$ . Then  $\text{Pot } \nu \leq \text{Pot } \lambda$  is bounded. Let  $\sigma = \inf\{t: J_t(\nu) > 0\}$ . We claim:

- (a)  $\sigma = \tau$ ,  $P^\lambda$ -a.e., for every  $x \in R^d$ , and
- (b)  $\nu(V) > 0$  for any finely open set  $V$  such that  $V \cap K$  is not polar.

Proof of (a).  $\text{Pot } \lambda = \text{Pot } \lambda_{\tau \wedge t}$ , quasi-everywhere on  $K$ , for all  $t$ . Thus, by (4.2),

$$(4.22) \quad E^\lambda[J_{\tau \wedge t}(\nu)] = 0 \text{ for all } t, \text{ so}$$

$$(4.23) \quad E^\lambda[J_\tau(\nu)] = 0.$$

Hence  $J_\tau(\nu) = 0$ ,  $P^\lambda$ -a.e., so  $\tau \leq \sigma$ ,  $P^\lambda$ -a.e..

Let  $\psi$  denote the distribution of  $B_\sigma$  on  $\{\sigma < \infty\}$  with respect to  $P^\lambda$ .

Then  $\text{Pot } \nu \geq \text{Pot } \psi$ . By (4.2),

$$\int (\text{Pot } \lambda - \text{Pot } \lambda_{t \wedge \sigma}) d\nu = E^\lambda[J_{t \wedge \sigma}(\nu)] = 0 \text{ for all } t, \text{ so } \int (\text{Pot } \lambda - \text{Pot } \psi) d\nu = 0.$$

Thus  $\text{Pot } \psi = \text{Pot } \lambda \geq \text{Pot } \nu$ ,  $\nu$ -a.e.. Hence by the domination principle,  $\text{Pot } \psi \geq \text{Pot } \nu$ . Thus  $\text{Pot } \psi = \text{Pot } \nu$ . Since  $E^\lambda[(\sigma - \tau)\chi_{\{\tau < \infty\}}] =$

$\int (\text{Pot } \nu - \text{Pot } \psi) dm = 0$ , this proves  $\tau = \sigma$ ,  $P^\lambda$ -a.e.. Since  $\tau = \lim_{t \downarrow 0} (t + \tau \circ \theta_t)$  and  $\sigma = \lim_{t \downarrow 0} (t + \sigma \circ \theta_t)$ , and  $\tau \circ \theta_t = \sigma \circ \theta_t$ ,  $P^X$ -a.e., for every  $x \in R^d$ , (a) is proved.

Proof of (b). Let  $V$  be finely open. As noted in the proof of Lemma 4.2, we can find measures  $\rho_1, \rho_2$  such that  $\text{Pot } \rho_1$  is bounded and continuous,  $\text{Pot } \rho_1 \geq \text{Pot } \rho_2$ , and  $U = \{\text{Pot } \rho_1 > \text{Pot } \rho_2\}$  differs from  $V$  by a polar set. Let  $\phi_i$  be the swept measure of  $\rho_i$  on  $K$ . Then  $\text{Pot } \phi_1 \geq \text{Pot } \phi_2$ . Let

$W = \{\text{Pot } \phi_1 > \text{Pot } \phi_2\}$ . Then  $W \cap K = V \cap K$  up to a polar set.  $\int (\text{Pot } \phi_1 - \text{Pot } \phi_2) d\nu = \int (\text{Pot } \phi_1 - \text{Pot } \phi_2) d\lambda$ .  $W$  is finely open, so  $\lambda(W) > 0$ . Thus  $\nu(W \cap K) > 0$ . This proves (b). (a) and (b) clearly imply the result.

A similar construction in  $R^2$  shows that  $M(\infty_K)$  = the first hitting time of  $K$  in this case also.

5. In this section we shall illustrate the earlier results by proving some facts relating to Theorem 5.10 of [8].

**Lemma 5.1** Let  $\mu_n$  and  $\nu_n$  be two sequences in  $\mathbf{M}_0$  such that  $\mu_n$   $\mathcal{X}$ -converges to  $\mu$  and  $\nu_n$   $\mathcal{X}$ -converges to  $\nu$ . Let  $U$  be a finely open set in  $R^d$ ,  $U$  bounded if  $d=2$ . Suppose that  $\mu_n = \nu_n$  on all finely open subsets of  $U$ . Then  $\mu = \nu$  on all finely open subsets of  $U$ .

**Proof** We may rephrase the lemma as follows: let  $\mu_n, \mu$  be in  $\mathbf{M}_0$ , such that  $\mu_n$   $\mathcal{X}$ -converges to  $\mu$ . Let  $U$  be a finely open set in  $R^d$ ,  $U$  bounded if  $d=2$ . Let  $\nu_n = \chi_U d\mu_n$ , and let  $\psi$  be any  $\mathcal{X}$ -limit point of  $\nu_n$ . Then  $\psi = \mu$  on finely open subsets of  $U$ .

