Pricing under Imperfect Awareness

A Dissertation

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Abstract

The study of price determination in markets has been a defining element of the science of economics. In this dissertation, I have developed models of strategic pricing under imperfect awareness. Imperfect awareness in this context means that decision makers are not aware of every trader operating in the market, instead, trading is constrained to the set of traders which the decision maker is aware of. The degree of trader's awareness can evolve over time. I study pricing dynamics in such markets. I show that prices and allocations approximate perfect competition as awareness increases in a variety of environments.

In Chapter 2, I study pricing in a dynamic duopoly. Buyers may be imperfectly aware of operating sellers, but they can gain awareness regarding sellers through a word-of-mouth matching mechanism. I show that there is a unique subgame perfect equilibrium. The unique equilibrium features price dispersion with asymmetric price posting strategies. I show that, depending on the parameters, the distribution of prices of one seller first order stochastically dominates the prices posted by the other seller. I also show that the price posting strategies of each seller depend on his or her relative degree of market experience.

In Chapter 3, I extend the model developed in Chapter 2 to an infinite horizon environment with a continuum of sellers. I show that a Markov perfect equilibrium exists, is unique, and features asymmetric price posting strategies. In this equilibrium entrants post prices that are strictly lower than prices posted by more mature competitors, average markups decline over time as the market for the product matures, and the distribution of prices features substantial price dispersion at both the individual and aggregate levels. This model explains a several deviations from competitive conditions that are empirically observed in product markets as being caused by imperfect awareness.

In Chapter 4, I study a market with a continuum of buyers and sellers (such as the model of Chapter 3). In this case, I focus on a static setting and introduce differentiated products. Consumers have imperfect awareness regarding product varieties. Even in this market with differentiated products, I show that the equilibrium approximates perfect competition when consumers are aware about a high number of product varieties. It concludes that, when unawareness about product varieties exist, markups increase when the degree of product differentiation is higher, but for any degree of product differentiation, markups vanish when unawareness about varieties vanish.

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Chapter 1

Introduction

The theory of value based on competitive equilibrium remains the cornerstone of modern economic theory (Debreu (1959)[13]). This theory of value assumes perfect competition, but real product markets are characterized by many deviations from perfectly competitive conditions. My objective in this dissertation is to develop a theory of pricing under trading frictions that can be consistent with these deviations but that also nests the model of perfect competition as its limiting case when trading frictions vanish.

To develop this theory, I study a novel type of frictions of trading: frictions exist if the decision makers have imperfect awareness of the other traders operating in the market. In this case, the set of trading opportunities available to a decision maker is constrained to the set of other traders in the market that the decision maker is aware of. In the models I study in the dissertation I represent this imperfect awareness through a random network determined by a pairwise random matching process, and I apply these analytical tools to shed light on a variety of problems.

A motivation for this novel way of modeling frictions of trading is that it allows for the degree of trading frictions to be endogenous and dynamic: With the passage of time decision makers learn about the other traders in the market. In older, well-established markets, traders have accumulated a larger network of contacts, so the degree of frictions is lower than in a newly established market. This theory implies that well-established markets more closely approximate competitive conditions than newly established markets.

The models studied here are closely related to the literature on random matching and bargaining[1]. In models of random matching and bargaining there are multiple rounds of matching. At each round, decision makers are divided into small coalitions and can interact only with other coalition members. Typically, these coalitions are pairs of traders. The costs of awaiting for additional matching rounds represents the frictions of trading.

In the class of models studied in this dissertation, at a given point in time, a trader can trade with any other traders they are currently aware of. Frictions of trading are represented by the fact that traders might not be aware of every other trader that is operating in the market. In dynamic versions of these models, decision makers can become aware of additional traders over time, accumulating more trading contacts. These contacts are available at any point in time after being discovered. In particular, I assume that buyers become aware of the sellers operating in the market through the word-of-mouth from other buyers that they randomly meet.

This novel approach allows for the study of problems that have not yet been studied in existing literature. In particular, it allows for the analysis of markets with frictions that change endogenously over time, as decision makers discover other buyers and sellers operating in the market. In addition, the added flexibility of this approach of describing frictions of trading is that it allows for the analysis of market frictions in a static game instead of a sequential model of matching and bargaining.

For example, the models studied here can explain the tendency for profit margins to decline over time in specific industries: the passage of time implies in decreasing frictions for trading as traders become more well informed about other market participants. As buyers become aware of a greater number of competing sellers, it becomes harder for individual sellers to extract surplus from their customers, which more easily shift their demand to other sellers. In addition, this class of models can explain several features of seller behavior observed in product markets such as the fact entrant sellers tends to undercut incumbents to accumulate demand, features replicated in the equilibrium of the models presented in chapters 2 and 3. In Chapter 4, I show that when trading frictions are low the equilibrium approximates perfect competition even in

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[1] E.g., Osborne and Rubinstein (1990),Gale (2000), Lauermann (2013).

a model of product differentiation, as long as there exists a large number of varieties and the degree of differentiation between varieties is low.

Chapter 2 presents a dynamic duopoly model where sellers compete by posting prices and acquire customers over time as buyers gain awareness about the sellers through word-of-mouth. There is an entrant and an incumbent seller. The entrant will arrive at the market later than the incumbent and so will have a smaller number of buyers who are aware of the seller, that is, a smaller customer base. The seller with smaller customer base will have more elastic demand because the fraction of its customers who are aware of the competitor is larger. This does not necessarily imply that the smaller seller will post lower prices. When the sellers grow the degree of overlap between the groups of buyers who are aware of each seller also grows, which lowers equilibrium prices. The seller with smaller customer base will have greater potential for growth which means that if he or she posts lower prices than the larger seller its customer base might grow so much that future prices will decrease profits in present value by a greater degree than the increase in present profits. In other words, the smaller entrant might post high prices to grow more slowly which decreases the intensity of competition in the future, leading to higher profits.

Chapter 3 extends the model developed in Chapter 2 to an environment with a continuum of sellers where large number of sellers enter and exit the market every period. I show that the model's Markov perfect equilibrium is unique and efficient. Because there is a large number of sellers, each seller has negligible impact on the distribution of prices in equilibrium. Therefore, the dynamic effects of customer base accumulation on equilibrium prices are not taken in consideration by individual sellers. As a result, in equilibrium entrant sellers who have smaller customer bases, post strictly lower prices than incumbent sellers. Over time, as sellers accumulate more customers, the degree of competition between sellers increases which drives markups down, eventually to marginal cost if seller exit rates are reduced to zero.

Chapter 4 studies a static setting, without customer capital accumulation. It extends the environment from markets for identical goods to a market where each seller is a producer with the monopoly of the technology to produce a particular variety of the good. I study an environment with product differentiation and private information regarding valuations and consumer awareness. I show that as the average number of producers that consumers are aware of increases to infinity then equilibrium prices converge to the competitive equilibrium. In addition, I perform welfare analysis and I show that the allocation in equilibrium is efficient when constrained to the set of feasible allocations (defined in this environment with private valuations as the set of allocations that can be implemented by a mechanism).

Chapter 2

Price dispersion in dynamic duopoly

2.1 Introduction

The purpose of this study is to define and characterize equilibrium in a product market in which two buyers compete by posting prices and buyers acquire information about the sellers through a process of word of mouth contagion. This study extents the class of models originating from Burdett and Judd (1983) [5]'s seminal equilibrium price dispersion results to an environment that features dynamic customer-capital accumulation.

I call a seller's customer base the set of buyers that are aware of the seller, which is the set of buyers that observe the seller's posted price and can choose to purchase from the seller. The word of mouth contagion process increases the set of buyers who are aware of a seller and it is determined by the seller's past sales activity. When formulating her price posting strategy a seller considers the effects of her strategy on future customer base accumulation besides the present effect on sales. When a seller considers posting a lower price that will imply in higher sales in the present and therefore higher degree of diffusion of awareness about the seller among the buyers and hence a larger number of potential customers in the future. However, an increase in the seller's customer base in the future will have ambivalent consequences for the expected profits of the seller in future periods.

In order to study competition between an entering seller and an incumbent seller I assume that one seller enters the market before the other. This implies in a dynamic competition between the two sellers that features asymmetry in their customer bases as the incumbent seller has a longer amount of time to accumulate customers. I show that any equilibrium in this environment is characterized by price dispersion, in the sense of randomized price posting strategy by the sellers. In this case the equilibrium features asymmetric price posting strategies by the two sellers as it is nearly always the case that one seller's randomized price posting strategy will first order stochastically dominate the strategy of the other seller. One particular feature of the equilibrium in this environment with a finite population of sellers is that varying the parameters the order of stochastic dominance in the equilibrium price posting strategies shifts between the entrant seller and the incumbent.

The empirical literature has shown that when a seller enters a specific product market it's demand grows slowly and takes a long time for its size to close relative to its more mature competitors (Foster et al (2008, 2016) [24, 25]). Other studies have also shown that in product markets in a given geographical area there is significant variation in the prices of transactions. In particular, Kaplan and Menzio (2015) [48] decompose each transaction price into three components: (i) a store component, the average price for transactions at a particular store relative to all other stores, (ii) a store-specific good component, defined as the average price for transactions of a particular good at that store relative to the prices of all goods in that store, (iii) a transaction component, defined as the price of a good at that particular transaction relative to the average price for the same good of all transactions made in that store. They find that the store-component contributes 10% of the overall variance in prices, the store-specific good component good component contributes 25-45% depending on how broadly they define what constitutes a "single good" and the transaction component accounts for the remaining variance.

The model presented in this paper replicates these features. The diffusion of awareness among the buyers caused by past sales activity explains the slow growth in seller specific demand while the equilibrium features asymmetric randomized price posting strategies between the two sellers which explains price dispersion between different sellers and among transactions with the same seller.

Contrasting the results of a related study, in Guthmann (2019) [36] the equilibrium also features price dispersion in an environment with a continuum of sellers. In this paper the supports of the distribution of the prices posted by each individual seller are identical while in the aforementioned study in equilibrium the supports of the price posting strategies for sellers with different customer bases have disjoint interiors and the support of the price posting strategy a seller type (given by it's customer base) converges to a point if the cumulative distribution of customer bases is continuous. This is a weakness of the theoretical framework used in that study because Kaplan and Menzio (2015) [48] found that 45-65% of the variance in prices for transactions of the same good is due to differences in prices for transactions made at the same store. Their findings are consistent with this study where sellers with different customer bases post prices according to a distribution that has the same support.

The paper is organized as follows: subsection 2.1.1 gives an overview of the related literature. Section 2.2 presents the model where customer capital accumulation occurs through contagion, subsection 2.2.1 presents the environment of the model, subsection 2.2.2 defines the equilibrium of the model. In subsection 2.2.2.1, I solve for the equilibrium in the period 2, in subsection 2.2.2.2, I solve for the equilibrium in period 1, and I finish characterizing the equilibrium in subsection 2.2.2.3, solving for the trivial case of period 0. Section 2.4 concludes.

2.1.1 Related Literature

The aforementioned Burdett and Judd (1983) paper is a seminal contribution for the concept of equilibrium price dispersion in a market for an physically identical good. Similar studies include Stahl (1989) [72] and Menzio and Thrachter (2015) [57], which tries to extent the concept of equilibrium with randomized price posting strategies to environments that allow for sequential search. This study extents the model by allowing sellers to accumulate a customer base, extending the concept of equilibrium price dispersion to a fully dynamic environment.

Previous studies also feature customer-capital accumulation. Gourio and Rudanko (2014) [34] and Gilbukh and Roldan (2018) [32] use directed search to analyze customer-capital accumulation. A major discrepancy between this study and these previously mentioned models is that customer accumulation mechanism used here consists of word of mouth contagion (which is explicitly defined subsection 2.2.1). Imperfect awareness has been previously used in model featuring imperfect competition in Perla (2017) [63]. The model presented here also has similarities to Fishman and Rob (2005) [23] as in both cases, from the perspective of the buyers search is assumed to be costless.

The motivation for developing a model that combines price dispersion and customer-capital is that the empirical evidence suggests that seller's specific demand exists and that it is accumulated over time. Recent empirical studies such as Foster et. al. (2008, 2016) [24] show that in product markets for homogeneous goods incumbent plants sell their output for higher prices than entering plants. They also find robust evidence that incumbent plants have higher plant specific demand than their entering competitors.

2.2 Model

2.2.1 Environment

Consider a market for a single perishable indivisible good. Time is discrete and there are three periods, indexed by $t = 0, 1, 2$. There is a continuum of identical buyers of unit of measure. Each buyer has a unit demand for the good in each period and reservation price equal to one. There are two sellers in this economy, which I call E and I (which stand for "entrant" and "incumbent" respectively). The sellers can produce any quantity $q \in \mathbb{R}_+$ of the indivisible good at constant marginal cost normalized to zero. So when a seller sells the indivisible good to a measure of buyers $m \in [0, 1]$, each buyer buys one unit and hence it means the quantity q produced and sold by the seller is $q = m$, and if the good was sold at a price p the seller's revenue is $p \times q$ which is also the seller's profit. See Gottardi and Serrano (2005) [33] for another example of a model that is characterized by finitely many sellers and a continuum of buyers.

Buyers may know zero, one or two sellers. Let M_t^j represent the set of buyers who are aware of seller j in period t and $m_t^j \in [0, 1]$ be the corresponding measure of this set of buyers. I call the set of buyers who know seller j the customer base of j. Let $m_t = (m_t^I, m_t^E)$ be the pair of measures of the customer bases of each seller in period t.

There is only one seller active in the market in period $t = 0$. That is, the other seller's customer base measure is assumed to be zero. When a seller begins operating the market it's initial customer base measure is $m_s \in (0, 1)$. I is the seller who is active in period 0 while seller E is not operating in the market. In period 1, seller E enters the market as it's customer base grows (exogenously) to the entry value m_s.

In this model the growth of the customer-base results from buyers discovering the seller through meeting other buyers through a random contagion mechanism.[1] That is, buyers learn about a seller through word of mouth from shoppers of that seller. For a concrete example, I am assuming that consumers discover different brands of smartphones from noticing the brands their friends are using. I do consider contagion through word of mouth as the main form of customer acquisition in this study as Trusov, Bucklin, & Pauwels (2009) [74] found that word of mouth effects in customer acquisition are much stronger relative to other marketing tools. In particular, the elasticity of customer acquisition to word of mouth effects was estimated to be 20 to 30 times higher than promotional events and media appearances respectively. See section 2.3 for extensions that characterize the equilibrium with other forms of customer accumulation.

I assume the awareness of each of the sellers by the buyers is independent across buyers. That means that the probability that a buyer is aware of one of the sellers is independent of being aware the other seller. Let $-j$ be the seller competing with seller j. Then the unconditional probability that a buyer knows j is m^j. The probability a buyer knows j conditional on knowing $-j$ is then $\frac{m^j m^{-j}}{m^{-j}} = m^j$ which is the same as the unconditional probability that a buyer is aware of j and the probability that a buyer knows both j and $-j$ is just the

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[1] This "discovery" is more broader than simple finding out about the existence of the sellers: As Foster, Haltiwanger and Syverson (2016) state: "Our read of the evidence is that the customer 'learning' that drives demand stock growth is much broader than the simple process of buyers finding out about the existence of a producer. While spotty information about mere existence might be consistent with the large gaps in idiosyncratic demand present at plants' births, it seems unlikely to explain why convergence takes upwards of 15 years. We posit that learning involves much deeper components, like details of producers' product attributes, the quality and quantity of their bundled services, the consistency of their operations, their expected longevity, and so on. Having to learn about these features can impart considerable inertia into producers' demand stocks."

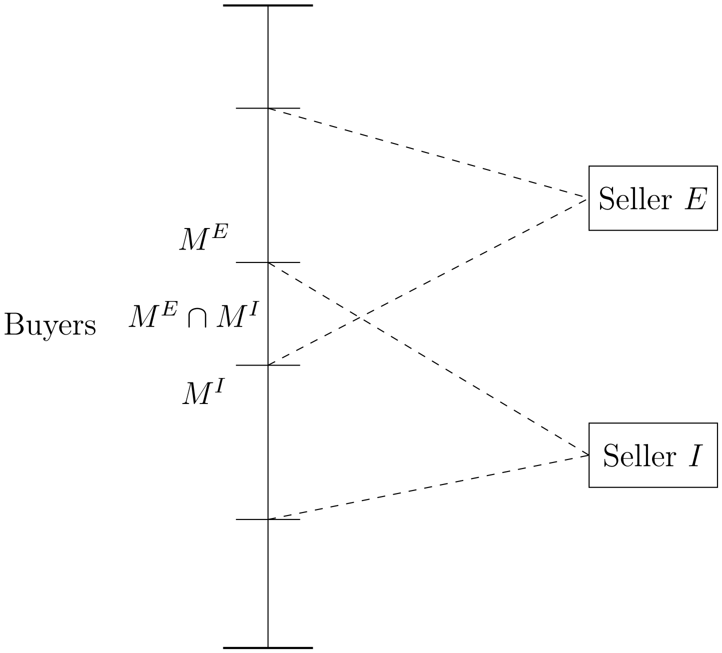

Figure 2.1: Customer bases of the two sellers when $m^j = .4, \forall j$.

product of the probabilities of knowing each seller, $m^j m^{-j}$. Independence is just an assumption made for simplicity of exposition. See the appendix subsection 5.1.4 for an explicit characterization of the (trivial) consequences of relaxing independence.

Let $q^j_{t,i} \in \{0\} \cup \mathbb{N}$ be the quantity of the indivisible good that buyer i purchases from seller j in period t. Let q^j_t be the total quantity purchased by customers of j in period t, it satisfies $q^j_t = \int_0^1 q^j_{t,i} di$, where buyers are indexed by $i \in [0,1]$. Each customer has a unit demand for the good, hence $q^j_{t,i} \in \{0,1\}$ which imply that q^j_t is equal to the measure of customers who know the seller j and choose to buy at j. I call the customers who choose to purchase from j in a period t the seller j's active customers in period t.

To provide random matching-based microfoundations to the law of motion for the customer base, suppose that buyers who are unaware of the seller can discover the seller by meeting with active customers of the seller. Let $d(q^j_t, 1 - m^j_t)$ be the probability that a buyer who does not know j in period t meets a customer of j by period $t+1$. The discovery probability $d(.,.)$ is a function of q^j_t, the number of currently active customers of the seller j, and $1 - m^j_t$, the population of buyers who do not know the seller. Hence, the total measure of new potential customers who discover the seller between period t to period $t+1$ is $\Phi(q_t^j, 1-m_t^j) = d(q_t^j, 1-m_t^j) \times (1-m_t^j)$. I assume Φ is a constant returns to scale matching function (as in DMP matching models) and it's increasing and concave in q_t^j and $1-m_t^j$: that is, the measure of matchings between active customers of j and unaware buyers is increasing in the number of active buyers and unaware buyers but with decreasing returns.

Foster, Haltiwanger and Syverson (2016) [25] finds robust empirical evidence that seller's demand depreciates over time. Hence, I include in the model a customer base depreciation factor $\delta \in [0,1)$ which implies that at a given period buyers exit the market and are be replaced by new buyers at a rate δ. [2]

Taking in consideration both the discovery of the seller by buyers and depreciation of the customer base the stock of potential customers of seller j evolves according to the following law of motion:

$$m_{t+1}^j(q_t^j, m_t^j) = (1-\delta)m_t^j + \Phi(q_t^j, 1-m_t^j). \tag{2.1}$$

Note that $\Phi(.,.)$ is only a function of j's sales and not of $-j$'s which implies that an increase in the competing seller sales does not affect the probability of discovery of firm j. However, an increase in the competing seller $-j$'s sales implies in a larger fraction of j's potential customers choosing to buy from $-j$. Hence j's sales will decrease. Thus the probability of a buyer discovering j decreases next period if $-j$ sales are higher in the current period.

Seller compete by posting prices. A seller j chooses at each period to post a price $p \in (-\infty, 1] = A$, so A is seller j's action set which is assumed to be identical across sellers. At each period sellers choose which price to post for that period while buyers can choose which seller among those they are aware of to purchase the perishable good from. I assume that buyers always purchase from the lowest-price seller they know. Sellers can also mix strategies in their action set A. Let σ_t^j be a cumulative distribution that characterizes j's mixed strategy, $\sigma_t^j : [0,1] \rightarrow \mathcal{F}([0,1])$, where $\sigma_t^j(m_t, p)$ specifies the probability that seller j will post a price equal or lower than p in period t and $m_t = (m_t^I, m_t^E) \in [0,1]^2$ is the pair of customer base measures in period t.

This is a game with finitely many players (2 players which are I and E), where

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[2] Or alternatively, it can be interpret that buyers forget about seller j at a rate δ.

each player's action set is a compact metric space and the payoff functions feature discontinuities (since equation 2.2 will be discontinuous at atom's of the competitor's distribution of prices posted). As Simon and Zame (1990) [70] shows that Nash equilibrium in games with discontinuous payoffs with endogenous sharing rules always exists. Here I assume a specific tie-breaking rule. In period 2, if both sellers post the same price the buyer will always purchase from the seller with the smallest customer-base. In period 1 if both sellers post the same price the buyer will always purchase from the seller with the highest $\underline{p}^j(m)$ as defined by equation 2.19.

As I will show, this tie-breaking rule is sufficient to guarantee the existence and uniqueness of equilibrium. It is a tie-breaking rule that only depends on the state of the market, which is described by the pair of customer bases.

I assume that each seller knows the measure of the customer bases of both sellers operating in the market but does not know whether their potential customers knows the competitor or not. I restrict attention to uniform pricing posting strategies which means the seller's posted price is not conditioned on individual customer types. Where "type" means whether the customer knows the competing seller of not. I do so because for this environment posting the same price for all buyers is an optimal truth telling mechanism.

Proposition 2.1. *An optimal sales mechanism for the seller is characterized by a uniform price for the buyers.*

Proposition 2.1 follows from the reasoning that in any type revealing mechanism where the price paid for the good varies with the type of customer some customers will have the incentive to lie to pay the lowest price in exchange for the good. Considering that valuations do not vary among buyers and the fact that the good is indivisible means that any form of discrimination of the buyers by the sellers is not feasible.

Let $P^j(p, \sigma_t^j, m_t)$ be the probability that a potential customer of the seller j purchases from the seller given j posts price p, $-j$ is posting prices according to a distribution σ_t^j and the state of the market is $m_t = (m_t^I, m_t^E)$. A buyer will always purchase from the seller j if that is the only seller he or she knows. If the buyer knows both sellers then he or she will always purchase from the lowest-price seller. Therefore $P^j_t(p)$ is equal to the probability that the potential customer is not part of the competing seller $-j$'s customer base measure m^{-j}_t plus the probability the buyer knows the competing seller and that the seller's posted price is lower than the competing seller's. [3] Therefore, assuming σ^{-j}_t doesn't have an atom at p, $P^j(p, \sigma^{-j}_t, m_t)$ satisfies

$$\begin{aligned} P^j(p, \sigma^{-j}_t, m_t) &= (1 - m^{-j}_t) + (1 - \sigma^{-j}_t(p))m^{-j}_t \qquad (2.2) \\ &= 1 - \sigma^{-j}(p)m^{-j}_t. \end{aligned}$$

If σ^{-j} has an atom at price p then $P^j(p, \sigma^{-j}_t, m_t)$ satisfies

$$P^j(p, \sigma^{-j}_t, m_t) = \begin{cases} 1 - \sigma^{-j}(p)m^{-j}_t & \text{if the tie-breaking rule favors -}j \\ 1 - \lim_{\hat{p}\uparrow p} \sigma^{-j}(\hat{p})m^{-j}_t & \text{if the tie-breaking rule favors } j \end{cases},$$

where "the tie-breaking rule favors j" means that buyers will buy the good from j in case of a tie in posted prices.

There are two possible quantities that the seller j can sell, $\overline{q}^j_t = m^j_t$ if j undercuts $-j$ and all it's customer base purchases from the seller, and $\underline{q}^j_t = (1 - m^{-j}_t) \times m^j_t$ if j is undercut by $-j$. For a continuous distribution σ^{-j}_t the probability of $\overline{q}^j_t$ being realized is $(1 - \sigma^{-j}_t(p))$ while the probability of $\underline{q}^j_t$ being realized is $\sigma^{-j}_t(p)$.

The expected quantity to be sold in the present period by a seller posting p, $E[q^j(p, \sigma^j_t, m_t)]$ is

$$\begin{aligned} E[q^j(p, \sigma^j_t, m_t)] &= (1 - \sigma^{-j}_t(p))m^j_t + \sigma^{-j}_t(p)(1 - m^{-j}_t)m^j_t \qquad (2.3) \\ &= m^i_t P^j(p, \sigma^j_t, m_t) \qquad (2.4) \\ &= m^{-j}_t(1 - m^{-j}_t \sigma^{-j}_t(p)). \end{aligned}$$

Let $\gamma^j(p, \sigma^{-j}, m_t)$ be the expected profits of the seller j by posting price p in period t given a vector of customer measures $m = (m^I_t, m^E_t)$ and that the competing seller is posting prices distributed according to a distribution σ^{-j}.

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[3] If σ^{-j}_t has an atom at p then there is a positive probability of a tie in posted prices and hence $P^j_t(p)$ will also depend on the specific tie-breaking rule.

Since marginal cost is zero expected profits are just the expected quantity sold multiplied by the posted price:

$$\gamma^j(p, \sigma^{-j}, m_t) = p \times E[q^j(p, \sigma_t^j, m_t)]. \tag{2.5}$$

Sellers care about profits in the future therefore I need to explicitly determine the seller's present value for actions taken in each period to determine the payoffs of each subgame of this game. I assume that sellers discount future profits according to a discount factor $\beta \in (0, 1)$. Let Eu_t^j be the present value of discounted expected profits of seller j in period t.

Period 2: The present value of expected profits in period 2 if j post a price p^j is the value of current profits since it is the last period. Hence, the present value of discounted expected profits $Eu_2^j(.)$ of a seller j in period 2 posting p_2^j is just the current profits

$$Eu_2^j(p_2^j, \sigma_2^{-j}, m_2) = \gamma^j(p_2^j, \sigma_2^{-j}, m_2),$$

where σ_2^{-j} is the competitor's distribution of prices and $m_2 = (m_2^I, m_2^E)$ is the vector of customer bases.

Period 1: From equation 2.1, we know that the client bases in period 2 depend on sales quantity in period 1. For a seller $j \in \{I, E\}$ there exist two period 2 nodes: if $p^j < p^{-j}$, than sales quantity is $q^j = m^j$ and if $p^j > p^{-j}$, then $q^j = m^j(1 - m^{-j})$. Hence, given a vector of customer bases m_1 in period 1 there exists only a pair of possible vectors of customer base measures in period 2, $(\overline{m}_2^I, \underline{m}_2^E)$ and $(\underline{m}_2^I, \overline{m}_2^E)$, where $\underline{m}_2^j$ and $\overline{m}_2^j$ are the customer base measures associated with being undercut or undercutting the competing seller, respectively. They satisfy:

$$\overline{m}_2^j = (1 - \delta)m_1^j + \Phi(m_1^j, 1 - m_1^j) \tag{2.6}$$

$$\underline{m}_2^j = (1 - \delta)m_1^j + \Phi(m_1^j(1 - m_1^j), 1 - m_1^j). \tag{2.7}$$

Let $P(\overline{m}_2^j | p_1^j, \sigma_1^{-j})$ be the probability of $(\overline{m}_2^j, \underline{m}_2^{-j})$ being realized in the second period conditional on j posting p and $-j$ posting prices according to a distribution σ_1^{-j}.

Suppose that seller $j \in \{E, I\}$ posts p_1^j in period 1, $p_2^j(\overline{m}_2^j)$ in period 2 if $(\overline{m}_2^j, \underline{m}_2^{-j})$ is realized and $p_2^j(\underline{m}_2^j)$ if $(\underline{m}_2^j, \overline{m}_2^{-j})$ is realized. Let

$$p^{j,1} = (p_1^j, (p_2^j(\overline{m}_2^j), p_2^j(\underline{m}_2^j)))$$

be this triple of prices.

That seller $-j$ prices posting strategy consists of distributions

$$\sigma^{-j,1} = (\sigma_1^{-j}, \sigma_2^{-j}(\overline{m}_2^j), \sigma_2^{-j}(\underline{m}_2^j))$$

be the corresponding triple for $-j$. Then given a pair of customer base measures m_1 in period 1 the present value of expected profits in period 1 of seller j satisfies

$$Eu_1^j(p^{j,1}, \sigma^{-j,1}, m_1) = \tag{2.8}$$

$$\gamma^j(p_1^j, \sigma_1^{-j}, m_1) + \beta[P(\overline{m}_2^j | p_1^j, \sigma_1^{-j})\gamma^j(p_2^j(\overline{m}_2^j), \sigma_2^{-j}(\overline{m}_2^j), \overline{m}_2) + P(\underline{m}_2^j | p_1^j, \sigma_1^{-j})\gamma^j(p_2^j(\underline{m}_2^j), \sigma_2^{-j}(\underline{m}_2^j), \underline{m}_2)]. \tag{2.9}$$

Period 0: In period 0 only seller I is operating in the market so buyers will always choose to purchase from I hence for any price $p \in A_I$ that I posts the quantity sold will be $m_0^I = m_s$, therefore it's period 0 I's profits will be

$$\gamma_0^I(p_0^I) = p_0^j \times m_s$$

and it's customer-base in period 1 will grow to a measure m_1^I which satisfies

$$m_1^I = (1 - \delta)m_s + \Phi(m_s, 1 - m_s). \tag{2.10}$$

Therefore the present value of expected profits in period 0 for $j = I$ given that that $-j$'s strategy profile when he enters the market in period 1 is $\sigma^{-j,1}$, posting a price p_0^j, a profile of prices $p^{j,1}$ beginning in period 1 is

$$Eu_0^j(p_0^j, p^{j,1}, \sigma^{-j,1}) = p_0^j \times m_s + \beta Eu_1^j(p^{j,1}, \sigma^{-j,1}, m_1)$$

where $m_1 = (m_1^I, m_1^E) = (m_1^I, m_s)$, where m_1^I satisfies 2.10.

If $j = E$, then he has not entered the market yet so he doesn't post prices in period 0 while $-j = I$ and so $-j$ posts prices according to σ_0^{-j} in period 0 and

$\sigma^{-j,1}$ beginning in period 1. Therefore j's present discounted payoffs are just

$$Eu_0^j(\sigma_0^{-j}, p^{j,1}, \sigma^{-j,1}) = \beta Eu_1^j(p^{j,1}, \sigma^{-j,1}, m_1),$$

with $m_1 = (m_1^I, m_s)$.

2.2.2 Equilibrium

The solution concept used here is subgame perfection. That is, at every period the sellers maximize the present value of expected profits.

Definition 2.1. An equilibrium is a set of strategy profiles $\{(\sigma_t^I, \sigma_t^E)\}_{t=0,1,2}$ where $\sigma_t^j : [0,1] \times [0,1] \to \mathcal{F}^+((-\infty, 1], [0,1])$, where $\mathcal{F}^+([-\infty, 1], [0,1])$ is the space of cumulative probability distributions on $(-\infty, 1]$, such that posting prices on the support of σ_t^j maximizes the present value of expected profits Eu_t^j for every $t \in \{0,1,2\}$ and $j \in \{I, E\}$.

In words, an equilibrium is a triple $\{(\sigma_t^E, \sigma_t^I)\}_{t=0,1,3}$, such that it is present discounted expected profit maximizing for the sellers to post prices in the supports of these distributions for each period. Since subgame perfection is the solution concept I characterize the equilibrium by solving it's last period's subgame and work our way by backward induction.

Consider period 2. Let $v_2^j(m_2)$ be the value for a seller $j \in \{I, E\}$ where $m_2 = (m_2^I, m_2^E)$. It satisfies:

$$v_2^j(m_2) = \max_{p \in (-\infty, 1)} \gamma^j(p, \sigma_2^{-j}, m_2), \tag{2.11}$$

where $m_2 = (m_2^E, m_2^I)$ is the vector of customer bases and σ_2^{-j} is the competitor's equilibrium price posting strategy.

Let $v_0^j(m_0)$ and $v_1^j(m_1)$ the seller's value function for periods 0 and 1. They satisfy

$$v_1^j(m_1) = \max_{p \in (-\infty, 1)} \gamma^j(p, \sigma_1^{-j}, m_1) + \beta E[v_2 | p, \sigma_1^{-j}, m_1] \tag{2.12}$$

$$v_0^j(m_0) = \max_{p \in (-\infty, 1)} \gamma^j(p, \sigma_0^{-j}, m_0) + \beta E[v_1 | p, \sigma_0^{-j}, m_0], \tag{2.13}$$

where $E[v_{t+1}|p, \sigma_t^{-j}, m_t]$ is the period $t+1$ expected value in subgame perfect equilibrium considering j posts p,$-j$ posts prices according to σ_t^{-j} and the present pair of customer base measures is m_t.

The equilibrium implied by this tie-breaking rule is unique. It is characterized by the following properties: In the period 0, the incumbent I posts the monopoly price 1. The entrant enters the market in period 1 and the equilibrium in that period is characterized by a pair of distributions with the same support but where the firm with the greatest potential for gain in profits in period 2 will post prices according to a distribution that is first order stochastically dominated by the distribution of prices posted by the competing seller. In period 2 the seller with the largest customer base will post prices that first order stochastically dominate the prices posted by the smaller seller.

In the next subsections I will explicitly characterize the unique equilibrium. Since it's subgame perfect I first characterize the equilibrium in the last subgame in period 2 and by backward induction characterize the equilibrium in earlier periods.

2.2.2.1 Period 2

Period 2 is the last period hence the solution concept in the subgame starting in this period is just the Nash equilibrium where each seller maximizes present revenues given a pair of customer bases $m_2 = (m_2^E, m_2^I) \in [0,1]^2$, accumulated from previous periods. Each seller maximizes profits taking in consideration the state of the market and the competitor's price posting strategy.

Proposition 2.2. *Suppose $m_2^E, m_2^I \in (0,1)$. Let $h \in \{E, I\}$ such that $m_2^h = \max\{m_2^E, m_2^I\}$. The unique equilibrium in the subgame starting in period 2 is characterized by:*

(i) A pair of distribution of prices $\{\sigma_2^h, \sigma_2^l\}$ such that

$$supp(\sigma_2^h(m_2)) = supp(\sigma_2^l(m_2)) = [\underline{p}_2(m_2), 1] \tag{2.14}$$

where

$$\underline{p}_2(m_2) = 1 - m_2^l. \tag{2.15}$$

(ii) For a price p in the interior of the support $[\underline{p}_2, 1]$ the pair of distributions satisfies:

$$\sigma_2^l(m_2, p) = \frac{1}{m_2^l}\left(\frac{p - \underline{p}_2(m_2)}{p}\right) \tag{2.16}$$

$$\sigma_2^h(m_2, p) = \frac{1}{m_2^h}\left(\frac{p - \underline{p}_2(m_2)}{p}\right), \tag{2.17}$$

and $\sigma_2^h(m_2)$ has an atom at 1 with mass $1 - \frac{m_2^l}{m_2^h}$.

Proposition 2.2 states that in the equilibrium of the period 2 the seller with the largest customer base will post prices according to a distribution that first order stochastically dominates the distribution posted by the seller with the smaller customer base (i.e. $\sigma_2^h(m_2, p) < \sigma_2^l(m_2, p), \forall p \in (\underline{p}_2(m_2), 1)$ if $m_2^h > m_2^l$). The economic reasoning underlying this result is that the smaller seller will have fewer potential customers hence a smaller fraction of the buyers will known the smaller seller. Therefore, if both sellers are posting prices according to a common distribution σ, when the large seller raises it's price the expected decrease in sales is smaller than for the smaller seller because a smaller proportion of their potential customers knows the competitor. Therefore, to sustain the equilibrium by making both sellers indifferent between any two prices on the support of the distribution each seller posts a different distribution from the other seller and one distribution will first order stochastically dominates the other. Since both distributions have the same support and first-order stochastic dominance doesn't vanish as $p \uparrow 1$ it is implied that there will be an atom of mass at the upper bound of the support for one of the distributions.

Proposition 2.2 implies, from equation 2.15 that the profits realized in the 2 period for a seller j with customer base measure m_2^j are $m_2^j(1 - m_2^l)$.

Therefore, the 2 period value v_2^j satisfies

$$v_2^j(m_2) = m_2^j(1 - \min(m_2^j, m_2^{-j})). \tag{2.18}$$

2.2.2.2 Period 1

The environment in period 1 will be similar to period 2's with the exception that sellers will take into account the dynamic gains from posting higher or lower prices. In this period, if a seller posts lower prices than his competitor the seller will obtain a larger customer base in period 2. This introduces an dynamic element that will affect the equilibrium prices.

The gains in future customers from undercutting prices in the present will be stronger for the smaller seller because the accumulation of new customers is logistical: If the number of customers who don't know the seller in period 1 is higher the growth rate of the customer base from posting lower prices in the present is also higher.

