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ON THE COMPUTATIONAL COMPLEXITY OF CODES IN GRAPHS*

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Abstract. This paper links to continuing research of the first author on codes in graphs [6-9]. Here codes are studied from the point of view of their computational complexity. It is shown that the problem of perfect code recognition is NP-complete even when restricted to $k-$regular graphs ($k \geq 4$) or to $3-$regular planar graphs. On the other hand in the case of trees and graphs of bounded tree-width an optimal $\Theta(n)$ algorithm is developed. Some optimization problems are also investigated.

I. Introduction. The theory of self-correcting codes belongs to thoroughly investigated parts of applied combinatorics. Special attention was paid to the most effective codes, so-called perfect codes. Such codes were shown to be fairly rare, namely in the case of the classical Hamming metrics [1,13]. The classical concept of perfect codes was generalized by Biggs [2] to perfect codes in graphs. However, Biggs and others [5,11] studied only distance regular graphs for which a strong necessary condition for the existence of perfect codes was derived [2].

Perfect codes in general graphs (and their cartesian powers) were studied in [6,8,9]. Though one can easily construct general graphs containing perfect codes, still typical graphs do not contain perfect codes. For example for every fixed $p, 0 < p < 1$, the random graph $G_{n,p}$ almost surely does not contain a 1-perfect code [7]. In subsequent sections computational problems concerning codes in a variety of graph families will be discussed.

II. Background. Through-out this paper we shall use the following notation and conventions:

a) notation from graph theory is standard [3]. Especially $N_1(u) \overset{def}{=} \{v; d(u,v) \leq 1\}$.

b) due to space limitation, figures are preferred in the proof of Theorem 8. (All figures are listed in the Appendix). In this respect a mapping $\Phi_C : V(G) \longrightarrow \{\bullet, \otimes, \circ\}$, where

$$\Phi_C(u) = \begin{cases} \bullet & \text{if } u \in C \\ \otimes & \text{if } u \text{ is covered by } C \text{ and } u \notin C \\ \circ & \text{if } u \text{ is uncovered by } C \text{ in } G, \end{cases}$$

will be widely used. The reader is encouraged to follow this notation in his own pictures while going through the proofs. Also the labels from our figures are referred to in the text without stating it explicitlly.

c) NP-completeness terminology is that of [4].

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*Extended abstract

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All technical details will appear in a forthcoming full paper.

We start with some necessary definitions:

Let G be a graph (undirected, without loops and multiple edges). The set $C \subset V(G)$ is said to be a

$$\textbf{code} \Leftrightarrow (\forall u, v \in C)\ d(u,v) \geq 3\ ;$$
$$\textbf{perfect code} \Leftrightarrow (\forall u \in V(G))(\exists! c \in C)\ d(u,c) \leq 1.$$

Thus C is a code iff the sets $N_1(u), u \in C$, are pair-wise disjoint, while C is a perfect code iff these sets form a partition of $V(G)$.

Let C be a perfect code in $G - v$. Then the vertex v is called **uncovered** by C in G if $d(v, C) > 1$.

First we shall be interested in the following optimization decision problems :

1. PERFECT CODE (PC) :

INSTANCE : A graph G;

QUESTION : Does G contain a perfect code?;

1a. PCvC :

INSTANCE : A graph G, a specified vertex v;

QUESTION : Does G contain a perfect code C such that $v \in C$?;

1b. PCvNC :

INSTANCE : see **1a**;

QUESTION : Does G contain a perfect code C such that $v \notin C$?;

1c. PCvU :

INSTANCE : see **1a**;

QUESTION : Is there a perfect code C in $G - v$ such that v is uncovered by C in G?;

2. PERFECT CODE COMPLETION (PCC) :

INSTANCE : A graph G, non-negative integer k;

QUESTION : Is there a sequence of at most k changes (will be specified later) that transforms G into a graph having a perfect code?;

2a. PCC-VERTEX ADDITION (VA) :

INSTANCE : see **2**;

QUESTION : see **2** where change $\equiv$ vertex addition with some of its incident edges;

2b. PCC-VERTEX DELETION (VD) :

INSTANCE : see **2**;

QUESTION : see **2** where change $\equiv$ vertex deletion;