Replacing  $\mu_n$  and  $\mu$  by equivalent measures, we may assume that  $\mu_n, \mu \in \mathbf{M}_1$ . Let  $M(n) = M(\mu_n)$ ,  $M = M(\mu)$ ,  $N(n) = M(\nu_n)$ ,  $N = M(\psi)$ . By relabelling, assume  $N(n) \rightarrow N$  stably. Let  $\tau$  = the first exit time of  $U$ . Fix  $t > 0$ , and let  $Y = \chi_{\{\tau > t\}}$ . Let  $\lambda$  be any probability measure with  $\lambda \ll m$ . By Lemma 3.5, for  $s > 0$ ,  $M(\{s\}) = 0$  and  $N(\{s\}) = 0$ ,  $P^\lambda$ -a.e., and hence, by Lemma 3.4(ii), for any  $H \in L^1(C, \mathcal{G}, P^\lambda)$ ,

$$\int YH\chi_{[0,s]} d(P^\lambda \times M(n)) \rightarrow \int YH\chi_{[0,s]} d(P^\lambda \times M), \text{ and } \int YH\chi_{[0,s]} d(P^\lambda \times N(n)) \rightarrow \int YH\chi_{[0,s]} d(P^\lambda \times N).$$

Since  $M_u(n) = N_u(n)$  for  $u < \tau$ , we have, for  $0 < s \leq t$ ,

$\int YH\chi_{[0,s]} d(P^\lambda \times M) = \int YH\chi_{[0,s]} d(P^\lambda \times N)$ . It follows that  $M_s = N_s$ ,  $P^\lambda$ -a.e., on  $\{\tau > t\}$ , for  $0 < s \leq t$ . Hence  $M_s = N_s$ ,  $P^\lambda$ -a.e., on  $\{\tau > s\}$ . Thus  $K_{s+}(\mu) = K_{s+}(\psi)$  for  $0 < s < \tau$ , in the notation of section 3. Hence  $K_s(\mu) = K_s(\psi)$  for  $0 \leq s \leq \tau$ . It follows from (4.11) that for every stopping time  $\sigma \leq \tau$ , such that

$\text{Pot } \lambda_\sigma \leq \text{Pot } \lambda$ ,  $\int (\text{Pot } \lambda - \text{Pot } \lambda_\sigma) d\mu = \int (\text{Pot } \lambda - \text{Pot } \lambda_\sigma) d\psi$ . Hence, by lemma 4.2,  $\mu = \psi$  on any finely open subset of  $U$ , so Lemma 5.1 is proved.

**Definition 5.1** For any measure  $\mu \in \mathbf{M}_0$ , the set of finiteness  $W(\mu)$  for  $\mu$  is the union of all finely open sets  $V$  such that  $\mu(V) < \infty$ .

**Lemma 5.2** Let  $\mu_n, \mu$  be in  $\mathbf{M}_0$ , such that  $\mu_n$   $\mathcal{X}$ -converges to  $\mu$ . Let  $W = W(\mu)$ . Let  $A$  be a Borel set in  $R^d$  such that  $\mu(\text{fine-}\partial A) = 0$ . Let  $U$  be the fine-interior of  $A$ . Suppose that  $(\text{fine-}\partial U) \cap (\text{fine-}\partial W)$  and  $(\text{fine-}\partial A - (\text{fine-}\partial U)) \cap W^c$  are polar. Let  $\nu_n = \chi_A d\mu_n$ ,  $\nu = \chi_A d\mu$ . Then  $\nu_n$   $\mathcal{X}$ -converges to  $\nu$ .

**Proof** Let  $G = \text{fine-interior of } A^c$ . Let  $\phi_n = \chi_{A^c} d\mu_n$ ,  $\phi = \chi_{A^c} d\mu$ . Let  $\psi, \lambda$  be any  $\mathcal{X}$ -limit points of  $\nu_n, \phi_n$ , respectively. By Lemma 5.1,  $\nu \leq \psi$  and  $\phi \leq \lambda$  on all finely open sets. Also, by Lemma 5.1, since  $\chi_G d\psi = \text{the limit of } \chi_G d\nu_n$  on finely open subsets of  $G$ ,  $\psi(G) = 0$ . Similarly  $\lambda(U) = 0$ .

Since  $\nu_n + \phi_n = \mu_n$   $\mathcal{X}$ -converges to  $\mu$ , we must have  $\psi + \lambda = \mu$ , so  $\psi + \lambda = \mu = \nu + \phi$  on all finely open sets. It follows that  $\psi = \nu$  and  $\lambda = \phi$  on all finely open subsets of  $W$ .