There are two channels by which that the accumulation of future customers affect seller's profits. First, profits depend on the number of potential customers the seller has. So more potential customers in the future implies in greater profits. Second, by undercutting the larger seller in period 1 and accumulating a larger customer base, the smaller seller will increase the intensity of competition in period 2 by a higher degree than if she posted higher prices than the larger seller. By accumulating more customer the smaller seller will lower the expected markup level from the prices posted in equilibrium in the period 2. Therefore, if the smaller seller undercuts the larger seller in period 1 will imply in lower equilibrium profit margins for both sellers in the future compared to a situation where the smaller seller posts higher prices.

The second effect can dominate the first for certain parameter values and make the entrant, who has the smaller customer-base, to post higher prices than if the dynamic effect wasn't present. That is, consider a pair of customer bases (m^I, m^E) with $m^I > m^E$ at the beginning of a period which might be either period 1 or 2. In equilibrium seller E might post higher prices in period 1 than in period 2. And for certain parameter values the seller with a smaller customer base will post prices according to a distribution that first order stochastically dominates the distribution of prices posted by the competing seller, which inverts the equilibrium outcome in period 2.

Let $\overline{m}_2^j$ be the measure of the customer base of seller $j \in \{E, I\}$ in period 2 if $q^j = m^j$, that is, if j sells to all it's potential customers. Let $\underline{m}_2^j$ be the measure of the customer base of j if $q^j = m^j(1 - m^{-j})$, that is, the seller j only sells to the potential customers that do not know the competing seller$-j$. Clearly, in period 2 we have only two possible pairs of customer base measures, $(\overline{m}_2^E, \underline{m}_2^I)$ and $(\underline{m}_2^E, \overline{m}_2^I)$.

Let $\bar{v}_1(p, \sigma^{-j}, (m_1^j, m_1^{-j}))$ be the first period value of a seller posting p, with a customer base of measure m^j and suppose the competing seller has a customer base of measure m^{-j} and post prices according to a distribution σ^{-j}. For non-atomic points of σ^{-j} (which have zero probability of a tie in prices), the discounted present value satisfies

$$\bar{v}_1^j(p, \sigma^{-j}, m_1) = p\times m_1^j\times(1-m^{-j}\sigma^{-j}(p))+\beta[(1-\sigma^{-j}(p))v_2^j(\overline{m}_2^j, \underline{m}_2^{-j})+\sigma^j(p)v_2^j(\underline{m}_2^j, \overline{m}_2^{-j})].$$

Let $\Delta^j(m_1)$ be the change in discounted period 2 profits for j normalized by the current measure of the seller j's customer base, that is

$$\Delta^j(m_1) = \frac{\beta}{m_1^j}(v_2^j(\overline{m}_2^j, \underline{m}_2^{-j}) - v_2^j(\underline{m}_2^j, \overline{m}_2^{-j})).$$

The assumed tie breaking rule implies that in case of a tie in posted prices the seller with the highest $m_1^{-j} + \Delta^j(m_1)$ will sell the good with probability one to the potential customers who also know the competitor. In the case that both sellers have the same $m_1^{-j} + \Delta^j(m_1)$ then each seller sells the good with the .5 probability if a tie occurs.

I will construct a pair of candidate equilibrium strategies and then I will show they are an equilibrium and that the equilibrium is unique.

Proposition 2.3. *For a pair of customer base measures* $(m_1^I, m_1^E) \in (0,1)^2$ *the unique equilibrium in the subgame beginning in the first period is characterized by a pair of distributions* (σ_1^E, σ_1^I) *posted by* E *and* I *respectively, with common support* $[\underline{p}_1, 1]$ *which satisfies the following properties:*

(i) The common support satisfies:

$$\begin{aligned} \underline{p}^j(m_1) =& (1 - m_1^{-j}) - \Delta^j(m_1), \qquad (2.19) \\ \underline{p}_1(m_1) =& \max\{\underline{p}^E, \underline{p}^I\}. \end{aligned}$$

(ii) Each of the distributions satisfies:

$$\sigma_1^j(m_1, p) = \frac{p - \underline{p}_1}{m_1^j(p + \Delta^{-j}(m_1))}. \tag{2.20}$$

for $p \in [\underline{p}_1, 1)$ and for $j \in \{E, I\}$.

2.2.2.3 Period 0

In the period 0 only seller I is operating in the market. She has a starting customer base m_s. Hence the buyers will always buy from the seller as long as the price is lower than the reservation price, hence the seller's profits are just $m_s p$ for $p \in A$ the quantity sold is equal to m_s. The seller's period zero value function is, for a $p \in (-\infty, 1]$:

$$v_0^I(m_0) = \max_{p \in A} m_s \times p + \beta v_1(m_1),$$

where

$$m_1^I = (1 - \delta)m_s + \Phi(m_s, 1 - m_s). \tag{2.21}$$

It is easy to see that the equilibrium in this period is for the seller to post the monopoly price.

We then arrive at the following proposition regarding the characterization of the equilibrium in this environment:

Theorem 2.1. *There is a unique subgame perfect equilibrium and it satisfies the following conditions*

(i) Period 0: Seller I posts a price equal to 1 while seller E is not operating in the market.

(ii) Period 1: The sellers have a pair of customer base measures (m_1^I, m_1^E) and sellers post prices according to Proposition 2.3.

(iii) Period 2: Given a pair of customer base measures (m_2^I, m_2^E), which can be $(\underline{m}_2^I, \overline{m}_2^E)$ or $(\overline{m}_2^I, \underline{m}_2^E)$, the seller's price posting strategy satisfies Proposition 2.2.

Note that if buyers do not exit the market and for a fast enough awareness diffusion, the description of the customer accumulation 2.1 implies that in period

2 both (m_2^I, m_2^E) converge to 1. That implies that in period 2 the distribution of posted prices converges in probability to marginal cost, as in Bertrand competition two firms compete by posting prices attain price equal to marginal cost in equilibrium. This fact implies the following corollary, which shows that, in the present environment, that perfect competition can be obtained as the limit of the non-cooperative equilibria of a sequence of markets.

Corollary 2.1. *Suppose that* $\delta = 0$*, and let* $\{d_n\}$ *be a sequence of discovery functions that satisfy* $\lim_{n\to\infty} d_n(x) = 1$ *for all* $x \in (0,1]^2$*, then the sequence of equilibrium price distributions posted by both sellers in period* 2 *converges in probability to the marginal cost.*

2.2.2.4 Properties of Equilibrium

For this subsection I am assuming that m_s and the function Φ are such so that $(1-\delta)m_s + \Phi(m_s, 1-m_s) > m_s$ is always satisfied. That is, buyers do not exit the market at a rate so fast that the seller's customer base cannot grow after the seller begins operating in the market.

In period 2 the equilibrium is always characterized by the property that the seller with the larger customer base will post prices according to a distribution that first order stochastically dominates the distribution of the smaller firm. Hence, if I, the incumbent seller, is larger than E, the entrant, by period 2 then I will post higher prices in the period 2.

Both sellers enter the market with the same customer base measure m_s and I's client base in period 1 will be $m_1^I = (1-\delta)m_s + \Phi(m_s, 1-m_s) = \overline{m}_2^E$. Therefore even if E undercuts I in period 1 in period 2, unless the customer base measure depreciation parameter δ is very large, I's customer base will grow and hence will be larger than E's, that is, in equilibrium $m_2^E < m_2^I$.

Example 2.1. In any case $m_2^E < m_2^I$ for customer depreciation parameter δ that is not very large, equation 2.18 implies that the expected value of period 2's profits will be $v_2(m_2) = m_2^E(1-m_2^E)$, hence

$$\Delta^E(m_1) = \beta\left(\frac{m_1^I}{m_1^E}\right)(\overline{m}_2^E(1-\overline{m}_2^E) - \underline{m}_2^E(1-\underline{m}_2^E)).$$

For example, suppose the matching function Φ satisfies $\Phi(q, 1-m) = q^{\alpha}(1-m)^{1-\alpha}$ with $\alpha = 2/3$, $\beta = .95$, $\delta = .05$ and starting customer base measure $m_s = .35$. Then m_1^I and $\overline{m}_2^E$ are both .668 while $\overline{m}_2^I = \underline{m}_2^I = 1$ and $\underline{m}_2^E = .497$. Hence,

$$\Delta^E(m_1) = -.11$$
$$\Delta^I(m_1) = .24,$$

so equation 2.19 implies in $(\underline{p}_1^I, \underline{p}_1^E) = (.408, .439)$.

Hence, proposition 2.3 implies that E's distribution of prices will first order stochastically dominate I's distribution in period 1 even though I's customer base is almost twice as large as E's. Note that in any equilibrium the sellers will post asymmetric distribution of prices however the relation of stochastic dominance is not always that the incumbent's distribution stochastically dominates the entrant's.

There are two channels that generate heterogeneity in the price posting strategies in this model. First, there is the static channel: the seller with with smaller customer base will have a more elastic demand for it's product. Second, the dynamic effect, which is the channel through the seller's future customer base alters the future equilibrium price distributions. This dynamic effect can have a positive or negative effect on the prices posted in the present equilibrium. On one hand, firms in the present have the incentive to post lower prices to increase their customer base and hence profits in the future. On the other hand, if sellers post a lower price in the present which entails a higher customer base in the future which could increase competition and lower future prices and hence future profit margins.

This negative effect can dominate the positive effect and imply in a positive effect on prices posted by the entrant in the period 1. That is, entrants post higher prices than they would otherwise to not accumulate too many potential new customers and increase the intensity of competition in the future. Under certain parameters (as shown in example 2.1) the positive dynamic effect of present prices can even overcome the static effect in equilibrium which implies that the entrant will post higher prices than the incumbent in the first period.

Hence, when the dynamic channel is taken into consideration the entrant seller does not always posts a distribution of prices that is first order stochastically dominated by the distribution posted by the incumbent seller.

There is an important assumption that makes this dynamic competition channel on prices significant. It is the assumption that each seller has a significant share of the total awareness. If sellers can attain a significant share of the awareness their customer accumulation dynamics will substantially affect the future prices posted in the equilibrium. It's simple to verify that if the entrant's initial customer base is small relative to the total measure of buyers then it's price posting strategy will be first order stochastically dominated by the incumbent for both period 1 and 2.

It's seems to be a trivial conjecture that in an environment where individual sellers customer bases consist of small share of the market the seller's effect on the market's aggregate distribution of prices will be small. In addition, studying a market with a large number of sellers might provide interesting insights regarding the assumptions required for the aggregate distribution of prices posted in equilibrium to approximate the distributions of prices observed in Kaplan and Menzio (2015). In a related paper (Guthmann (2019) [36]) I show that when we have a large number of sellers the sellers that have been operating in the market for a smaller amount of time will always post lower prices than the veteran sellers and that under reasonable assumptions the density of the distribution of posted prices for transactions in the market is continuous and approximately symmetric.

2.3 Extensions

2.3.1 Extension: Exogenous awareness process

In the previous sections I assumed that awareness of a seller evolves according to a random matching process between buyers who are unaware and buyers who are active customers of the seller. Following Luttmer (2006) and Perla (2017), here I postulate a customer accumulation process where all buyers who know the seller randomly meet and tell other buyers about the seller. With such type of contagion process it's relatively simple to compute the value functions and the equilibrium strategies in this environment. The equilibrium price posting strategies are the same as in period 2 of the finitely many periods environment as the evolution of the seller customer bases does not depend on the current price posting strategy.

Given a pair of customer bases $m = (m^I, m^E) \in [0,2]^2$. The buyers who are unaware of seller j match with buyers who are aware of j according to the matching function $\Phi(1-m^j, m^j)$, where $1-m^j$ is the population of buyers who are unaware of seller j and m^j is the population of buyers aware of seller j. The customer base measure evolves according to the law of motion:

$$m^{j+} = (1-\delta)m^j + \Phi(m^j, 1-m^j). \tag{2.22}$$

The seller's future customer base does not vary relative to the price posted in the present. Hence the seller's equal profit condition can be written with just the current profits. To construct the equilibrium I first construct a pair of candidate supports for the equilibrium strategies. Let $[\underline{p}^j(m), \overline{p}^j(m)]$ for $j \in \{I, E\}$ be a candidate support with $\overline{p}^j(m) = 1$ for each $j \in \{I, E\}$ and pair of client bases m. For each seller j, let the lower bound $\underline{p}^j$ satisfy the following equal profit condition:

$$m^j \times \underline{p}^j(m) = m^j \times 1 \times (1 - m^j).$$

Let $\underline{p}(m) = \max\{\underline{p}^I(m), \underline{p}^E(m)\}$ and let $\sigma^{-j}(m)$ be a $-j$ price posting distribution when the pair of customer bases is m, then the equal profit condition satisfies:

$$m^j \times p \times (1 - \sigma^{-j}(m)m^{-j}) = m^j \times \underline{p}(m).$$

This equal profit condition is identical to the equal profit condition of the period 2 in the finite horizon model 5.1 and hence the equilibrium distribution of prices is identical.

The seller's value functions satisfies for $p \in [\underline{p}(m), \overline{p}(m)]$,

$$v^j(m) = \gamma^j(p, \sigma^{-j}(m), m) + \beta v^j(m^+) = \underline{p}(m) \times m^j + \beta v(m^+),$$

where m^+ is the next period pair of customer bases measures implied by the law of motion for customer accumulation 2.22.

2.3.2 Extension: Advertising

Instead of assuming that awareness diffuses across decision makers through random contagion or through an exogenous awareness process here I assume awareness of seller j among buyers expands due to advertising efforts made by the seller. Customer accumulation occurs due to advertising activities, and are not affected by the magnitude of present sales, hence there is no change in intertemporal payoffs from changing prices posted in the present. Therefore, in this case the equilibrium price posting strategies are also the same as in period 2 of the finitely many periods environment.

Let $A^j \geq 0$ be the expenditures on advertising, they discovery of the seller by buyers is a function Φ^A of the expenditures on advertising A^j and population of buyers who don't know the seller $1 - m^j$. I assume that Φ^A is monotone increasing in both arguments and satisfies constant returns to scale. Therefore the law of motion for the customer bases satisfies

$$m^{j+} = (1-\delta)m^j + \Phi^A(A^j, 1-m^j).$$

With advertising the seller's problem becomes

$$v^j(m) = \max_{p, A^j} \gamma^j(p, \sigma^{-j}, m) + \beta v^j(m^+(A^j, A^{-j})) - A^j.$$

The FOC for the seller's advertising is

$$\beta \frac{\partial v^j}{\partial m^{j+}} \frac{\partial m^{j+}}{\partial A^j} = 1.$$

2.4 Concluding Remarks

This is a study of a market for a homogeneous perishable good characterized by identical buyers and ex-ante identical sellers. The sellers are only differentiated by the size of their customer base measures which evolve over time thanks to diffusion of knowledge about the sellers through a random matching mechanism between informed and uninformed buyers. I have succeeded in computing the equilibrium and showing it is unique. The unique equilibrium is characterized by the following two features: dispersion of prices for transactions and asymmetric price posting strategies with one of the sellers posting higher expected prices than the other seller, this typically being the incumbent seller.

The finite horizon environment has three periods in order to exhibit all dynamic channels that affect the equilibrium. The first channel is the static competition involving price dispersion in the equilibrium of the period 2. The second channel is the inclusion for the dynamic effects of prices posting policies when we have competition between sellers that occurs in the period 1. Thirdly, to include incumbent sellers and entrant sellers in the model I include period 0 and assume that one seller, I enters the market in period 0 while E enters the market in period 1.

I show that it is possible to find an equilibrium in this environment that features mixed pricing strategies. In equilibrium, the price posting strategy of the incumbent seller can first order stochastically dominate the price posting strategy of entering sellers but the inverse can occur as well if the effects of the second channel are strong enough. The smaller seller will post higher prices to not grow too much in the future as that will reduce future profits due to the increased intensity of competition.

Chapter 3

Price dispersion in dynamic equilibrium

3.1 Introduction

In the textbook model of perfect competition each decision maker of the economy has access to the same linear price vector that allows him or her to trade any quantity of goods desired at fixed prices, which are equal to the marginal costs of production. The current state of the empirical evidence paints a very distinct picture in product markets: There exists substantial dispersion in the prices for transactions of identical goods [1]. After entering a market, sellers slowly accumulate demand for their product and as a result prices for transactions with incumbent sellers tend to be higher than with entrants [2]. Further, markups over marginal cost are substantial, and in a given a product category, average markups tend to decline over time [3]. This chapter develops a theory of dynamic industry pricing that explains these deviations from competitive conditions as results from a single fundamental friction and it nests the model of perfect competition as its frictionless limit.

Standard models of equilibrium price dispersion in product markets (as in Burdett and Judd (1983) [5] and Stahl (1989) [72]) feature symmetric price posting

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[1] Sorensen (2000) [71].

[2] Foster, Haltiwanger and Syverson (2008,2016) [24, 25].

[3] De Loecker and Eeckhout (2018) [10], Perla (2017) [63].

strategies for all sellers and a cross-sectional distribution of prices posted with a strictly decreasing density. In the data, price posting depends on the seller's market experience and the cross-sectional distribution of prices is roughly symmetric around the mode (see Kaplan and Menzio (2015) [48]). Fitting the predictions of such models to the empirical data on price dispersion can only be made using implicit assumptions of product heterogeneity, such as postulating that buyers assign different reservation prices to the good if purchased from different sellers.[4]

There are directed search models that do explain why incumbent sellers post higher prices than entrants (such as Paciello, Pozzi and Trachter (2019) [62] and Gilbukh and Roldan (2018) [32]). But these models do not feature price dispersion at the firm level and do not allow for customers to accumulate information about the sellers over time. As a result, this class of models cannot explain how average markups in a product category can be decreasing over time while the average age (measured as time since entry) of sellers operating in the market is increasing.

The model of dynamic price formation presented here is essentially a dynamic version of Burdett and Judd (1983) [5] based on a novel theoretical approach to customer-capital accumulation. I consider a product market for a single indivisible good populated by a continuum of buyers and sellers. Buyers have imperfect awareness of the sellers, that is, each buyer is aware of only a subset of the sellers from which she can purchase the good. Over time the buyers gain awareness of additional sellers through a process of word of mouth contagion: a matching process between the customers of the seller and buyers who are unaware of the seller. Past sales activity of a seller determines the evolution of its customer base. The model's equilibrium features price dispersion, with mixed pricing strategies for all sellers, where incumbent sellers post higher prices than entrants. In equilibrium, the cross-sectional distribution of prices for transactions closely approximates the form of the price dispersion observed in the data. As in Paciello, Pozzi and Trachter (2019) [62], incumbents have incentives to extract higher surplus from their customers while entrants have lower customer

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[4] Some authors explain this assumption of heterogeneity by using the concept of "amenities" provided by individual sellers, which can be used to explain different average levels of expensiveness of different sellers (see, e.g., Sorensen (2000) [71] and Kaplan and Menzio (2016) [49]).

base and so they post lower prices to increase their customer-capital. In equilibrium, the average distribution of prices for transactions is characterized by declining markups as the product market matures. That is because, the average customer base of the sellers is increasing over time and so awareness of different sellers by their customers is also increasing, which implies that the intensity of competition is increasing over time.

3.1.1 Summary of contributions

There exists a well-developed literature of random matching and bargaining games that provide noncooperative foundations for perfectly competitive equilibrium [5], but this literature is not primarily concerned with explaining the observed deviations from competitive conditions, instead they ask if competitive equilibrium arises as the limit of the set of equilibria in a sequence of noncooperative games as frictions are reduced to zero. In this article, I am concerned both with convergence to perfect competition in noncooperative games and to explain how these deviations observed in the data can arise in equilibrium.

This study considers a product market for a single good. There is a continuum population of buyers and sellers. Buyers are infinitely lived, and sellers randomly exit the market at each period and are replaced by entering sellers. All buyers have identical reservation price for the good and all sellers have the same constant returns to scale production technology. Buyers have unit demand for the perishable good at each period and at a point in time they might be aware only of a subset of the sellers. I call the population of buyers who are aware of a particular seller that seller's customer base. Over time sellers expand their customer base through their sales activity while buyers gain awareness of additional sellers.

Foster, Haltiwanger and Syverson (2008,2016) [24, 25] use the term 'demand accumulation by doing' to mean the effect of past sales activity on the present demand for a seller's product, as opposed to 'demand accumulation by being',

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[5] E.g., Novshek (1985) [61], Rubinstein Wolinski (1985) [66], Gale (1987, 2000) [29, 30], Lauermann (2013) [51] and more recently Lauermann et. al. (2018) [52]. For a more specific example, Duffie et. al. (2005) [19] proves convergence to Walrasian equilibrium for over the counter markets while providing broad analysis of the effects of market frictions on the model's equilibrium.

which represents the accumulation of demand due to the continued presence of the seller in the market. They show that seller's 'demand accumulation by doing' is the dominant factor in customer-capital accumulation. This assertion is also corroborated by studies in the marketing literature such Trusov et. al. (2009) [74]. The customer-capital accumulation mechanism used here is similar to the mechanism used in other studies such as Luttmer (2006) [54] and Perla (2017) [63] but in this study the seller's previous sales activity directly influences the awareness process (that is, the diffusion of information regarding sellers in a product market among the buyers in the market) to incorporate the empirical finding of FHS that what they call seller's 'demand accumulation by doing' is the dominant factor in customer-capital accumulation.

The particular trading mechanism that is used in this study proceeds as follows: At each period sellers choose which prices to post and buyers choose whether to purchase the good among the set of sellers that they are aware off. I only examine rational expectations symmetric Markov perfect equilibria where sellers know the distribution of the number of other sellers that their customers might know. Sellers choose prices to post as to maximize expected discounted lifetime profits at every period and the sellers price posting strategies are symmetric for sellers with the same customer base size.

In this model sellers have access to more information than buyers. Buyers only observe the vector of prices among the sellers that they know (a subset of measure zero of the sellers in the market) while sellers know the distribution of seller's customer bases across the whole market as well as their price posting strategies. Still, I find it is a reasonable assumption to suppose that firms in an industry know more about their competitors than their customers do. As done in a search model in Lauermann et. al. (2018) [52], an interesting avenue for further research is, assuming that upon entry that sellers do not know the population and the distribution of customer bases of competitors operating in the market, to describe the equilibrium evolution of their beliefs, and the resulting allocation and prices.

I demonstrate that the model's equilibrium is unique. It is characterized by price dispersion as the sellers randomize the prices they post (as in Burdett and Judd (1983) [5]) and the distribution of prices posted depends on the seller's customer base: sellers with larger customer bases post prices according to a distribution that always first order stochastically dominates the distribution of prices posted by the sellers with smaller customer bases. Well established incumbents post higher prices because the scope for further growth in their customer bases is smaller than for entering sellers, instead they focus on maximizing current profits while entering sellers with smaller customer bases focus on growing their customer base by posting lower prices which will entail higher sales and hence, higher profits in the future. Finally, I show that the non-steady-state equilibrium converges to the steady-state equilibrium as the number of trading periods converges to infinity.

This study describes several equilibrium properties of the model. I show that any symmetric equilibrium is constrained efficient in the sense that equilibrium strategies maximize aggregate surplus among feasible allocations at each period (as Gilbukh and Roldan (2018) [32], this model can also be interpreted as a theory of efficient markups). It is shown that the average markup for transactions is strictly decreasing over time in the non-steady-state equilibrium as it converges to the steady-state. Finally, it is shown that as market frictions vanish the distribution of prices for transactions in the steady-state equilibrium converges weakly in probability to the competitive price. In addition, the model is extended with free entry of sellers: the population of sellers is monotone decreasing on the cost of entry and hence there is a unique steady-state equilibrium for each entry cost. As the entry cost decreases the steady-state equilibrium approximates the perfectly competitive equilibrium.

I also perform quantitative analysis of the model to see how substantial its predictions are when its parameters are calibrated to match moments found in empirical studies. The model predicts that from the date of its creation a product market for a physically identical good takes several decades to converge to it's steady-state, where it more closely approximates the conditions of perfect competition. Even in it's steady-state the equilibrium still features significant deviations from competitive conditions with price dispersion and substantial markups. The distribution of prices for transactions converges to a distribution that is roughly symmetric around the mode which replicates the data, while other models do not exhibit that property without the addition of specific assumptions. The average markup for transactions in a industry starts at ca. 110% above marginal cost when the industry has just been created and they slowly decrease as the industry converges to its steady-state. Yet, in the steady-state of the industry, after a convergence process that takes over 50 years, the model still predicts average markups at 40% above marginal cost in market for a homogeneous good with a continuum of identical buyers and sellers.

For comparison, De Loecker and Eeckhout (2018) [10] estimate the average global markups in 2016 at 59% above marginal cost, De Loecker et al (2018) [11] estimate then at 61% for the US in 2016. According to Perla (2017) the median industry age in the US is a little less than 20 years, which implies in average markup of 60% in the equilibrium of the calibrated model for an industry aged at 20 years. This means that this model can replicate the substantial markups observed in the data through this single friction. De Loecker et al (2018) [11] also argue that markups in the US have increased substantially since 1980. This increase in average markups can be explained by this model as being due to increase in frequency of product creation and, hence, a lowering in the average age of product markets, a point also made in Perla (2017) [63], in addition to an increase in market-frictions (which can be represented in the model as increasing costs of entry, which is consistent with the reported decrease in entrepreneurship rates in the US economy). Finally, this study has an important normative implication. It implies that the positive level of markups found in the data does not imply that there is inefficiency: given the existence of frictions, markets will always feature prices that deviate from marginal cost but they still can be efficient relative to the set of feasible allocations.

3.1.2 Related literature

A recent empirical literature has argued that product markets are characterized by the following features: (1) There exists significant dispersion in the prices for transactions for physically identical goods over a small period of time in a geographical area. (2) The distribution of prices for transactions tends to be approximately symmetric and leptokurtic. (3) A substantial fraction of the variance in prices (ca. 45%) is due to transactions involving the same seller. (4) When a seller enters a market the demand for his or her product grows slowly, and it takes a long time for the gap to close relative to the seller's more mature competitors. (5) Active demand accumulation driven by seller's past supply decisions quantitatively dominates passive demand accumulation. (6) Entrant sellers also typically post lower prices than well-established competitors. (7) In an industry, the growth rate of a seller tends to decline with the size of the seller's sales. (8) Finally, product markets tend to exhibit higher markups when the respective product category has been recently introduced and markups tend to decline over time as the market matures.[6]

Kaplan and Menzio (2015) [48] studies the shape and structure of the distribution of prices at which an identical good is sold in a given market and time period. They conclude that the distribution of prices for transactions is approximately symmetric and leptokurtic, and only 10% of the variation in prices is due to variation in the expensiveness of the stores at which a good is sold while 90% is due in approximately equal parts, first, to variation in the average price of a good sold across equally expensive stores and, second, due to variation in prices of a good sold across transactions in the same store. In addition, Gilbukh and Roldan (2018) [32] have found that across all stores selling the good at the week and geographical market of interest the standard deviation of prices at the barcode level is 15.7% of the average price. Corroborating this evidence, using a different source, Kaplan et al (2016) [50] find that the standard deviation of prices is 15.3% of the average price for the same good in the same week and market.

There is a recent empirical literature tying productivity to firm survival by explicitly accounting for demand side effects (Das et. al. (2007) [8], Eslava et. al. (2008) [21], Foster et. al. (2008) [24], Roberts et. al. (2011) [64], De Loecker and Goldberg (2014) [12] and Foster et. al. (2016) [25] are examples of this approach). Foster, Haltiwanger and Syverson (2008,2016) have studied US census data of industries for physically identical goods where they observe producer-level quantities and prices separately. They found that in markets for physically identical goods entrant producers charge lower prices than incumbents. They also found that entering producers have slightly higher physical productivity and hence slightly lower marginal costs than incumbent firms. They conclude that the role of demand variation in explaining plant size is paramount in in-

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[6] (1) is reported in many studies, Sorensen (2000) [71] for example, (2) and (3) are reported in Kaplan and Menzio (2015) [48], (4), (5) and (6) are presented in Foster, Haltiwanger and Syverson (2008,2016) [24, 25], (7) are presented in Almus, Nerlinger (2000) [1], Yatsuda (2005) [75], Perla (2017) shows evidence for (8).

dustries with physically homogeneous products, where one would expect that demand variation would be of smaller consideration than in industries with highly differentiated products.

These empirical findings challenge the underlying assumptions and many of the implications of the theoretical literature inspired by the seminal studies of Jovanovich (1982) [47] and Hopenhayn (1992) [43] which explain firm selection on the basis of differences in productivity (for more recent examples, refer to Melitz (2003) [56] and Asplund and Nocke (2006) [2]). In these studies, price dispersion for identical products does not exist and hence differences in productivity among industry producers can be measured solely by input costs and revenues. However, later empirical studies have shown that there exists substantial variation in prices posted by different firms in the same industry, so revenue measures of productivity will differ from physical measures of productivity (see Haltiwanger (2016) [37]).

Perla (2017) [63] provides evidence that, as an industry ages, it's markups tend to decline over time. He argues that when a new product category is introduced, the firms have small absolute demand but high markups. Over time competition intensifies and markups are driven to razor thin margins and simultaneously market concentration increases. That study assumes frictions in the consumer's choice sets for new products to explain firm growth, industry life cycle, and aggregate profits. Consumers are aware of a fraction of the products and hence have incomplete choice sets in a given product category. The model replicates the empirical finding that average markups decrease as industries age while the industry concentration increases. I also borrowed from Perla the use of term "awareness" in this dissertation.

Theoretically, the model described in this study is also related to search models that feature equilibrium with price dispersion generated by randomized price posting strategies. These include the aforementioned Burdett and Judd (1983) and Stahl (1989) [72] seminal contributions and more recently by Menzio and Trachter (2015) [57]. The studies of Head et. al. (2012) [41], Kaplan and Menzio (2016) [49], Kaplan et. al. (2016) [50] and Burdett and Menzio (2018) [6] represent some recent applications of such models.

Paciello, Pozzi and Trachter (2019) [62] and Gilbukh and Roldan (2018) [32] have also studied a model of customer-capital accumulation using a search framework. In both studies, buyers can either be matched to one seller or be searching for a seller. While this study allows buyers to trade with multiple sellers simultaneously. In Gilbukh and Roldan (2018), their equilibrium yields that the price a seller posts increases with firm's customer-capital. A result that this study also replicates.

There are important differences from the previous literature on equilibrium price dispersion in that this study incorporates dynamic considerations through a distinct mechanism of information diffusion. These differences lead to novel equilibrium results. In this study awareness regarding the sellers is diffused through buyers through a random matching process between customers (who are the buyers who are aware of a particular seller) and uniformed buyers. The customer accumulation mechanism I use in this study has some parallels with Luttmer (2006) [54], who uses consumer search to understand firm growth dynamics.

The use of the expression "buyer awareness of the sellers" borrows from the theoretical literature on unawareness (Modica and Rustichini (1994,1999) [58, 59], Dekel, Lipman and Rustichini (1998), [15], Heifetz, Meier and Shipper (2006) [42], Li (2009) [53] and Galanis (2013) [26]). Conventional decision theory under uncertainty interprets uncertainty to mean ignorance about the particular realization of the state of the world but it assumes that the description of the world in the mind of the decision makers is complete. Unawareness means ignorance regarding the state space itself. In this study stating that the buyers are unaware of all sellers means that they do not known all sellers and don't know that they don't know them. Hence, the discovery of additional sellers by the buyers is not the outcome of a conscious decision to search but exogenous to the buyer's decisions. The practical implication of buyers being unaware of the sellers is that it justifies the assumption of the buyer's search for sellers being exogenous [7] as the buyers are not aware that the description of the market they have in their minds is incomplete. Instead of expressing the discovery of the sellers by the buyers as a result of conscious search effort here it is assumed that the past sales activity of the sellers are the main drivers in the diffusion

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[7] Which is commonly made in many studies, e.g., Fishman and Rob (2003, 2005) [22, 23] and the aforementioned Kaplan and Menzio (2016) [49] and Burdett and Menzio (2018) .

of awareness about the sellers among the buyers. The assumption of exogenous search is consistent with the empirical literature on demand accumulation by the sellers.

Another contribution to understanding search as being partly expansion of awareness of the decision makers instead of being fully controlled by conscious decisions is that it solves Diamond's (1971) [16] paradox in an elegant manner. Diamond's paradox is the conclusion that, in a simple price posting sequential search model, if search costs are strictly greater than zero the equilibrium will feature monopoly pricing. In equilibrium all sellers will post monopoly price and buyers will only search for one seller. If sellers posted lower prices than the monopoly price the other sellers would find optimal to deviate by posting higher prices than the market average but still low enough to keep their customers from searching for additional sellers. But, if search costs are equal to zero, then buyers search for many sellers and Bertrand competition holds which implies that equilibrium is competitive. That is, equilibrium prices move discontinuously from monopoly price to competitive price as search costs converge to zero. Instead, if we relax the assumption that buyers have full control of their search activity then even if buyers don't choose to search more than one seller some of them end up discovering multiple sellers. In this case, even with strictly positive search costs, there is no discontinuity in the equilibrium as search costs converge to zero.

3.1.3 Paper organization

The description of the formal model beings in section 3.2 which describes the environment of the model. Section 3.3 describes the game and the Markov perfect solution concept used by this model. Section 3.4 contains the existence proof for equilibrium and subsection 3.4.1 characterizes the properties of the unique symmetric steady-state equilibrium. In subsection 3.4.2 I construct an equilibrium outside of the steady-state that converges to the steady-state equilibrium. In section 3.5 I show that the equilibrium is efficient, that the equilibrium converges to a steady-state and I describe and prove a set of sufficient conditions for the equilibrium in steady-state to converge to perfect competition. Section 3.6 presents a quantitative analysis of the model. Section 3.7.1 discusses an extension for the model to incorporate endogenous entry of sellers. While section 3.8 presents concluding remarks.

3.2 Environment

3.2.1 Physical environment

Consider a market for a single, indivisible and perishable good. Time is discrete with periods indexed by $t = 0, 1, 2, \ldots$. There is a continuum of buyers and sellers in the market. Let I be the set of buyers and J be the set of sellers. Both sets are represented by closed unit intervals with Lebesgue measure (naturally) of 1. At each period each buyer $i \in I$ has unit demand for the good and reservation price equal to 1. At any period each seller $j \in J$ can produce any quantity $q \in \mathbb{R}_+$ of the good with constant marginal cost which is normalized to 0.

Buyers have imperfect awareness of the sellers: at a given period t, an individual buyer $i \in I$ can only be aware of a finite subset of the sellers, with cardinality $k \in \mathbb{N} \cup \{0\}$, from which she can purchase the good. Let $\pi_t^k \in [0,1]$ be the measure (or fraction, as the set of buyers is measure 1) of the buyers aware of k different sellers. The seller awareness profile $\{\pi_t^k\}_{k=0}^{\infty}$ satisfies $\sum_{k=0}^{\infty} \pi_t^k = 1$. Let $c_t^j \in [0, \overline{c}]$ be the size of j's customer base which represents the population of buyers who are aware of seller $j \in J$ in period t. This population is bounded above by $\overline{c}$. [8]

Let $\hat{c}_t$ be the average customer base of all sellers in period t, that is,

$$\hat{c}_t = \int_0^1 c_t^j dj. \tag{3.1}$$

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[8] Which represents the idea that demand accumulation by a seller is bounded. Note that without this condition the customer base of an individual seller would be unbounded since individual sellers have zero weight on the average customer base of all sellers.

I assume the following consistency condition[9]

$$\hat{c}_t = \sum_k k\pi_t^k. \tag{3.2}$$

Equation 3.2 states that the average number of customers of the sellers $\hat{c}_t$ is equal to the average number of sellers that the customers know $\sum_k k\pi_t^k$. This condition is satisfied in a finite economy when the number of buyers and sellers is the same: with $n^s \in \mathbb{N}$ sellers and $n^b \in \mathbb{N}$ buyers, if each seller has on average $\hat{c} \in \mathbb{R}_+$ customers then each buyer is aware of $\hat{c}n^s/n^b$ sellers. The state of the market in period t is described as a pair $(\{c_t^j\}_{j\in J}, \{\pi_t^k\}_{k=0}^{\infty})$, profile of customer bases $\{c_t^j\}_{j\in J}$ and a profile of buyer populations by number of sellers that they are aware of $\{\pi_t^k\}_{k=0}^{\infty}$, that satisfy the consistency condition 3.2.

In subsection 3.2.3 I construct the random matching mechanism between buyers and sellers that satisfies the consistency condition 3.2. The mechanism yields a unique awareness profile $\{\pi_t^k\}_{k=0}^{\infty}$ from the profile of customer bases $\{c_t^j\}_{j\in J}$ and $\{\pi_t^k\}_k$ distributed according to a Poisson with parameter $\hat{c}_t$ (so that in period t the distribution of $\{\pi_t^k\}_k$ is generated by a Poisson process where buyers discover sellers at arrival rate $\hat{c}_t/t$ as in Diamond (1987) [17]). That is, the higher the average number of customers is the higher is the expected number of sellers that buyers know, hence, $\hat{c}_t$ is a measure of the degree of market friction.