2c. PCC-EDGE ADDITION (EA) :

INSTANCE : see **2**;

QUESTION : see **2** where change $\equiv$ edge addition;

2d. PCC-EDGE DELETION (ED) :

INSTANCE : see **2**;

QUESTION : see **2** where change $\equiv$ edge deletion;

2e. PCC-MIXED (VDEAD) :

INSTANCE : see **2**;

QUESTION : see **PCC** where change $\equiv$ vertex addition and/or vertex deletion and/or edge addition and/or edge deletion;

3. DEFECT :

INSTANCE : A graph G, non-negative integer k;

QUESTION : Does it hold that $def(G) \overset{def}{=} \min\{j;$ there exists a code in G such that exactly j vertices are left uncovered$\} \leq k$?

Suppose that $VA(G)$, $EA(G)$, $VD(G)$, $ED(G)$, $VDEAD(G)$ denote the minimum number of changes that are required in corresponding computational problems **2a-2e**.

Now, we shall present several NP-completeness results for problems **1-3**. Their proofs originate in the following fundamental theorem :

THEOREM 1. *The problem* **PC** *is NP-complete in connected graphs.* □

The proof is postponed to section III. As it is customary with the NP-completeness proofs we omit the trivial verification of membership in the class NP. Our polynomial transformations start in the following preliminary assertion:

PROPOSITION. *The two following problems are NP-complete :*

(1) **kRkP** *: INSTANCE : Finite set of elements $X = \{x_1, x_2, ..., x_{kq}\}, (k \geq 3,$ q is a positive integer), and a collection $\mathcal{C}$ of k-element subsets of X such that each element of X appear in exactly k subsets.*

QUESTION : Is there a subcollection $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{C}'$ forms a partition of X ?

(2) **13p3S** *: INSTANCE : A formula in conjunctive normal form with the set of clauses C over the set of variables X such that :*

(i) $|c| = 3$ *for each clause c of C,*

(ii) *The bipartite graph $G = (C \cup X, E)$, where*

$E = \{\{x, c\}; either\ x \in c\ or\ \neg x \in c\}$, *is planar;*

QUESTION : Is there a satisfying truth assignment for C such that each clause in C has exactly one **true**-*literal ?*

Proof. By standard local replacement transformation technique from the well-known NP-complete problems **3DM, PLANAR 3-SAT**, c.f.[4]. □

THEOREM 2. *The problems* **PCvC, PCvNC, PCvU** *are NP-complete in connected graphs.*

Proof.

(1) **PC** $\propto$ **PCvU.**
Given a connected graph G, take an arbitrary vertex $u \in V(G)$ and construct G' as follows :

$$V(G') = V(G) \cup \{w, z, x, y\}, \quad \text{and}$$

$$E(G') = E(G) \cup \{\{w, z\}, \{x, z\}, \{u, z\}, \{y, x\}\}.$$

Then G contains a perfect code $\Leftrightarrow$ G' contains a perfect code uncovering the vertex w.

(2) **PCvU** $\propto$ **PCvC.**
Given a connected graph G, take an arbitrary vertex $u \in V(G)$ and construct G' as follows :

$$V(G') = V(G) \cup \{w\}; \quad E(G') = E(G) \cup \{\{u, w\}\}.$$

Then G' contains a perfect code C with $w \in C$ $\Leftrightarrow$ G contains a perfect code uncovering the vertex u.

(3) **PCvC** $\propto$ **PCvNC.**
...similar construction... $\square$

THEOREM 3. *$VA(G) \leq 1$ for all graphs G.*

Proof. If there is no perfect code in G then define

$$G' = (V(G) \cup \{u\}, E(G) \cup \{\{u, v\}; v \in V(G)\}),$$

where there exists the perfect code $C' = \{u\}$. $\square$

COROLLARY. *The problem* **VA** *is NP-complete for $k = 0$. On the other hand it becomes trivial for $k \geq 1$.* $\square$

THEOREM 4. $EA(G) = def(G)$.