Now let  $S$  be any finely open set. Let  $Z = S - \{((\text{fine-}\partial U) \cap (\text{fine-}\partial W)) \cup ((\text{fine-}\partial A - \text{fine-}\partial U) \cap W^c)\}$ . Let  $x \in Z$ . We consider four cases. Case (i):  $x \in U$ . Then  $\nu(Z \cap U) = \psi(Z \cap U)$  by Lemma 5.1. Case (ii):  $x \in G$ . Then  $\nu(Z \cap G) = \psi(Z \cap G) = 0$ . Case (iii):  $x \in W$ . Then  $\nu(Z \cap W) = \psi(Z \cap W)$ . The remaining possible case is  $x \in \text{fine-}\partial U$ ,  $x$  not in fine-closure  $W$ . Let  $D$  be a finely open set,  $D \subset Z$ ,  $x \in D$ , such that  $D \cap W = \emptyset$ .  $D \cap U \neq \emptyset$ , so  $\nu(D \cap U) = \mu(D \cap U) = \infty = \psi(D \cap U)$ . We have shown that in every case,  $x$  is contained in a finely open subset  $D$  of  $Z$  with  $\nu(D) = \psi(D)$ . Since the fine topology has the quasi-Lindelof property, and  $\nu$  and  $\psi$  are in  $\mathbf{M}_0$ ,  $\nu(Z) = \psi(Z)$ , so  $\nu(S) = \psi(S)$ . Thus  $\nu = \psi$ , and Lemma 5.2 is proved.

As a corollary, we see that if  $\mu$  is Radon, so that  $W^c = \emptyset$ , then  $\nu_n$   $\mathcal{X}$ -converges to  $\nu$  whenever  $\mu(\text{fine-}\partial A) = 0$ , in particular when  $\mu(\partial A) = 0$ . This is a special case of a more general criterion obtained in [6], section 5.

For a general  $\mu \in \mathbf{M}_0$ , we note that the condition that  $(\text{fine-}\partial A - \text{fine-}\partial U) \cap W^C$  be polar is trivially satisfied when  $A$  is the fine closure of its fine interior, for example when  $A$  is an open or closed ball.

If  $\mu$  happens to be radially symmetric, then  $W$  is a union of open annuli, and  $\partial W$  is a countable union of spheres. The hypothesis of Lemma 5.2 is then especially easy to check.

**6.** In this section we give some results relating to the probabilistic solution of the  $\mu$ -Dirichlet problem.

Let  $\mu \in \mathbf{M}_2$ . Let  $M$  denote  $M(\mu)$ , and let  $T$  denote the randomized stopping time corresponding to  $M$ . For brevity we will often write  $M(\omega, dt) = -dM_t$ . (The measure  $-dM_t$  thus has some mass at  $t = \infty$ .) Let  $\tau$  be a stopping time,  $\tau \leq$  the first exit time of some bounded set. Let  $\nu$  be any probability measure on  $\mathbb{R}^d$ , and let  $\lambda$  denote  $\nu_{\tau \wedge T}$ , the distribution of  $B_{\tau \wedge T}$  with respect to  $P^\nu \times m_t$ . Thus  $\lambda$  is that probability measure such that for any bounded Borel function  $h$  on  $\mathbb{R}^d$ ,

$$(6.1) \quad \int h d\lambda = \int h \circ B_{\tau \wedge T} d(P^\nu \times m_t) = E^\nu \left[ - \int_{[0, \infty)} h \circ B_{\tau \wedge t} dM_t \right]$$

Let  $\psi$  denote the distribution of  $B_\tau$  on  $\{T > \tau\}$  with respect to  $P^\nu \times m_t$ . That is, for any bounded Borel function  $h$  on  $\mathbb{R}^d$ ,

$$(6.2) \quad \int h d\psi = \int_{\{T > \tau\}} h \circ B_\tau d(P^\nu \times m_t) = E^\nu [h \circ B_\tau M_\tau].$$

Clearly

$$(6.3) \quad \int h d\lambda = E^\nu \left[ - \int_{[0, \tau]} h \circ B_t dM_t \right] + \int h d\psi.$$

**Lemma 6.1** Under the preceding assumptions,

$$(6.4) \quad \lambda = \text{Pot}(\nu - \lambda) d\mu + \psi.$$

**Proof** Let  $J_t = J_t(\mu)$ , so that  $M_t = \exp(-J_t)$ . Then  $dM_t = -\exp(-J_t) dJ_t = -M_t dJ_t$ .

For any bounded Borel function  $h$  on  $\mathbb{R}^d$ ,  $E^\nu \left[ - \int_{[0, \tau]} h \circ B_t dM_t \right] =$

$E^\nu \left[ \int_{[0, \tau]} h \circ B_t M_t dJ_t \right] = E^\nu \left[ \int_{[0, \tau]} M_t dL_t \right]$ , where  $L_t = J_t(\rho)$ ,  $\rho = h d\mu$ .

$E^\nu \left[ \int_{[0, \tau]} M_t dL_t \right] = E^\nu \left[ \int_{[0, \infty)} M_t dL_{\tau \wedge t} \right] = E^\nu \left[ - \int_{[0, \infty)} L_{\tau \wedge t} dM_t \right] =$

$\int L_{\tau \wedge T} d(P^\nu \times m_t) = \int \text{Pot}(\nu - \lambda) d\rho$ .