At each period a fraction α of sellers exits the market and are replaced by new sellers, keeping the seller population constant at 1 while buyers stay in the market forever. When a new seller enters the market it's starting customer base size is $\underline{c} \in (0, \overline{c})$. I assume all sellers in period 0 have customer base of zero. At period 1 the product market is "created" and the first cohort of sellers with non-zero customer base enters the market, replacing a population α of sellers with customer base 0. In period t there will be t seller cohorts $\{J_t^1, J_t^2, \dots, J_t^t\}$ of ages $\{1, \dots, t\}$ with non-zero customer bases, as described in table 3.1 for an example of 5 periods.

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[9] Note that each seller j customer base is an infinitesimal fraction of all customer bases in the market as there is a continuum of sellers, so each individual c_t^j does not affect π_t^k. Therefore, c_t^j represents the "density" of buyers which have j in their choice sets.

	Cohorts operating				
Date	age = 1	2	3	4	5
1	J_1^1				
2	J_2^1	J_2^2			
3	J_3^1	J_3^2	J_3^3		
4	J_4^1	J_4^2	J_4^3	J_4^4	
5	J_5^1	J_5^2	J_5^3	J_5^4	J_5^5

Table 3.1: Seller cohorts over 5 periods

Seller's exit rates is constant at α. As a measure α of new sellers enter, therefore, in a period t the measure of sellers of age $a \in \{1, \ldots, t\}$[10], is

$$n^a = \alpha(1-\alpha)^{a-1}. \tag{3.3}$$

Therefore, for $t \to \infty$ the population of sellers of ages $a \in \{1, 2, 3, \ldots\}$ converges to n^a for each "age" $a \in \{1, 2, 3, \ldots\}$, and the population of sellers is in a steady-state.

3.2.2 Dynamics of seller's customer bases

The customer base c_t^j is a variable of seller j's demand, the assumption that c_t^j represents a continuum of customers allows both seller demand and the evolution of customer bases to be deterministic. That is, if each seller has a large number of customers and if the customers purchase with some probability ι then by the Law of Large Numbers there will be a deterministic quantity $q_t^j = \iota c_t^j$ of purchases from seller j in period t.

This subsection introduces a law of motion for customer accumulation that determines the growth of a seller's customer base by the current sales of the seller, its customer base size and the degree of market congestion (represented here by the average size of the seller's customer bases). I assume that customers of a buyer retain the memory of the seller they discovered in the past and can shop there as long as the seller is operating in the market hence $c_{t+1}^j \geq c_t^j$ if j doesn't exit, when the seller exits it is replaced by a new seller who starts

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[10] Age is what I call the total number of periods that the seller has been operating in the market.

with a customer base size $\overline{c}$. Hence, the seller's customer base follows the law of motion

$$c_{t+1}^{j} = \begin{cases} c_t^j + \Phi(q_t^j, \overline{c} - c_t^j) & \text{if } j \text{ doesn't exit at } t+1 \\ \underline{c} & \text{if } j \text{ exits and is replaced,} \end{cases} \tag{3.4}$$

where $\Phi : \mathbb{R}_+ \times [0, \overline{c}] \rightarrow \mathbb{R}_+$ is a awareness diffusion function that specifies the population of buyers who discovers a seller in a given period. Φ is a function of the seller's sales q_t^j, the size of his or her customer base c_t^j. Note also that c_{t+1}^j is bounded above by $\overline{c}$, the upper bound in the customer base size.

The customer-capital accumulation function Φ satisfies the following assumptions:

Assumption (1). Φ is non-decreasing in q_t^j and $\overline{c} - c_t^j$ and satisfies $\Phi(q_t^j, \overline{c} - c_t^j) > 0$ for $c_t^j \in (0, \overline{c})$ and $q_t^j \in (0, c_t^j)$.

Assumption (2). Φ is strictly increasing and strictly concave in q_t^j and concave in $\overline{c} - c_t^j$.

Assumption (3). For any $\hat{c} \in [0, \overline{c}]$, any $c \in [0, \overline{c})$ and $q \in [0, c]$, $\Phi(q, \overline{c} - c) < (\overline{c} - c)$.

The first condition states that as the number of customers who purchase from the seller [11] in the current period increases the number of people who became aware of the seller in the next period increases. That represents the "word of mouth" between buyers who are aware of the sellers and the buyers who are not (as in Fishman and Rob (2003,2005) [23, 22] buyers learn about the seller through word of mouth). The second condition states that the scope for increase in the seller's customer base has diminishing returns on the quantity sold and is reduced when the seller's customer base is larger, that is, the larger the seller already is, the smaller is the number of new customers he or she can acquire. This is consistent with the empirical evidence that small firms grow faster than larger ones. [12] Finally, the third condition states that the upper bound of customer base is never reached. This last condition is imposed to keep the equilibrium unique, but it's not required for existence as it is not satisfied in the calibration of the model in quantitative analysis section 3.6.

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[11] Which is the same as the quantity sold since individual buyers have unit demand.

[12] See, e.g., Caves (1998) [7] and Rossi-Hansberg and Wright (2007) [65].

To construct a matching process that microfounds the customer accumulation function Φ consider the interval $[0, \bar{c}]$ that represents the population of buyers who can become aware of seller j. Let $c_t^j \in (0, \bar{c})$ and $q_t^j \in (0, c_t^j]$. As buyers have unit demand for the good the quantity sold by the seller is the same as the population of buyers who decide to purchase from j. The population who is not aware of the seller is a set of buyers of measure $\bar{c} - c_t^j$. Consider a matching process where buyers who do not know j randomly meet customers of j who just purchased from j and then become aware of j through word of mouth (as in Fishman and Rob (2005) [23]). Suppose the probability that a buyer i who is not aware of j meets a customer is a strictly increasing and concave function $f(q_t^j) \in [0, 1]$ of the number of customers of j. Then the measure of buyers in $[c_t^j, \bar{c}]$ who meet customers who purchased the good from j is described by a logistical matching function

$$\mu(f(q_t^j), \bar{c} - c_t^j) = (\bar{c} - c_t^j) f(q_t^j),$$

which is concave and strictly increasing on both arguments, and f satisfies $f(q_t^j) < 1$ for all $q \in [0, \bar{c})$. Let Φ be $\Phi(q_t^j, \bar{c} - c_t^j) = \mu(f(q_t^j), \bar{c} - c_t^j)$, then it is easy to see that Φ satisfies assumptions (1)-(3).

Example 3.1. Consider f with functional form

$$f(q_t^j) = A \left(q_t^j / \bar{c}\right)^{\zeta}, \tag{3.5}$$

where $A \in (0, 1), \zeta \in (0, 1)$. Assumptions (1) and (2) are trivially meet. Note that $q_t^j \in [0, c_t^j]$ hence $q_t^j < \bar{c}$ if $c_t^j < \bar{c}$, therefore $\left(q_t^j / \bar{c}\right)^{\zeta} < 1$ and hence $\mu(q_t^j, \bar{c} - c_t^j) < (\bar{c} - c_t^j)$. Therefore $\Phi(q_t^j, \bar{c} - c_t^j) = \mu(f(q_t^j), \bar{c} - c_t^j)$ described by 3.5 satisfies assumption (3).

3.2.3 Distribution of buyer's awareness

Subsection 3.2.2 described the determination of the evolution of the seller's customer base profile. In this subsection, from the sellers customer base profile I construct a random matching mechanism that determines the degree in which customer bases of different sellers overlap, which yields the buyer's awareness profile $\{\pi_t^k\}_k$. That is, it determines the probability customers of a seller also know $k \in \{0, 1, 2, \ldots\}$ other sellers from the average customer base size of the sellers.[13]

To provide a description of the matching process that yields a unique awareness profile from the profile of customer bases in the environment with a continuum of buyers and sellers I consider an environment with finitely many buyers and sellers and replicate it, postulating that its properties at the limit to hold in the continuum. Let I and J be the sets of buyers and sellers and $n^b, n^s \in \mathbb{N}$ be the number of buyers and sellers, respectively. Consider a vector $(c_t^j)_{j=1}^{n^s}$ that represents a customer base profile which assigns a value $c_t^j \in \mathbb{N} \cup \{0\}$ for each seller $j \in \{1, \ldots, n^s\}$, let $\hat{c}_t = \sum_{j=1}^{n^s} c_t^j / n^s$ and consider the matching process described as follows.

Let ν be a positive integer such that

$$\nu n^b > \sum_{j=1}^{n^s} c_t^j.$$

Consider a ν-replica of the market with set of buyers I^ν with νn^b buyers and a set of sellers J^v with νn^s sellers. J^v consists of ν identical replicas $\{J_1, \ldots, J_\nu\}$ of J (in the sense that the profile of customer bases of the sellers in each subset is $\{c_t^j\}_{j=1}^{n^s}$).

Consider a matching process with ν rounds. In each round $n \in \{1, \ldots, \nu\}$, as each seller $j \in J_n$ is matched with c_t^j buyers. Each seller in J_n in sequential order takes a simple random sample of c_t^j buyers from the set of buyers I^ν without replacement. Hence, in each round buyers can match with at most one seller and therefore the sellers of subset J_n are matched with $\sum_{j=1}^{n^s} c_t^j$ different buyers among the νn^b buyers.

More formally, let S_ν be the set of states that specifies a state for each profile of customer bases $\{C_t^j\}_{j \in J^\nu}$ drawn by the set of sellers J^ν. Let $m : I^\nu \times J^\nu \times S_\nu \to \{0, 1\}$ be a matching function (as in Duffie and Sun (2012) [20]) where for each seller $j \in J_n$, $m(i, j, s) \in \{0, 1\}$ indicates whether buyer i and seller j were

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[13] This mechanism shares similarities with Satterthwaite and Shneyerov (2007,2008) [67, 68] in being a multilateral matching procedure that is characterized by one side of the market being matched with a deterministic number of individuals from the other side while the number of agents matched to each agent in the other side is stochastic and follows a Poisson distribution.

matched in round n if state s is realized (with 0 not matched and 1 if matched). The matching function m satisfies for every state $s \in S_\nu$, for each $n \in \{1, \ldots, \nu\}$ and $j \in J_n$, that

$$\sum_{i \in I^\nu} m(i, j, s) = c_t^j, \tag{3.6}$$

and for every $i \in I^\nu$ and $n \in \{1, \ldots, \nu\}$,

$$\sum_{j \in J_n} m(i, j, s) \in \{0, 1\}. \tag{3.7}$$

Consider the probability space $(S_\nu, \mathcal{S}_\nu, P_\nu)$ where $\mathcal{S}_\nu$ is a σ-algebra on the set of states S_ν and P_ν is a probability measure on $\mathcal{S}_\nu$. Let $i \in I^\nu$, consider round $n \in \{1, \ldots, \nu\}$ and the event $\pi_{i,n} \in \mathcal{S}_\nu$ such that for every $s \in \pi_{i,n}$, there exists a seller $j^s \in J_n$ such that $m(i, j^s, s) = 1$ (note that equation 3.7 implies that $\forall j \in J_n \diagdown \{j^s\}, m(i, j, s) = 0$). The probability $P_\nu(\pi_{i,n})$ is

$$P_\nu(\pi_{i,n}) = \sum c_t^j / \nu n^b. \tag{3.8}$$

That is, $P_\nu(\pi_{i,n})$ is the probability that a (arbitrary) buyer i is matched to a seller in round n. Equation 3.8 implies that $P_\nu(\pi_{i,n})$ is identical for all $n \in \{1, \ldots, \nu\}$ and for all $i \in I^\nu$. To simplify notation, let $\pi^\nu = P_\nu(\pi_{i,n})$, hence, the probability a seller is not matched to a buyer in any round $n \in \{1, \ldots, \nu\}$ is just $1 - \pi^\nu$.

Since the random matching process involving each set $J_n, n \in \{1, \ldots, \nu\}$ is a different round they are independent by construction. Let $i \in I^\nu$, $k \in \{0, 1, \ldots, \nu\}$. Let $E_i(k) \in \mathcal{S}$ be the event where $\forall s \in E_i^k$, $\sum_{j \in J^\nu} m(i, j, s) = k$, then independence of matching between rounds and equation 3.8 implies

$$P_\nu(E_i(k)) = \left(\begin{array}{c} \nu \\ k \end{array} \right) (\pi^\nu)^k (1 - \pi^\nu)^{\nu - k}, \tag{3.9}$$

that is, $P_\nu(E_i(k))$ is the unconditional probability that a (arbitrary) buyer is matched with k different sellers. Where $\left(\begin{array}{c} \nu \\ k \end{array} \right)$ is the binomial coefficient which means that the probability i is matched to k sellers is distributed according to the binomial distribution with parameters π^ν and ν.

Let $j \in J_n$ be a seller and $P_\nu(E_i(k+1)|i \in C_t^j)$ be the probability that buyer $i \in I^\nu$ has matched with $k+1$ sellers over the ν matching rounds conditional on being matched to seller j in round n. Equation 3.9 and the independence between the matching rounds imply that $P_\nu(E_i(k+1)|i \in C_t^j)$ satisfies

$$P_\nu(E_i(k+1)|i \in C_t^j) = \begin{pmatrix} \nu - 1 \\ k \end{pmatrix} (\pi^\nu)^k (1-\pi^\nu)^{(\nu-1)-k}. \tag{3.10}$$

Note that $P_\nu(E_i(k+1)|i \in C_t^j)$ is the same for all sellers letting $\nu \to \infty$ the Poisson limit theorem implies that both $P_\nu(E_i(k))$ and $P_\nu(E_i(k+1)|i \in C_t^j)$ converge to the probability mass function at k of a Poisson distribution with parameter $\sum_{j=1}^{n^s} c_t^j/n^b$[14]. Setting $n^b = n^s$ implies that both $P_\nu(E_i(k))$ and $P_\nu(E_i(k+1)|i \in C_t^j)$ converges to a Poisson distribution with parameter $\hat{c}_t$.

Example 3.2. Consider $J = \{j_1, j_2\}$ where $c_t^{j_1} = 1, c_t^{j_2} = 2$ and $I = \{i_1, i_2\}$. Consider the ν-replica with $\nu = 2$, of the market with set of sellers $J^\nu = \{J_1, J_2\} = \{(j_{11}, j_{21}), (j_{12}, j_{22})\}$ and set of buyers $I^\nu = \{I_1, I_2\} = \{(i_{11}, i_{21}), (i_{12}, i_{22})\}$ hence there are 4 buyers and 4 sellers.

In round $n \in \{1, 2\}$, a total of 3 buyers are matched to sellers, hence the probability that a buyer is matched to a seller in round n is 3/4. Hence, there are $(4 \times 4 \times 3)^2$ states of the world where a state $s \in S$ specifies a matching function between buyers and sellers such as the one depicted in the graph in figure 3.1.

The matching process just described generates a Poisson distribution for buyer's awareness with parameter $\hat{c}_t$ for a finite set of seller types each with a finite customer base. To approximate the environment where each seller has a continuum of customers that is studied in the rest of the paper, consider a profile of customer bases $\{c_t^j\}, c_t^j \in \mathbb{R}_+$ with $j \in [0,1]$, with $n^s \in \mathbb{N}$ types of sellers $\{c_t^a\}_{a=1}^{n^s}$, each type with the same measure. Hence, $\hat{c}_t = \sum_a c_t^a/n^s$.

The profile of customer bases $\{c_t^a\}$ can be approximated by the profile of customer bases of a sequence of sequences of finite markets $\{I^{\mu,\nu}, J^\nu, \{c_t^{a,\mu}\}_\mu\}_\nu$ where $I^{\mu,\nu}$ is a $\mu \times \nu$-replica o I and $J^\nu = \{J_1, \ldots, J_\nu\}$ is a ν-replica o J such

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[14] It's trivial to note that the sequence $\left\{\nu \times \sum_{j=1}^{n^s} c_t^j/\nu n^b\right\}_\nu$ converges to $\sum_{j=1}^{n^s} c_t^j/n^b$ as $\nu \to \infty$, so the conditions to apply the Poisson limit theorem are satisfied.

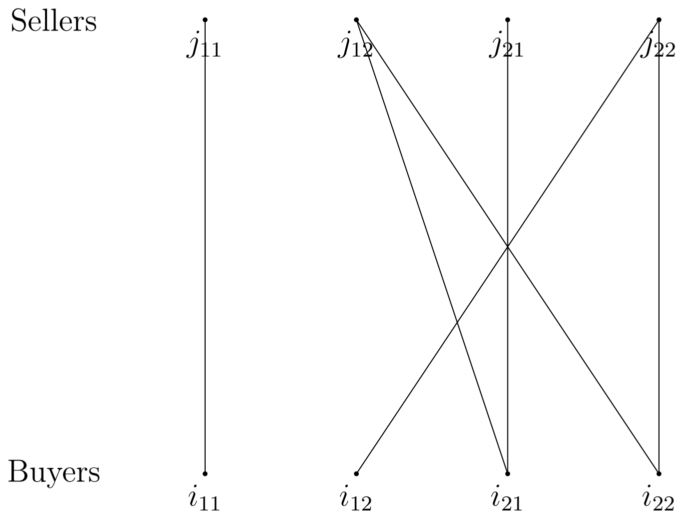

Figure 3.1: Example of a graph that represents buyer's awareness

that, for each $a \in \{1, \ldots, n^s\}$, the sequence of customer bases $\{c_t^{a,\mu}\}$ satisfies

$$\lim_{\mu \to \infty} \frac{c_t^{a,\mu} n^s}{\mu n^b} = c_t^a. \tag{3.11}$$

Note that as $\mu \to \infty$, $c_t^{a,\mu}$ converges to infinity.

This $\mu \times \nu$-replica of the market has $\mu\nu n^b$ buyers and νn^s sellers and each seller has $c_t^{a,\mu}$ customers. By analogous argument to the matching process described above, by taking ν to infinity, the distribution of buyers matched to $k \in \{1, \ldots, \nu\}$ different sellers converges to a Poisson with parameter

$$\hat{c}_t^{\mu} = \sum_a c_t^{a,\mu} / \mu n^b, \tag{3.12}$$

which converges to

$$\sum_a c_t^a / n^s = \hat{c}_t$$

as $\mu \to \infty$.

3.3 The game and equilibrium

The section describes the game that is played in this market environment.

3.3.1 Players

The players in this game are the sellers. Buyers are assumed to be not strategic. That is, in each period they always purchase from the lowest-priced seller that they are aware of if this seller's price is lower than their reservation price.

3.3.2 Strategies and action sets

At the beginning of each period sellers choose prices to post. A buyer who knows k different sellers observes a vector of prices posted by the buyers he or she knows. The buyers choose whether to purchase their unit of the good from among these sellers. The good is produced by the sellers and sold to the buyers who purchase the good and is consumed. By the end of period the profile of quantities sold $\{q_t^j\}_j$ and the entry and exit of sellers determine the profile of customer bases of the next period $\{c_{t+1}^j\}_j$.

No rational seller would post a price higher than the customer's reservation price of 1 so attention is restricted to prices posted in $(-\infty, 1]$. Sellers can post prices according to a randomization mechanism. Hence a seller j's price posting strategy for period t is a cumulative distribution $\sigma_t^j : (-\infty, 1] \to [0, 1]$ hence the action set of the sellers is $\Delta((-\infty, 1], [0, 1])$, the set of cumulative distribution functions on $(-\infty, 1]$.

I assume that when sellers offers a non-degenerate price posting distribution each customer of the seller will observe sequentially prices drawn from the distribution σ_t^j. Hence, a buyer who is aware of j observes a price drawn according to σ_t^j in period t. Note that prices posted might be different for each customer of seller j. A buyer i who knows a set of k sellers in period t will observe a vector of prices $(p_{j^1}^{i,t}, \ldots, p_{j^k}^{i,t})$, where $p_j^{i,t}$ is a price drawn from the distribution of prices posted by seller j ,σ_t^j. The buyer i chooses to purchase at the lowest price from the vector of k prices he or she observes from those k different sellers.

The set of sellers that buyers know is private information. Therefore, sellers do not know which and how many other sellers their customers know. However, I assume that sellers know the profile of customer bases $\{c_t^j\}_{j \in J}$ of the sellers in the market. A seller form beliefs on the probability that their customers to be in contact with other sellers based on the average customer base $\hat{c}$ implied by

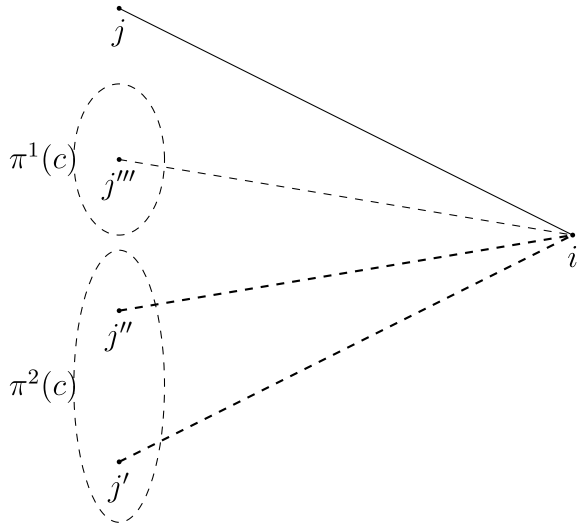

Figure 3.2: Beliefs of seller j about the number of competing sellers that it's customer i might be aware off.

$\{c_t^j\}_{j\in J}$ and they know the random matching mechanism that determines $\{\pi_t^k\}_k$ as a function of $\hat{c}_t$ (that is described in subsection 3.2.3). The matching mechanism implies that for a given customer i of seller j, j assigns the probability $\pi^k(\hat{c}_t)$ that buyer i is aware of k other sellers besides j in period t.

Let i be a buyer in seller j's customer base. Seller j also does not know who are the particular competing sellers that i might be aware of so j forms beliefs regarding the prices posted by the other sellers that i might know. Let $\sigma_t^{-j} = \{\sigma_t^h\}_{h\neq j}$ be the profile of distributions of prices posted by j's competitors. Let $\hat{\sigma}_t$ be the weighted average of the profile of price posting strategies of the other sellers σ_t^{-j}, weighted by their customer base size. With a continuum of sellers $\hat{\sigma}_t(.)$ is expressed as

$$\hat{\sigma}_t(p) = \int_0^1 \left(\frac{c_t^h}{\hat{c}_t}\sigma_t^h(p)\right) dh. \tag{3.13}$$

In a market with a continuum of sellers, each seller has negligible share of the aggregate customer base and hence $\hat{\sigma}_t(.)$ is independent of j. I call $\hat{\sigma}_t$ the average distribution of prices and it's the distribution of prices that each seller assigns to prices posted to its customers by competing sellers.

The weights of the other sellers in $\hat{\sigma}_t$ is determined by the relative size of each seller's customer base. Since sellers do not know which specific competing seller their customers might be aware off they form beliefs that assign a weight to the seller based on relative size of each seller's customer base. For example, consider a market with only three sellers, j_1, j_2, j_3. For a customer i of j_1, if i is aware of another seller the beliefs of seller j_1 assign a probability $c^l/(c^1 + c^2)$ that the competing seller is j_l for $l \in \{2, 3\}$. That is, j_1 beliefs assign a probability that conditional on j_1's customer knowing another one seller that seller is l with probability determined by l's relative fraction of the customer bases of all competing sellers.

3.3.3 Payoffs

To determine payoffs from strategies I need to determine the quantity sellers will sell in a given period considering their price posting strategy and the strategies of the other sellers. Note that since A is bounded above by 1, I assume that in any period a buyer that is aware of at least one seller will always purchase the good as no seller will post a price above their reservation price.

Let $\xi(p, \hat{\sigma}_t, \hat{c}_t)$ be the probability that a customer chooses to purchase from a seller that posts price p, if the average distribution of prices posted by the sellers is $\hat{\sigma}_t$ and the average customer base is $\hat{c}_t$. The probability of purchase if $\hat{\sigma}_t$ does not have an atom at p is

$$\xi(p, \hat{\sigma}_t, \hat{c}_t) = \pi^0(\hat{c}_t) + \pi^1(\hat{c}_t)\left[1 - \hat{\sigma}_t(p)\right] + \pi^2(\hat{c})\left[1 - \hat{\sigma}_t(p)\right]^2 + ..., \tag{3.14}$$

where $\pi^k(\hat{c})\left[1 - \hat{\sigma}_t(p)\right]^k$ is the probability a customer of seller j knows k other sellers times the probability that the price posted by j is lower than all k other prices. Note that $\{\pi^k(\hat{c}_t)\}_k$ is a Poisson probability mass function, therefore the expression 3.14 can be written as

$$\xi(p, \hat{\sigma}_t, \hat{c}_t) = \exp(-\hat{c}_t \hat{\sigma}_t(p)). \tag{3.15}$$

I assume a tie-breaking rule such that when there is a positive probability of a tie in prices between two or more sellers each seller has equal probability that a buyer will purchase from him. Then, the probability of sale at p with a probability of a tie in posted prices (that is, $\hat{\sigma}_t$ has an atom at p) is

$$\xi(p, \hat{\sigma}_t, \hat{c}_t) = \sum_k \pi^k(\hat{c}_t) \left[1 - \left(\lim_{\hat{p}\uparrow p} \hat{\sigma}_t(\hat{p}) + \frac{1}{2}\left(\hat{\sigma}_t(p) - \lim_{\hat{p}\uparrow p} \hat{\sigma}_t(\hat{p})\right)\right)\right]^k . \tag{3.16}$$

Since sellers have a continuum of customers the law of large numbers implies that the quantity sold is deterministic and given by c_t^j times $\xi(p, \hat{\sigma}_t, \hat{c}_t)$. Let $q(p, \hat{\sigma}_t, c_t^j, \hat{c}_t)$ be the quantity sold if the seller posts p, the average distribution of prices is $\hat{\sigma}_t$, the seller's customer base is c_t^j and the average customer base is $\hat{c}_t$. As marginal costs are normalized to zero, profits are equal to the quantity sold times the price. Let $\gamma_t(p, \hat{\sigma}_t, c_t^j, \hat{c}_t)$ be seller j's profit if for the corresponding variables.

Hence the realized quantity sold and profit in period t for seller j are given by

$$q(p, \hat{\sigma}_t, c_t^j, \hat{c}_t) = c_t^j \xi(p, \hat{\sigma}_t, \hat{c}_t), \tag{3.17}$$

$$\gamma(p, \hat{\sigma}_t, c_t^j, \hat{c}_t) = pq(p, \hat{\sigma}_t, c_t^j, \hat{c}_t). \tag{3.18}$$

The payoff for a seller is the present discounted profit stream, with discount factor $\beta \in (0,1)$. In the beginning of period t, the sequential problem of a seller j of age a is to choose a sequence of prices $\{p_k\}_k$ in A that maximizes the discounted present value of the future profits

$$\sup_{\{p_k\}_k \in A} \sum_{k=t}^{\infty} [\beta(1-\alpha)]^{k-t} \gamma_k(p_k, \hat{\sigma}_k, c_k^j, \hat{c}_k) \tag{3.19}$$
$$\text{s.t. } c_{k+1}^j = c_k^j + \Phi(q_k(p_k, \hat{\sigma}_k, c_k^j, \hat{c}_k), \bar{c} - c_k^j).$$

Note that the seller's payoff function incorporates the exogenous probability of exit after $k-t$ periods in operation $\prod_{z=1}^{k-t}(1-\alpha^{a+(k-t)})$ as the seller cannot make profits after it exits the market.

Let $v_t(c_t^j)$ be the value of the solution to the sellers problem 3.19 for a seller with customer base c_t^j in period t. The value function has a t subscript because the state of the market in which the seller operates, given by the profile of customer bases and price posting strategies, can change over time.

3.3.4 Equilibrium

The solution concept used here is Markov perfect equilibrium. It is implicit in the model that sellers are anonymous: as sellers have a negligible fraction of the aggregate customer bases the probability that their customers might know any particular competing seller is zero. That implies that seller's equilibrium strategies do not depend on the past actions of any individual seller. Instead price posting policies can only depend on the current state of the market which is described by the profile of customer bases.

I study symmetric equilibrium where sellers with the same customer base post prices according to the same distribution.

Definition 3.1. A symmetric equilibrium is a sequence of customer base profiles $\{(c_t^j)_{j\in[0,1]}\}_t$ and sequence of strategy profiles $\{(\sigma_t^j)_{j\in[0,1]}\}_t$ such that:

(i) For each period t and for each seller $j \in [0,1]$ posting a sequence of prices $\{p_k^j\}_{k=t}^{\infty}$ such that for each $k \geq t$, p_k^j is in the support of the distribution of prices σ_k^j, solves the seller's maximization problem 3.19 given the sequences of average distribution of prices and customer bases $\{\hat{\sigma}_k, \hat{c}_k\}_{k=t}^{\infty}$ implied by the sequence of strategies and customer bases profiles.

(ii) The sequence of customer base profiles $\{(\sigma_t^j)_{j\in[0,1]}\}_t$ satisfies 3.4.

From the definition of symmetric equilibrium and the assumption that entrants start with the same customer base the following proposition follows:

Proposition 3.1. *In symmetric equilibrium sellers of the same cohort have the same customer base.*

3.4 Equilibrium: Existence and characterization

To construct and characterize equilibrium, this section first proves existence, uniqueness and characterizes a steady-state equilibrium where the distribution of seller's population by cohorts is in steady state and the profiles of strategies and customer bases are both stationary. Then, using the unique value functions from the steady-state equilibrium, the non-steady-state equilibrium is characterized, which is an equilibrium where the profile of seller's population by cohorts is evolving towards a steady-state.

To articulate the characterization of the equilibrium I introduce additional notation. Proposition 3.1 states that sellers of the same cohort have the same customer base and in symmetric equilibrium sellers with the same customer base post the same prices so from this point onward I distinguish sellers only by age. Let c_t^a be the customer base of a seller of age a in period t. In a symmetric equilibrium, all sellers of the cohort of age a have the same customer base and therefore post prices according to the same distribution σ_t^a. In this case, the average distribution $\hat{\sigma}_t$ is the weighted average of the price posting strategies of all cohorts weighted by the customer share of each cohort. The customer share of each cohort is the population of each cohort n^a multiplied by the customer base of sellers in the cohort c_t^a. Let

$$\omega_t^a = \frac{n^a c_t^a}{\sum_{h=1}^t n^h c_t^h} \tag{3.20}$$

be the weight of cohort a in period t. Equation 3.20 implies that the average distribution of prices $\hat{\sigma}_t(.)$ can be written as

$$\begin{aligned} \hat{\sigma}_t(.) &= \frac{\sum_{a=1}^{\infty} n^a c_t^a \sigma_t^a(.)}{\hat{c}_t}, & (3.21) \\ &= \sum_a \omega_t^a \sigma_t^a(.). & (3.22) \end{aligned}$$

Let

$$\overline{\omega}_t^a = \sum_{h=1}^{a} \omega_t^h \tag{3.23}$$

be the cumulative customer share of all cohorts up to age a.

3.4.1 Symmetric steady-state equilibrium

In period t the population of each age cohort $(n_t^a)_{a=1}^t$ is, by construction, the same as the next period $(n_{t+1}^a)_{a=1}^t$. As $t \to \infty$ the population of sellers reaches a steady-state distribution by age. The steady-state equilibrium is the equilibrium that corresponds to the steady-state of the seller's population.

Definition 3.2. A steady-state symmetric equilibrium is a stationary profile of customer bases $\{c^a\}_{a=1}^{\infty}$, and a stationary profile of strategies $\{\sigma^a\}_{a=1}^{\infty}$ such that $\{c^a, \sigma^a\}_{a=1}^{\infty}$ is a symmetric equilibrium in the environment with a steady-state population of sellers by cohort $\{n^a\}_{a=1}^{\infty}$.

In a steady-state symmetric equilibrium the price posting strategies and the customer bases of the sellers are stationary and hence the distribution of prices posted and prices for executed transactions are also both stationary. Since the average distribution of prices and the average customer base is constant for variables in the steady-state I drop the t subscripts from $\hat{c}$, π^k, the probability of purchase from a customer $P(.,.)$, the quantity sold to cohort a, q^a and the associated profit function γ.

Note that when t is finite in the environment under study (with fixed exogenous entry and exit rates) we cannot have a steady state equilibrium. That is because the customer base of sellers with age $a > t$ is "zero" as no seller has reached this age yet but in future periods there are sellers of this age with non-zero customer bases. In the next subsection it is shown that the steady state equilibrium is approximated by the non-steady-state equilibrium as t becomes large.

Theorem 3.1. *Suppose that Assumptions (1)-(3) are satisfied. Then there exists a unique steady-state symmetric equilibrium* $\{c^a, \sigma^a\}_{a=1}^{\infty}$ *and it has the following properties*

(i) The price posting strategies of sellers $\{\sigma^a\}_a$ *are atomless.*

(ii) For cohorts of ages a, b *if* $a > b$*, then* $c^a > c^b$ *and every seller of cohort* a *posts higher prices than every seller of cohort* b *with probability* 1.

Theorem 3.1 states that in equilibrium entrant sellers have smaller customer bases, post lower prices than incumbents and post prices according to non-degenerate distributions. The proof of theorem 3.1 is constructive: I construct a steady-state equilibrium and show that it is the unique steady-state equilibrium.

The following lemmas are used in the proof of existence and the characterization of symmetric steady-state equilibrium. First, I show that for an arbitrary average distribution of prices $\hat{\sigma}$ there exists a corresponding value function v that satisfies the recursive formulation of the seller's problem. Value function

$v : [0, \overline{c}] \to \mathbb{R}$ satisfies

$$v(c) = \sup_{p \in (-\infty, 1]} \gamma(p, \hat{\sigma}, c, \hat{c}) + \beta(1-\alpha) v(c^+(p, c)), \tag{3.24}$$

$$\text{s.t. } c^+(p, c) = c + \Phi(q(p, \hat{\sigma}, c, \hat{c}), \overline{c} - c). \tag{3.25}$$

The law of motion for customer bases 3.25 implies that Φ maps $[0, \overline{c}]$ into itself, combined with $\beta(1-\alpha) \in (0,1)$ and $\gamma(p, \hat{\sigma}, c, \hat{c}) \leq \overline{c}$ implies that the sequential problem 3.19 with stationary $\hat{\sigma}$ and $\hat{c}$ and recursive problem above are equivalent (see Theorems 4.2-4.4 of Stokey and Lucas (1989) [73]).

Next there are the statements of three lemmas regarding the properties of the value function v, properties which I will use to show existence and describe the properties of the steady-state equilibrium. These properties are that, in symmetric equilibrium, v is non-decreasing. If $\hat{\sigma}$ is atomless, then v is strictly increasing and strictly concave. These properties are intuitive enough: the expected profits of a seller are increasing in the size of his or her customer base and they are strictly concave because the growth of the customer base is logistical.

Lemma 3.1. *If assumption (1) is satisfied, then, for every stationary average distribution of prices $\hat{\sigma}$ and stationary average customer base $\hat{c} \in [0, \overline{c}]$, the value function v exists and is unique.*

The next pair of lemmas characterizes the value function.

Lemma 3.2. *If assumptions (1)-(3) are satisfied, then, for every stationary average distribution of prices $\hat{\sigma}$ and stationary average customer base $\hat{c} \in [0, \overline{c}]$, the value function v is non-decreasing and strictly concave.*

The next lemma states that if $\hat{\sigma}$ is atomless then v is also strictly increasing and continuous:

Lemma 3.3. *If assumptions (1)-(2) are satisfied, then, for every stationary and atomless average distribution of prices $\hat{\sigma}$ and stationary average customer base $\hat{c} \in [0, \overline{c}]$, the value function v is continuous and strictly increasing.*

The next lemma states properties that any steady-state equilibrium $\hat{\sigma}$ has convex support, sellers with larger customer bases post higher prices with probability 1.

Lemma 3.4. *If v is strictly increasing, continuous and strictly concave, then the steady-state symmetric equilibrium has the following properties:*

(i) The price posting strategies for all sellers are atomless.

(ii) $\hat{\sigma}$ has a convex support.

(iii) The supports of the price posting strategies for sellers with different customer bases have disjointed interiors and sellers with larger customer bases post prices equal or higher than smaller sellers.

(iv) The upper bound of the support of the average distribution of prices $\hat{\sigma}$ is equal to 1.

Now we are ready to prove the main proposition that characterizes the steady-state equilibrium of this model:

Proposition 3.2. *If assumptions (1)-(3) are satisfied, the unique steady-state symmetric equilibrium has the following properties*

(i) A profile of sales for sellers of each cohort $\{q(c^a)\}_a$ satisfying for each $a \geq 1$,

$$q(c^a) = c^a \exp\left[-\hat{c}\left(\overline{\omega}^{a-1} + \frac{1}{2}\omega^a\right)\right]. \tag{3.26}$$

(ii) A profile of price posting strategies for each cohort $\{\sigma^a\}_{a=1}^{\infty}$ with support $[\underline{p}^a, \overline{p}^a]$ for each seller cohort a, where $\overline{p}^a = \underline{p}^{a+1}$ for all $a \geq 1$ (so the interiors of the supports are disjointed), and

$$\lim_{a\to\infty} \overline{p}^a = 1, \tag{3.27}$$

where each distribution σ^a satisfies for each $p \in [\underline{p}^a, \overline{p}^a]$,

$$c^a p\xi(p, \hat{\sigma}, \hat{c}) + \beta(1-\alpha)v\left[c^a + \Phi(q(p, \hat{\sigma}, c^a, \hat{c}), c^a)\right] = v(c^a). \tag{3.28}$$

(iii) The average price distribution $\hat{\sigma}$ is atomless and has support $[\underline{p}, \overline{p}] = \cup_{a=1}^{\infty}[\underline{p}^a, \overline{p}^a]$, such that $\overline{p} > 0$ and satisfies for $p \in [\underline{p}^a, \overline{p}^a]$,

$$\hat{\sigma}(p) = \overline{\omega}^{a-1} + \sigma^a(p)\omega^a.$$

Proposition 3.2 implies in theorem 3.1. Note that the steady-state symmetric equilibrium is characterized by price dispersion across cohorts of sellers, each cohort of sellers randomizing the prices they post and the price posting strategy of incumbent sellers first-order stochastically dominates the price posting strategy of the entering sellers. Properties that are found in the data by several empirical studies.