Proof. Let C be a code in G such that $def(G)$ vertices are left uncovered by C. Each of these vertices can be joined by an edge with some $u \in C$, i.e. $def(G) \geq EA(G)$. On the other hand let G' be a graph arising from G by adding $EA(G)$ edges and let C' be a perfect code in G'. By deleting these $EA(G)$ edges the cardinality of the set of vertices covered by C' is diminished at most by $EA(G)$, i.e. $EA(G) \geq def(G)$. $\square$

THEOREM 5.

(i) *The problems* **VD, EA, ED, VDEAD** *are* NP*-complete in connected graphs for every fixed* $k \geq 0$;

(ii) *The problem* **DEFECT** *is* NP*-complete in connected graphs for every fixed* $k \geq 0$;

(iii) *There is no polynomial approximation algorithm for problems* **VD,EA,ED,VDEAD** *in connected graphs.*

Proof. Let G be a connected graph, $v \in V(G)$. Define G_k as follows

$$V(G_k) = V(G) \cup \{u_1, u_2\} \cup \bigcup_{i=1}^{k} \{r_i, s_i, x_i, y_i, z_i, t_i, w_i\},$$

$$E(G_k) = E(G) \cup \{\{v, u_1\}, \{v, u_2\}\} \cup$$

$$\cup \bigcup_{i=1}^{k} \{\{v, r_i\}, \{r_i, s_i\}, \{v, x_i\}, \{x_i, y_i\}, \{y_i, z_i\}, \{z_i, t_i\}, \{z_i, w_i\}\}.$$

Now for every $X \in \{VD, ED, EA, VDEAD\}$ we have $X(G_k) \geq k$ and $X(G_k) = k \Leftrightarrow G$ contains perfect code C with $v \in C$. Then part (i) follows from Theorem 2.

(ii) follows from Theorem 4.

As all before-mentioned problems are NP-complete even for $k = 0$ there is no polynomial approximation algorithm A solving them within a finite ratio $\frac{A(I)}{OPT(I)}$, where $A(I)$ is the value found by A and $OPT(I)$ is the optimum value for a given instance I. Therefore the part (iii) is concluded. □

For the problem of computing a defect in "perfect-code-free" graphs we have another refinement.

THEOREM 6. *The problem* **DEFECT** *is NP-complete in connected graphs for* $k = cn$*, where* $n = card(V(G))$ *and* $c = 1 - \frac{1}{r}$*, r is an arbitrary positive integer.*

Proof. We use the polynomial transformation from the problem **PCvU**. Given a connected graph G we choose an arbitrary vertex $v \in G$ and construct G' as follows

$$V(G') = V(G) \cup \bigcup_{i=1}^{m} (\{u_i\} \cup \bigcup_{j=1}^{k} \{u_{ij}\}),$$

$$E(G') = E(G) \cup \bigcup_{i=1}^{m} (\{\{v, u_i\} \cup \bigcup_{j=1}^{k} \{\{u_i, u_{ij}\}\}).$$

Now $def(G') \geq (k-1)(m-1)$ and the equality holds iff G contains a perfect code not covering a vertex v. Let us put $k = 2r, m = (r-1)n + 2r^2 - r$. For $n' \overset{def}{=} \text{card}(V(G'))$ we have $cn' = (1 - \frac{1}{r})(n + [(r-1)n + (2r-1)r](2r+1)) = (r-1)((2r-1)n + (2r-1)(2r+1))$. Consequently $(k-1)(m-1) = (2r-1)(r-1)(n+2r+1)$. □

III. Regular graphs. This section is devoted to the problem **PC** considered for k-regular graphs. The investigations of codes in graphs are interesting both from practical and theoretical points of view. See [4] for the discussion of formally very similar problems on dominating sets. The main result of this section is read as the following

THEOREM 7. *The problem* **PC** *is NP-complete even when restricted to k-regular graphs, $k \geq 4$.*

Proof. The proof of Theorem 7 is technically complicated and thus divided into several steps. In each step one auxiliary graph is introduced. Graph G_1 has

$$V(G_1) = \bigcup_{i=1}^{k}(\{a_i, b_i, c_i\} \cup \bigcup_{j=1}^{k-1}\{x_{ij}\}) \quad \text{and}$$

$$E(G_1) = \bigcup_{i=1}^{k}(\{\{a_i, b_i\}\} \cup \bigcup_{j=1}^{k-1}\{\{b_i, x_{ij}\}, \{x_{ij}, c_i\}\}) \cup \{\{x_{ij}, x_{rs}\}; \quad i+j \equiv r+s \mod k\}.$$