Equation (6.4) follows at once, so Lemma 6.1 is proved.

**Lemma 6.2** Let  $\mu \in \mathbf{M}_2$ . Let  $D$  be open in  $\mathbb{R}^d$ ,  $u \in H^1_{\text{loc}}(D)$ ,  $u$   $\mu$ -harmonic on  $D$ .

Let  $\tau$  be a stopping time,  $\tau \leq$  the first exit time of some compact subset of  $D$ . Then for quasi every  $x \in D$ ,

$$(6.5) \quad u(x) = E^x[u \circ B_{\tau} M_{\tau}].$$

**Proof** By Proposition 2.6,  $u$  can be made continuous on  $D$  by changing the values of  $u$  on a polar set. Thus we assume that  $u$  is continuous.

For any  $v$  which is  $C^\infty$  with compact support in  $D$ , since  $u$  is  $\mu$ -harmonic we have

$$(6.6) \quad \int u(-\Delta v) d\mu = - \int u v d\mu.$$

Let  $\nu$  and  $\lambda$  be probability measures with compact support in  $D$ , such that  $\text{Pot } \nu = \text{Pot } \lambda$  outside a compact subset of  $D$ . Let  $\phi$  be  $C^\infty$  with compact support on  $\mathbb{R}^d$ ,  $\phi$  nonnegative and radially symmetric,  $\int \phi d\mu = 1$ . Define  $\phi_\delta$  for  $\delta > 0$  by  $\phi_\delta(x) = \phi(x/\delta)/\delta^d$ . Then for  $\delta$  small,  $\nu_\delta \equiv \phi_\delta * \nu$  and  $\lambda_\delta \equiv \phi_\delta * \lambda$  have compact support in  $D$ , and  $\text{Pot } \nu_\delta = \text{Pot } \lambda_\delta$  outside a compact subset of  $D$ . Letting  $v = \text{Pot}(\nu_\delta - \lambda_\delta)$  in (6.6),

$$(6.7) \quad \int u d\nu_\delta - \int u d\lambda_\delta = - \int u \text{Pot}(\nu_\delta - \lambda_\delta) d\mu.$$

Letting  $\delta \rightarrow 0$ , since  $\nu_\delta \rightarrow \nu$ ,  $\lambda_\delta \rightarrow \lambda$  weakly, we have  $\int u d\nu - \int u d\lambda$  as the limit of the left side of (6.7).

$\text{Pot } \nu_\delta \uparrow \text{Pot } \nu$ ,  $\text{Pot } \lambda_\delta \uparrow \text{Pot } \lambda$  pointwise, and  $\int \text{Pot } \nu d\mu < \infty$ ,  $\int \text{Pot } \nu d\mu < \infty$  since  $\text{Pot } \mu$  is bounded. Since  $|u|$  is bounded, we have  $-\int u \text{Pot}(\nu - \lambda) d\mu$  as the limit of the right side of (6.7), by the dominated convergence theorem. Thus

$$(6.8) \quad \int u d\nu - \int u d\lambda = - \int u \text{Pot}(\nu - \lambda) d\mu.$$

In particular, when  $\nu = \delta_x$ , and  $\lambda$  is defined by (6.1), for a given  $\tau$ , (6.8)

holds. By lemma 6.1 we then have  $\int u d\nu = \int u d\psi$ , where  $\psi$  is defined by (6.2), and thus (6.5) holds, proving Lemma 6.2.

**Lemma 6.3** Let  $\mu \in \mathbf{M}_2$ . Let  $M = M(\mu)$ . Let  $D$  be open in  $\mathbb{R}^d$ ,  $u \in H^1_{loc}(D) \cap L^2_{loc}(D, \mu)$ . Suppose that for any open ball  $K$  with compact closure in  $D$ ,

$$(6.10) \quad u(x) = E^x[u \circ B_{\tau} M_{\tau}], \text{ for } m\text{-a.e. } x \in K, \text{ where } \tau = \tau_K \text{ denotes the first}$$

exit time of  $K$ .

Then  $u$  is  $\mu$ -harmonic on  $D$ .

**Proof** Let  $K$  be fixed. Let  $w$  be the solution to the  $\mu$ -Dirichlet problem on  $K$  with data  $u$  on  $\partial K$ . Let  $K_n$  be a sequence of balls with the same center as  $K$  such that  $K_n \subset K$  and  $K_n \uparrow K$ . Let  $\tau_n$  be the first exit time of  $K_n$ . Then

$\tau_n \uparrow \tau$ . Define  $\psi_x, \psi_x(n)$  by  $\int h d\psi_x = E^x[h \circ B_{\tau} M_{\tau}]$ ,  $\int h d\psi_x(n) = E^x[h \circ B_{\tau_n} M_{\tau_n}]$ .