3.4.2 Non-steady-state equilibrium

Consider a product market that is created in period 1 when the first cohort of sellers enters the market and such that Φ satisfies assumptions (1)-(3). This subsection describes the construction and the uniqueness of the symmetric equilibrium of this product market. The equilibrium is such that at any period t, prices posted by sellers entrant sellers are lower than prices of posted by incumbents with probability 1, the randomized price posting strategies of the sellers are non-degenerate, individual sellers accumulate customers over time, increasing the prices they post while (as shown in section 3.5) average prices for transactions in the market are decreasing over time.

To construct and characterize the non-steady-state equilibrium I proceed according to the following strategy: I construct a candidate equilibrium in subsection 3.4.2.1, a sequence of strategy profiles $\{(\sigma_t^a)_{a=1}^t\}_t$ and it's corresponding sequence of customer base profiles from a sequence of (arbitrary) value functions $\{v_t\}$. In subsection 3.4.2.2 it is shown that this sequence of strategy profiles and corresponding sequence of customer bases profiles implies in a unique sequence of value functions $\{v_t\}$ that converges to the steady-state value function v and that the strategy and customer base profiles $\{(\sigma_t^a, c_t^a)_{a=1}^t\}_t$ are consistent with symmetric equilibrium. Finally, in subsection 3.4.2.4 it is shown that the symmetric non-steady-state equilibrium is unique.

3.4.2.1 Candidate equilibrium strategy profile

Consider period t and seller cohort of age $a \in \{1, \ldots, t\}$ and let $[\underline{p}_t^a, \overline{p}_t^a]$ be the support of a price posting strategy $\sigma_t^a(.)$. Hence, in period t there is a profile of supports $\{[\underline{p}_t^a, \overline{p}_t^a]\}_{a=1}^t$ for the profile of strategies $\{\sigma_t^a\}_{a=1}^t$. Suppose the profile of supports of the randomized strategies $\{[\underline{p}_t^a, \overline{p}_t^a]\}_{a=1}^t$ have disjoint interiors and assume that $\{\sigma_t^a\}_{a=1}^t$ are atomless. In this case equations 3.21 and 3.23 imply that $\hat{\sigma}_t(.)$ satisfies

$$\hat{\sigma}_t(p) = \overline{\omega}_t^{a-1} + \omega_t^a \sigma_t^a(p). \tag{3.29}$$

To construct the profile of supports and distributions the following equal profit conditions are used. Let $\overline{p}_t^t = 1$ and for each $a \geq 1$, $\overline{p}_t^a = \underline{p}_t^{a+1}$ and $\underline{p}_t^a$ satisfies

$$\underline{p}_t^a c_t^a \xi(\underline{p}_t^a, \hat{\sigma}_t, \hat{c}_t) + [\beta(1-\alpha)]\, v_{t+1}\left[c_{t+1}(\underline{p}_t^a, c_t^a)\right] = \overline{p}_t^a c_t^a \xi(\overline{p}_t^a, \hat{\sigma}_t, \hat{c}_t) + [\beta(1-\alpha)]\, v_{t+1}\left[c_{t+1}(\overline{p}_t^a, c_t^a)\right], \tag{3.30}$$

where

$$c_{t+1}(p, c_t^a) = c_t^a + \Phi(c_t^a \xi(p, \hat{\sigma}_t, \hat{c}_t), \overline{c} - c_t^a), \tag{3.31}$$

$$\xi(p, \hat{\sigma}_t, \hat{c}_t) = \exp\left[-\hat{c}_t\left(\overline{\omega}_t^{a-1} + \omega_t^a \sigma_t^a(p)\right)\right], \tag{3.32}$$

where the right hand side of 3.32 follows from the definition of $\xi(p, \hat{\sigma}_t, \hat{c}_t)$ and 3.29.

Let $\overline{p}_t^a = \underline{p}_t^{a+1}$ for $a < t$ which implies that the profile of supports has disjoint interiors and that the average distribution $\hat{\sigma}_t$ satisfies

$$\operatorname{supp}(\hat{\sigma}_t) = [\underline{p}_t^1, 1].$$

For $p \in (\underline{p}_t^a, \overline{p}_t^a)$ the distribution σ_t^a satisfies the following equal profit condition

$$p c_t^a \xi(p, \hat{\sigma}_t, \hat{c}_t) + [\beta(1-\alpha)]\, v_{t+1}(c_{t+1}(p, c_t^a)) = \overline{p}_t^a c_t^a \xi(\overline{p}_t^a, \hat{\sigma}_t, \hat{c}_t) + [\beta(1-\alpha)]\, v_{t+1}(c_{t+1}(\overline{p}_t^a, c_t^a)). \tag{3.33}$$

Since all sellers of cohort of age a will post prices according to the same distribution these sellers will sell the same quantity. Let q_t^a be the quantity sold by seller of age a in period t, it satisfies

$$q_t^a = c_t^a \exp\left[-\hat{c}_t\left(\overline{\omega}_t^{a-1} + (1/2)\omega_t^a\right)\right]. \tag{3.34}$$

Equation 3.34 follows from: (1) Cohorts older than a will post strictly higher prices than the prices in the support of the distribution of cohort a, so the probability of sale is at least $\exp\left[-\hat{c}_t \overline{\omega}^{a-1}\right]$. (2) Cohorts younger than a will post strictly lower prices than cohort a's support, making the probability of sale not higher than $\exp\left[-\hat{c}_t \overline{\omega}^{a}\right]$. Sellers of cohort a will be only competing against sellers from the same cohort. Since sellers in a post the same distribution of prices the probability of undercutting other sellers is .5. Hence, the probability of sale satisfies $\exp\left[-\hat{c}_t\left(\overline{\omega}^{a-1}+(1/2)\omega^a\right)\right]$. See the proof of Proposition 3.2 for a formal derivation of equation 3.34.

Equation 3.34 implies in a sequence of customer bases profiles $\{(c_t^a)_{a=1}^t\}_t$ where $c_t^1=\underline{c}, \forall t$ and for all t and $a\in\{1,\dots t\}$,

$$c_{t+1}^{a+1}=c_t^a+\Phi(q_t^a,\overline{c}-c_t^a). \tag{3.35}$$

To show that this strategy profile and implied sequence of customer bases is in fact an equilibrium it is first required to show that there exists sequence of $\{v_t\}$ implied by problem 3.19 that is consistent with this profile of strategies.

3.4.2.2 Candidate equilibrium value functions

To construct an equilibrium when the distribution of customer bases and prices posted changes in each period using the candidate equilibrium strategies and customer bases constructed above I construct a sequence of value functions $\{v_t\}_t$ that converges to v as $t\to\infty$. Then I show that this candidate equilibrium strategies and customer bases $\{(\sigma_t^a,c_t^a)_a\}_t$ is indeed a Markov-perfect equilibrium, that is, it is optimal for sellers of cohort a to post prices according to σ_t^a in period t.

Note that in subsection 3.4.1 that a steady-state equilibrium and the corresponding steady-state value function v exists and is unique. I construct a sequence of value functions $\{v_t\}$ that as $t\to\infty$ converges to the value function v that corresponds to the steady-state. Suppose that the value function in period $t+1$, v_{t+1}, exists and is concave, continuous and strictly increasing. Let v_t satisfy for $c_t\in\mathbb{R}_+$,

$$v_t(c_t)=\max_{p\in A}\gamma(p,\hat{\sigma}_t,c_t,\hat{c}_t)+\beta(1-\alpha)v_{t+1}\left[c_{t+1}(p,c_t)\right], \tag{3.36}$$

where $\hat{\sigma}_t$ is constructed according to equations 3.30 and 3.33. Since v_{t+1} is well defined, concave, continuous and strictly increasing, then σ_t is atomless and so γ_t is continuous. Therefore the function v_t exists and is continuous, concave and strictly increasing.

Pick a $T > 1$ and suppose that

$$v_T = v. \tag{3.37}$$

It's easy to see that the profile of seller cohort populations converges to $\{n^a\}_{a=1}^{\infty}$ as $t \to \infty$.

If seller cohorts follow the symmetric price posting strategy $\{\sigma_t^a\}_{a=1}^{t}$ equation 3.34 implies that the profile of customer bases converges to $\{c^a\}_{a\geq 1}$ and $\hat{c}_t \to \hat{c}$ as $t \to \infty$. Consider v_t and v, note that v_t and v are bounded, then since $\hat{c}_t \to \hat{c}$ and customer base profiles converges to $\{c^a\}_{a\geq 1}$, then equation 3.36 for $\epsilon > 0$ there is a T large enough such that

$$||v_{T-1} - v_T|| = \frac{\max_{m\in\mathbb{R}_+} |v_{T-1}(c) - v_T(c)|}{c} < \epsilon. \tag{3.38}$$

Let $(v_1, \ldots, v_T)$ be a profile of value functions defined by 3.37 and 3.36, for $T \to \infty$ equation 3.38 implies that $\lim_T v_T = v$. Let $\{v_t\}_t$ be the sequence of value functions implied by taking the limit $\lim_{t\to\infty}\{(v_1, \ldots, v_T)\}$. Note that since v_{T-1} converges to v such a sequence exists. This will be our candidate sequence of equilibrium value functions for each period.

To see that $\{v_t\}_t$ is consistent with the seller's problem defined in 3.19 note that: γ is bounded above by $\bar{c}$, $\beta(1-\alpha) \in (0,1)$ and $c_{t+1} \in [0, \bar{c}]$. The conditions for applying Theorems 4.2-4.4 of Stokey and Lucas (1989) are satisfied which imply that the sequential problem 3.19 and the functional equation 3.36 are equivalent if posting prices in the support of $\{(\sigma_t^a)_a\}_t$ is optimal for the sellers.

3.4.2.3 Consistency of candidate strategies and value functions with the non-steady-state equilibrium

It remains to show that this construction of strategy profiles $\{(\sigma_t^a)_a\}_t$, value functions $\{v_t\}$ and its associated sequence of customer base profiles are in fact a symmetric equilibrium.

Theorem 3.2. *The sequences of strategy profiles and associated customer base profiles* $\{(\sigma_t^a)_a, (c_t^a)_a\}_t$ *described by 3.30 and 3.33 are an equilibrium.*

Note that the desired properties of the symmetric equilibrium for a general environment have not been proven yet. I have constructed a sequence of value functions given a equilibrium sequences of price posting strategies and customer base profiles and showed that this sequence of value functions implies in an unique equilibrium sequence of price posting strategies and customer base profiles. It was not shown yet that this is the only Markov perfect symmetric equilibrium of this model.

3.4.2.4 Uniqueness

The following theorem states that the equilibrium we have just constructed is the unique symmetric equilibrium for this environment.

Theorem 3.3. *Given assumptions (1)-(3), the non-steady-state equilibrium is unique.*

The proof of theorem 3.3 utilizes additional notation which is introduced here. It's possible to represent the state of the market in any period t by a sequence of customer bases $\{c^a\}_{a=1}^{\infty}$ such that $c^a = c_t^a$ for all $a \in \{1, \dots t\}$ and $c^a = 0$ for $a > t$ (that is, cohorts which haven't entered the market yet). Let $K^{\mathbb{N}}([0, \overline{c}])$ be the set of all sequences in $[0, \overline{c}]$. Let $V(c, \{c^a\}_{a=1}^{\infty})$ be a value function that maps the space of customer base profiles $\{c^a\}_{a=1}^{\infty} \in K^{\mathbb{N}}([0, \overline{c}])$ and the seller's customer base $c \in [0, \overline{c}]$ into a real number. I want to show that in symmetric equilibrium the value function V is unique. Subsection 3.4.1 already has shown that in a steady-state symmetric equilibrium, when $\{c^a, \sigma^a\}_{a=1}^{\infty}$ are constant, then V is unique, here I generalize this proof.

Let $\mathcal{F}([0, \overline{c}] \times K^{\mathbb{N}}([0, \overline{c}]))$ be the space of functions that map $[0, \overline{c}] \times K^{\mathbb{N}}([0, \overline{c}])$ into $\mathbb{R}_+$ that are strictly increasing and concave on $[0, \overline{c}]$. Consider the operator T that maps $\mathcal{F}([0, \overline{c}] \times K^{\mathbb{N}}([0, \overline{c}]))$ into a functional space. $T(V)$ for $V \in \mathcal{F}([0, \overline{c}] \times K^{\mathbb{N}}([0, \overline{c}]))$ satisfies

$$T(V)(c, \{c^a\}_{a=1}^{\infty}) = \sup_p \gamma(p, \hat{\sigma}, c, \hat{c}) + \beta(1-\alpha) V(c^+(p, c), \{c^{a+}\}_{a=1}^{\infty}), \quad (3.39)$$

where $\hat{\sigma}$ is an average distribution of prices determined from a symmetric profit maximizing strategy profile $\{\sigma^a\}_{a=1}^{\infty}$, where $p \in \operatorname{supp}(\sigma^a)$ satisfies

$$\gamma(p, \hat{\sigma}, c^a, \hat{c}) + \beta(1-\alpha) V(c^{a+}, \{c^{a+}\}_{a=1}^{\infty}) = T\left[V(c^a, \{c^a\}_{a=1}^{\infty})\right],$$

and $c^{+}(p,c), \{c^{a+}\}_{a=1}^{\infty}$ are, respectively, the next period customer base of the seller with customer base c if she posts p and $\{c^{a+}\}_{a=1}^{\infty}$ is the next period profile of customer bases that is determined by the strategy profile $\{\sigma^{a}\}_{a=1}^{\infty}$.

A fixed point of T is a value function V that is consistent with equilibrium for any profile of customer bases. The following lemma states that such value function will imply in a unique profile of strategies for the sellers.

Lemma 3.5. *For a fixed point V of T that is strictly increasing and strictly concave on c, then there is a unique equilibrium symmetric strategy profile that is consistent with V.*

Theorem 3.3 implies that the only non-steady-state equilibrium is the equilibrium constructed here, which converges to the unique steady-state equilibrium. Therefore, Theorem 3.3 implies in the following corollary:

Corollary 3.1. *The non-steady-state equilibrium converges to the steady-state equilibrium.*

3.5 Properties of the equilibrium

This section provides a characterization of the dynamic behavior of the distribution of prices posted, the volume of transactions and prices for transactions in the symmetric equilibrium as it converges to the steady-state equilibrium from an initial state where all sellers have zero customer base and cohorts of new sellers enter at the rate α with positive customer base $\underline{c}$ and where Φ satisfies assumptions (1)-(3). It also shows the symmetric equilibrium is efficient and that it converges to perfect competition as frictions vanish.

3.5.1 Decreasing markups

In the symmetric non-steady-state equilibrium that is being studied the average price for transactions is strictly decreasing over time as the equilibrium converges to the steady-state. To show that, first I show that the average number of sellers that buyers know is strictly increasing on time. Second, I show that that the distribution of prices for transactions is monotone decreasing on time if the average number of sellers that buyers know is strictly increasing on time.

Lemma 3.6. *In symmetric equilibrium the average number of sellers that buyers are aware of $\hat{c}_t$ is strictly increasing on the "age" t of the product market.*

Let AM_t be average price or markup (as marginal cost is normalized to zero) for transactions in equilibrium in period t. The average markup AM_t in period t is a function of $\hat{\sigma}_t$ and $\hat{c}_t$, as a buyer who knows k sellers buys at the lowest price among these k sellers who post prices according to $\hat{\sigma}_t$. Note that the average markup for transactions is the average price paid in transactions in this model since marginal costs are normalized to zero. Let F_t^k, be the the cumulative distribution of prices paid by a buyer who knows k sellers in period t, the expected markup paid by all the buyers in the market is

$$AM_t = \left(\frac{\sum_{k=1}^{\infty} \pi^k(\hat{c}_t)}{1 - \pi^0(\hat{c}_t)} \right) \int p dF_t^k(p), \tag{3.40}$$

where

$$\begin{aligned} F_t^k(p) &= \text{Prob}_t(p \geq \min\{p_1, \ldots, p_k\}) \\ &= 1 - \text{Prob}_t(p \leq \min\{p_1, \ldots, p_k\}) \\ &= 1 - \text{Prob}_t(p \leq p_1) \times \ldots \times \text{Prob}_t(p \leq p_k) \\ &= 1 - [1 - \hat{\sigma}_t(p)]^k . \end{aligned}$$

The following theorem states that the prices for transactions are decreasing over time.

Theorem 3.4. *In symmetric equilibrium the average markup for transactions is strictly decreasing in t.*

3.5.2 Efficiency

A property of the symmetric equilibrium is that it is efficient. Efficiency here means that at any period t, the aggregate flow surplus is maximized for any feasible trajectory of customer bases and sales quantities. That is, the seller customer base growth is such that it maximizes the possible aggregate market surplus to be realized while at any period all opportunities to realize the surplus from transactions are realized.

When a buyer and seller exchange the good they generate 1 unit of surplus. The aggregate market surplus is the average quantity sold in the market, $\int_J q_t^j dj$. In equilibrium buyers always purchase the good if they know at least one seller therefore the aggregate market surplus in period t is the population of buyers who know at least one seller, $1 - \pi_t^0$.

First I define the set of feasible allocations then I define efficiency. An allocation is a sequence of sales profiles $\{\{q_t^j\}_{j \in J}\}_{t=1}^{\infty}$.

Definition 3.3. An allocation is feasible if $\int_J q_t^j dj \leq 1 - \pi_t^0$, each buyer purchases from only one seller in each period (so that two sellers cannot sell to the same buyer), $q_t^j \in [0, c_t^j], \forall j \in J$ and the profile of customer bases for each period $\{c_t^j\}_{j \in J}$ is consistent with the profile of sales and customer bases in the previous period and the law of motion for customer bases described by equation 3.4.

Example 3.3. Consider a market in period t where

$$c_t^j = \begin{cases} 1 & \text{if } j \in [0, 1/2] \\ 2 & \text{if } j \in (1/2, 1] \end{cases},$$

hence $\hat{c}_t = 3/2$ and therefore

$$\pi_t^k = \frac{(3/2)^k \exp(-3/2)}{k!},$$

consider a profile of sales

$$q_t^j = \begin{cases} 1 \left(\sum_k \pi_t^k \left[\frac{2}{3} + \frac{1}{3(k+1)} \right] \right) & \text{if } j \in [0, 1/2] \\ 2 \left(\sum_k \pi_t^k \left[\frac{1}{3} + \frac{2}{3(k+1)} \right] \right) & \text{if } j \in (1/2, 1] \end{cases},$$

this profile of sales is feasible: if customers of seller j know only j they shop at j, if customers knows multiple sellers they shop first at sellers with customer base of 1 then at sellers with customer base of 2. Hence, the sales profile $\{q_t^j\}_{j\in J}$ satisfies all the conditions for being a feasible allocation in period t.

An efficient allocation in this environment is a feasible trajectory of sales profiles for each seller such that that it maximizes the sequence of aggregate market surpluses.

Definition 3.4. A sequence of sales profiles $\{\{q_t^j\}_{j\in J}\}_{t=1}^{\infty}$ is efficient if it is feasible and yields a sequence of market surpluses $\{\int_J q_t^j dj\}_{t=1}^{\infty}$ such that there is no feasible sequence of sales profiles that implies in equal or higher market surpluses for all periods with strict inequality in at least one period.

Theorem 3.5. *The symmetric equilibrium is efficient.*

Efficiency is obtained by a feasible profile of sales among sellers that maximizes $\hat{c}$. The equilibrium is efficient because price posting strategies by the sellers implies that buyers always purchase from the sellers with smallest customer base among the sellers that they know. In an efficient allocation, buyers should always purchase from these sellers as they have the greatest potential for growth and in equilibrium it is such that sellers with smaller customer base never grow to become larger than sellers with larger customer base (hence, there is no excessive growth of smaller sellers). The symmetry of equilibrium is essential for the efficiency result: efficiency requires that sellers with the same customer base should sell the same quantity, otherwise the strict concavity of Φ implies that it's possible to improve efficiency by reallocating sales between sellers with the same customer base, increasing the size of the average customer base next period.

In standard search models efficiency is not so easily obtained (Diamond (1982) [18] and Mortensen and Wright (2002) [60]). The reason why the efficiency result is more robust here is that only the ordering of the pricing strategies matter for determining whether the equilibrium is efficient or not: there exists a continuum of efficient pricing strategies in this environment. On the other hand, in a standard random search model efficiency can only be obtained in the knife-edge case when the relative bargaining power of each side of the market exactly matches the relative marginal gains from search effort, as there are congestion externalities otherwise.

3.5.3 Convergence to perfect competition in the steady-state symmetric equilibrium

The competitive equilibrium that corresponds to the physical environment described in section 3.2 satisfies the following properties: First, in every period, all sellers post a price equal to the marginal cost, 0, second, all buyers purchase each a unit of the good, hence, the aggregate quantity sold is 1.

Consider a steady-state equilibrium with $\pi^0 = 0$. In this case it's simple to check that the steady-state symmetric equilibrium has the same properties as the competitive equilibrium as it is Bertrand competition: All customers of a seller will know a competing seller thus for any non-degenerate price distribution $\hat{\sigma}$ with support above marginal cost, no seller would like to post a price at the upper bound of the support of the distribution as they would make zero sales while they could make non-zero expected profits by posting a price in the interior of the distribution. Which implies a non-degenerate $\hat{\sigma}$ is inconsistent with equilibrium. Therefore, in equilibrium, the distribution of prices posted is degenerate and for any price above 0, sellers will have an incentive to undercut each other.

Proposition 3.3 states that as $\alpha \to 0$ and $\overline{c} \to \infty$ then in the steady-state $\hat{c} \to \infty$ (which implies that $\pi^0 \to 0$). Theorem 3.6 states that as $\hat{c} \to \infty$ then the symmetric steady-state equilibrium converges to the competitive equilibrium: the prices posted in the symmetric equilibrium converge in probability to the competitive price while the fraction of buyers that purchase the good converges to 1.

Proposition 3.3. *Let* $\{\overline{c}_n\}_{n=1}^{\infty}$ *be a strictly increasing sequence in* $\mathbb{R}_{++}$ *that diverges to infinity,* $\{\alpha_n\}_{n=1}^{\infty}$ *be a monotone decreasing sequence in* $(0,1)$ *such that* $\lim_n \alpha_n = 0$*, and let* $\{\{c_n^a\}_a\}_n$ *and* $\{\hat{c}_n\}$ *be respectively the sequence of steady-state equilibrium profiles of customer bases by seller age and the average customer base corresponding to the sequences of upper bound on customer bases* $\{\overline{c}_n\}$ *and seller entry/exit rates* $\{\alpha_n\}$*. Then* $\{\hat{c}_n\}$ *converges to infinity and* $\lim_{n\to\infty} \pi^0(\hat{c}_n) = 0$.

That is, in a sequence of environments where the upper bound of the customer base diverges to infinity and the exit rates converge to zero the sequence of average customer bases in the steady-state equilibrium converges to infinity.

The following theorem states that as the average customer base of sellers increases to infinity then the equilibrium distribution of prices for transactions converges weakly in probability to the marginal cost.

Theorem 3.6. *For a steady-state distribution of seller by age cohorts* $\{n^a\}_{a=1}^{\infty}$, *let* $\{\{c_k^a, \sigma_k^a\}_{a=1}^{\infty}\}_k$ *be a sequence of steady-state equilibria such that*

$$\lim_k \hat{c}_k = \infty$$

then for all $p \in (0,1)$, $\lim_{k\to\infty} p\xi(p, \hat{\sigma}^k, \hat{c}_k) = 0$, *where* $\xi(p, \hat{\sigma}^k, \hat{c}_k)$ *is the probability that a customer purchases from a seller posting* p *in the steady-state equilibrium corresponding to* $\{(c_k^a, \sigma_k^a)\}_a$ *and for* $p \in (0,1)$ *the equilibrium average distribution of prices satisfies*

$$\lim_k \sum_l \pi^l(\hat{c}_k) \left[1 - \hat{\sigma}^k(p)\right]^l = 0.$$

That is, in equilibrium, when the average customer base increases to infinity the probability that the prices for transactions are higher than marginal cost converges to 0 *as* $k \to \infty$ *and the fraction of buyers who do not purchase the good from any seller converges to zero.*

Theorem 3.6 is intuitive: when the average customer base increases the average number of sellers that customers know also increases, which increases the intensity of competition between sellers, driving prices down. When $\hat{c} \to \infty$, the increased intensity of competition between sellers drives prices down to marginal cost, the discounted value of the profit stream converges to zero and the price dispersion equilibrium converges to perfect competition.

3.6 Quantitative analysis

This section brings the model to the data by performing exercises comparative statics using parameter values derived from empirical studies to better understand the implications of the model. The quantitative analysis is exploratory, as this model is too stark to capture a substantial portion of the features of the data. Nevertheless, I believe that these findings may still be useful as back-of-the-envelope calculations. The next subsection shows the results of a simulation for the evolution of customer bases, distribution of prices for transactions and the distribution of prices posted at different points in time and the evolution of average markups.

3.6.1 Parameter values

Let the period length represent a quarter. I assume a constant seller exit rate $\alpha = .02$ to be consistent with the empirical evidence that average exit rate is approximately 8% per year (Siegfried and Evans (1994) [69]), a discount factor $\beta = 1 - .0125$, corresponding to a annual discount rate of about 5%. Set the upper bound of the customer base $\overline{c} = 5$ and the initial size of the customer base $\underline{c} = .001$.

The functional form for the customer-capital accumulation Φ function is described by 3.5 but removing the term $(\overline{c} - c)$, which is a Cobb-Douglas,

$$\Phi(q, \overline{c} - c) = A \left(q^{\zeta} (\overline{c} - c)^{1-\zeta} \right). \tag{3.41}$$

Note that a Cobb-Douglas does not satisfy assumptions (2)-(3) but the simulation still yields a continuous, strictly concave and increasing value function which makes the equilibrium computed unique. This implies that Assumptions (1)-(3) are sufficient conditions for uniqueness but not all are necessary conditions. There is an upper bound for the customer base, so assume that c_t^j follows the law of motion

$$c_{t+1}^j = \min\{c_t^j + \Phi(q_t^j, \overline{c} - c_t^j), \overline{c}\},$$

if the seller does not exit the market.

I assume $\zeta = 3/4$ and removed the term $(\overline{c} - c)$ from 3.5 to reflect the empirical findings that active demand accumulation through sales effort is the dominant factor in demand accumulation (Foster, Haltiwanger and Syverson (2016)). I choose $A = .06$ so that sellers reach 90% of the upper bound of their customer base after 15 years (Foster, Haltiwanger and Syverson (2016) [25] find as the approximated expected duration for the process of demand accumulation after entry).

In this calibration, I target the standard deviation of prices measured in Kaplan et al. (2016) [50], who measured price dispersion for consumer goods using the Kielts-Nielsen Homescan Dataset. They find that the standard deviation of prices is 15.3% and the variance is 2.34%. They also find that 85% of the variance in prices is due to differences in prices for transactions for stores which are equally expensive on average, which means that standard deviation of prices excluding the store-expensiveness component is approximately 14.1%, which is the value that I target.

According to Perla (2017) [63], the median age of US industries is approximately 20 years, so I set a marginal cost that implies that the standard deviation of prices for transactions is approximately 14.1% in the equilibrium of a product market after 20 years. Which implies in a marginal cost of .929.

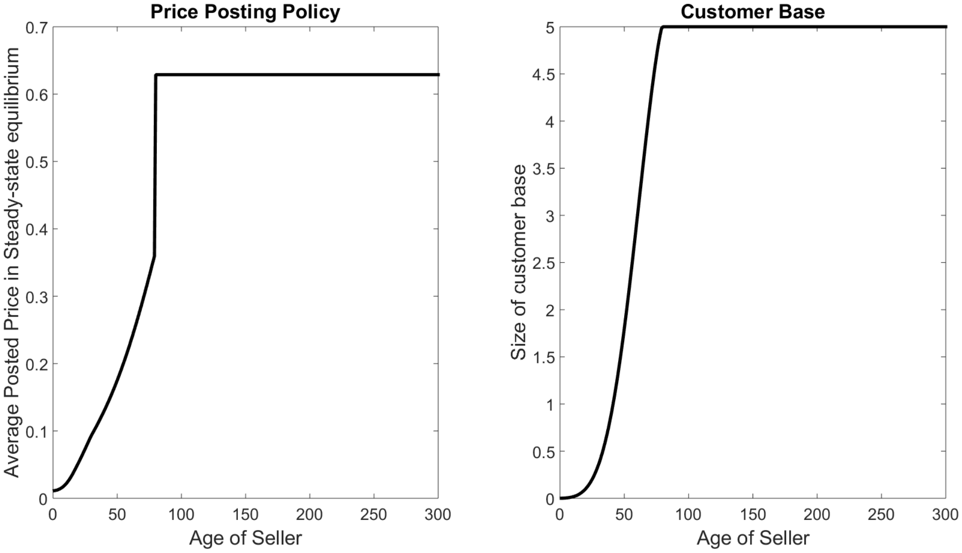

Figure 3.3: Average price posted and customer base of sellers by cohort in steady state equilibrium

3.6.2 Results

Expected prices posted and the evolution of the customer bases by cohort age in the steady-state. Note that the price posting policy is the same for all cohorts who reached the upper bound of the customer base.

Note also that the distribution of prices in the steady-state equilibrium is roughly symmetric around the mode of the distribution, such a result is not easily attained in other models. Empirical aggregate data on price dispersion shows a similar pattern (Kaplan and Menzio (2015) [48]). Also, since all sellers have the same constant marginal cost functions the distribution of prices for transactions in equilibrium approximates the distribution of average markups for sellers if the number of seller cohorts is large. In this simulation with a few hundred cohorts it is already possible to approximate the empirical evidence for the distribution of markups as shown by De Loecker and Eeckhout (2018a,b) [11, 10].

The distribution of prices in the steady-state equilibrium and the average markup for transactions is very close to the steady-state level after about $65-70$ years.

The average standard deviation (in relation to the average price) for prices for transactions first increases as prices move away from monopoly and then it starts

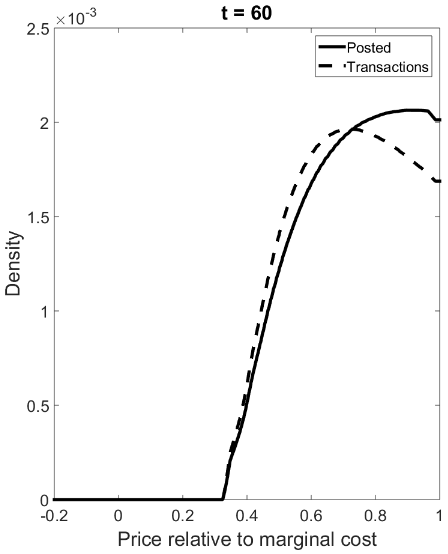

Figure 3.4: Distribution of prices in the equilibrium converging to a steady-state distribution, t = 60,120

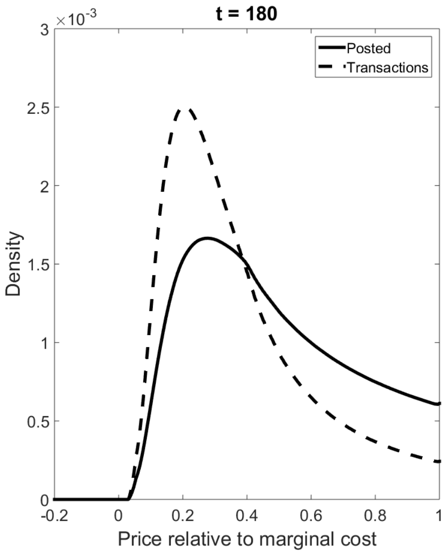

Figure 3.5: Distribution of prices in the equilibrium converging to a steady-state distribution, t = 180,240

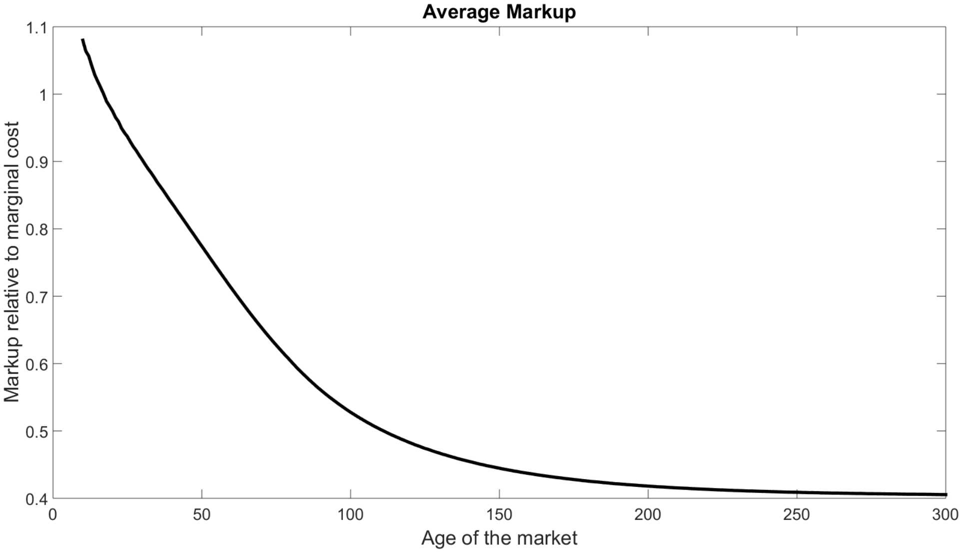

Figure 3.6: The average markup level for transactions in the market

to decrease as competition intensifies and prices are pressed in the direction of marginal costs.

Lowering market frictions, for example, by increasing A (keeping every other variable constant), moves the distribution of prices of the steady-state equilibrium closer to perfect competition.