It is easy to see that for each  $x \in K$ ,  $\text{Pot } \psi_x(n) \rightarrow \text{Pot } \psi_x$  in such a way that

for each  $x \in K$ ,  $\int w d\psi_x(n) \rightarrow \int u d\psi_x$  as  $n \rightarrow \infty$ . Indeed, we see that  $\psi_x(n)$  converges to  $\psi_x$  in energy norm, and so  $\psi_x(n) \rightarrow \psi_x$  in  $H^{-1}(R^d)$ . But

$\int w d\psi_x(n) = w(x)$  for q.e.  $x$ , by Lemma 6.2, while  $\int u d\psi_x = u(x)$  for m-a.e.  $x$  by (6.10). Thus  $w = u$ , m-a.e.. Thus  $w = u$  q.e., and Lemma 6.3 is proved.

**Theorem 6.1** Let  $\mu \in \mathcal{M}_0$ ,  $M = M(\mu)$ . Let  $D$  be open in  $R^d$ . Let  $u$  be in  $H^1_{loc}(D) \cap L^2_{loc}(D, \mu)$ . The following statements are equivalent:

- (i)  $u$  is locally  $\mu$ -harmonic on  $D$ ;
- (ii) if  $\tau$  is a stopping time,  $\tau \leq$  the first exit time of a compact subset  $W$  of  $D$ , then for quasi every  $x \in D$ , if  $\text{Pot } \lambda \in H^1_{loc}(D)$ , where  $\lambda$  is the distribution of  $B_{\tau}$  with respect to  $P^x$ , then

$$(6.18) \quad u(x) = E^x[u \circ B_{\tau} M_{\tau-}];$$

- (iii) for every  $\nu \in \mathcal{M}_2$  with  $\nu(D^c) = 0$ , and every open ball  $K$  with compact closure in  $D$ ,

$$(6.19) \quad \int u d\nu = E^{\nu}[u \circ B_{\tau} M_{\tau-}], \text{ where } \tau = \tau_K, \text{ the first exit time of } K.$$

**Proof** (i)  $\Rightarrow$  (ii): Let  $W, \tau$  be given. Let  $U$  be a finite union of open balls with compact closure in  $D$ , such that  $W \subset U$ . We may assume  $\mu \in \mathcal{M}_1$ . There exist measures  $\mu_n \in \mathcal{M}_2$  with  $\mu_n \uparrow \mu$ .

Let  $u_n^{\pm}$ ,  $n=1,2,3$ , be the solutions to the  $\mu_n$ -Dirichlet problem on  $U$  with fixed data  $u^{\pm}$  on  $\partial W$ . By Lemma 6.2, if  $M(n) = M(\mu_n)$ , for quasi every  $x$  in  $U$  we have

$$(6.20) \quad u_n^{\pm}(x) = E^x[u^{\pm} \circ B_{\tau} M_{\tau}(n)].$$

By Proposition 2.7,  $u_n^+ - u_n^-$  converges to  $u$  q.e. on  $U$ . Consider  $x \in U$  such that  $u_n^+(x) - u_n^-(x) \rightarrow u(x)$  and such that (6.20) holds for all  $n$ . Suppose  $\text{Pot } \lambda$  is in  $H^1_{loc}(D)$ , where  $\lambda$  is the distribution of  $B_{\tau}$  with respect to

$P^x$ . Then  $E^x[u^\pm \circ B_{\tau_n} M_{\tau_n}(1)] < \infty$ . Since  $M_{\tau_n} \downarrow M_{\tau_-}$  as  $n \rightarrow \infty$ , the dominated convergence theorem gives  $\lim_{n \rightarrow \infty} E^x[u^\pm \circ B_{\tau_n} M_{\tau_n}(n)] =$

$E^x[u^\pm \circ B_{\tau} M_{\tau_-}]$ . Thus (ii) holds.

(ii)  $\Rightarrow$  (iii), clearly.

(iii)  $\Rightarrow$  (i): Given  $K$ ,  $\tau = \tau_K$ , define  $w(x) = E^x[u \circ B_{\tau} M_{\tau_-}]$  for every  $x \in D$ . Let

$A = \{w > u\}$ . If  $A$  is not polar, we can find  $\nu \in \mathcal{M}_2$  with compact support in  $K$

and  $\nu(A^c) = 0$ . Then  $\int w d\nu > \int u d\nu$ . But  $\int w d\nu = E^\nu[u \circ B_{\tau} M_{\tau_-}] = \int u d\nu$  by

(6.19), contradiction. Thus  $A$  is polar. Similarly  $\{w < u\}$  is polar. Hence  $w = u$  quasi everywhere. Thus, for every choice of  $K$ , letting  $\tau = \tau_K$ , for quasi every  $x \in D$ ,

$$(6.21) \quad u(x) = E^x[u \circ B_{\tau} M_{\tau_-}].$$

Now let  $K$  be fixed,  $\tau = \tau_K$ . Let  $\mu_n \in \mathcal{M}_2$ ,  $\mu_n \uparrow \mu$ . Let  $u_n^\pm$  solve the  $\mu_n$ -Dirichlet problem on  $K$  with data  $u^\pm$  on  $\partial K$ . Again  $u_n^+ - u_n^- \rightarrow f$  quasi everywhere, where  $f$  is the solution of the  $\mu$ -Dirichlet problem on  $K$  with data  $u$  on  $\partial K$ . As in the earlier argument, for every  $x$  in  $K$ , the dominated convergence theorem shows  $\lim_{n \rightarrow \infty} E^x[u^\pm \circ B_{\tau_n} M_{\tau_n}(n)] = E^x[u^\pm \circ B_{\tau} M_{\tau_-}]$ . By (6.21) we then have  $u = f$  quasi everywhere on  $K$ . Thus  $u$  is  $\mu$ -harmonic locally on  $D$ . This proves Theorem 6.1.