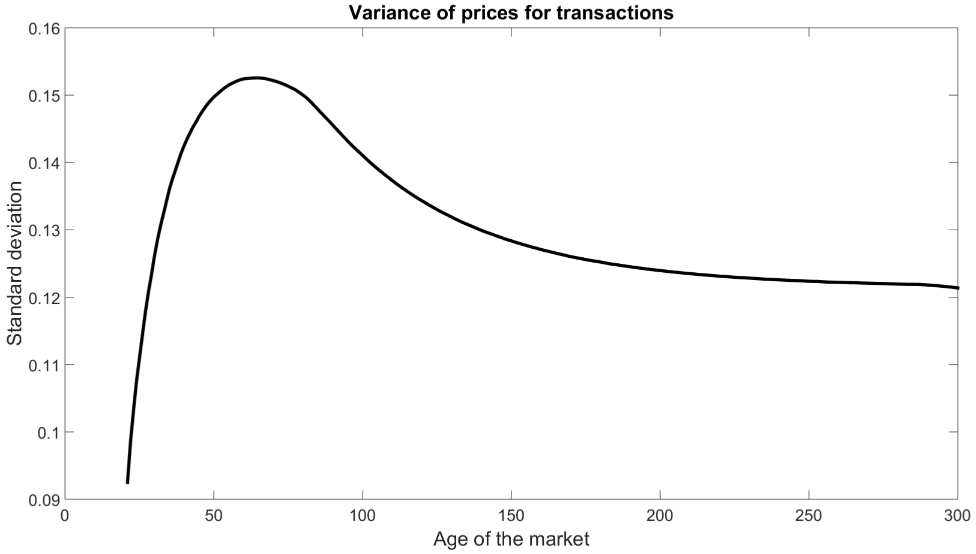

Figure 3.7: Variance of prices for transactions

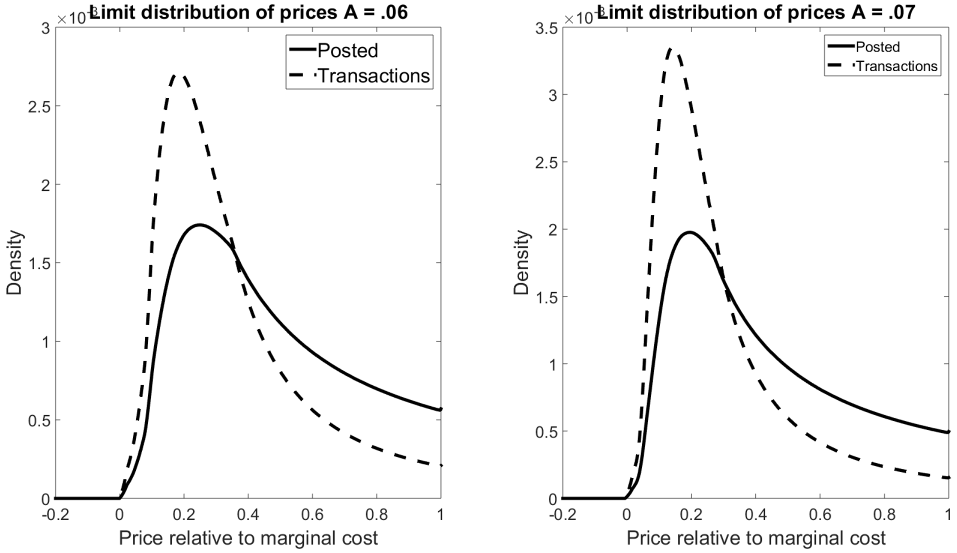

Figure 3.8: Limiting distributions (1)

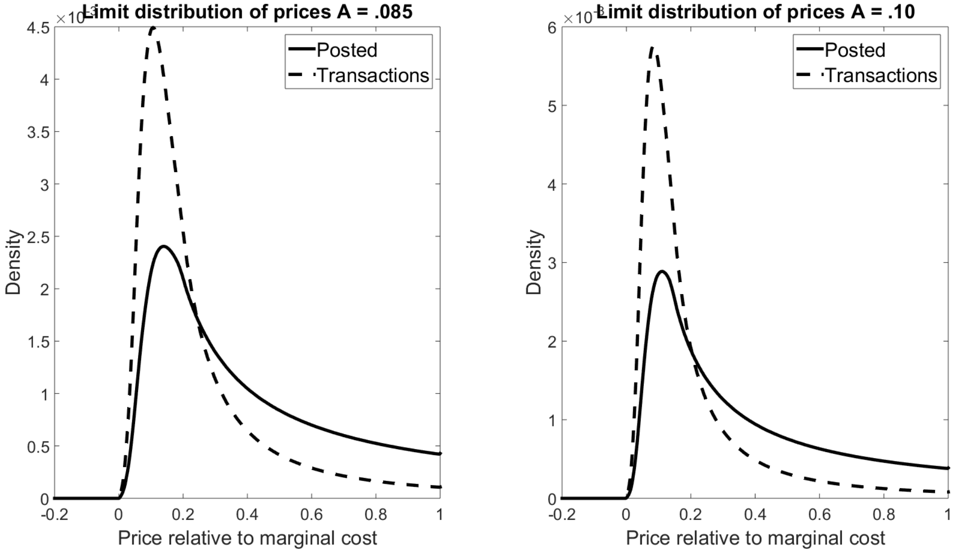

Figure 3.9: Limiting distributions (2)

3.7 Extensions

3.7.1 Endogenous entry

In previous sections seller entry was assumed to be exogenous. In this subsection, I relax that assumption, instead allowing sellers to choose whether to enter the market by spending $e > 0$ in start-up costs. Allowing for endogenous entry implies that the population of sellers is endogenous and so new assumptions for the properties for the accumulation of new customers have to be made to take into consideration market congestion. Let $n_s \in \mathbb{R}_+$ be the measure of sellers entering the market at each period. The measure of sellers operating in the market in the steady-state of the seller's population is $s = n_s/\alpha$.

The new assumptions should imply that if a market is well established with many sellers that have well established customer bases then it becomes harder for a new seller to grow their customer bases. The original set of assumptions is already characterized by this property in regards to the average size of the competitor's customer bases but not explicitly in regards to the potential measure of sellers that could operate in the market since that parameter was exogenously fixed at 1. Let $\hat{c}_t$ be the average size of the customer base, s be the aggregate measure of sellers in the market and

$$\hat{s}_t = s\hat{c}_t, \tag{3.42}$$

be the average number of sellers that buyers know, hence $\pi^k_t = \pi^k(\hat{s}_t)$ is distributed according to a Poisson with parameter $\hat{s}_t$. In addition, the seller's beliefs regarding the distribution of prices posted competing sellers that their customers might be in contact, that is the "average distribution of prices" $\hat{\sigma}_t(.)$, satisfies

$$\hat{\sigma}_t(p) = \frac{\int_0^s c^j_t \sigma^j_t(p)dj}{s\hat{c}}.$$

The steady-state equilibrium has the property that in each period the measure of sellers that enters the market α is such that it equalizes the entry cost e with the expected lifetime profits of the seller at the entry period which are $v(\underline{c})$. For example, the results in the quantitative exercise 3.6 imply in $v(\underline{c}) = 5.27$ so setting entry costs $e = 5.27$ implies that the steady-state equilibrium of population of sellers equal to one.

Theorem 3.7. *For a set of parameter values and* $e > 0$ *is there always a unique entry rate* n_s *that is consistent with a steady-state equilibrium where* $v(\underline{c}) = e$ *if* $n_s > 0$ *or* $v(\underline{c}) < e$ *if* $n_s = 0$.

Consider the parameters from the simulation 3.6, with $A = .06$. In this case the average markup and the initial value of a seller in the market in the steady-state equilibrium as a function of the population of sellers s is described by the figure below. Note that average markups can become negative if the population of sellers in the market is large enough, even though the entry value remains positive. The reason for that is that entrants will price below marginal cost and the majority of transactions will be with entering sellers but older sellers will turn out a profit.

As described in figure 3.11 and figure 3.12, the entry cost converges to zero the population of sellers in the steady-state symmetric equilibrium increases and the distribution of prices approximates the perfectly competitive price.

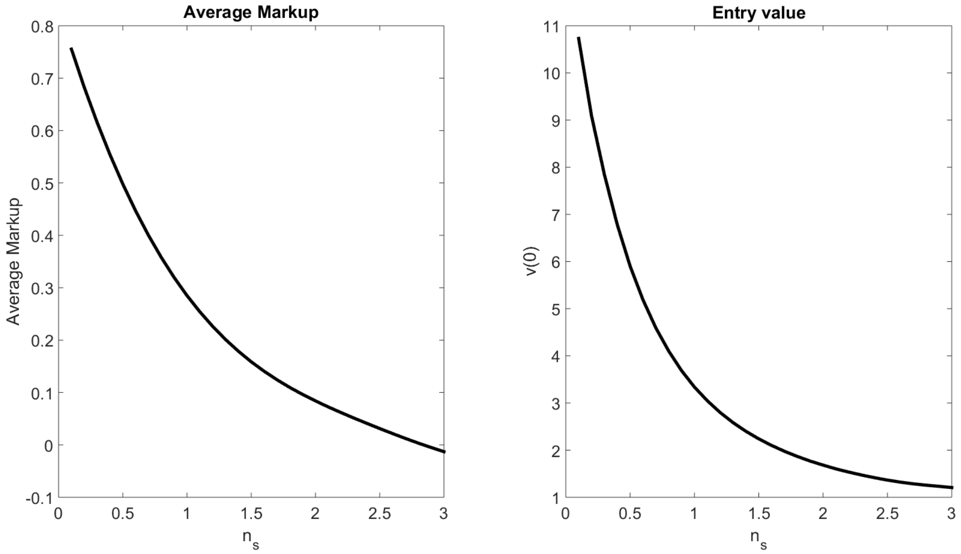

Figure 3.10: Markups and present value of entry as functions of n_s

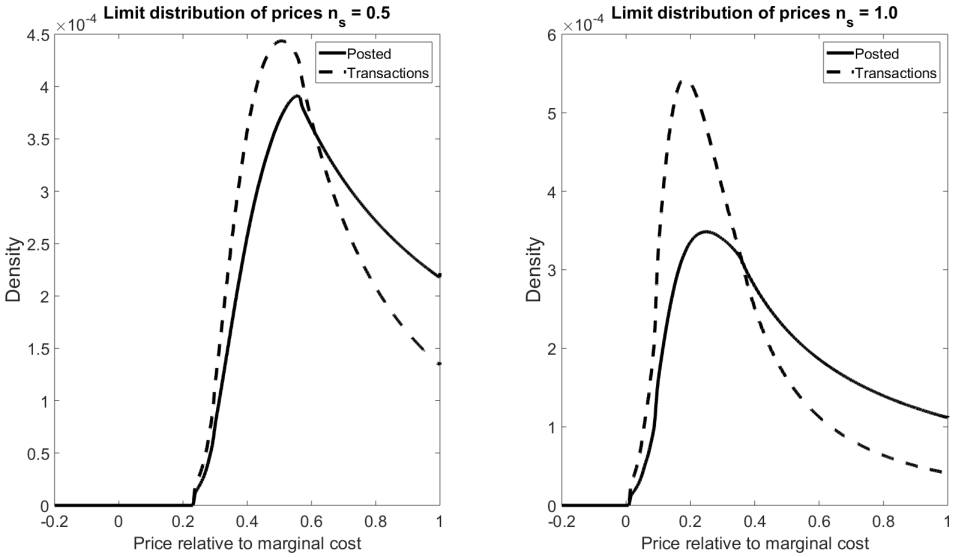

Figure 3.11: Limiting distributions of prices as functions of n_s (1)

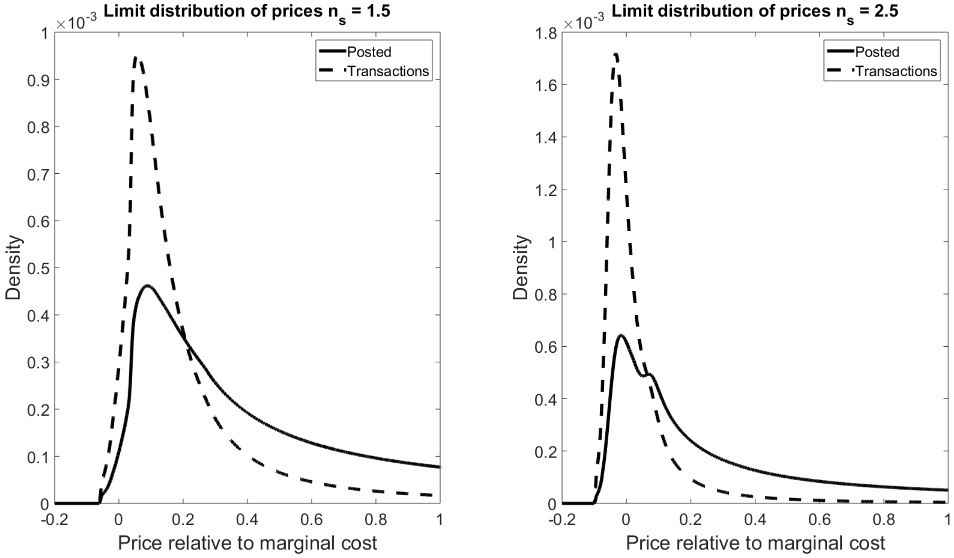

Figure 3.12: Limiting distributions of prices as functions of n_s (2)

3.7.1.1 Endogenous entry and efficiency

Based on proposition 3.5 we know that at a given population of sellers in the market the distribution of the quantity sold among different seller cohorts is efficient. However, is the equilibrium population of sellers efficient?

An entry cost of e implies in the aggregate flow cost of entrants of $n_s e = \alpha s e$ which in turn implies in a steady-state equilibrium level $\hat{s}$ and hence in $\pi^0(\hat{s})$. The aggregate flow surplus in the market in the steady-state equilibrium is just $1 - \pi^0(\hat{c})$. Therefore, the efficient population of sellers s^* is obtained when the marginal gain in surplus is equal to the marginal cost of increasing the population of sellers. The marginal cost of seller entry is constant at e.

While the marginal increase in surplus in some period t in increasing the seller population s satisfies:

$$\frac{\partial(1-\pi^0(\hat{s}_t))}{\partial \hat{s}_t}\frac{\partial \hat{s}_t}{\partial s} = \frac{\partial(1-\pi^0(\hat{s}_t))}{\partial \hat{s}_t}\left[s\left(\partial \hat{c}_t/\partial s\right)+\hat{c}_t\right],$$

where

$$\frac{\partial(1-\pi^0(\hat{s}))}{\partial \hat{s}} = \exp(-\hat{s}) = \pi^0(\hat{s}).$$

If the customer bases are exogenous (they do not depend on sales) then WLOG assume that seller's customer bases are constant, hence $\hat{c} = \underline{c}$, then the equilibrium price posting strategies of all cohorts have the same support and hence the initial value of a seller operating in the market satisfies

$$v(\underline{c}) = \frac{\pi^0(\hat{s})\underline{c}}{1 - \beta(1-\alpha)},$$

and the marginal increase in aggregate market surplus from the entry of a additional seller is $\pi^0(\hat{s})\underline{c}$, therefore for a central planner the discounted flow value of entry of a seller is also $v(\underline{c})$ while the cost is e. This leads me to conjecture that in steady-state equilibrium the population of sellers is efficient.

Conjecture 3.1. *For an entry cost $e > 0$, the resulting steady-state equilibrium population of sellers is efficient for a central planner with the same discount factor as the sellers.*

3.7.2 Seller heterogeneity

So far I have assumed all sellers have identical marginal costs. In this extension I extend the model by allowing for heterogeneous production technology. A fraction $\epsilon \in (0,1)$ of the sellers have high cost c^h and a fraction $1-\epsilon$ of sellers have low cost c^l with $1 > c^h > c^l = 0$.

In this case profits of seller j with high costs if he or she posts a price p are

$$\gamma(p, m_t^j, \hat{\sigma}_t, \hat{m}_t) = (p - c^h) m_t^j \xi(p, \hat{\sigma}_t, \hat{m}_t).$$

In equilibrium, sellers with cost $c \in \{c^h, c^l\}$ and age $a \in \{1, \ldots, t\}$ post prices according to a non-degenerate distribution $\sigma_t^{a,c}$ with support $[\underline{p}_t^{a,c}, \overline{p}_t^{a,c}]$, where $\overline{p}_t^{a,c} > \underline{p}_t^{a,c}$ and $\underline{p}_t^{a,c^h} \geq \overline{p}_t^{a,c^l}$. These equilibrium properties imply that sellers with higher costs will post higher prices with probability 1. The reason for that for a distribution σ_t^{a,c^l} that makes low cost sellers indifferent between prices in his or her support means that the increase in quantity by posting lower prices that makes a low cost seller indifferent between prices is smaller than the required increase in order to make a high cost seller indifferent. Therefore, the profile of equilibrium price posting strategies $\{\sigma_t^{a,c}\}_{a \in \{1,\ldots,t\}, c \in \{c^h, c^l\}}$ satisfies the property that all sellers post prices according to non-generate distributions, older sellers post higher prices than younger sellers with probability 1, and that higher cost sellers with the same age post higher prices than lower cost sellers also with probability 1. A formal proof would be very similar to the proof of theorem 3.1 and its extension to this environment is left as an exercise for the reader.

3.7.3 Heterogeneity of exit rates

Empirical evidence suggests exit rates fall as firms become older and more established, so allowing exit rates to vary depending on seller age in this model is an important extension. Suppose that seller exit rates are not constant over time but given by a sequence $\{\alpha^a\}_{a=2}^{\infty}$ that determines the fraction of sellers who exit based on age: a fraction α^a of sellers age a exits the market by the beginning of the current period. In this case a seller's value function depends on his or her age besides the size of customer base, $v_t(c,a)$ satisfies

$$v_t(c,a) = \max_p \gamma(p,c,\sigma,\hat{c}) + \beta(1-\alpha^{a+1})v_t(c^+(p,c),a+1).$$

It is a simple reproduction of the proofs by replacing α with α^{a+1}. Nearly all theorems still hold except Lemma 3.4 which might not hold if $\alpha^a > \alpha^{a+1}$ by a high enough to degree for some cohort a which would imply that price posting strategies of cohort a will first order stochastically dominate the distribution of cohort $a+1$.

3.8 Concluding remarks

This is a theoretical study of dynamic industry pricing that takes in consideration the effects of imperfect awareness and learning through word of mouth among buyers in the market. It extends the theory of equilibrium price dispersion of Burdett and Judd (1983) [5] by incorporating dynamic customer-capital accumulation. This study approaches customer-capital accumulation in a distinct way from other studies such as Gorio and Rudanko (2014) [34] and Paciello, Pozzi and Trachter [62], which both use directed search mechanisms, where customers can only interact with a single seller at each point in time. Instead, here

I use a mechanism for the accumulation of potential customers through word of mouth contagion.

The environment of this model implies in a unique symmetric Markov perfect equilibrium. The properties the equilibrium include price dispersion as sellers randomize their price posting strategies, entrant sellers post lower prices than incumbents and in the non-steady-state equilibrium the average markup level falls over time as it converges to the steady-state level of markup. Quantitative analysis of the model shows that the model closely replicates the empirical data on price dispersion for transactions without recourse to specific assumptions.

In this environment market frictions are represented by the seller's exogenous exit rates, upper bound in the size of the seller's customer base and the existence of entry costs. As the market frictions converge to zero (either by lowering costs of entry to zero or by simultaneously lowering the exit rates to zero and the upper bound of the customer bases to infinity) the steady-state equilibrium converges to the allocation that corresponds to perfect competition. This is not a trivial result as demonstrated in numerous studies such as Diamond (1971) [16], Rubinstein and Wolinsky (1985) [66], Gale (1986,1987,2000) [30, 27, 29] and Mortensen and Wright (2002) [60]. These studies show that the choice of trading mechanisms and particular details of the environment can imply that markets might not converge to perfect competition as matching frictions vanish. In particular, Gale (2000) [30] shows that the assumption of anonymity, that is, that traders do not condition their strategies on a particular individual agent in the market, is critical for the convergence to perfect competition to occur when market frictions converge to zero. Lauermann (2013) [51] provides a general approach to the set of conditions required for random matching and bargaining games to converge to the competitive outcome as frictions vanish.

This study assumes a trading mechanism where the sellers post prices to anonymous buyers who purchase from the cheapest priced seller they know, which is similar to the trading mechanism used in the aforementioned studies of Satterthwaite and Shneyerov (2007, 2008) [67, 68] and Lauermann et al (2018) [52], where agents trade through first price auctions: buyers post offers, and the seller sells to the highest posted offer. A candidate avenue for further research would be to extend the model to other trading mechanisms beyond price posting by the sellers. However, classic results obtained in cooperative game theory (e.g., Debreu and Scarf (1963) [14] and Aumann (1964) [3]) and the mechanism design literature (e.g., Hammond (1979) [38]), suggest that in large (i.e. continuum of agents) economies without frictions and restrictions on specific trading mechanisms will arrive at allocations that correspond to competitive equilibria. These results appear to suggest that satisfactory models that describe market frictions should utilize a trading mechanism that implies in an equilibrium allocation that converges to the competitive allocation as frictions vanish, as already is the case in this paper.

It is important to emphasize that this study does not present a model for firm growth (unlike Luttmer (2006) [54]). For example, instead of representing firms, the sellers in the model can be interpreted to be individual stores or brands in a market for a homogeneous good while firms can own several stores or brands in the same market as well as the fact that firm grow tends to involve expansion of the firms activities into additional markets besides the expansion of their sales in markets in which they currently operate. However, this study provides a model of market consolidation and entry that can be used to construct a model of firm growth.

Finally, in the endogenous entry extension of the model, I showed existence and uniqueness of the equilibrium that corresponds to a steady-state with constant population of sellers. An empirical application for the model is that is establishes a positive relationship between the average level of markups and entry costs. The increase in average markups globally since 1980 (shown in De Loecker and Eeckhout (2018) [10]) can also be explained, besides an shift to shorter product life cycles, as result of the shift from industries such as commodity like industries, where most costs are variable, to industries such as information technology and pharmaceuticals where R&D costs are a substantial fraction of the total costs of a product while marginal costs are relatively low. An avenue for future research would be to solve the non-steady state version of model with free entry: as profit margins are higher when an industry is young, one would expect initially a high entry rate that would decrease as the industry converges to the steady-state equilibrium.

A shortcoming of the model presented in this study is that the degree of randomization of the price posting strategy of individual sellers depends on the relative size of the seller's type to the aggregate customer base of sellers operating. This is an issue because it implies that if the cumulative distribution of customer bases is continuous the model then it does not yield an equilibrium in randomized price posting strategies for individual sellers and the empirical evidence is that price dispersion among prices with transactions with individual sellers is substantial in product markets. In a related study, Guthmann (2019) [35] constructs an equilibrium in a dynamic duopoly game where the incumbent and the entrant sellers post prices according to a pair of distributions that have the same support even though one distribution first order stochastically dominates the other. However, this kind of asymmetric price posting strategies can only be implemented when some sellers have a non-negligible share of the aggregate market. I conjecture that another avenue to expand the degree of individual seller price randomization in equilibrium would be to allow for different cost functions. Decreasing marginal costs might result in an equilibrium where sellers with smaller customer bases have, simultaneously, incentive to post prices that are higher and lower than prices posted by a competitor with a larger customer base.

Another important shortcoming of this model is the assumption that individual sellers can produce arbitrary quantities of the good with constant marginal cost. This implicitly means that sellers have access to frictionless factor markets that enable then to replicate the scale of supply to any desired level. Hence, this model describes the equilibrium with frictions in a single market but implicitly assumes that its intermediary input markets are frictionless.

Chapter 4

Vanishing market power

In many markets no two suppliers are identical: even two shops that sell identical products are located at different addresses and might offer slightly different selections of goods, which differentiates then to a certain degree. The prevalence of differentiation among sellers in many markets might be a problem for the broad applicability of the model of competitive equilibrium. For the competitive model to have broad empirical relevance it must be provided with strategic foundations from models where terms of trade are not taken as given, that incorporate differentiated sellers, and that converge to perfect competition as market frictions vanish. Hart (1979) [39] provides strategic foundations for perfect competition in an economy with differentiated commodities. He shows that prices converge to marginal cost as each firm becomes small relative to the aggregate economy but does so without incorporating explicit market frictions. Models that do provide convergence results to perfect competition and incorporate explicit trading frictions study markets for identical commodities.[1]

This chapter argues that monopolistic competition and market frictions are related: Decision makers might be small relative to the aggregate economy but large relative to the small coalition of other decision makers that are in contact with them. As market frictions become smaller traders can more easily contact each other and market power vanishes, even in an environment where firms have

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[1] See, e.g., Noshek (1985) [61], Rubinstein and Wolinsky (1985) [66], Gale (1986a,1986b, 1987, 2000) [27, 28, 29, 30], Sonnenschein and McLennan (1991) [55], De Fraja and Sákovics (2001) [9], Satterthwaite and Schneyerov (2007, 2008) [67, 68], Lauermann (2013) [51], Lauermann, Merzyn and Virag (2018) [52].

a monopoly on the variety they supply. I show this result holds as long as the degree of product differentiation varies continuously and there is a large number of varieties available.

I study a market for a product category where there are many producers and consumers, and each producer holds a monopoly on the supply of a variety of a good. There are many imperfect substitutes whose degree of substitutability varies continuously.[2] Each consumer in this market has different tastes for different varieties. There are market frictions that might prevent consumers and producers from trading. Market frictions are represented here by consumers being imperfectly aware of the varieties available. The number of varieties that a consumer is aware of is determined by a random matching process.

This study develops a way of representing market frictions in a fully static setting. Usually, random matching games are used to explicitly represent market frictions as a dynamic sequential of process of random matching and bargaining. In this study, there is a random matching process which yields a network, and all trading is done simultaneously through the network. It is a mechanism similar to Perla (2017) [63], and has similarities with models that assume exogenous search [3]. This makes the model easily tractable. In addition, both the degree of awareness of individual consumers as well as their preferences are private information.

Under certain conditions this model features a unique Nash equilibrium in pure strategies, where all producers post a price between the marginal cost and the monopoly price for their product. As the frictions of trading decreases, the equilibrium price converges to a Walrasian price and each consumer purchases a variety offered in the market that is arbitrarily close to the variety most

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[2] Note that it is common in macroeconomics to assume that every producer holds a monopoly on the supply of its product and the output of all other producers has exactly the same elasticity of substitution (as in Gali's (2015) [31] popular textbook). It is more accurate to say that product differentiation is better represented by a continuum of degrees of differentiability instead of the assumption of constant elasticity of substitution between any pair of product varieties. For example, economy class flights from a particular airline can be imperfectly substituted by bus or train rides, more closely substituted by a flight from a low-cost airline or high-speed rail and almost perfectly substituted by a flight from another airline of the same category. This is the case in Hart (1979), Jones (1984) as well as in this paper.

[3] E.g., Diamond (1987) [17], Kaplan and Menzio (2016) [49], Burdett and Menzio (2018) [6].

congruent to their tastes. That is, the equilibrium allocation converges to the perfectly competitive allocation.

The space of varieties is a compact metric space. The degree of product differentiation is the degree of variation in utility relative to distance between two varieties. In a specific example where the degree of product differentiation is fixed, I show that the degree of product differentiation affects market prices when there are frictions: the higher the degree of product differentiation, the higher is the level of markups over marginal cost. However, for any degree of product differentiation, when market frictions vanish market power vanishes as well. An important implication of this study is that imperfectly competitive behavior observed in markets is not fundamentally caused by product differentiation but by the existence of market frictions. However, the degree of deviation from competitive outcomes caused by market frictions are magnified by the existence of product differentiation.

In addition, I study the welfare implications of equilibrium. When there are frictions the equilibrium is not necessarily Pareto efficient due to the presence of asymmetric information. But, by performing the optimal mechanism design for this environment, I show that the equilibrium in this market is constrained efficient.

An important observation on the equilibrium concept of this model is that, unlike in standard search models (see Lauermann et. al. (2018) [52]), equilibrium is not premised on the assumption that consumers and producers have information about the aggregate supply capacity of the producers in the market, which determines the competitive equilibrium price. Producers formulate their best response strategies from the distribution of prices posted by other producers (described by s^*) and the distribution of customer types in their customer base. The fact that in this model the Nash equilibrium strategy profile approximates the Walrasian equilibrium when frictions are low provides strategic foundations for the model of competitive equilibrium without assuming that decision makers know ex-ante what the competitive equilibrium price should be.

In this case decision makers do not need to form beliefs regarding what the competitive equilibrium price must be. That is, they do not need to form beliefs about aggregate supply capacity of the whole market to formulate their price posting strategies, instead they only need "local" information which is the distribution of types of their customers and the prevailing distribution of prices s^*. Therefore, this model provides another example of Hayek's (1945) [40] idea that markets prices allow decision makers to act as if they had information about the relative scarcity of a good without being in actual possession of this information. As proven by Jordan (1982) [46], competitive equilibrium prices are uniquely informationally efficient, this study extends the logic that prices transmit information from the competitive environment to a strategic environment, where prices are chosen by the decision makers instead of being taken as given.

4.1 Environment

I study a market for a product category. This product category is represented by a one-dimensional compact metric space of varieties, the closed unit interval. Two varieties in $[0,1]$ which are close will be regarded as having similar properties and yield similar utility to the consumers.

I study a market for a product category. There is a continuum of producers and consumers, both sets of measure 1 and are uniformly distributed on the unit interval. Each producer $j \in [0,1]$ produces variety j at marginal cost normalized to 0 and has a capacity constraint $q > 0$. Each consumer i has unit demand for the product category with valuation $v(i,l) \in [0,1]$ for variety l. The function $v : [0,1]^2 \to [0,1]$ is continuous and satisfies $v(i,i) = 1, v(i,l) < 1, \forall l \neq i$. That is consumers have a favorite variety and valuation across different varieties and consumers varies continuously.[4]

Each consumer is aware of a finite subset of producers A^i, that she can trade with. Let $a^i = |A^i|$. Each producer j has a continuum $a^j \in \mathbb{R}_+$ of consumers who are aware of him (which I call j's customer base), which satisfies the following consistency condition with the cardinality (i.e., a^i) of consumer's awareness

$$\int_0^1 a^i di = \int_0^1 a^j dj. \tag{4.1}$$

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[4] These preferences are a simple example of preferences the one-dimensional case of Hart (1979) [38] or Jones (1984) [45].

The cardinality of the sets $\{A^i\}_i$ is distributed according to a Poisson distribution with parameter $a = \int_0^1 a^j dj$, with probability mass function $\pi^k(a)$ for $k \in \{0, 1, 2, \ldots\}$. The following subsection describes the random matching process between consumers and producers that generates the sets $\{A^i\}_i$ and $\pi^k(a)$ from a profile of consumer bases $\{a^j\}_j$.

4.1.1 Matching process that determines buyers' awareness

The distribution of the cardinality of A^i is determined by a random matching mechanism where each producer's customer base is a simple random sample from the interval $[0, 1]$. As in Guthmann (2019) [36], to provide a description of the matching process in the environment with a continuum of consumers and producers, we consider an environment with finitely many buyers and producers and replicate it to infinity, postulating that its properties at the limit to hold in the continuum economy. Let I and J be the set of consumers and producers and n^c and n^p be its cardinalities, respectively.

Let $\mu, \nu \in \mathbb{N}$. Consider a $\mu\nu$-replica of this economy, with set of customers $I^{\mu\nu}$ which is $\mu \times \nu$ replica of I and set of producers $J^{\mu\nu}$, a ν-replica of J, with each replica indexed as $J^{\mu\nu} = \{J_1, \ldots, J_\nu\}$, and each producer is known by $a^{j\mu} \in \mathbb{N}$ consumers. Let ν be large enough so that

$$\nu n^c > \sum_{j=1}^{n^p} a^{j\mu}$$

and suppose that the sequence $\{a^{j\mu}/\mu\}_\mu$ converges to some $a^j \in \mathbb{R}_+$ for each $j \in \{1, \ldots, n^p\}$. Therefore, the sequence $\{\sum a^{j\mu}/\mu n^p\}_\mu$ converges to $a = \sum_{j=1}^{n^p} a^j$.

Consider a matching process of $\{1, \ldots, \nu\}$ rounds between consumers in I^ν and producers in J^ν such that in round $n \in \{1, \ldots, \nu\}$ matching is described by a matching function $g : I^\nu \times J^\nu \times S_\nu \to \{0, 1\}$, where S_ν is the set of states of the world, number 0 denotes matched and 1 not matched. The matching function satisfies

$$\sum_{j \in J_n} g(i,j,s) \in \{0,1\}, \tag{4.2}$$

$$\sum_{i \in I^{\mu\nu}} g(i,j,s) = \mu a^j, \forall j \in J_n. \tag{4.3}$$

Condition 4.2 states that consumers can only match with at most one producer in J_n. Condition 4.3 states that the number of consumers matched to producer j is the size of j's customer base.

The matching probabilities are described by the probability space $(S_\nu, \mathcal{S}_\nu, P_\nu)$, where $\mathcal{S}_\nu$ is a σ-algebra on $S_\nu$and P_ν is a probability measure on $\mathcal{S}_\nu$. Consider the event $\pi_{i,n} \in \mathcal{S}_\nu$, where consumer i is matched to a producer in round $n \in \{1, \ldots, \nu\}$, the probability measure P_ν satisfies

$$P_\nu(\pi_{i,n}) = \frac{\sum_{j=1}^{n^P} a^{j\mu}}{\nu n^c}. \tag{4.4}$$

Note that $P_\nu(\pi_{i,n})$ is invariant to $i \in I^{\mu\nu}$ and $n \in \{1, \ldots, \nu\}$. Consider event $\pi_{i,j}$ where i is matched to a producer $j \in J_n$, P_ν satisfies

$$P_\nu(\pi_{i,j}) = \frac{a^{j\mu}}{\nu n^c}. \tag{4.5}$$

Let $A^i(s) = \{j \in J^{\mu\nu} : g(i,j,s) = 1\}$ be the set of producers that i is matched with, that is, the set of producers that i is aware of given state of the world s. The matching technology 4.4 implies in that after all ν rounds the distribution of the number of producers known to a consumer (the distribution of $|A^i|$) follows a binomial distribution with parameters $P_\nu(\pi_{i,n})$ for the probability of success and ν for the number of experiments. Let $n^c = n^p$ and taking $\nu \to \infty$, by the Poisson limit theorem, the binomial distribution converges to a Poisson distribution with parameter a, and A^i is sampled from $[0,1]$ according to a Poisson point process.

In addition, consider the probability that an arbitrary consumer i is aware of n other producers conditional on being matched to j in some round, this probability follows a binomial distribution with parameters $P_\nu(\pi_{i,n})$ or the probability of success and $\nu - 1$ for the number of experiments. Taking $\nu \to \infty$, it is ob-

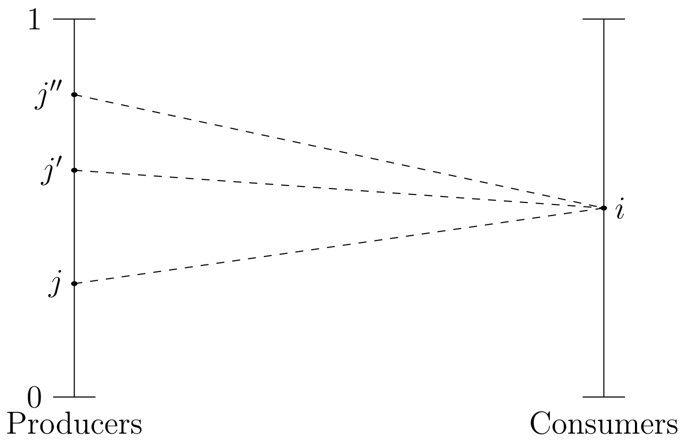


Figure 4.1: Diagram represents a consumer i is aware of 3 different sellers so $A^i = \{j, j', j''\}$.

vious that this distribution converges to the same Poisson as the unconditional probability.

From the perspective of the consumers, a can be interpreted as the discovery rate as they are searching in a market in continuous time, where each consumer discovers a producer in the market at Poisson rate of a for one period, while producer j accumulate potential customers at the rate a^j, but instead of a accumulating a discrete number of consumers, a^j is a continuum flow rate. After the search and matching period, consumers can trade with any producers they meet during their search.

4.2 The Game

Producer $j \in [0, 1]$ strategy is a contract specified by a price $p \in \mathbb{R}$ in exchange for a unit of variety j of consumption good. A strategy profile $s : [0, 1] \to \mathbb{R}$ assigns a price $s(j)$ posted by producer j. Since consumer's types are private information, the producers cannot make contracts that discriminate among consumers in their customer base.

The consumer chooses whether to trade or not. If she chooses to trade, she can choose among contracts posted among producers that she is aware of, that is contracts posted by producers in A^i. I assume that consumers choose the contract that yields the highest valuation, conditional on it being preferred to no trade.

A producer does not observe the valuations of the consumers who are aware of him and also does not observe the set of other sellers their consumers are aware of (the A^i of their customers). They form beliefs about the consumer's types based on the distribution of consumers and producers on the unit interval and on the distribution of the cardinality of the consumer's awareness sets A^i. Producers' beliefs are consistent with rational expectations: producer j's belief of a random consumer's valuation is $v(i,j)$ with i uniformly distributed over the unit interval. A producer j's beliefs regarding the number of other producers a consumer might be aware of is a Poisson with parameter a. The producer assumes that the awareness set of the consumer, A^i, is a random sample from the unit interval with distribution weighted by a^j/a (see 4.5) as producers are distributed uniformly in the unit interval but a^j varies on j. Which means that for an arbitrary consumer i, the probability that producer $j \in A^i$ is in $[0,x]$ is described by the cumulative distribution G, given by

$$G(x) = \frac{\int_0^x a^j dj}{a}, \tag{4.6}$$

as producers more well known by consumers have a higher probability of being in A^i.

Let $P_j(p,s)$ be the probability that a random consumer accepts producer j's contract, given posted price p, strategy profile s of other producers, and $a > 0$. P_j satisfies

$$P_j(p,s) = \pi^0(a)P_j^0(p,s) + \pi^1(a)P_j^1(p,s) + \pi^2(a)P_j^2(p,s) + \dots, \tag{4.7}$$

where $P_j^n(p,s)$ be the probability that a random consumer prefers j's contract over the contracts posted by n competing producers and to no trading, and $\pi^n(a)$ is the producer j's belief that a consumer who knows j is aware of n other producers. As each producer has a continuum of consumers who are aware of then, producer's profits are deterministic and given by

$$\Pi_j(p,s) = p\min\{a^j P_j(p,s), q\}, \tag{4.8}$$

where q is the producer's capacity constraint.

4.2.1 Derivation of a closed form P_j

Consumers are distributed uniformly over the unit interval, so $P^n(p,s)$ is given by

$$P_j^n(p,s) = \int_{[0,1]^{n+1}} I_{\{v(i,j)-p\geq \max\{(v(i,j^k)-s(j^k))_{k=1}^n,0\}\}}(i,j^1,\dots,j^n)d(i,G(j^1),\dots,G(j^n)), \tag{4.9}$$

where

$$I_{\{v(i,j)-p\geq \max\{(v(i,j^k)-s(j^k))_{k=1}^n,0\}\}}(i,j^1,\dots,j^n) \in \{0,1\} \tag{4.10}$$

is the indicator function of whether the contract (j,p) is preferred by i over not trading (i.e., $v(i,j)-p\geq 0$), and over the profile of contracts $(j^k,s(j^k))_{k=1}^n$ posted by producers $j^1,\dots,j^n$ (i.e., $v(i,j)-p \geq v(i,j^k)-s(j^k), \forall k\in\{1,\dots,n\}$). As the indicator function satisfies

$$I_{\{v(i,j)-p\geq \max\{(v(i,j^k)-s(j^k))_{k=1}^n,0\}\}}(i,j^1,\dots,j^n) \tag{4.11}$$

$$= I_{\{v(i,j)-p\geq 0\}}(i)\times I_{\{v(i,j)-p\geq v(i,j^1)-s(j^1)\}}(i,j^1)\times\dots\times I_{\{v(i,j)-p\geq v(i,j^n)-s(j^n)\}}(i,j^n), \tag{4.12}$$

then $P^n(p,s)$ satisfies

$$P_j^n(p,s)$$

$$= \int_0^1 I_{\{v(i,j)-p\geq 0\}}(i)\left[\int_0^1 I_{\{v(i,j)-p\geq v(i,j^1)-s(j^1)\}}(i,j^1)dG(j^1)\times\dots\times\int_0^1 I_{\{v(i,j)-p\geq v(i,j^n)-s(j^n)\}}(i,j^n)dG(j^n)\right]di \tag{4.13}$$

$$= \int_0^1 I_{\{(j,p)\succeq_i R\}}(i)\left[\int_0^1 I_{\{v(i,j)-p\geq v(i,j')-s(j')\}}(i,j')dG(j')\right]^n di, \tag{4.14}$$

from Fubini–Tonelli theorem.