Let  $D$  be a bounded open set. Consider the  $\mu$ -Dirichlet problem for  $u$  on  $D$  with data  $g$  on  $\partial D$ . As usual we assume  $g \in H^1_{loc}(R^d)$  and  $u - g \in H^1_0(D)$ . Let  $D_n$  be open,  $\bar{D}_n \subset D_{n+1}$ ,  $D_n \uparrow D$ . Let  $\tau_n, \tau$  be the first exit times of  $D_n, D$

respectively. Then  $\tau_n \uparrow \tau$ . Fix  $x \in D$ . Let  $\psi_x(n), \psi_x$  be defined by  $\int h d\psi_x =$

$$E^x[h \circ B_{\tau_n} M_{\tau_n}], \quad \int h d\psi_x(n) = E^x[h \circ B_{\tau_n} M_{\tau_n}], \text{ for any } h \text{ bounded Borel on } R^d.$$

Since  $M_{\tau_n} \downarrow M_{\tau_-}$ , it is easy to see that  $\psi_x(n) \rightarrow \psi_x$  in  $H^{-1}(R^d)$ , so that

$$\int u d\psi_x(n) \rightarrow \int g d\psi_x \text{ as } n \rightarrow \infty. \text{ For q.e. } x, u(x) = \int u d\psi_x(n) \text{ for all } n. \text{ Thus } u(x) =$$

$$\int g d\psi_x = E^x[g \circ B_{\tau} M_{\tau_-}], \text{ so we have shown:}$$

**Remark 6.1** The solution  $u$  of the  $\mu$ -Dirichlet problem on  $D$  with data  $g$  is given, for quasi every  $x \in D$ , by

$$(6.22) \quad u(x) = E^x[g \circ B_{\tau} M_{\tau_-}], \text{ where } \tau \text{ is the first exit time of } D.$$

We recall that a point  $x \in R^d$  is called a regular Dirichlet point for  $\mu$  if every  $u$  which is  $\mu$ -harmonic near  $x$  is continuous at  $x$  and vanishes there.

**Theorem 6.2** A point  $x$  is regular for  $\mu$  if and only if  $M_0(\mu)=0$ ,  $P^x$ -a.e..

**Proof** (i) Suppose  $x$  is regular. Let  $K$  be a small ball centered at  $x$ . Let  $u$  be the solution of the  $\mu$ -Dirichlet problem with data  $\equiv 1$  on  $\partial K$ . For quasi every  $y$  in  $K$ ,

$$(6.22) \quad u(y) = E^y[M_{\tau-}], \text{ where } \tau \text{ denotes the first exit time of } K.$$

For any  $t > 0$ , and any  $z \in K$ ,  $E^z[\chi_{\{\tau > t\}} u \circ B_t] = E^z[\chi_{\{\tau > t\}} E^{B_t}[M_{\tau-}]] = E^z[\chi_{\{\tau > t\}} E[M_{\tau-} \circ \theta_t | \mathcal{G}_t]] \geq E^z[E[\chi_{\{\tau > t\}} M_t M_{\tau-} \circ \theta_t | \mathcal{G}_t]] = E^z[E[\chi_{\{\tau > t\}} M_{(t+\tau) \circ \theta_t} | \mathcal{G}_t]] = E^z[E[\chi_{\{\tau > t\}} M_{\tau-} | \mathcal{G}_t]] = E^z[\chi_{\{\tau > t\}} E[M_{\tau-} | \mathcal{G}_t]] = E^z[E[M_{\tau-} | \mathcal{G}_t]] - E^z[\chi_{\{\tau \leq t\}} E[M_{\tau-} | \mathcal{G}_t]]$ . Thus for  $z \in K$ ,  $t > 0$ ,

$$(6.23) \quad E^z[\chi_{\{\tau > t\}} u \circ B_t] \geq E^z[M_{\tau-}] - P^z(\tau \leq t).$$

Since  $u$  is continuous and  $u(y) \rightarrow 0$  as  $y \rightarrow x$ , for q.e.  $y$ , and  $0 \leq u \leq 1$ , we see that  $E^x[\chi_{\{\tau > t\}} u \circ B_t] \rightarrow 0$  as  $t \rightarrow 0$ . Since  $P^x(\tau \leq t) \rightarrow 0$  as  $t \rightarrow 0$ , taking  $z = x$  in

(6.23) we have  $E^x[M_{\tau-}] = 0$ , so  $M_{\tau-} = 0$ ,  $P^x$ -a.e.. Since this is true for all  $K$ , we must  $M_0 = 0$ ,  $P^x$ -a.e..