Substituting P^n in equation 4.7 by the left hand side of 4.14, implies that the probability of sale to a random consumer $P_j(p,s,a)$ satisfies

$$P_j(p,s) = \int_0^1 I_{\{v(i,j)-p\geq 0\}}(i)\exp\left[-a\left(1-\int_0^1 I_{\{v(i,j)-p\geq v(i,j')-s(j')\}}(i,j')dG(j')\right)\right]di. \tag{4.15}$$

4.3 Equilibrium

Definition 4.1. A Nash equilibrium is a strategy profile $s^* : [0,1] \to \mathbb{R}_+$ such that for each producer j posting $s^*(j)$ is a best response to s^*, that is $s^*(j)$ maximizes $\Pi_j(p, s)$ over p given strategy profile s^*.

4.3.1 Convergence to competitive equilibrium

Before characterizing properties of the equilibrium, I define a competitive equilibrium for this market to compare it with the Nash equilibrium. A competitive equilibrium is a profile of prices for varieties such that demand for varieties is equal to supply, where consumers have full awareness of all varieties in the market and can choose any variety from the unit interval. Demand for varieties is defined as the limiting ratio of the measure of consumers that choose to purchase a subset of varieties divided by the measure of the subset when that subset converges to a point.

Formally, a profile of prices for varieties is a function $p : [0,1] \to \mathbb{R}_+$. Consumers can randomize their consumption: for example, consume variety l with probability α and not-consume any variety with probability $1-\alpha$. The reason for why I allow for randomization is to convexify demand: although there is a continuum of consumers in this economy (Aumann (1966) [4]) there is only one consumer i with preferences given by valuation function $v(i, .)$ and i's demand is indivisible.

Let $\Delta = \{x \in [0,1]^l : \sum_k x(k) \leq 1, l \in \mathbb{N}\}$ be the consumption set of consumers, Δ is the set of probability distributions with assign positive probability mass of consuming a finite subset of $l \in \mathbb{N}$ different varieties and probability zero for all other varieties. The consumer problem is to choose a finite subset of varieties which maximizes their utility, consumer i's Walrasian demand x_i satisfies

$$x_i \in \arg\max_{x \in \Delta} \sum_{\{j : x_i(j) > 0\}} x(j)[v(i,j) - p(j)].$$

Let X be the aggregate demand implied by a profile of individual consumption bundles $(x_i)_{i\in[0,1]}$, it satisfies

$$X(j) = \begin{cases} \sum_{\{i:x_i(j)>0\}} x_i(j) & \text{if } \{i : x_i(j) > 0\} < \infty \\ \infty & \text{else.} \end{cases} \tag{4.16}$$

Producer j choice of output $Y(j)$ of variety j satisfies

$$Y(j) \in \arg\max_{z\in[0,q]} p(j)z. \tag{4.17}$$

The pair $((x_i)_{i\in[0,1]}, Y)$ represents an allocation of consumption profiles $(x_i)_{i\in[0,1]}$ and output profile Y.

Definition 4.2. A competitive equilibrium is an allocation $((x_i)_{i\in[0,1]}, Y)$ profile of prices $p^W : [0,1] \to \mathbb{R}_+$ such that $X(l) = Y(l)$ for all varieties $l \in [0,1]$.

Proposition 4.1. *The unique competitive equilibrium price* $p^W : [0,1] \to \mathbb{R}_+$ *satisfies for all* $l \in [0,1]$*,*

$$p^W(l) = \begin{cases} 0 & \textit{if } q > 1 \\ 1 & \textit{if } q < 1. \end{cases} \tag{4.18}$$

Note that in competitive equilibrium the prices are the same for almost every variety. The reason for that property is that supply is by construction symmetric across varieties and if prices are identical then all consumers will purchase their favorite variety (which is, for consumer i, variety $j = i$, which is consistent with equilibrium if $p^W(l) \leq 1$) or will not purchase anything (which is consistent with equilibrium if $p^W(l) \geq 1$), which is consistent with equilibrium.

Consider the case where $q > 1$, then supply for varieties will be always strictly greater than 1 for strictly positive prices while demand for at least a some variety will be less or equal to 1. Consider the case where $q < 1$, if prices are higher than 1 for some variety j then demand for j will be zero and supply will be $q > 0$, not consistent with equilibrium, if prices are lower than 1 for a positive measure of varieties demand for those varieties will be higher than supply, which is also not an equilibrium.

Theorem 4.1. *Consider a sequence* $\{\boldsymbol{a}_n\}_n$ *of producer awareness profiles* $\boldsymbol{a}_n = (a_n^j)_{j\in[0,1]}$ *such that* $\lim_{n\to\infty} a_n^j = \infty$ *for all* $j \in [0,1]$ *with* $\max\{a_n^j/a_n^{j'} : j, j' \in [0,1]\} < A$ *for some* $A > 0$ *for all* n*, then the Nash equilibrium pricing strategy* $s^*(j, \boldsymbol{a}_n)$ *must satisfy for almost every* $j \in [0,1]$

$$\lim_{n\to\infty} s^*(j, \boldsymbol{a}_n) = \begin{cases} 1 & \text{if } q < 1 \\ 0 & \text{if } q > 1, \end{cases}$$

and the competitive equilibrium allocation converges in probability, for almost every variety, to a competitive equilibrium allocation $((x_i)_{i\in[0,1]}, Y)$*.*

Theorem 4.1 shows that the Nash equilibrium converges to the competitive equilibrium as the parameter a converges to infinity (that is, as market frictions vanish and the market network becomes perfectly "thick"). Even though each producer is a monopolist of their specific variety the elasticity of demand converges to infinity as a converges to infinity. The reason is that the probability that customers of a producer j are aware of competing producers producing a variety at a distance $\epsilon > 0$ converges to 1 as $a \to \infty$, therefore undercutting competing producers becomes profitable. The competition between producers implies that as $a \to \infty$ the Nash equilibrium pricing strategy $s^* : [0,1] \to \mathbb{R}_+$ must converge to 0 almost everywhere.

4.3.2 Existence

In this subsection I show a set of sufficient conditions for the existence of equilibrium. In particular sufficient conditions for symmetric equilibrium to exist: a price p^* such that posting p^* is a best response for any producer if all other producers are also posting p^*.

Let

$$\Lambda_{ji}(x) = m(\{i \in [0,1] : v(i,j) \geq x\})$$

(where m is the Lebesgue measure) for $x \in [0,1]$, be the fraction of consumers whose reservation price for j's product is higher than x. Let

$$\Lambda_{jij'}(x) = \int_0^1 I_{\{v(i,j)-v(i,j')\geq x\}}(i,j')dG(j')$$

be the consumer base weighted fraction of producers such that j's variety is preferred conditional on its price being x higher than other producer's prices.

The assumptions below state "symmetry" conditions on v which are part of sufficient conditions for the existence of symmetric equilibrium.

Condition 1: For all $j, \tilde{j} \in [0,1]$, $x \in [0,1]$,

$$\frac{\Lambda_{ji}(x)}{\Lambda_{ji}(x')} = \frac{\Lambda_{\tilde{j}i}(x)}{\Lambda_{\tilde{j}i}(x')}.$$

Condition 2: For all $j, \tilde{j} \in [0,1]$, $x \in [0,1]$,

$$\frac{\Lambda_{jij'}(x)}{\Lambda_{jij'}(x)} = \frac{\Lambda_{\tilde{j}ij'}(x)}{\Lambda_{\tilde{j}ij'}(x)}.$$

Condition 3: For all $j \in [0,1]$, $a^j = a$.

Let $\overline{p}$ be the monopoly price which solves

$$\max_{p \in \mathbb{R}} p\Lambda_{ji}(p). \tag{4.19}$$

Note that $\overline{p}$ that solves 4.19 is the same for all producers if Condition 1 is satisfied.

Theorem 4.2. *If v satisfies conditions 1, 2 and 3, is piecewise concave on i and j, and satisfies $P_j^n(p', p) > 0$ for all $p', p \in (0, \overline{p})$, and $n > 0$, then there exists a symmetric equilibrium price p^*.*

4.4 Analytical example

Consider the functional form for the valuation function of each $i \in [0,1]$,

$$v(i,j) = 1 - \lambda\delta(i,j) \tag{4.20}$$

where

$$\delta(i,j) = \min\{|i-j|, 1-|i-j|\}$$

and $\lambda \in (0, 1]$. That is $[0, 1]$ represents a circle and δ is the distance between two points in the circle and consumers prefer varieties closer to their location in the circle (as in Hotelling's (1929) [44] model of geographical differentiation). Suppose that all producers have the same a^j so $G(x) = x$.

As $\lambda \leq 1$, the monopoly price is $\overline{p} = 1 - \lambda/2$, which implies all consumers prefer to trade at the monopoly price rather than not trade. Consider a price $p \leq \overline{p}$. Then, suppose other producers are all posting a price p' (that is, $s(j) = p', \forall j \in [0, 1]$). Let

$$\epsilon = \frac{p' - p}{\lambda},$$

ϵ is the undercut factor, that is the degree that j's price is above or below the price posted by the other producers relative to the degree of product substitutability. In this case a contract (j, p) if preferred over a competing contract (j', p') if and only if $\delta(i, j) - \epsilon \leq \delta(i, j')$. The symmetry of the circle implies that

$$\int_0^1 I_{\{v(i,j)-v(i,j')\geq -\epsilon\}}(i, j')dj' = \min\{1, 1 - 2\left(\delta(i, j) - \epsilon\right)\}.$$

I drop the j subscripts from P^n and P as the valuation function implies that P^n and P are the same for all j. Which implies that the probability that a consumer purchases conditional on being aware of n other producers satisfies for $n \geq 0$,

$$\begin{aligned} P^n(p, p') &= \int_0^1 \left[\int_0^1 I_{\{(j,p)\succeq_i (j', s(j'))\}}(i, j')dj'\right]^n di \\ &= \int_0^1 \left\{\left[1 - 2\left(\delta(i, j) - \epsilon\right)\right] \cap [0, 1]\right\}^n di \\ &= 2\int_0^{1/2} \left\{\left[1 - 2\left(x - \epsilon\right)\right] \cap [0, 1]\right\}^n dx \\ &= \begin{cases} 2\left\{\int_\epsilon^{1/2} [1 - 2\left(x - \epsilon\right)]^n dx + \epsilon\right\} & \text{if } \epsilon > 0 \\ 2\left\{\int_0^{1/2+\epsilon} [1 - 2\left(x - \epsilon\right)]^n dx\right\} & \text{if } \epsilon < 0 \end{cases} \\ &= \begin{cases} \frac{1}{n+1}(1 - 2\epsilon)^{n+1} + 2\epsilon & \text{if } \epsilon > 0 \\ \frac{1}{n+1}(1 + 2\epsilon)^{n+1} & \text{if } \epsilon < 0. \end{cases} \end{aligned} \tag{4.21}$$

Substituting in 4.8,

$$P(p,p') = \begin{cases} 2\epsilon + \frac{1}{a}[\exp(-2\epsilon a) - \exp(-a)] & \text{if } \epsilon > 0 \\ \frac{1}{a}[\exp(2\epsilon a) - \exp(-a)] & \text{if } \epsilon < 0. \end{cases} \tag{4.22}$$

The proposition below states that the symmetric Nash equilibrium exists for $a \in \mathbb{R}_+ \diagdown [\underline{a}, \overline{a}]$ and is unique. Where $\underline{a}$ and $\overline{a}$ are positive real numbers such that $\underline{a} < \overline{a}$.

Theorem 4.3. *If consumer valuations satisfy 4.20 there is a pair* $\underline{a}, \overline{a} \in \mathbb{R}_+$ *where* $0 < \underline{a} < \overline{a}$*, such that if* a *satisfies*

$$a \notin [\underline{a}, \overline{a}],$$

then there exists a corresponding unique symmetric Nash Equilibrium price $p^* \in (0, 1]$*. The quantity sold by each producer in equilibrium is*

$$\min\{q, 1 - \pi^0(a)\}. \tag{4.23}$$

If $p^* < \overline{p}$ *then* p^* *is given by*

$$p^* = \frac{\lambda}{2a}. \tag{4.24}$$

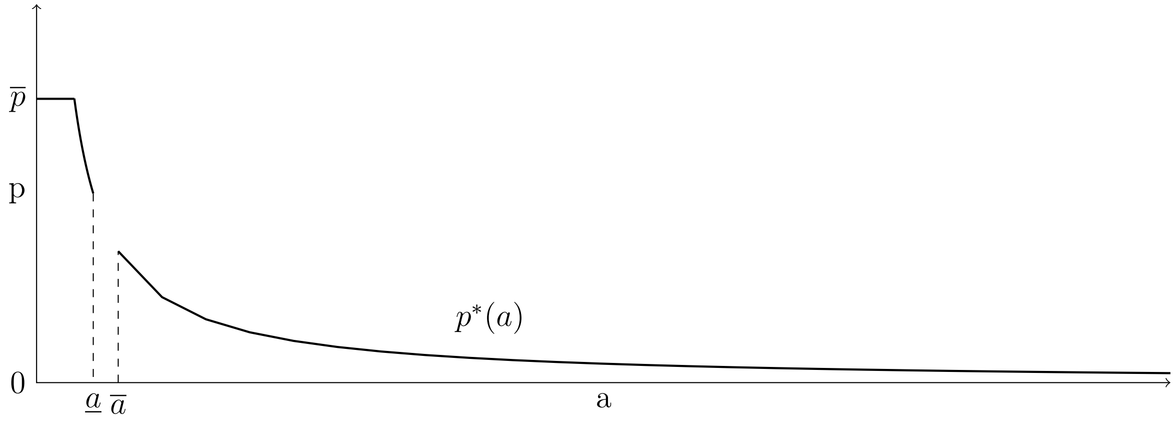
If the quantity sold is smaller than $1 - \pi^0(a)$*, the capacity constraint is hit and the equilibrium price is given by*

$$p^* = 1 - \frac{\lambda}{a}\left(\log(1 - \exp(-a)) - \log(1 - \exp(-a) - q)\right). \tag{4.25}$$

Note that equilibrium does not always exist here. The reason is that, under some values of a, the condition $P_j(p', p) > 0$ is not satisfied if p' is too high relative to p.

4.4.1 Numerical example

Let $\lambda = .5$ then for $a \in \mathbb{R}_+ \diagdown [0.50, 0.72]$ and $q \geq 1$, $p^*(a)$ satisfying equation 4.24 is the unique equilibrium. For awareness parameter a in $[0.50, 0.72]$ a symmetric pure-strategy Nash equilibrium does not exist. The figure below plots the equilibrium price as a function of a. Note that it converges to marginal cost as a converges to infinity.



Suppose that the capacity constraint is smaller than 1, for example, $q = 3/4$, then for $a > 1.39$ the capacity constraint is binding, so the equilibrium price jumps from 0.18 to 0.75 at $a > 1.39$ and from equation 4.25 converges to 1 as $a \to \infty$. This discontinuity in price posting arises when the supply constraint is hit and producers cannot undercut their prices as they lack the capacity to supply the additional demand.

4.5 Welfare analysis

In this section I use mechanism design techniques to define constrained efficiency in this environment and to compare it to the equilibrium found in the model. There are two purposes for the welfare analysis performed in this section. First, note that it is easy to see that the trading mechanism that was assumed in this environment (sellers post prices and buyers choose their favorite contract among the set of contracts they are aware of) is a incentive-compatible direct mechanism. Second, I show that a pure strategy equilibrium is a constrained efficient mechanism. Equilibrium allocations are not necessarily efficient in the sense that they maximize market surplus on the set of allocations constrained by imperfect awareness. They are constrained efficient in the sense that there does not exist any implementable mechanism that Pareto dominates it. That is, if agents were free to choose any trading protocol, there does not exist an alternative trading protocol that, ex-post, makes some decision makers better off without making others worse off. Therefore, the trading protocol of price posting by the producers/sellers that is observed in most product markets with product differentiation is consistent with an optimal allocation mechanism that is constrained by asymmetric information.

I show that the allocation implied by an equilibrium price posting strategy $s^* : [0,1] \to \mathbb{R}_+$ is always efficient[5]. Constrained efficiency is obtained because the pricing strategies yield an ex-post profile of producer and consumer utilities that is not Pareto dominated by any other feasible allocation. The set of feasible allocations is defined here to be the allocations that can be implemented by a feasible allocation mechanism.

By the revelation principle if an allocation can be implemented by an arbitrary mechanism in this static environment it can be implemented by a incentive-compatible-direct-mechanism, so I restrict attention to direct revelation mechanisms. Let $\mathcal{A}$ be the set of finite subsets of $[0,1]$, that is the set of possible awareness profiles of a particular consumer. Let $\theta \in \Theta$ be a consumer of type θ, who has preferences indexed by $\succeq(\theta) \in [0,1]$, and is aware of a set of producers $A(\theta) \in \mathcal{A}$, where $\Theta = [0,1] \times \mathcal{A}$ is the set of consumer types.

I define a direct revelation mechanism F as

$$F : \Theta \to [0,1] \times \mathbb{R} \tag{4.26}$$

that maps a consumer of announced type θ into a variety $j(\theta) \in A(\theta) \subset [0,1]$ the consumer consumes (where $j(\theta) = \emptyset$ represents not trade) and numeraire transfers $t(\theta) \in \mathbb{R}$ the consumer receives. Given a profile of announced types, F also implements an allocation where producer of variety j receives transfer $t(j)$ and produces quantity $q(j)$.

Definition 4.3. A mechanism F is feasible if and only if F satisfies:

(i) (Incentive compatibility) For a consumer of type $\theta \in [0,1]$,

$$F(\theta) \succeq_{i(\theta)} F(\hat{\theta}), \forall \hat{\theta} \in \Theta \text{ s.t. } j(\hat{\theta}) \in A(\theta). \tag{4.27}$$

---

[5] I note that this result holds without assumptions 1 and 2, which are used to provide sufficient conditions for existence.

(ii) (Physical feasibility) For each producer j, the quantity produced is given by

$$q(j) = a \int_0^1 \left[I_{\{j=j(\theta(i))\}}(i) \right] di,$$

where $\theta(i)$ is the type of $i \in [0,1]$ and satisfies the capacity constraint $(q(j) \leq q)$.

(iii) (Non-negative net balance) The transfers add up to zero

$$\int_0^1 t(\theta(i))di + \int_0^1 t(j)dj = 0.$$

Definition 4.3 states that a direct revelation mechanism F is feasible if it is physically feasible, so each consumer consumes only the product from one producer that she actually knows (that is, $j(\hat{\theta}) \in A(\theta)$), producers produce a feasible quantity, the transfers involving consumers and producers add-up, and it is incentive compatible.

Pareto-dominance is defined ex-post for the economy: given a profile $\{A^i\}_i$ of awareness among consumers an allocation Pareto-dominates another if there exists another feasible allocation that Pareto dominates it.

Definition 4.4. A mechanism F is Pareto-dominated by another mechanism F' if F' is feasible, and for every consumer type θ, $F(\theta) \succeq F'(\theta)$ and for all producer $j \in [0,1]$, profits are equal or higher under F' than F, with at least a strict inequality for a subset of consumers and or producers of strictly positive measure.

From definitions 4.3 and 4.4, Pareto-efficiency is defined

Definition 4.5. A mechanism F is constrained efficient if it is feasible and there is no mechanism that Pareto-dominates it.

Theorem 4.4. *The incentive-compatible-direct-mechanism F^* which is implied by a Nash equilibrium price posting profile s^* is constrained efficient.*

Constrained efficiency follows from the fact that in equilibrium producers post the same price to all their consumers and maximize profits. Any incentive compatible mechanism must satisfy the property that for any set of consumers who consume the same variety the transfers are the same for all these consumers as otherwise they will choose to misreport their type. Any mechanism that implements an allocation that yields a higher aggregate market surplus than the allocation implied by the Nash equilibrium will have to pay higher information rents to certain consumers and, hence, to all the consumers who consumer the same varieties, which either reduces profits of the producers of these varieties or violates the non-negative net balance condition.

4.6 Concluding Remarks

This chapter presented a theory of industry pricing with a continuum of degrees of product differentiation, market frictions (represented by imperfect product awareness), and asymmetric information. I show that, under certain assumptions, the model has a unique symmetric Nash equilibrium in pure strategies and the equilibrium price depends both on the degree of product differentiation as well as on the degree of market frictions.

The equilibrium price was shown to be strictly increasing in the degree of product differentiation and on the degree of market frictions. But, for any degree of product differentiation, I also show that when market frictions converge to zero the equilibrium price converges to a price consistent with Walrasian equilibrium. This is an important result because this study provides a justification of the use of the model of competitive equilibrium even in markets which are characterized by product differentiation as long as market frictions are low. It also suggests that the presence of product differentiation amplifies the effect of market frictions on market prices. Finally, I perform welfare analysis and I show that the trading mechanism used in the model (price posting by the producers) is an efficient incentive compatible mechanism.

The work done here can be extended to include endogenous matching costs (for instance, advertising costs such as in Perla (2017)). In addition, providing proof of convergence to perfect competition in more general differentiated product spaces (as described by Jones (1984) [45]) and the characterization of more general conditions for existence and uniqueness could be helpful.

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Chapter 5

Appendix

5.1 Appendix to Chapter 2

5.1.1 Proof of Proposition 2.1

Proof. Consider the following mechanism: let I be the set of buyers who know seller j and they can participate in a mechanism with the seller. The buyer $i \in I$ of type θ_i chooses to report a type $\hat{\theta}_i$ to the mechanism and the mechanism allocates the perishable good produced by the seller to the buyers $x_i(\hat{\theta}) \in \{0, 1\}$ and transfers to the seller and the buyers $\{t_j(\hat{\theta}), \{t_i(\hat{\theta})\}_{i\in I}\}$ where $t_j = -\sum t_i(\hat{\theta})$ given a profile of reported types $\hat{\theta} = \{\hat{\theta}_i\}$. In such mechanism the buyer would report it's type $\hat{\theta} \in \{0, 1\}$ where $\hat{\theta} = 0$ means the buyer claims it doesn't know the competitor and $\hat{\theta} = 1$ means that the buyer claims it knows the competitor.

The seller's marginal cost of production is normalized to zero and the seller designs the mechanism. So I assume that the mechanism maximizes the seller's revenue. Since the seller wants to maximize revenue it's easy to see she will set $x_i = 1$ for all buyers to induce the highest fraction of buyers into participating in the mechanism and hence to maximize her net transfers $t_j(\theta)$. Suppose that $t_i(0, \hat{\theta}_{-i}) > t_i(1, \hat{\theta}_{-i})$, then it's obvious that all buyers who participate in the mechanism would choose to report 1, by analogous argument, $t_i(0, \hat{\theta}_{-i}) < t_i(1, \hat{\theta}_{-i})$ is also not IC. Hence the seller has to post the same price for every buyer for the mechanism to be IC. The revelation principle them implies that an optimal sales mechanism for the seller is characterized by the seller posting a uniform price for the buyers. □

5.1.2 Proof of Proposition 2.2

Proof. First, I show that the pair of distributions that satisfy equations 2.16, 2.17 and 2.14 is an equilibrium. Then I will show there does not exist any other pair of distributions that satisfy the conditions for equilibrium.

If h posts the lower bound of the support of the distribution the probability of sale is 1 so the expected profit per unit measure of customer base is $\underline{p}_2$. Hence, equilibrium implies in following equal profit condition for a price $p \in (\underline{p}_2, 1)$ for seller h:

$$p(1 - m^l \sigma_2^l(m,p)) = \underline{p}_2(m) = 1 - m_2^l. \tag{5.1}$$

To keep h indifferent between prices in $(\underline{p}_2, 1)$, the distribution σ_1^l has to satisfy 5.1, so for $p \in (\underline{p}_2, 1)$,

$$\sigma_1^l(m,p) = \frac{1}{m^l}\left(\frac{p - \underline{p}_2}{p}\right).$$

And the equal profit condition for the seller with smaller customer base is for prices in the interior of $[\underline{p}_2, 1]$,

$$\begin{aligned} p(1 - m^h \sigma_2^h(m,p)) &= \underline{p}_2 \\ \Rightarrow \sigma_2^h(m,p) &= \frac{1}{m^h}\left(\frac{p - \underline{p}_2}{p}\right). \end{aligned}$$

Note that if $m^h > m^l$ for all $p \in [\underline{p}_2, 1]$,

$$\sigma_1^l(m,p) > \sigma_1^h(m,p),$$

$$\lim_{p \uparrow 1} \sigma_2^h(m,p) < 1.$$

Hence, $\sigma_2^h(.)$ has an atom at $p = 1$ with mass

$$\begin{aligned}\sigma_2^h(m,1) - \lim_{p\uparrow 1}\sigma_2^h(m,p) &= 1 - \frac{1}{m^h}\left(\frac{1-\underline{p}_2(m)}{1}\right)\\ &= 1 - \frac{m^l}{m^h}.\end{aligned}$$

Then seller l will be indifferent between posting 1 or prices in the support but lower than 1 because the endogenous tie-breaking rule implies that l will be selling the good with probability

$$1 - m_2^h \lim_{p\uparrow 1}\sigma_2^h(p) = 1 - m_2^h\frac{m_2^l}{m_2^h} = 1 - m_2^l$$

if l posts 1.

To see that this is an equilibrium note that sellers will decrease their profits by posting prices lower than $\underline{p}_2$ and that both sellers are indifferent between posting prices over the support.

To show uniqueness I first show that the equilibrium distributions don't have atoms in the interior of the support, then I show that they will share the same support and that it is connected. Then given that they share the same support and feature mixed strategies I show that the equilibrium support must satisfy equation 2.14. Given that a pair of equilibrium strategies with the property of being in the support $[\underline{p}_2, 1]$ it is easy to see it implies that the pair (σ_2^h, σ_2^l) will satisfy 2.16 and 2.17 respectively.

Lemma: In equilibrium any strategy cannot have an atom in the interior of the support.

Proof: Note that for any pair of distributions (σ_2^h, σ_2^l), if one of them has an atom of mass in the interior of the unit interval $[0, 1]$ it is easy to use a standard undercutting argument to show that it is profit maximizing for any seller to deviate and post slightly lower prices than the atom. Posting prices equal to 0 is also not optimal because a seller can always make a profit by posting the monopoly price 1 as $\max\{m_2^E, m_2^I\} < 1$. Hence, in any equilibrium, the distribution of prices will not be characterized by atoms.

Lemma: And in mixed strategy equilibrium the supports of the pair of distributions must be the same.

Proof: It suppose $\text{supp}(\sigma_2^h) \neq \text{supp}(\sigma_2^l)$. Without loss of generality suppose $\min\{\text{supp}(\sigma_2^h)\} = \underline{p}^h < \min\{\text{supp}(\sigma_2^l)\} = \underline{p}^l$, then if h posts $\underline{p}^h$ will yield strictly lower profits than posting a $p \in (\underline{p}^h, \underline{p}^l)$. Contradiction.

Therefore, an equilibrium must feature mixed strategies and a common support.

Lemma: The connected common equilibrium support is $[\underline{p}_2, 1]$.

Proof: The common support must be connected otherwise it is easy to see it wouldn't be optimal for sellers to post in that support.

The support must include the monopoly price, otherwise the sellers will find optimal to post the monopoly price over a price at the upper bound of the support that is strictly lower than the monopoly price.

Suppose that the support of σ_2^h, σ_2^l is $[\underline{p}_2, 1]$ with $\underline{p}_2 < 1 - m_2^l$, then seller h will find it optimal to post the monopoly price rather than $\underline{p}_2$. If $\underline{p}_2 > 1 - m_2^l$ to keep h indifferent between prices in the support will imply that seller l will post 1 with an atom of probability strictly greater than zero which would imply that h will not be indifferent between posting 1 and lower prices due to the tie breaking rule. A contradiction. □

5.1.3 Proof of Proposition 2.3

Proof. The equal profit condition in period 1 for prices in the interior of the support $[\underline{p}_1, 1]$ is:

$$\begin{aligned} \bar{v}_1^j(\underline{p}_1, \sigma_1^{-j}, m_1) =& \underline{p}_1 \times m_1^j + \beta v_2^j(\overline{m}_2^j, \underline{m}_2^{-j}) = \\ p \times m_1^i \times (1 - \sigma_1^{-j}(p) m_1^{-j}) +& \beta \left((1 - m_1^{-j}\sigma_1^{-j}(p)) v_2^j(\overline{m}_2^j, \underline{m}_2^{-j}) + m_1^{-j}\sigma_1^{-j}(p) v_2^j(\underline{m}_2^j, \overline{m}_2^{-j}) \right). \end{aligned}$$

Solving the equation above for σ_1^{-j} characterizes the pair of equilibrium price distributions (σ_1^I, σ_2^E) in the interior of the support,

$$\begin{aligned} &\Rightarrow \sigma_1^{-j}(p) \times m_1^{-j} \times p \times m_1^j + \beta m_1^{-j}\sigma_1^{-j}(p)(v_2(\overline{m}_2^j, \underline{m}_2^{-j}) - v_2(\underline{m}_2^j, \overline{m}_2^{-j})) = (p - \underline{p}_1) m_1^j \\ &\quad \Rightarrow \sigma_1^{-j}(p) \left[m_1^{-j} \times p \times m_1^j + \beta m_1^{-j}(v_2(\overline{m}_2^j, \underline{m}_2^{-j}) - v_2(\underline{m}_2^j, \overline{m}_2^{-j})) \right] = (p - \underline{p}_1) m_1^j \\ &\quad \Rightarrow \sigma_1^{-j}(p) = \frac{p - \underline{p}_1}{\left(m_1^{-j} \times p + \beta \left(\frac{m_1^{-j}}{m_1^j} \right) (v_2(\overline{m}_2^j, \underline{m}_2^{-j}) - v_2(\underline{m}_2^j, \overline{m}_2^{-j})) \right)} = \frac{p - \underline{p}_1}{m_1^{-j}(p + \Delta^j(m_1))}. \end{aligned}$$

Let j be the seller such that $\underline{p}^j = \max\{\underline{p}^E, \underline{p}^I\}$ if $\underline{p}^E \neq \underline{p}^I$. Then note that 2.20 implies that

$$\lim_{p\uparrow 1} \sigma_1^j(p) < 1.$$

Therefore the distribution σ_1^j has an atom at 1. Hence a tie occurs with positive probability if the seller j's competitor,$-j$, considers posting a price of 1. However$-j$ is indifferent between posting 1 and prices in the interior of the support because the tie breaking rule implies that $-j$ sells the good with probability one in case of a tie with j.

It is easy to see that present value from posting a price below $\underline{p}_1$ will be lower than posting $\underline{p}_1$ and (σ_1^E, σ_1^I) makes I and E indifferent between posting prices in the supports of σ_1^I and σ_1^E, respectively. Therefore, equations 2.19 and 2.20 characterize a pair of equilibrium price distributions for the first period.

It remains to show that (σ_1^E, σ_1^I) is the only distribution of prices consistent with equilibrium. First, by analogous argument as in 2.2, the equilibrium distribution of prices will not feature atoms in the interior of the support. And by analogous argument the equilibrium will feature a common connected support.

It it is easy to see that if $[\underline{p}_1, 1]$ is the support of equilibrium strategies then (σ_1^E, σ_1^I) are the only distributions that make E and I indifferent among prices posted in the support given the tie-breaking rule. It remains to show that $[\underline{p}_1, 1]$ is the only support consistent with equilibrium.

To see that note that for any upper bound lower than 1 it is easy to see a seller would strictly prefer to post 1 over that lower bound. While for a lower bound $\underline{p} < \underline{p}_1$, would impossible to maintain the equal profit condition with posting 1 for both sellers.

And for a lower bound $\underline{p} > \underline{p}_1$, it would imply that given the tie-breaking rule, for one of the sellers it will not be possible to maintain the equal profit condition between $\underline{p}$ and 1 for any distribution of prices with support $[\underline{p}, 1]$: as posting the monopoly price 1 would strictly dominate posting $\underline{p}$ for at least one of the sellers.

The equilibrium is unique given the assumptions we have made including the specific tie-breaking rule. $\square$

5.1.4 Relaxing Independence of Discovery

I have assumed that discovery is independent. That is, the probability that a buyer becomes aware of a seller j is independent of the buyer knowing the competing seller $-j$. How does the model's equilibrium change when we relax this assumption? One simple way of relaxing this assumption is by assuming that there are two types of buyers. A fraction $\alpha \in (0,1)$ of the buyers are "shoppers" (following Stahl (1989)) who can more easily discover sellers than the other buyers. I assume that the probability that shoppers discover sellers in a given period is $\Xi > 1$ times the probability that non-shoppers do. I call a shopper by a buyer of type s and a non-shopper as ns. Discovery is independent across shoppers: that is, the probability that a shopper discovers seller j is independent of the shopper already knowing seller $-j$.

Let $m = (m^1, m^2)$ be the pair of customer base measures. Let m^{jl} be the measure of j's customers of type $l \in \{s, ns\}$. With the two types the law of motion for m^{jl} satisfies:

$$m^{js}_{t+1} = (1-\delta)m^{js}_t + \Phi(q^j_t, 1-m^j_t) \times \frac{\Xi \times (\alpha - m^{js})}{\Xi \times (\alpha - m^{js}) + 1 \times (1-\alpha-m^{jns})},$$
$$m^{jns}_{t+1} = (1-\delta)m^{jns}_t + \Phi(q^j_t, 1-m^j_t) \times \frac{1 \times (1-\alpha-m^{jns})}{\Xi \times (\alpha - m^{js}) + 1 \times (1-\alpha-m^{jns})},$$

where $\frac{\Xi\times(\alpha-m^{js})}{\Xi\times(\alpha-m^{js})+1\times(1-\alpha-m^{jns})}$ and $\frac{1\times(1-\alpha-m^{jns})}{\Xi\times(\alpha-m^{js})+1\times(1-\alpha-m^{jns})}$ are respectively the fractions of new customers of types s and ns.

Let M^j_t be the set of customers of seller j in period t and let ζ^j_t be the fraction of customers of seller j in period t of type s. Clearly $\zeta^j_t = \frac{m^{js}}{m^j}$.

Given that the period was arbitrary, a fraction ζ of m^j will be shoppers while a fraction $1-\zeta$ will not. Hence the probability that a customer in m^j is in m^{-j} is

$$\begin{aligned}
&P(i \in M^{-j} : i \in M^j)\\
=&\zeta^j_t \times P(s \in M^{-j} : s \in M^j) + (1-\zeta^j_t) \times P(ns \in M^{-j} : s \in M^j)\\
=&\zeta^j_t \times P(s \in M^{-j}) + (1-\zeta^j_t) \times P(ns \in M^{-j})\\
=&\zeta^j_t \times m^{-js} + (1-\zeta^j_t) \times m^{-jns}\\
=&\frac{m^{js}}{m^j} \times m^{-js} + \frac{m^{jns}}{m^j} \times m^{-jns}.
\end{aligned}$$

While

$$P(i \in M^{-j}) = \frac{m^{jt}}{\alpha} \neq P(i \in M^{-j} : i \in M^j).$$

In this environment the only change is on the probability that a current customer of a seller will know the competing seller. Hence equation 2.2 becomes

$$P^j(p, \sigma_t^j, m_t) = 1 - \left(\frac{m_t^{js}}{m_t^j} \times m_t^{-js} + \frac{m_t^{jns}}{m_t^j} \times m_t^{-jns} \right) \sigma(p).$$

It's clear that no major property of equilibrium changes besides the greater difficulty in computing equilibrium.

5.2 Appendix to Chapter 3

5.2.1 Proof of Proposition 3.1

Proof. Sellers of age 1 enter the market with customer base $\underline{c}$ at any period. In symmetric equilibrium sellers with the same customer base post prices according to the same distribution. This implies that the quantity sold by all sellers of age 1 is the same. The law of motion for the customer base implies that all sellers of age 2 have the same customer base. Following the same argument, for any period t, sellers of age $a \in \{1, \dots, t\}$ have the same customer bases. $\square$

5.2.2 Proof of Lemma 3.1

Proof. The proof is a verification that the operator T satisfies Blackwell's sufficient conditions for a contraction. Let $c^+(p, c) = c + \Phi(q(p, \hat{\sigma}, c, \hat{c}), c)$, to simplify notation.

Let T be the operator defined by

$$T(v)(c) = \sup_{p \in (-\infty, 1]} \gamma(p, \hat{\sigma}, c, \hat{c}) + \beta(1 - \alpha) v(c^+(p, c)), \tag{5.2}$$

for $\beta(1-\alpha) \in (0, 1)$, for a function $v \in \mathcal{F}([0, \overline{c}], \mathbb{R})$, where $\mathcal{F}([0, \overline{c}], \mathbb{R})$ is the space of functions mapping $[0, \overline{c}]$ into $\mathbb{R}$.

Does T satisfy Blackwell's sufficiency conditions for a contraction?

(Monotonicity) Let $f : [0,\overline{c}] \to \mathbb{R}, g : [0,\overline{c}] \to \mathbb{R}$ such that $f(c) \leq g(c), \forall c \in [0,\overline{c}]$, then for all $p \in \mathbb{R}$,

$$\begin{aligned}
\gamma(p,\hat{\sigma},c,\hat{c}) + \beta(1-\alpha)f(c^+(p,c)) &\leq \gamma(p,\hat{\sigma},c,\hat{c}) + \beta(1-\alpha)g(c^+(p,c)),\\
\Rightarrow \sup_{p\in(-\infty,1]} \left(\gamma(p,\hat{\sigma},c,\hat{c}) + \beta(1-\alpha)f(c^+(p,c))\right) &\leq \sup_{p\in(-\infty,1]} \left(\gamma(p,\hat{\sigma},c,\hat{c}) + \beta(1-\alpha)g(c^+(p,c))\right),\\
&\Rightarrow T(f)(c) \leq T(g)(c).
\end{aligned}$$

(Discounting) Let $f : [0,\overline{c}] \to \mathbb{R}$ and $b > 0$, $c \in [0,\overline{c}]$, then

$$\begin{aligned}
T(f+b)(c) &= \sup_{p\in\mathbb{R}} \left(\gamma(p,\hat{\sigma},c,\hat{c}) + \beta(1-\alpha)\left(f(c^+(p,c)) + b\right)\right),\\
&= \sup_{p\in\mathbb{R}} \left(\gamma(p,\hat{\sigma},c,\hat{c}) + \beta(1-\alpha)f(c^+(p,c))\right) + \beta(1-\alpha)b,\\
&= T(f)(c) + \beta(1-\alpha)b.
\end{aligned}$$

Therefore, T is a contraction and we can apply the contraction mapping theorem implies that the value function v exists and is unique. □

5.2.3 Proof of Lemma 3.2

Proof. Part (i). v is non-decreasing in c.

Let $c, c' \in [0,\overline{c}]$ such that $c' > c$. We know that v satisfies:

$$\begin{aligned}
v(c) &= \sup_{\{p_l\}} \sum_{l=0}^{\infty} \left[\beta(1-\alpha)\right]^l \gamma(p_l,\hat{\sigma},c_l,\hat{c}), \qquad (5.3)\\
\text{s.t. } c_{l+1} &= c_l + \Phi(q(p_l,\hat{\sigma},c_l,\hat{c}), \overline{c} - c_l),\\
c_0 &= c.
\end{aligned}$$

Since $\hat{\sigma}$ is not necessarily continuous, therefore γ is not necessarily continuous and hence an optimal price sequence satisfying the problem 5.3 might not exist. However, by the definition of v as the solution to 5.3 there exists a sequence

$\{\{p_l^n\}_l\}_n$ of price sequences satisfying:

$$\lim_n \sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \gamma(p_l^n, \hat{\sigma}, c_l^n, \hat{c}) = v(c),$$

where

$$c_l^n = \begin{cases} c_{l-1}^n + \Phi(q(p_{l-1}^n, \hat{\sigma}, c_{l-1}^n, \hat{c}), \bar{c} - c_{l-1}^n) & \text{if } l \geq 1, \\ c & \text{if } l = 0. \end{cases}$$

Let $\{\{c_l'^n\}_l\}_n$ and $\{\{c_l^n\}_l\}_n$ be the corresponding sequences of sequences of customer bases with $c_0'^n = c'$ and $c_0^n = c$.

Clearly, for all l and n,

$$\gamma(p_l^n, \hat{\sigma}, c_l'^n, \hat{c}) \geq \gamma(p_l^n, \hat{\sigma}, c_l^n, \hat{c}).$$

So, for all n,

$$\begin{aligned} &\sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \gamma(p_l^n, \hat{\sigma}, c_l'^n, \hat{c}) \geq \sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \gamma(p_l^n, \hat{\sigma}, c_l^n, \hat{c}), \\ \Rightarrow &\lim_n \sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \gamma(p_l^n, \hat{\sigma}, c_l'^n, \hat{c}) \geq \lim_n \sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \gamma(p_l^n, \hat{\sigma}, c_l^n, \hat{c}) = v(c). \end{aligned}$$

As $v(c') \geq \sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \gamma(p_l^n, \hat{\sigma}, c_l'^n, \hat{c})$ for all n by definition, therefore $v(c') \geq v(c)$.

Part (ii). The function v is strictly concave in c.

By definition, $v(0)$ is given by

$$\begin{aligned} v(0) = &\sup_{\{p_l\}} \sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \gamma(p_l, \hat{\sigma}, c_l, \hat{c}), \\ \text{s.t. } &c_{l+1} = c_l + \Phi(q(p_l, \hat{\sigma}, c_l, \hat{c}), \bar{c} - c_l), \\ &c_0 = 0, \end{aligned}$$

clearly, for any p_l, $\hat{c}$, $\hat{\sigma}$ and feasible sequence $\{c_l\}$, $\gamma(p_l, \hat{\sigma}, c_l, \hat{c}) = 0$. Hence $v(0) = 0$.

Let $\lambda \in (0,1)$ and $c \in (0,\overline{c})$. Since v is non-decreasing and $v(0)=0$ to show it's strictly concave in c it suffices to show that $v(\lambda c) > \lambda v(c)$. Consider the case $\hat{c}=0$ the strict concavity of v in c follows trivially from the strict concavity of Φ on q. For the case $\hat{c}>0$, let $\{\{p_l^n\}_l\}_n$ be a sequence of sequence of prices satisfying

$$v(c) = \lim_{n\to\infty} \sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \, \gamma(p_l^n, \hat{\sigma}, c_l, \hat{c}),$$

since $v(c)$ satisfies 5.3 such a sequence of sequences exists and $v(c)>0$ since $\operatorname{supp}(\sigma)\cap(0,1] \neq \emptyset$. Let $\{\{c_l^n\}_l\}_n$ be the corresponding sequence of customer bases.

Let $\{\{c_l^n(\lambda)\}_l\}_n$ be a sequence of sequences of customer bases satisfying

$$c_l^n(\lambda) = \begin{cases} c_{l-1}^n(\lambda) + \Phi(q(p_l^n, \hat{\sigma}, c_{l-1}^n(\lambda), \hat{c}), \overline{c} - c_{l-1}^n(\lambda)) & \text{if } l>0, \\ \lambda c & \text{if } l=0. \end{cases}$$

Since Φ is strictly concave in q and nonincreasing in c while γ is linear on c it's easy to see that for every n,

$$v(\lambda c) \geq \sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \, \gamma(p_l^n, \hat{\sigma}, c_l^n(\lambda), \hat{c}) > \lambda \left[\sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \, \gamma(p_l^n, \hat{\sigma}, c_l^n, \hat{c}) \right].$$

Note that by construction, $\lambda v(c) = \lim_n \lambda \left[\sum_{l=0}^{\infty} [\beta(1-\alpha)]^l \, \gamma(p_l^n, \hat{\sigma}, c_l^n, \hat{c}) \right]$. Therefore $v(\lambda c) > \lambda v(c)$.

QED. □

5.2.4 Proof of Lemma 3.3

Proof. First note that v exists and if σ is continuous in c then the set $\{\gamma(p,\sigma,c,\hat{c}) : p \in (-\infty,1)\}$ is an interval $(-\infty, \overline{\gamma}(c)] \subset (-\infty, c]$ for $c \in [0,\overline{c}]$ and $\{0\}$ for $c=0$ while $\gamma(p,\sigma,c,\hat{c})$ and $c^+(p,c)$ are continuous on p. Therefore, for any customer base $c \in [0,\overline{c}]$ there is an optimal sequence of prices $\{p_k\}_k$ that solves the sequential problem:

$$\begin{aligned}
&\max_{\{p_l\}_l} \sum_{l=0}^{\infty} [\beta(1-\alpha)]^k \gamma(p_l, \hat{\sigma}, c_l, \hat{c}),\\
&\text{s.t.} c_{l+1} = c_l + \Phi(q(p_l, \hat{\sigma}, c_l, \hat{c}), \overline{c} - c_l),\\
&\quad c_0 = c.
\end{aligned}$$

The proof is divided into two parts, first I show that v is continuous then I show it is strictly increasing.

Part (i). The function v is continuous in c.

To prove continuity suppose v is not continuous so there exists a c and sequence $\{c_n\}_n$ in $[0, \overline{c}]$ with $c = \lim_n c_n$ such that $\lim_n v(c_n) \neq v(c)$.

Let $\{p_k\}_k$ be the sequence of prices satisfying

$$v(c) = \sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \gamma(p_k, \hat{\sigma}, c_k, \hat{c}),$$

where

$$c_k = \begin{cases} c_{k-1} + \Phi(q(p_{k-1}, \hat{\sigma}, c_{k-1}, \hat{c}), \overline{c} - c_{k-1}) & \text{if } k > 0, \\ c & \text{if } k = 0. \end{cases}$$

Let $\{c_n\}$ a sequence in $[0, \overline{c}]$ converging to c. Since γ is linear on c and c^+ is continuous on c,

$$\lim_n \left(\sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \gamma(p_k, \hat{\sigma}, c_k^n(\{p_k\}_k), \hat{c}) \right) = v(c),$$

where $\{c_k^n(\{p_k\}_k)\}$ is the sequence of customer bases implied by $\{p_k\}_k$ since

$$v(c_n) \geq \sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \gamma(p_k, \hat{\sigma}, c_k^n(\{p_k\}_k), \hat{c})$$

for each n, then if $\lim_n v(c_n) \neq v(c)$ implies

$$\lim_n v(c_n) > v(c).$$

Let $\{\{p_k^n\}_k\}_n$ the sequence of sequences of prices satisfying

$$v(c_n, a) = \sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \, \gamma(p_k^n, \hat{\sigma}, c_k^n, \hat{c}),$$

where $\{c_k^n\}$ is the sequence of customer bases implied by $\{p_k^n\}_k$, taking a subsequence if necessary let $p_k^* = \lim_n p_k^n$, for each $k \in \{0, 1, 2, \ldots\}$. Since profits are linear on the customer base and next period's customer base is continuous on present period customer base it follows that

$$\lim_n v(c_n) = \sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \, \gamma(p_k^*, \hat{\sigma}, c_k(\{p_k^*\}_k), \hat{c}) > v(c)$$

which is a contradiction.

QED.

Part (ii). The function v is strictly increasing.

Since $\gamma(p, \hat{\sigma}, c, \hat{c})$ is linear on c, for a continuous average distribution of prices $\hat{\sigma}$ and $\pi \in \Delta^3_{++}$, it's easy to see that there exists a $p \in (0, 1]$ such that $\gamma(p, \hat{\sigma}, c, \hat{c}) > 0$ for $c > 0$. It's easy to prove that v is strictly increasing on c: for $c, c' \in [0, \overline{c}]$ such that $c' > c$, let $\{p_k\}_k$ be the sequence of prices satisfying

$$v(c) = \sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \, \gamma(p_k, \hat{\sigma}, c_k, \hat{c}),$$

where

$$c_k = \begin{cases} c_{k-1} + \Phi(q(p_{k-1}, \hat{\sigma}, c_{k-1}, \hat{c}), \overline{c} - c_{k-1}) & \text{if } k > 0, \\ c & \text{if } k = 0. \end{cases}$$

Let $\{c'_k\}$ be the sequence of customer bases with $c'_0 = c'$ and

$$c'_k = c'_{k-1} + \Phi(q(p_{k-1}, \hat{\sigma}, c'_{k-1}, \hat{c}), \overline{c} - c'_{k-1}),$$

then it's easy to see that

$$v(c') \geq \sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \, \gamma(p_k, \hat{\sigma}, c'_k, \hat{c}) > \sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \, \gamma(p_k, \hat{\sigma}, c_k, \hat{c}) = v(c).$$

QED. $\square$

5.2.5 Proof of Lemma 3.4

Proof. (i) To see that the support of $\hat{\sigma}$ is convex, suppose it's not: so there is an open interval not in the support of $\hat{\sigma}$ but in the convex hull of $\hat{\sigma}$. It implies that there is a $p \in \text{Conv}(\text{supp}(\hat{\sigma}))$, $p > p'$ for some $p' \in \text{supp}(\hat{\sigma})$ and the quantity sold under p is the same as p'. Laying out the argument more explicitly, there is an interval $(\underline{p}, \overline{p}) \subset \text{Conv}(\text{supp}(\hat{\sigma})) \cap (\text{supp}(\hat{\sigma}))^C$ and where $\underline{p} \in \text{supp}(\hat{\sigma})$, it follows that posting a $p \in (\underline{p}, \overline{p})$ yields the same quantity sold as posting $\underline{p}$ and hence strictly higher payoffs. Which is a contradiction with $\hat{\sigma}$ being consistent with equilibrium.

(ii) Consider two sellers with customer bases c, c' such that $c' \neq c$. Suppose $\text{Int}(\text{supp}(\hat{\sigma}(c))) \cap \text{Int}(\text{supp}(\hat{\sigma}(c'))) \neq \emptyset$, then the distribution of prices $\hat{\sigma}$ must make seller with customer base c indifferent between posting prices in $\text{Int}(\text{supp}(\hat{\sigma}(c))) \cap \text{Int}(\text{supp}(\hat{\sigma}(c')))$, however, due to the concavity of v (Lemma 3.3) and strict concavity of Φ in respect to sales quantity, a seller with customer base $c' \neq c$ will not be indifferent in posting prices in the same support as a seller with customer base c. A contradiction with $\hat{\sigma}$ being consistent with equilibrium.

To see that the seller with the larger customer base posts higher prices let $c' > c$ and suppose that some price p in $\text{supp}(\hat{\sigma}(c'))$ is strictly smaller than any price in $\text{supp}(\hat{\sigma}(c'))$. We know that $\hat{\sigma}$ is such that it makes seller c' indifferent between those prices which implies that combined with strict concavity of v and Φ in the sales quantity that the seller with customer base c will enjoy higher payoffs by posting p rather than prices in $\text{supp}(\hat{\sigma}(c'))$. Which is a contradiction with $\hat{\sigma}$ being consistent with equilibrium.

(iii) Let $\overline{p}$ be the upper found of the support, that is:

$$\overline{p} = \sup(\text{supp}(\hat{\sigma}))$$

Where $j \in [0, 1]$ indexes all sellers. Suppose that $\overline{p} < 1$, then a seller j where $\overline{p}_j$ is arbitrarily close to $\overline{p}$, if seller j chooses to post 1 instead of $\overline{p}_j < 1$ the quantity sold decreases by an amount that is arbitrarily small for $\overline{p}_j$ is arbitrarily close to $\overline{p}$ since $\hat{\sigma}$ is continuous. While profits per unit sold increases by at least $1-\overline{p}$, therefore for some sellers j, posting 1 yields strictly higher payoffs than $\overline{p}_j$ close to $\overline{p}$. A contradiction with σ being consistent with equilibrium. □

5.2.6 Proof of Lemma 3.5

Proof. The proof is organized in steps. Let $\{\sigma^a\}_a$ be an equilibrium strategy profile generated by the fixed point $V(., \{c^a\}_a)$.

Step 1: For each cohort a, σ^a is atomless.

To note that σ^a is atomless suppose that it has an atom at some $p > 0$, then sellers will have incentive to undercut by posting a price $p - \epsilon$ for some $\epsilon > 0$.

Step 2: For cohorts $a, b \in \{1, \ldots, t\}$, if $a < b$ then the supports of σ^a and σ^b have disjoint interiors and $\max(\text{supp}(\sigma^a)) \leq \min(\text{supp}(\sigma^b))$.

To see that the supports of σ^a and σ^b have disjoint interiors note that for a seller of cohort a to find optimal to post prices in the support of σ^a then σ^a must satisfy an equal profit condition that makes sellers of a indifferent. Given that $\Phi(q(p, \hat{\sigma}, c, \hat{c}), \overline{c} - c)$ is strictly concave in q, non-increasing and concave in c and $V(c, \{c^a\}, a)$ is strictly concave in c, by an argument analogous to step 1 of the proof of proposition 3.2, sellers in b will not find optimal to post prices in the support of σ^a.

To see that $\max(\text{supp}(\sigma^a)) \leq \min(\text{supp}(\sigma^b))$ note that since $\Phi(q(p, \hat{\sigma}, c, \hat{c}), \overline{c} - c)$ is strictly concave in q, non-increasing and concave in c and $V(c, \{c^a\})$ is strictly concave in c, if the support of σ^a includes prices strictly higher than prices in the support of σ^b then, by an argument analogous to step 1 of the proof of proposition 3.2, sellers in cohort b will find that posting prices in the interior of the support of σ^a will yield higher profits than in the interior of the support of σ^b, which is a contradiction.

Step 3: For each cohort a, the support of σ^a is connected.

Suppose that the support of σ^a is not connected then there is an open interval $(\underline{p}, \overline{p})$ such that $\exists p, p' \in \text{supp}(\sigma^a)$ and $p < \underline{p}$ while $\overline{p} < p'$. But Step 2 and the equal profit condition implies that posting a price $z \in (\underline{p}, \overline{p})$ will yield strictly higher profits than posting p.

Step 4: The upper bound of the support of $\hat{\sigma} = \sum_a n^a \sigma^a$ is 1.

To see that note that if the upper bound was lower than 1 it's easy to see that sellers will find it strictly dominant to post 1 instead of the upper bound. A contradiction.

Step 5: Then the profile of equilibrium strategies $\{\sigma^a\}_a$ is unique.

The upper bound 1, and the continuity, monotonicity of γ and V in c implies that for each cohort a, for a given upper bound $\overline{p}^a$ the equal profit condition for prices in the support of the equilibrium strategy plus the fact the supports are connected, disjoint and monotone increasing on the seller's age implies in a unique support $[\underline{p}^a, \overline{p}^a]$ and in a unique distribution of prices σ^a that maintains the equal profit condition. If the support $[\underline{p}^a, \overline{p}^a]$ is determined, then $[\underline{p}^{a-1}, \overline{p}^{a-1}]$ is also determined, note that Step 4 implies that $\overline{p}^t = 1$ and following this reasoning the profile of supports $\{[\underline{p}^a, \overline{p}^a]\}_a$ is determined and hence the profile of equilibrium strategies $\{\sigma^a\}_a$. □

5.2.7 Proof of Theorem 3.1

Proof. The proof is divided into three parts. First I prove that $\{\sigma^a\}$ is consistent with the steady-state equilibrium. Then I show the equilibrium exists by showing that there exists a value function v consistent with the equilibrium strategies. Then I show that this equilibrium is the unique symmetric equilibrium. To simplify notation let $c^+(p, c) = c + \Phi(q(p, \hat{\sigma}, c, \hat{c}), \overline{c} - c)$ be the next period customer base of a seller who posts price p and has customer base c.

Part 1: The price posting strategies $\{\sigma^a\}_a$ are consistent with equilibrium.

Note that the profile of customer bases $\{c^a\}_a$ implies in a $\hat{c} = \sum_a n^a c^a > 0$ and that $\hat{\sigma}$ is continuous and includes strictly positive prices in it's support. Therefore, Lemma 3.3 imply that v is a strictly increasing, strictly concave and continuous function.

For a seller of age a, σ^a satisfies the following equal profit condition, for any $p \in (\underline{p}^a, \overline{p}^a)$:

$$\begin{aligned} c^a p\xi(p,\hat{\sigma},\hat{c}) + \beta(1-\alpha)v\left[c^+(p,c^a)\right] &= c^a\underline{p}^a\xi(\underline{p}^a,\hat{\sigma},\hat{c}) + \beta(1-\alpha)v\left[c^+(\underline{p}^a,c^a)\right], \qquad (5.4)\\ &= c^a\overline{p}^a\xi(\overline{p}^a,\hat{\sigma},\hat{c}) + \beta(1-\alpha)v\left[c^+(\overline{p}^a,c^a)\right]. \end{aligned}$$

Since v is non-decreasing from Lemma 3.2, it's easy to see that $\overline{p}^a > \underline{p}^a$ (as $P(\overline{p}^a,\sigma,\hat{c}) < P(\underline{p}^a,\sigma,\hat{c})$) and σ^a satisfies

$$\lim_{p\uparrow\overline{p}^a}\sigma^a(p) = 1,$$
$$\lim_{p\downarrow\underline{p}^a}\sigma^a(p) = 0.$$

Further since v is continuous, then σ^a is also continuous.

Therefore, since $\overline{p}^a = \underline{p}^{a+1}$, the supports of $\{\sigma^a\}$ have disjoint interiors for any two cohorts, hence it's easy to check that $\hat{\sigma}(p)$ satisfies

$$\hat{\sigma}(p) = \overline{\omega}^{a-1} + \omega^a\sigma^a(p),$$

for $p \in [\underline{p}^a,\overline{p}^a] \subset [\underline{p},\overline{p}]$, and so $\hat{\sigma}$ is also continuous on $[\underline{p}^a,\overline{p}^a]$.

To see that the profile of strategies $\{\sigma^a\}$ is consistent with equilibrium, note that it's obvious that no seller would find it advantageous to post a price $p < \underline{p}^1 = \underline{p}$, that is, strictly lower than the lower bound of the support of prices posted in equilibrium since the quantity sold would be the same as posting $\underline{p}$ and profits would be lower. It remains to show that each cohort will not find advantageous to post prices outside of their supports.

Consider a pair of sellers, one of age a and another of age b with $b > a$. Note that σ^a by satisfying equation 3.28 it implies that if a seller of age a is indifferent between posting $p \in [\underline{p}^a,\overline{p}^a)$ and $\overline{p}^a$. Then for a seller of age $b > a$, the fact $c^b > c^a$ and the strict concavity of the discovery function $\Phi(q(p,c),c)$ in c implies that for such $p \in (\underline{p}^a,\overline{p}^a)$,

$$\frac{c^+(p,c^b) - c^+(\overline{p}^a,c^b)}{c^b} < \frac{c^+(p,c^a) - c^+(\overline{p}^a,c^a)}{c^a}.$$

Note that v is strictly increasing and concave (by lemmas 3.2 and 3.3) implies that for $p \in (\underline{p}^a, \overline{p}^a)$,

$$\begin{aligned}
v(c^a) &= c^a p \xi(p, \hat{\sigma}, \hat{c}) + \beta(1-\alpha) v\left[c^+(p, c^a)\right] \\
\Rightarrow c^a p &\left\{\sum_{k=0}^{\infty} \pi^k(\hat{c}) \left[1 - (\overline{\omega}^{a-1} + \omega^a \sigma^a(p))\right]^k\right\} + \beta(1-\alpha) v\left[c^+(p, c^a)\right] \\
&= c^a \overline{p}^a \left\{\sum_{k=0}^{\infty} \pi^k(\hat{c}) \left[1 - \overline{\omega}^{a-1}\right]^k\right\} + \beta(1-\alpha) v\left[c^+(\overline{p}^a, c^a)\right], \qquad (5.5) \\
&\Rightarrow \frac{\beta(1-\alpha)\left\{v\left[c^+(p, c^a)\right] - v\left[c^+(\overline{p}^a, c^a)\right]\right\}}{c^a} \\
= p &\left\{\sum_{k=0}^{\infty} \pi^k(\hat{c}) \left[1 - (\overline{\omega}^{a-1} + \omega^a \sigma^a(p))\right]^k\right\} - \overline{p}^a \left\{\sum_{k=0}^{\infty} \pi^k(\hat{c}) \left[1 - \overline{\omega}^{a-1}\right]^k\right\}, \qquad (5.6) \\
&\Rightarrow \frac{\beta(1-\alpha)\left\{v\left[c^+(p, c^a), a+1\right] - v\left[c^+(\overline{p}^a, c^a)\right]\right\}}{c^a} \qquad (5.7) \\
&= p\left\{\exp\left[-\hat{c}\left(\overline{\omega}^{a-1} + \omega^a \sigma^a(p)\right)\right]\right\} - \overline{p}^a \left[\exp(-\hat{c}\overline{\omega}^{a-1})\right] \qquad (5.8) \\
&\Rightarrow \frac{\beta(1-\alpha)\left\{v\left[c^+(p, c^b)\right] - v\left[c^+(\overline{p}^a, c^b)\right]\right\}}{c^b} \\
&< p\left\{\exp\left[-\hat{c}\left(\overline{\omega}^{a-1} + \omega^a \sigma^a(p)\right)\right]\right\} - \overline{p}^a \left[\exp(-\hat{c}\overline{\omega}^{a-1})\right], \qquad (5.9) \\
&\Rightarrow c^b p\left\{\exp\left[-\hat{c}\left(\overline{\omega}^{a-1} + \omega^a \sigma^a(p)\right)\right]\right\} + \beta(1-\alpha^{a+1}) v\left[c^+(p, c^b)\right] \qquad (5.10) \\
&< c^b \overline{p}^a \left[\exp(-\hat{c}\overline{\omega}^{a-1})\right] + \beta(1-\alpha^{a+1}) v\left[c^+(\overline{p}^a, c^b)\right]. \qquad (5.11)
\end{aligned}$$

For $p \in (\underline{p}^b, \overline{p}^b)$, by analogous reasoning,

$$\begin{aligned}
v(c^b) &= c^b p\left\{\exp\left[-\hat{c}\left(\overline{\omega}^{b-1} + \omega^b \sigma^b(p)\right)\right]\right\} + \beta(1-\alpha) v\left[c^+(p, c^b)\right] \\
&\Rightarrow c^a p\left\{\exp\left[-\hat{c}\left(\overline{\omega}^{a-1} + \omega^a \sigma^a(p)\right)\right]\right\} + \beta(1-\alpha) v\left[c^+(p, c^a)\right] \qquad (5.12) \\
&> c^a \overline{p}^b \left[\exp(-\hat{c}\overline{\omega}^{b-1})\right] + \beta(1-\alpha) v\left[c^+(\overline{p}^b, c^a)\right]. \qquad (5.13)
\end{aligned}$$

Let $b = a + 1$ by induction it follows that sellers of age a will not find it profit maximizing to post prices in $[\underline{p}, \overline{p}] \diagdown [\underline{p}^a, \overline{p}^a]$ and the equal profit condition 5.4 implies that sellers of cohort a will be indifferent to prices in the support $[\underline{p}^a, \overline{p}^a]$.

Since $\lim_a \overline{p}^a = 1$, therefore $[\underline{p}, \overline{p}] = [\underline{p}, 1]$. Hence, posting prices in $[\underline{p}^a, \overline{p}^a]$ is optimal for sellers of age a. Therefore, the profile of price posting strategies $\{\sigma^a\}_a$ is consistent with equilibrium.

To see that $q(c^a)$ satisfies 3.26, it's simple algebra:

$$\begin{aligned} q(c^a) &= c^a \int \xi(p, \hat{\sigma}, \hat{c}) d\sigma^a \\ &= c^a \int \exp\left[-\hat{c}\left(\overline{\omega}^{a-1} + \omega^a \sigma^a(p)\right)\right] d\sigma^a \\ &= c^a \exp(-\hat{c}\overline{\omega}^{a-1}) \int \exp(-\hat{c}\omega^a\sigma^a(p)) d\sigma^a \\ &= c^a \exp\left[-\hat{c}\left(\overline{\omega}^{a-1} + \frac{1}{2}\omega^a\right)\right]. \end{aligned}$$

Part 2: Existence.

To show existence if suffices to show that a strictly increasing concave value function v that is consistent with the equilibrium strategy profile and customer base profiles exists. I show that by constructing an operator in the space of strictly increasing concave functions and showing it is a contraction and hence has a unique fixed point.

Let $\mathcal{F}$ be the space of functions on $[0, \overline{c}]$ that are strictly increasing and concave. Let T be an operator that maps $\mathcal{F}$ into a functional space.

Consider a $v \in \mathcal{F}$. Let $\{\underline{p}^{a,v}, \overline{p}^{a,v}\}$ be a sequence with $\lim_{a\to\infty} \overline{p}^{a,v} = 1$ satisfying $\overline{p}^{a,v} = \underline{p}^{a+1,v}$ and $\underline{p}^{a,v}$ satisfies

$$\underline{p}^{a,v} c^a \xi(\underline{p}^{a,v}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha)v(c^+(\underline{p}^{a,v}, c^a)) = \overline{p}^{a,v} c^a \xi(\overline{p}^{a,v}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha)v(c^+(\overline{p}^{a,v}, c^a)),$$

for each a. The operator $T(v)$ satisfies the following conditions. For $c \in [c^{a-1}, c^a)$ for some $a > 1$, $T(v)$ satisfies

$$T(v)(c) = \underline{p}^{a,v} c \xi(\underline{p}^{a,v}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha)v(c^+(\underline{p}^{a,v}, c)), \tag{5.14}$$

for $c \in [0, \underline{c}] = [0, c^1]$, $T(v)$ satisfies

$$T(v)(c) = \underline{p}^{1,v} c \xi(\underline{p}^{1,v}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha) v(c^+(\underline{p}^{1,v}, c)). \tag{5.15}$$

Note that since $v(c)$ is a strictly increasing concave function in c, then equations 5.14 and 5.15 imply that $T(v)(c)$ is a strictly increasing and strictly concave function in c, hence T maps the space of strictly increasing concave functions in c into itself, that is $T : \mathcal{F} \to \mathcal{F}$.

I want to show that T is a contraction. To show that it suffices to show that T satisfies Blackwell's sufficient conditions for a contraction: monotonicity and discounting.

Monotonicity: Let $f, g \in \mathcal{F}$ such that $f(c) \geq g(c)$ for all $c \in [0, \overline{c}]$. I want to show that $T(f) \geq T(g)$, let $c \in [0, \overline{c}]$, $\{\underline{p}^{a,f}, \overline{p}^{a,f}\}$ and $\{\underline{p}^{a,g}, \overline{p}^{a,g}\}$, then there is some a such that either $c \in [c^{a-1}, c^a)$ or $c < \underline{c}^1$, then

$$\begin{aligned} T(f)(c) &= \underline{p}^{a,f} c \xi(\underline{p}^{a,f}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha) f(c^+(\underline{p}^{a,f}, c)) \\ &\geq \underline{p}^{a,g} c \xi(\underline{p}^{a,g}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha) f(c^+(\underline{p}^{a,g}, c)) \qquad (5.16) \\ &\geq \underline{p}^{a,g} c \xi(\underline{p}^{a,g}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha) g(c^+(\underline{p}^{a,g}, c)) \qquad (5.17) \\ &= T(g)(c). \qquad (5.18) \end{aligned}$$

where the inequality 5.16 follows from 5.10.

Discounting: Let $f \in \mathcal{F}$ and $b > 0$, for $c \in [0, \overline{c}]$,

$$\begin{aligned} T(f+b)(c) &= \underline{p}^{a,f} c \xi(\underline{p}^{a,f}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha)[f(c^+(\underline{p}^{a,f}, c)) + b] \\ &= \underline{p}^{a,f} c \xi(\underline{p}^{a,f}, \hat{\sigma}, \hat{c}) + \beta(1-\alpha) f(c^+(\underline{p}^{a,f}, c)) + \beta(1-\alpha) b \\ &= T(f)(c) + \beta(1-\alpha) b \\ &< T(f)(c) + b. \end{aligned}$$

Therefore, T is a contraction and hence has a unique fixed point. It's easy to check that the fixed point of T is a value function v that is consistent with the profile of supports$\{[\underline{p}^{a,v}, \overline{p}^{a,v}]\}$ for the mixed pricing strategies in the steady steady equilibrium. Hence, the equilibrium exists.

Part 3: Uniqueness.

Part 2, lemmas 3.2 and 3.3 imply that to show uniqueness it suffices to show that a symmetric equilibrium with v continuous, strictly increasing and concave must be characterized by a unique mixed price posting strategy profile. Thus, to show that the strategy profile is unique first it must be shown that the average distribution of prices $\hat{\sigma}$ is continuous which means that lemmas 3.2, 3.3 and 3.4 can be applied. If satisfied, these lemmas imply in uniqueness of the symmetric steady-state equilibrium.

To see that in equilibrium $\hat{\sigma}$ must be continuous suppose $\hat{\sigma}$ it's not continuous (i.e. it has at least one atom). Then there exists a price $p^A > 0$ such that $\lim_{p \uparrow p^A} \hat{\sigma}(p) < \hat{\sigma}(p^A)$.

In that case posting p^A is not consistent with profit maximization. To see that consider a seller posting a price $p^A - \epsilon$ with $\epsilon > 0$ instead of p^A. Lowering the price by ϵ will yield an increase in quantity sold at least as large as the customer base times $\hat{\sigma}(p^A) - \lim_{p \uparrow p^A} \hat{\sigma}(p)$. Hence, lowering the price by ϵ implies that present period profits increases by at least

$$c \left\{ (p^A - \epsilon)[\hat{\sigma}(p^A) - \lim_{p \uparrow p^A} \hat{\sigma}(p)] - \epsilon \hat{\sigma}(p^A) \right\}$$

which is a positive change for ϵ small enough and also that the future's customer base is higher with higher sales quantity. Note that Lemma 3.2 states that v is always non-decreasing. A contradiction with $\hat{\sigma}$ being consistent with equilibrium.

To show that in equilibrium each cohort a's strategy σ^a satisfies equation 3.28 note that Lemma 3.4 states that any steady-state equilibrium strategy profile $\{\sigma^a\}$ satisfies the following properties: (i) convexity of the support, (ii) disjoint interior of the supports for the strategies of sellers with different customer bases, (iii) sellers with larger customer bases always post equal or higher prices than sellers with lower customer bases and (iv) that the upper bound of the support of $\hat{\sigma}$ is 1. Therefore, in equilibrium cohort a's strategy σ^a satisfies conditions 3.28 and 3.27. Which implies that 5.4 holds.

Lemma 3.4 implies that $\lim_{a \uparrow \infty} p^a = 1$ with 5.4 and a value function v implies in an unique profile of price posting strategies $\{\sigma^a\}_a$. $\square$

5.2.8 Proof of Theorem 3.2

Proof. Suppose that v_{t+1} continuous, strictly increasing and strictly concave then the proof of existence and uniqueness of the equilibrium distribution of prices in period t is analogous to the proof in proposition 3.2.

It only remains to show that for any sequence $\{v_t\}_t$ converging to v if v_{t+1} is continuous, strictly increasing and strictly concave then v_t is also continuous, strictly increasing and strictly concave.

To show that v_t is continuous, strictly increasing and strictly concave note that

$$\begin{aligned} v_t(c) &= \max_{p\in A} \gamma(p, \sigma_t, c, \hat{c}) + \beta(1-\alpha)v_{t+1}(c_{t+1}(p, c)) \qquad (5.19) \\ &= \gamma(p(c), \sigma_t, c, \hat{c}) + \beta(1-\alpha)v_{t+1}(c_{t+1}(p(c), c)), \end{aligned}$$

where $p(c)$ is the value maximizing p, it exists because v_{t+1} and γ_t are continuous on c, clearly γ_t is continuous on c and γ_t is continuous on p. Since $c \in [0, \overline{c}]$, it can be shown that $v_t(c) \leq \frac{\bar{c}}{1-\beta(1-\alpha)}$ for all $c \in [0, \overline{c}]$ and $t \geq 1$. Hence, there exists an $\underline{p}$ such that $p(c) \in [\underline{p}, 1]$ for all $c \in [0, \overline{c}]$ and $t \geq 1$. Therefore by the maximum theorem v_t is continuous.

To see that v_t is strictly increasing first check that γ_t is strictly increasing on c hence if v_{t+1} is strictly increasing then v_t is strictly increasing. Since v is strictly increasing and $\lim v_t = v$, then there exists a T such that v_T is strictly increasing and by induction v_t is strictly increasing for all $t < T$.

To check that v_t is strictly concave note that v is bounded by $v(\overline{c})$ and $v_{t+k} \to v$ as $k \to \infty$. Let $\{p_k(c_t), c_k(c_t)\}_{k=t}^{\infty}$ be the value maximizing price posting policies and customer bases corresponding to each period $k \geq t$. Substituting 5.19 iteratively and that $\lim_{k\to\infty} \beta^k v(\overline{c}) = 0$ we arrive at:

$$\begin{aligned} v_t(c_t) &= \sum_{k=0}^{T} [\beta(1-\alpha)]^k \gamma(p_k(c_t), \hat{\sigma}_{t+k}, c_k(c_t), \hat{c}_{t+k}) + [\beta(1-\alpha)]^T v_{t+T}(c_{t+T}(c_t)), \\ &= \sum_{k=0}^{\infty} [\beta(1-\alpha)]^k \gamma(p_k(c_t), \hat{\sigma}_{t+k}, c_k(c_t), \hat{c}_{t+k}), \qquad (5.20) \end{aligned}$$

hence by analogous proof to Lemma 3.3, v_t is strictly concave. Hence, v_t is continuous, strictly increasing and strictly concave. □

5.2.9 Proof of Theorem 3.3

Proof. To show that the equilibrium is unique it suffices to show that T maps $\mathcal{F}\left[[0,\overline{c}]\times K^{\mathbb{N}}([0,\overline{c}])\right]$ into itself, that it is a contraction and that it's fixed point implies in a unique symmetric strategy profile.