(ii) Suppose that  $M_0 = 0$ ,  $P^x$ -a.e.. Let  $K$  be a fixed open ball centered at  $x$ ,  $\tau = \tau_K$ . Let  $\psi_y$  be defined by  $\int h d\psi_y = E^y[h \circ B_{\tau} M_{\tau-}]$ , for  $h$  bounded Borel on  $R^d$ ,  $y \in K$ . Let  $u(y)$  be defined for all  $y \in R^d$  by (6.22). For any  $z \in K$ ,  $u(z) \leq P^x(\tau \leq t) + E^z[\chi_{\{\tau > t\}} M_{\tau-}]$ .  $E^z[\chi_{\{\tau > t\}} M_{\tau-}] \leq E^z[\chi_{\{\tau > t\}} M_{(t+\tau) \circ \theta_t}] \leq$

$$E^z[M_{(t+\tau) \circ \theta_t}] = E^z[M_t M_{(\tau) \circ \theta_t}] \leq E^z[(M_{\tau-}) \circ \theta_t]. \text{ Thus}$$

$\limsup_{z \rightarrow x} u(z) \leq P^x(\tau \leq t) + E^x[(M_{\tau-}) \circ \theta_t]$ , for every  $t > 0$ ,  $P^x$ -a.e.. Let  $\sigma$  be a positive, measurable function taking rational values such that  $\sigma < \tau$ ,  $P^x$ -a.e.. Then  $M_{\tau-} \circ \theta_t \leq M_{\sigma} \circ \theta_t$ . By (3.31),  $M_{\sigma} \circ \theta_t \rightarrow M_{\sigma}$ ,  $P^x$ -a.e.. Thus

$\limsup_{t \rightarrow 0} E^x[(M_{\tau-}) \circ \theta_t] \leq E^x[M_{\sigma}] \leq E^x[M_0] = 0$ . Thus  $\limsup_{z \rightarrow x} u(z) = 0$ , so  $\psi_z \rightarrow 0$  in total variation norm as  $z \rightarrow x$ . The measures  $\psi_z$  are dominated by a finite uniform measure on  $\partial K$ , so  $\psi_z \rightarrow 0$  in  $H^{-1}(R^d)$ , so that for any  $v$  in  $H^1$

near  $\partial K$ ,  $\int v d\psi_z \rightarrow 0$  as  $z \rightarrow x$ . Now let  $u$  be an arbitrary  $\mu$ -harmonic function

near  $x$ . For  $K$  small, and quasi every  $z$  near  $x$ ,  $u(z) = \int u d\psi_z$ . Thus for these  $z$ ,  $u(z) \rightarrow 0$  as  $z \rightarrow x$ . This proves Lemma 6.4.

We now give one more criterion for regularity at a point. To avoid trivial details, we assume  $d \geq 3$ .

**Lemma 6.4.** Let  $d \geq 3$ ,  $\mu \in \mathbf{M}_0$ ,  $x \in \mathbb{R}^d$ . Then  $x$  is regular for  $\mu$  if and only if for every finely open set  $V$  containing  $x$ ,  $\text{Pot } \rho(x) = \infty$ , where  $\rho = \chi_V d\mu$ .

**Proof** We may take  $\mu \in \mathbf{M}_1$ . Suppose  $\text{Pot } \rho(x) < \infty$  for some  $V$ . Let  $\tau$  be the first hitting time of  $V^c$ . By (4.10),  $E^x[K_\tau(\mu)] < \infty$ . Thus  $K_{0+}(\mu) < \infty$ ,  $P^x$ -a.e., so  $M_0(\mu) > 0$ ,  $P^x$ -a.e., and so  $x$  is not regular.

Conversely, suppose  $M_0(\mu) > 0$ ,  $P^x$ -a.e.. Let  $\tau$  be the first time  $K_t(\mu) \geq 1$ .  $\tau > 0$ ,  $P^x$ -a.e.. Since Brownian motion has predictable  $\sigma$ -fields, we can find a stopping time  $\sigma$  with  $\sigma < \tau$ ,  $P^x$ -a.e., and  $P^x(\sigma > 0) = \alpha > 0$ .

$E^x[K_\sigma(\mu)] < \infty$ , so by (4.10),  $\int (\text{Pot } \lambda - \text{Pot } \lambda_\sigma) d\mu < \infty$ , where  $\lambda = \delta_x$ . By [12], I.XII.4,  $\text{fine-limit}_{y \rightarrow x} (\text{Pot } \lambda - \text{Pot } \lambda_\sigma)(y) / (\text{Pot } \lambda)(y) = 1 - \alpha$ , so we can find a fine-open set  $V$  containing  $x$  such that on  $V$ ,  $(\text{Pot } \lambda - \text{Pot } \lambda_\sigma) \geq \beta \text{Pot } \lambda$ , where  $\beta > 0$ . Thus  $\int_V \text{Pot } \lambda d\mu < \infty$ , or  $\text{Pot } \rho(x) < \infty$ , for  $\rho = \chi_V d\mu$ . This

proves Lemma 6.4.