Note that $\gamma(p,\hat{\sigma},c,\hat{c})$ is concave and strictly increasing on c, V is concave and strictly increasing on c and $c^{+}(p,c)$, equation 3.39 implies that $T(V)$ is strictly increasing and strictly concave on c. To show that T is a contraction it suffices to show that it satisfies Blackwell's sufficiency conditions for a contraction. This proof is analogous to the second step of the proof of proposition 3.2.

Finally, to show that the fixed point of T implies in a unique strategy profile note that for V fixed point of T, then V is strictly increasing and strictly concave on c. We can apply Lemma 3.5 which implies that a fixed point of T also implies in a unique strategy profile for each customer base profile. Which concludes the proof of Theorem 3.3. □

5.2.10 Proof Lemma 3.6

Proof. First note that the average age of the active sellers is always strictly increasing in t. Let $\hat{a}_t$ be the average age of sellers that are operating in the market in period t. It satisfies:

$$\hat{a}_t = \frac{\sum_{k=1}^{t} kn^k}{\sum_{k=1}^{t} n^k}. \tag{5.21}$$

Clearly, equation 5.21 is strictly increasing in t. The fraction of the population of sellers who are active in the market, $\sum_{k=1}^{t} n^k$ is also strictly increasing in k. Hence, if the average customer base of active sellers is increasing then the average number of sellers that buyers know is also increasing.

In symmetric equilibrium the customer bases of sellers are increasing with the seller's tenure in the market, however the state of the market is also changing. I must show that a seller of age $a \in \{1,\ldots,t\}$ will have an equal or larger customer base in period $t+1$ compared to period t to show that the average customer base of the sellers is increasing in t.

The quantity sold by a seller of age a in period t,

$$\begin{aligned} q_t^a &= c_t^a \exp\left[-\hat{c}_t\left(\overline{\omega}_t^{a-1} + (1/2)\omega_t^a\right)\right] \\ &= \exp\left[-\left(\sum_{k=1}^{a-1} c_t^k n^k + \frac{1}{2}c_t^a n^a\right)\right], \end{aligned}$$

note that $c_t^1 = \underline{c}$ for all $t \geq 1$, hence $q_t^1 = \underline{c}\exp(-\frac{1}{2}\underline{c}n^1)$ is independent in $t \geq 1$, which implies that c_t^2 is independent for $t \geq 2$, hence

$$q_t^2 = \underline{c}\exp\left[-\left(c_t^1 n^1 + \frac{1}{2}c_t^2 n^2\right)\right]$$

is independent for $t \geq 2$, therefore c_t^3 is the same for all $t \geq 3$, following this reasoning q_t^a is independent for $t \geq a$. Therefore, $\hat{c}_t$ is strictly increasing in t. □

5.2.11 Proof of Theorem 3.4

Proof. Suppose that $AM_t \leq AM_{t+1}$ for some t. Lemma 3.6 implies that $c_t < c_{t+1}$. The facts that $c_t < c_{t+1}$ and the definition of average markups 3.40 implies that the average distribution of posted prices $\hat{\sigma}_{t+1}$ is not first order stochastically dominated by $\hat{\sigma}_t$. Hence, there exists a $p \in [\underline{p}_t, \overline{p}_t]$ such that $\hat{\sigma}_t(p) > \hat{\sigma}_{t+1}(p)$.

Note that proposition 3.6 implies that that monopoly profits in $t+1$ will be strictly lower than in t. Therefore, it's easy to see that in the equilibrium of a static version (as in Burdett and Judd (1983)) of the pricing game that $\hat{\sigma}_t$ is first order stochastically dominated by $\hat{\sigma}_{t+1}$. If $\hat{\sigma}_t(p) > \hat{\sigma}_{t+1}(p)$ for some $p \in [\underline{p}_t, \overline{p}_t]$, then as the static losses from posting lower prices are larger, higher dynamic gains are needed to satisfy the equal profit condition therefore the dynamic gains of posting lower prices must be higher in t than in $t+1$. Additionally, note that $\hat{c}_{t+1} > \hat{c}_t$ and equation 3.4 implies that the increase in customer base growth of a seller with customer base c from being in period t posting the lower bound of the support of $\hat{\sigma}_t$, $\underline{p}_t$ vis the upper bound of the support, $\overline{p}_t$ is smaller than the gain in customer base in period $t+1$ from posting $\underline{p}_{t+1}$ vis $\overline{p}_{t+1}$, that is

$$\left[c_{t+2}(\underline{p}_{t+1}, c) - c_{t+2}(\overline{p}_{t+1}, c)\right] > \left[c_{t+1}(\underline{p}_t, c) - c_{t+1}(\overline{p}_t, c)\right]. \tag{5.22}$$

Equation 5.22 and the fact that the dynamic gains of posting lower prices must be higher in t than in $t+1$, implies that the expected discounted profit stream per customer of a seller in equilibrium must be strictly decreasing from t to $t+1$. From the perspective of period t and $t+1$ future markups are the same for periods after $t+1$, which in turn implies that AM_{t+1} must be substantially higher than AM_{t+2} to incentivize sellers to post substantially lower prices in t than in $t+1$. To support such substantial higher price posting in $t+1$ than in $t+2$ in turn implies that AM_{t+3} must be smaller than AM_{t+2} by also a substantial degree. Following this reasoning, $AM_q - AM_{q+1} > \epsilon$ for some $\epsilon > 0$ for all $q > t$. Therefore, the average markup must diverge to minus infinity to support an increase in average markups from $t+1$ to $t+2$ in equilibrium, which contradicts the fact that the non-steady-state equilibrium converges to the steady-state equilibrium (which occurs by construction). $\square$

5.2.12 Proof of Theorem 3.5

Proof. If the buyer knows at least one seller the buyer is going to purchase the good. Thus, the realized surplus in the market in a given period is just 1 minus the population of buyers who do not know a single seller, π_t^0. Therefore, an efficient profile of sales among sellers is such that it minimizes π_t^0, which implies that it maximizes the average size of the seller's customer bases in each period.

Therefore, given a profile of customer bases $\{c_t^a\}_a$ in period t, efficiency in that period is obtained when the profile of sales quantities $\{q_t^a\}_a$ maximizes $\sum_{a=1}^{t+1} n^a c_{t+1}^a$, where c_{t+1}^a satisfies 3.35.

Note that sellers with smaller customer bases have greater potential for growth than sellers with larger customer bases therefore the efficient distribution of sales among sellers is such that buyers will always purchase from the smallest seller that they know as long as the marginal increase in that seller's customer base is higher than the marginal increase in the larger seller's customer base.

Note that in equilibrium between any two sellers, the seller with smaller customer base will post lower prices than the seller with the larger customer base with probability 1 and in equilibrium smaller sellers will always be smaller than larger sellers (as $c_t^a < c_t^{a+1}, \forall a \in \{1, \ldots, t-1\}, \forall t \geq 2$). Thus, the marginal increase in their next period customer bases to equilibrium sales is always strictly higher than for larger sellers which implies that the equilibrium allocation is efficient. $\square$

5.2.13 Proof of Proposition 3.3

Proof. Let $\overline{\omega}_{n,a}$ be the cumulative customer share of seller cohorts up to age a in the steady-state equilibrium corresponding to seller exit rate α_n and upper bound on customer base $\overline{c}_n$.

Note that $\lim_n \overline{c}_n = \infty$ implies that in steady-state equilibrium

$$\lim_{n\to\infty} \lim_{a\to\infty} c_n^a = \infty. \tag{5.23}$$

Let $\hat{a}_n$ be the average seller age at n and $\hat{c}_n$ be the average customer base of a seller at n. I want to show that

$$\lim_{n\to\infty} \hat{c}_n = \infty. \tag{5.24}$$

As this condition implies that $\lim_{n\to\infty} \pi_n^0 = 0$.

Let $\epsilon > 0$ from equation 5.23 it's possible to pick a sequence $\{a_n\}$ such that $c_n^{a_n} > \epsilon$.

Note that for $\zeta \in (0,1)$ there is a sequence $\{N_n\}$ such that $\sum_{a\geq N_n} \alpha_n(1-\alpha_n)^n > (1-\zeta)$ for all n and is such that $N_n \to \infty$.

I want to show that for $\epsilon > 0$ and $\zeta \in (0,1)$, $\exists\{a_n(\epsilon)\}$ and $\{N_n(\zeta)\}$ such that $\exists K \in \mathbb{N}$ such that $\forall n \geq K$,

$$a_n \leq N_n.$$

If that's true then it's easy to see that 5.24 holds.

To show that 5.24 holds, construct a sequence $\{a_n\}$ such that for each $n \geq 1$,

$$a_n = \min\{a : c_n^a \geq \epsilon\},$$

such a_n exists since c_n^a is strictly increasing and converges to ∞, and a sequence

$\{N_n\}$ such that for each $n \geq 1$,

$$N_n = \min\{N : \sum_{a \geq N_n} \alpha_n(1-\alpha_n)^n > (1-\zeta)\}.$$

It's easy to see that $\{N_n\}$ is a non-decreasing sequence such that $\lim_n N_n = \infty$ and $\lim_n a_n \leq \infty$.

Suppose that π_n^0 doesn't converge to 0. WLOG assume that $\lim \pi_n^0 = \overline{\pi} > 0$. Note that π_n^0 bounded below by $\overline{\pi}$ implies that the probability of sale to a customer of seller of age a will be at least as high as

$$\sum_{k=0} \pi_n^k [1 - \overline{\omega}_{n,a-1}]^k > \pi_n^0 \geq \overline{\pi}.$$

Since the probability of a sale is bounded below by $\overline{\pi}$ it implies that for $\epsilon < \overline{c}$, there exists a $\overline{a}$ such that $\forall n, a_n \leq \overline{a}$, and, as N_n converges to infinity, that implies that $\exists K \in \mathbb{N}$ such that $\forall n \geq K$,

$$a_n \leq N_n.$$

Hence, condition 5.24 implies that π_n^0 converges to 0. A contradiction. □

5.2.14 Proof of Theorem 3.6

Proof. The proof that the fraction of buyers who do not purchase the good from any seller converges to zero follows trivially from $\lim_k \hat{c}_k = \infty$ and $\pi^0(.)$ being a Poisson probability mass function.

In a sequence of steady-state equilibrium characterized by the sequence of customer base profiles $\{\{c_k^a\}_a\}_k$ we know that very old sellers with a large enough will have customer base close enough to $\overline{c}_k$ will find it optimal to post price arbitrarily close to 1, in which case the expected profits would be arbitrarily close to $\overline{c}_k \pi_k^0$ as $\hat{\sigma}^k(1) = 1$, while posting $p \in [\pi_k^0, 1)$, expected profits would be strictly smaller than $p\overline{c}_k$, since the equilibrium condition implies that for a seller to find it optimal to post a price outside the lower bound of the support of $\hat{\sigma}$ the additional gains in future expected profits from customer base accumulation from posting lower prices must be compensated by higher present profits from higher prices. This implies that $\hat{\sigma}^k(p) > 0$ and therefore that $[\pi_k^0, 1) \subset [\underline{p}^k, \overline{p}^k] = \text{supp}(\hat{\sigma}^k)$.

Fix k and $p \in [\pi_k^0, 1)$. Note that $\hat{c}^k \to \infty$ as $k \to \infty$ hence

$$\lim_k P_k(1, \hat{\sigma}^k) = 0.$$

Therefore expected profit margin per customer for seller of age a from posting a price $p \in [\pi_k^0, 1)$ which are strictly lower than $\xi(1, \hat{\sigma}^k, \hat{c}_k)$ for a will also converge to zero.

Note also that for $\hat{c}^k \to \infty$ as $k \to \infty$ implies that any $p \in (\pi_k^0, 1)$, $\hat{\sigma}^k(p) > 0$ therefore

$$\lim_k \exp\left[-\hat{c}_k \hat{\sigma}^k(p)\right] = 0.$$

Finally, that $\hat{c}^k \to \infty$ as $k \to \infty$ implies that $\lim_k \pi_k^0 = 0$ and it follows that for any $p \in (0, 1)$

$$\lim_k \exp\left[-\hat{c}_k \hat{\sigma}^k(p)\right] = 0,$$

hence the distribution for prices of transactions converges in probability to 0. □

5.2.15 Proof of Theorem 3.7

Proof. Theorem 3.6 implies that as $\hat{c} \to \infty$, the equilibrium expected price for transactions converges to 0. Hence, if $\hat{c} \to \infty$ then $v(\underline{c}) \to 0$.

For any entering population of sellers an exit rate $\alpha \in (0, 1)$ implies that

$$v(\underline{c}) \le v(\overline{c}) \le \frac{\overline{c}}{\beta(1-\alpha)},$$

therefore, $v(\underline{c}) \in \left[0, \frac{\overline{c}}{\beta(1-\alpha)}\right]$.

Let $n_s = 0$ and sequence of customer bases $\{c_t\}_t$ such that $c_0 = \underline{c}$ and $c_{t+1} = c_t + \Phi(c_t, c_t, 0)$ for each $t \ge 0$. Then, with zero seller density the value of entry is just posting the monopoly price and selling to all customers,

$$\overline{v}_{entry} = \sum_{t=0}^{\infty} (\beta(1-\alpha))^t c_t.$$

Clearly, $\overline{v}_{entry}$ is the upper bound of the value at entry.

From Lemma 3.4, we know that the expected price for equilibrium transactions is monotone decreasing on $\hat{c}$ and it is easy to see that in the steady-state equilibrium $\hat{c}$ is strictly increasing on n_s. Therefore, if there exists a $n_s \in \mathbb{R}_+$ such that $v(\underline{c}) = e > 0$, then it's unique. If $e \geq \overline{v}_{entry}$ then $n_s = 0$. Finally, theorem 3.6 implies that as $n_s \to \infty$, $v(\underline{c}) \to 0$, therefore if $e > 0$ then there exists $n_s > 0$ such that $v(\underline{c}) < e$. $\square$

5.3 Appendix to Chapter 4

5.3.1 Proof of Proposition 4.1

Proof. I will show that the prices specified 4.18 are a competitive equilibrium, and are the only equilibrium if $q \neq 1$.

Sufficiency: Let $q < 1$, if $p^W(l) = 1$ for all l. Then demand for each variety l will be $[0, 1]$, as consumer i will be almost always indifferent between purchasing variety i or not (randomizing between the two) so demand for each variety is $[0, 1]$, while all producers will supply $q \in (0, 1)$ as prices are strictly greater than zero. Hence, its an equilibrium.

Let $q > 1$, if $p^W(l) = 0$ for all l then demand for each variety l will be 1, as almost every consumer $i \in [0, 1]$ will purchase a unit of variety $i \in [0, 1]$, while supply is given by $S(l) = [0, q]$. Hence, it is an equilibrium.

Necessity: Consider the case $q < 1$, I need to show that any pricing profile different from $p^W(l) = 1$ for all $l \in [0, 1]$ is inconsistent with equilibrium.

Consider a profile of prices p^W with $p^W(l) < 1$ for some subset of varieties $Z \subset [0, 1]$ and $p^W(j) = 1, \forall j \notin Z$. Then, continuity of v implies that for some consumers i near Z will find varieties in Z that yield higher utility than not-trading, therefore x_i is such that $\sum_{j \in Z} x_i(j) = 1$, then it is easy to see that $X(l) \geq 1$ for at least some $l \in Z$, a contradiction with equilibrium.

Consider the case where $q > 1$ and a pricing profile such that $p^W(l) > 0$ for subset of varieties $Z \subset [0, 1]$ and $p^W(j) = 0$. As prices are strictly greater than 0, demand for some varieties in Z are, at most, 1, as consumers will not want to purchase more than one unit of the consumption good and if multiple consumers purchase the same good other varieties in Z will have demand less than 1, but supply of any variety in Z will be equal to $q > 1$, a contradiction. $\square$

5.3.2 Proof of Theorem 4.1

Proof. Part 1: Convergence in prices

Let $G_n(x) = \int_0^x a_n^j / a_n$, and first note that $a_n^j \to \infty$ for every $j \in [0, 1]$ and $\max\{a_n^j / a_n^{j'} : j, j' \in [0, 1]\} < A$ implies that $a_n = \int a_n^j$ converges to infinity and that the density of G_n in any subset point of $[0, 1]$ does not converge to zero. As $a_n \to \infty$ and G_n converges to a strictly strictly increasing cumulative distribution function, the expected a_n^i converges to infinity and the expected valuation for an arbitrary consumer i's preferred variety among producers in A_n^i converges to 1 as $n \to \infty$. That is, as consumers becomes aware of more varieties, the closer the expected valuation of the preferred variety is to 1. Second, note that in equilibrium, $s^*(j, \boldsymbol{a}_n) \in [0, 1), \forall j, \forall n$, as negative pricing is never consistent with equilibrium: it is easy to see that a producer post negative prices then its profits will be strictly negative since $\pi^0(a) > 0$ for any $a \in \mathbb{R}_+$ and pricing above all consumer's reservation price yields zero profits.

The proof proceeds by contradiction. I first consider the case where $s^*(j, \boldsymbol{a}_n)$ converges to a symmetric pricing profile, that is $\lim_{n\to\infty} s^*(j, \boldsymbol{a}_n) = p^*$ for all varieties j. In an equilibrium where all producers post the same price, consumers will purchase their preferred variety. Therefore, the reservation price of the customers of a producer j converges in probability to 1. Which implies that the population of consumers who purchases some variety also converges to 1 if $\lim_{a\to\infty} s^*(j, \boldsymbol{a}_n) < 1$. Which implies that the average quantity sold by producers converges to 1 in such equilibrium if the equilibrium price converges (taking a subsequence if necessary) to a number that is less than 1.

Suppose that $q > 1$ and suppose that $p^* > 0$. Note that the average quantity sold in equilibrium by producers converges to 1 which means that for some producer j, the quantity sold converges to a quantity equal or smaller than 1 as $a \to \infty$, which is below capacity. If producer j consider reducing price posted

p^* by some $\epsilon > 0$ equation 4.15 implies that quantity sold is given by

$$\begin{aligned}
&a_n^j P_{n,j}(p^* - \epsilon, p^*) \\
&= a_n^j \int_0^1 I_{\{v(i,j) \geq p^* - \epsilon\}}(i) \exp\left[-a_n \left(1 - \int_0^1 I_{\{v(i,j) \geq v(i,j') - \epsilon\}}(i,j') dG_n(j')\right)\right] di \qquad (5.25) \\
&= a_n^j \int_0^1 I_{\{v(i,j) \geq p^* - \epsilon\}}(i) \exp\left[-a_n \left(1 - \left(\int_0^1 \left[I_{\{v(i,j) \geq v(i,j')\}}(i,j') + I_{\{v(i,j) \in [v(i,j') - \epsilon, v(i,j'))\}}(i,j')\right] dG_n(j')\right)\right)\right] di \qquad (5.26) \\
&\geq a_n^j \int_0^1 I_{\{v(i,j) \geq p^* - \epsilon\}}(i) \exp\left[-a_n \left(1 - \left(\int_0^1 I_{\{v(i,j) \geq v(i,j')\}}(i,j') dG_n(j') + \frac{1}{A} \int_0^1 I_{\{v(i,j) \in [v(i,j') - \epsilon, v(i,j'))\}}(i,j') dj'\right)\right)\right] di, \qquad (5.27)
\end{aligned}$$

where $P_{n,j}$ has the n subscript as it changes for each $n \in \mathbb{N}$, and as $a_n^j/a_n > 1/A, \forall n$, the density of G_n is bounded below by $1/A > 0$. As $p^* - \epsilon < 1$ the continuity of v_i implies that $v(i,j) > p^* - \epsilon$ for all i in some neighborhood of j and that the set $\{j' \in [0,1] : v(i,j) \in [v(i,j') - \epsilon, v(i,j'))\}$ has strictly positive measure if $i \neq j$, together with G_n having density higher than $1/A$ implies that $P_{n,j}(p^* - \epsilon, p^*) > P_{n,j}(p^*, p^*)$. In words, the quantity sold increases as there is a population of buyers in j's customer base who prefer some other variety j' among the a different varieties they know but the difference in reservation price between those other varieties and j is smaller than ϵ. Note that taking n to infinity, as implied by expression 5.27, the marginal log increase in quantity sold will increase to infinity thanks to the power of "compound interest". Therefore, for n large enough producer j will have incentives to undercut the competition, as the increase in sales will be larger than the decrease in profit margin, a contradiction with equilibrium. Therefore $s^*(j, \boldsymbol{a}_n)$ does not converge to a price higher than 0, hence, if the Nash equilibrium strategy profile converges to a symmetric pricing profile it will converge to the price of 0 for every variety.

Suppose $q < 1$ and suppose that $p^* < 1$. Note as the average quantity sold in equilibrium by producers converges to 1 in symmetric equilibrium if $p^* < 1$ then the demand for some producer j posting $s^*(j, \boldsymbol{a}_n)$ converges to a quantity greater or equal than 1 as $n \to \infty$, which is above capacity. Continuity of v implies demand (that is $a_a^j P_{n,j}(p, s, a)$) is continuous, hence, producers can increase profits by increasing prices a contradiction with $s^*(j, \boldsymbol{a}_n)$ being Nash equilibrium. Hence, in equilibrium, as a converges to infinity, in the limit the producers cannot post a price below 1. Since $p^* \in [0,1]$, therefore, if the Nash equilibrium strategy profile converges to a symmetric pricing profile it will converge to the price of 1 for every variety.

It remains to show that the case where $s^*(j, a)$ does not converge to a symmetric pricing profile almost everywhere is not consistent with equilibrium. Note that if $s^*(j, a)$ converges almost everywhere to a symmetric pricing profile the arguments above still applies.

Case 1: Suppose that $q > 1$. Consider the case where for a positive measure of varieties $Z \subset [0, 1]$, $s^*(j, \boldsymbol{a}_n)$ does not converge to 0, $\forall j \in Z$. Then (taking subsequences if necessary) $\{s^*(j, \boldsymbol{a}_n) : j \in Z\}$ converges to a profile of prices $p^*(j) > 0, \forall j \in Z$. Since Z has positive measure there is a j in the interior of Z such that $\inf\{s^*(B_\alpha(j))\} > 0$ for some $\alpha > 0$, then by analogous argument as in the symmetric pricing case (from equation 5.26), a small undercut by producer j of the price $\underline{p} = \inf\{p^*(B_\alpha(j))\}$ will yield a marginal log increase in quantity sold that increases to infinity as $n \to \infty$. Which implies that as $n \to \infty$, posting a $p < \underline{p} \leq p^*(j)$ yields higher profits than posting $p^*(j)$. A contradiction with $s^*(j, \boldsymbol{a}_n)$ being Nash equilibrium for every n.

Case 2: Suppose that $q < 1$. Consider the case where for a positive measure of varieties $Z \subset [0, 1]$, $s^*(j, \boldsymbol{a}_n)$ does not converge to 1, $\forall j \in Z$. Then (taking subsequences if necessary) $\{s^*(j, \boldsymbol{a}_n) : j \in Z\}$ converges to a profile of prices $p^*(j) < 0, \forall j \in Z$. Since Z has positive measure there is is a j in the interior of Z such that $\sup\{p^*(B_\alpha(j))\} < 1$. It is easy to see that as $a_n^j/a_n > 1/A, \forall n$ for producers j in $B_\alpha(j)$, demand for their output will exceed capacity as $n \to \infty$. Continuity of v implies demand (that is $a_a^j P_{n,j}(p, s, a)$) is continuous, hence, producers in $B_\alpha(j)$ can increase profits by increasing prices a contradiction with $s^*(j, \boldsymbol{a}_n)$ being Nash equilibrium.

Part 2: Convergence in allocation

To see that the equilibrium allocation converges for almost every variety to the competitive equilibrium allocation first consider producers. As equilibrium output converges to 1 almost everywhere if $q > 1$, and to q if $q < 1$, then equilibrium output trivially converges almost everywhere to Y.

Converge in probability in the consumption profile follows from the fact that as $a_n^j \to \infty$ for all j as $n \to \infty$, then for any subset of varieties Z with positive measure, the probability a random consumer i is aware of a variety in this subset converges to 1. Consider a consumer $i \in [0, 1]$, continuity of v and it being single-peaked in $v(i, i)$ implies that for a $\epsilon > 0$ small enough, an open ϵ-neighborhood B_ϵ of i is such that $v(i, x) > v(i, z), \forall x \in B_\epsilon(i), \forall z \in B_\epsilon(i)$.

Therefore, as $n \to \infty$ then the probability that i's preferred variety in A^i is in B_ϵ converges to 1. As prices converge to the same price almost everywhere, the probability that individual consumer's chosen variety is in B_ϵ or not-trading converges to 1. Therefore, the equilibrium consumption profile converges in probability to a competitive equilibrium consumption bundle. □

5.3.3 Proof of Theorem 4.2

Proof. First note that if $p > \overline{p}$ then profits will be always lower than at the monopoly price $\overline{p}$, if $p < 0$ then profits will be strictly negative as $\pi^0(a) > 0$. Then any candidate equilibrium price is in $p \in [0, \overline{p}]$.

Let Φ_j be the best response correspondence of producer j, it is given by

$$\Phi_j(p') = \arg\max_{p \in [0,\overline{p}]} \Pi_j(p, p'),$$

note that conditions 1 and 2 imply, given equation 4.15 that $\Phi_j(p') = \Phi_{j'}(p'), \forall j, j' \in [0,1]$, so we can write $\Phi(p')$ as the best response of all producers to p'.

A symmetric equilibrium is a fixed point of $\Phi : [0, \overline{p}] \to [0, \overline{p}]$. Note that $[0, \overline{p}]$ is a compact set, since v is continuous then Λ_{ji} and $\Lambda_{jij'}$ are both continuous which implies that ξ_j is continuous, hence by the Maximum Theorem, $\Phi(.)$ is a non-empty upper-hemicontinuous correspondence.

As v is continuous and piecewise concave, it is easy to show that Λ_{ji} and $\Lambda_{jij'}$ are concave on X, X' such that $\Lambda_{ji}(x) > 0, \forall x \in X$ and $\Lambda_{jij'}(x) > 0, \forall x \in X'$ respectively. As

$$\int_0^1 I_{\{v(i,j) \geq x\}}(i) di = \Lambda_{ji}(x),$$

$$\int_0^1 I_{\{v(i,j) - v(i,j') \geq x\}}(i, j') dj' = \Lambda_{jij'}(x),$$

if $P_j^n(p, p') > 0$ for all $p, p' \in (0, \overline{p})$, and $n > 0$, then P_j^n is concave on $[0, \overline{p}]$, and therefore P_j is concave on $p \in [0, \overline{p}]$.

Suppose $p_0, p_1 \in \Phi(p')$, with $p_1 > p_0$, let $\tilde{p} = \alpha p_0 + (1-\alpha)p_1$, then

$$\begin{aligned}
\tilde{p}P_j(\tilde{p},p') &= \left[\alpha p_0 + (1-\alpha)p_1\right] P_j(\tilde{p},p') \\
&= \alpha p_0 P_j(\tilde{p},p') + (1-\alpha)p_1 P_j(\tilde{p},p') \\
&> \alpha p_0 P_j(p_0,p') + (1-\alpha)p_1 P_j(p_1,p') \qquad (5.28) \\
&= p_0 P_j(p_0,p'),
\end{aligned}$$

where inequality 5.28 is a consequence that $p_1 > p_0$, that $P_j(p_0,p') \geq P_j(p_1,p')$, and

$$P_j(\tilde{p},p') \geq \alpha P_j(p_0,p') + (1-\alpha)P_j(p_1,p').$$

Therefore $\Pi_j(\tilde{p},p') > \Pi_j(p_0,p') = \Pi_j(p_1,p')$, a contradiction with $p_0, p_1 \in \Phi(p')$, therefore $\Phi(p')$ is single valued. Hence Φ is a continuous function. Since Φ is a continuous function that maps a compact set into itself by Brouwer's fixed-point theorem it has a fixed point which is the symmetric equilibrium. □

5.3.4 Proof of Theorem 4.3

Proof. Case 1: The capacity constraint q is not binding in equilibrium.

Let $\Phi : [0,\overline{p}] \rightrightarrows [0,\overline{p}]$ be the best response correspondence when the capacity constraint is not binding, it satisfies

$$\Phi(p) = \arg\max_{p\in[0,\overline{p}]} pP(p,p'). \qquad (5.29)$$

Clearly, a price $p^* = \Phi(p^*)$ is a symmetric Nash equilibrium.

To characterize Φ note that equation 4.21 implies that the derivative of P^n at the point $p = p'$ does not exist but there are left and right derivatives and they satisfy

$$\frac{\partial_- P^n(p,p')}{\partial p}\,|_{p=p'} = 0, \qquad (5.30)$$

$$\frac{\partial_+ P^n(p,p')}{\partial p}\,|_{p=p'} = -\frac{2}{\lambda}. \qquad (5.31)$$

Note that the left derivative is larger than the right derivative. This implies that the gain in probability of sale from a marginal decrease in price at p' is lower than the loss from a marginal increase in price. In equilibrium $p = p'$, the inequality 5.30 implies that the marginal increase in profits from decreasing price are always negative with $p = p'$ so in the equilibrium first order condition that holds is the right derivative.

As P^n is continuously left differentiable then ξ is continuously left differentiable. Note that an interior solution p^* to 5.29 satisfies

$$\frac{\partial_+ \Pi(p, p')}{\partial p} |_{p=p^*} = 0, \tag{5.32}$$

as the right hand side derivative always exists for ξ and is smaller than the left hand side if $p = p'$. Which implies that if the return in profits from decreasing the posted price when it is above the competitors' price is zero then the returns from decreasing the price below the competitors' is strictly negative.

Solving 5.32 for p^* yields the interior solution

$$p^* = \frac{\lambda}{2a} \tag{5.33}$$

therefore, Φ is single valued for interior best responses.

It remains to rule out corner solutions that might occur together with an interior solution, which occurs if a is not too large or small. Posting 0 yields zero profits while posting $p \in (0, \overline{p})$ always yields strictly positive profits, so 0 cannot be a best response. It only remains to find conditions such that the monopoly price $\overline{p}$ is not a best response when equation 5.32 is satisfied for $p' = p^*$. In the case where

$$\frac{\partial_- \Pi(p, \overline{p})}{\partial p} |_{p=\overline{p}} \geq 0,$$

then the equilibrium is clearly the monopoly price, which occurs if a is relatively small. The case where

$$\frac{\partial_- \Pi(p, \overline{p})}{\partial p} |_{p=\overline{p}} < 0$$

can be problematic since it implies that the monopoly price is not an equilibrium but it is not necessarily true that a price p^* satisfying equation 5.32 with $p' = p^*$ is also an equilibrium.

To rule out this case consider a candidate interior equilibrium $p^* \in (0, \overline{p})$ that satisfies 5.32, consider the profits from posting p^* compared to profits from posting the monopoly price

$$\begin{aligned}\Pi(p^*,p^*) &= ap^*P(p^*,p^*)\\ &= \frac{\lambda}{2a}[1-\exp(-a)],\\ \Pi(\overline{p},p^*) &= a\overline{p}P(\overline{p},p^*)\\ &= \begin{cases}\left(1-\frac{\lambda}{2}\right)\exp(-2a) & \text{if } \overline{p}-p^* > \lambda/2\\ \left(1-\frac{\lambda}{2}\right)[\exp(2\epsilon a)-\exp(-a)]. & \text{if } \overline{p}-p^* \leq \lambda/2\end{cases}\end{aligned}$$

If $\overline{p}-p^* \leq \lambda/2$ then the marginal gain in lowering prices is higher than posting the monopoly price $\overline{p}$, then trivially profits from posting p^* are higher than $\overline{p}$. Therefore if p^* is close enough to $\overline{p}$ it is an equilibrium.

If $\overline{p}-p^* > \lambda/2$ then a sufficient condition for p^* to be a Nash equilibrium is that a that satisfies

$$\frac{\lambda}{2a}[1-\exp(-a)] > \left(1-\frac{\lambda}{2}\right)\exp(-2a). \tag{5.34}$$

Let $\overline{a}$ that satisfies

$$\frac{\lambda}{2\overline{a}}[1-\exp(-\overline{a})] = \left(1-\frac{\lambda}{2}\right)\exp(-2\overline{a}),$$

for $a > \overline{a}$ a unique Nash equilibrium exists given by 5.33.

Let $\underline{a}$ such that $\overline{p}-p^* = \lambda/2$, solving for $\underline{a}$,

$$\underline{a} = \frac{\lambda}{2(1-\lambda)}.$$

Then for $a < \underline{a}$ the symmetric Nash equilibrium exists and is unique, given by

$$p^* = \min\left\{\overline{p}, \frac{\lambda}{2a}\right\}.$$

Case 2: The capacity constraint q is binding.

As the quantity sold by producers must be equal or lower than q in equilibrium, the producers price above the monopoly price to discourage potential customers from purchasing.

To compute demand for $p^* > \overline{p}$, then the quantity sold by a producer in equilibrium is given by

$$aP(p^*, p^*) = P(v(i,j) - p^* \geq 0 | v(i,j) \geq v(i,j'), \forall j' \in A^i)[1 - \pi^0(a)], \quad (5.35)$$

where $P(v(i,j) - p^* \geq 0 | v(i,j) \geq v(i,j'), \forall j' \in A^i)$ is the probability that j's contract is preferred by i over not trading conditional on it being preferred over all other contracts that i is aware of. Let $\overline{\delta}(p)$ be the distance between i and j that makes i indifferent between trading with j and not trading [1].

Let F^n be a cumulative distribution function that describes the probability that the valuation of the highest valuation consumer is above x conditional on the consumers being aware of n different producers. As producers are distributed uniformly F^n is given by

$$F^n(x) = 1 - \left[1 - \left(\frac{1-x}{\lambda}\right)\right]^n.$$

Then,

$$\begin{aligned} P(v(i,j) - p^* \geq 0 | v(i,j) \geq v(i,j'), \forall j' \in A^i) &= \sum_{n=0}^{\infty} \pi^n(a) F^{n+1}(p^*) \\ &= 1 - \exp\left(-\frac{1-p^*}{\lambda} a\right). \end{aligned}$$

In equilibrium with binding capacity constraint the equilibrium price satisfies p^*

$$aP(p^*, p^*) = q,$$

solving for p^* yields

$$p^* = 1 - \frac{\lambda}{a}\left(\log(1 - \exp(-a)) - \log(1 - \exp(-a) - q)\right).$$

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[1] Given by $\overline{\delta}(p) = \frac{1-p}{\lambda}$

□

5.3.5 Proof of Theorem 3.5

Proof. Feasibility:

Physical feasibility and zero net balance are properties of equilibrium.

To see that the F^* is incentive compatible note that reported consumer type θ implies that the consumer consumes the bundle $(j, -s^*(j))$, of variety j and transfers $-s^*(j)$, that maximizes utility among bundles in $A(\theta) \times \{-s^*(j) : j \in A(\theta)\}$, so the consumer has no incentive to lie.

Efficiency:

There is only one possible deviation from physical pareto efficiency in the equilibrium: if the consumer i does not purchase his or her preferred variety in A^i (as the surplus is from a transaction is $v(i,j)$). Otherwise aggregate surplus is automatically maximized.

Consider the situation where aggregate surplus is not maximized in equilibrium. That means there is some set of consumers I of positive measure such that for $i \in I$, i's preferred variety in A^i is j but i does not purchase from producer j, whose capacity constraint is not binding under F. Therefore either the consumer does not purchase any variety ($v(i,j) < s^*(j)$) or the consumer purchases from another competitor ($v(i,j) - s^*(j) < v(i,j') - s^*(j')$). In the first case consider a mechanism F' that implements trade between i and j, in this mechanism i receives a transfer which satisfies $-t(\theta(i)) < s^*(j)$, that implies that all other consumers who consume j's variety will also receive the same (higher) transfer in this mechanism, otherwise this mechanism would not satisfy incentive compatibility (described in equation 4.27).

In the second case consider a mechanism F' that makes i trade with j instead of j', then transfers t' to i trading with j should satisfy $t' \geq v(i,j') - v(i,j) - s^*(j')$, so $-t' < s^*(j)$, that implies that all other consumers who consume j's variety will also receive the same (higher) transfer in the mechanism F', otherwise F' would not satisfy incentive compatibility.

As F implements the allocation that follows from profit maximizing price posting strategies, in both cases, the net balance constraint implies either that the profits of the set of such producers j decreases or that some other set of decision makers will have to pay for these higher transfers to j's customers. So a feasible mechanism F' that implements trade between i and j does not pareto dominate F. $\square$