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| 251 | Suzanne M. Lenhart,                                      | Viscosity Solutions for Weakly Coupled Systems of First Order PDEs   | 287 | Lj Kalfal, Yan Ningning,                                 | The Extrapolation for Boundary Finite Elements  |
| 252 | M. Cranston, E. Fabes, Z. Zhao,                          | Condition Gauge and Potential Theory for the Schrödinger Operator  | 288 | R. Durrett, R.H. Schonmann,                              | Stochastic Growth Models  |
| 253 | H. Brezis, J.M.Coron, E.H. Lieb,                         | Harmonic Maps with Defects   | 289 | David Kinderlehrer,                                      | Remarks about Equilibrium Configurations of Crystals  |
| 254 | A. Carverhill,   | Flows of Stochastic Dynamical Systems: Nontriviality of the Lyapunov Spectrum                              | 290 | D.G. Aronson, J.L. Vazquez,                              | Eventual $C^\infty$ -Regularity and Concavity for Flow In One-Dimensional Porous Media                                  |
| 255 | A. Carverhill,   | Conditioning A 'Lifted' Stochastic System in a Product Case  | 291 | L.R. Scott, J.M. Boyle, B. Bagheri,                      | Distributed Data Structures for Scien Computation   |
| 256 | R.J. Williams,   | Local Time and Excursions of Reflected Brownian Motion   | 292 | J. Douglas, Jr., P.J.Pees Lemo, T. Arbogast, T. Schmitt, | Simulation of Flo. In Naturally Fractured Petroleum Reservoirs  |
| 257 | H. Föllmer, S. Orey,                                     | Large Deviations for the Empirical Field of a Gibbs Measure  | 293 | D.G. Aronson, L.A. Caffarelli,                           | Optimal Regularity for One-Dimensional Por. Medium Flow   |
| 258 | A. Leizarowitz,  | Characterization of Optimal Trajectories On an Infinite Horizon  | 294 | Haim Brezis,   | Liquid Crystals and Energy Estimates for $S^2$ -Valued Maps   |
| 259 | Y.Gige, T. Miyakawa, H. Osada,                           | Two Dimensional Navier Stokes Flow with Measures As Initial Vorticity                                      |     |  |   |
| 260 | M. Chipot, V. Ollker,                                    | Sur Une Propriete Des Fonctions Propres De L'Operateur De Laplace Beltrami                                 |     |  |   |
| 261 | V. Perez-Abreu,  | Decompositions of Semimartingales On Duals of Countably Nuclear Spaces                                     |     |  |   |
| 262 | J.M. Ball,   | Does Rank-One Convexity Imply Quasiconvexity   |     |  |   |
| 263 | B. Cockburn,   | The Quasi-Monotone Schemes for Scalar Conservation Laws. Part I  |     |  |   |
| 264 | K.A. Perlick-Spector,                                    | On Radially Symmetric Simple Waves in Elasticity   |     |  |   |
| 265 | P.N. Shivakumar, Chi-Sing Man, Simon W. Rabkin,          | Modelling of the Heart and Pericardium at End-Diastole   |     |  |   |
| 266 | Jose-Luis Menaldi,                                       | Probabilistic View of Estimates for Finite Difference Methods  |     |  |   |
| 267 | Robert Hardt, Harold Rosenberg,                          | Open Book Structures and Unicity of Minimal Submanifolds   |     |  |   |
| 268 | Bernardo Cockburn,                                       | The Quasi-Monotone Schemes for Scalar Conservation Laws  |     |  |   |
| 269 | H.R. Jauslin, W. Zimmerman, Jr.,                         | Dynamics of a Model for an Ac Josephson Effect In Superfluid $^4He$  |     |  |   |
| 270 | A.K. Kapila,   | Introductory Lecture on Reacting Flows   |     |  |   |
| 271 | J.C. Taylor,   | Do Minimal Solutions of Heat Equations Characterize Diffusions?  |     |  |   |
| 272 | J.C. Taylor,   | The Minimal Eigenfunctions Characterize the Ornstein-Uhlenbeck Process                                     |     |  |   |
| 273 | Chi-Sing Man, Quan-Xin Sun,                              | On the Significance of Normal Stress Effects In the Flow of Glaciers                                       |     |  |   |
| 274 | Omar Hijab,  | On Partially Observed Control of Markov Processes  |     |  |   |
| 275 | Lawrence Gray,   | The Behavior of Processes with Statistical Mechanical Properties   |     |  |   |
| 276 | R. Hardt, D. Kinderlehrer, M. Luskin,                    | Remarks About the Mathematical Theory of Liquid Crystals   |     |  |   |