Now we proceed to the definition of a graph G_2:

$$V(G_2) = \bigcup_{i=1}^{k}\{c_i, d_i, e_i\} \cup \{u, v\}.$$

The edge set $E(G_2)$ depends on the parity of k. If k is even then

$$E(G_2) = \bigcup_{i=1}^{\frac{k}{2}}\{\{c_i, u\}, \{d_i, u\}, \{c_{i+\frac{k}{2}}, v\}, \{d_{i+\frac{k}{2}}, v\}\} \cup \bigcup_{1 \leq i,j \leq \frac{k}{2}} \{\{d_i, e_j\}, \{d_{i+\frac{k}{2}}, e_{j+\frac{k}{2}}\}\} \cup$$

$$\cup \bigcup_{1 \leq i \neq j \leq \frac{k}{2}} \{\{d_i, d_{k+1-j}\}, \{e_i, e_{k+1-j}\}\}$$

else

$$E(G_2) = \bigcup_{i=1}^{\frac{k-1}{2}}\{\{c_i, u\}, \{d_i, u\}, \{c_{k+1-i}, v\}, \{d_{k+1-i}, v\}\} \quad \cup$$

$$\cup \bigcup_{1 \leq i,j \leq \frac{k-1}{2}} \{\{d_i, e_j\}, \{d_{k+1-i}, e_{k+1-j}\}\} \quad \cup$$

$$\cup \bigcup_{1 \leq i \neq j \leq \frac{k-1}{2}} \{\{d_i, d_{k+1-j}\}, \{e_i, e_{k+1-j}\}\} \quad \cup \quad \{\{c_{\frac{k+1}{2}}, v\}, \{u, d_{\frac{k+1}{2}}\}\} \quad \cup$$

$$\cup \bigcup_{i=1}^{\frac{k-1}{2}}\{\{d_{\frac{k+1}{2}}, e_i\}, \{d_{\frac{k+1}{2}}, d_{k+1-i}\}, \{e_{\frac{k+1}{2}}, e_i\}, \{e_{\frac{k+1}{2}}, e_{k+1-i}\}\}.$$

Further we need graph $G_3 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ supposing that $V(G_1) \cap V(G_2) = \{c_i; i = 1, ..., k\}$. Concerning G_3 we have a simple observation :

LEMMA 7.1. *Let C be a code in G_3 covering all vertices of degree k. Then exactly one of the two following cases occurs :*

(i) $\Phi_C(a_i) = \otimes, \quad \Phi_C(e_i) = \circ, \quad i = 1, ..., k;$

(ii) $\Phi_C(a_i) = \circ, \quad \Phi_C(e_i) = \otimes, \quad i = 1, ..., k.$ □

Finally, k-regular graph $G_{\mathcal{M}}$ is introduced for $\mathcal{M} = (M, \mathcal{T})$, where $\mathcal{T} \subset \binom{M}{k}$, and $\mathrm{card}(\{T; m \in T \in \mathcal{T}\}) = k, \quad \forall m \in M$.

First we denote by G_T a graph which is isomorphic to G_3 in such a way that its vertices a_i are renamed by vertices from T. Similarly the vertices e_i are renamed as e_T^m, $m \in T$. Moreover $V(G_T) \cap V(G_{T'}) = T \cap T'$ for $T \neq T' \in \mathcal{T}$, and $\overline{M} = \{\overline{m}; m \in M\}$. Further, "new" vertices $\{f_T^m, g_T^m; m \in T \in \mathcal{T}\}$ have to be considered. Finally we put

$$V(G_{\mathcal{M}}) = \bigcup_{T \in \mathcal{T}} V(G_T) \cup \overline{M} \cup \bigcup_{m \in M, m \in T} \{f_T^m, g_T^m\},$$

$$E(G_{\mathcal{M}}) = \bigcup_{T \in \mathcal{T}} E(G_T) \cup \bigcup_{m \in M} \left(\bigcup_{T \ni m} \{\{\overline{m}, f_T^m\}, \{g_T^m, e_T^m\}\} \cup \bigcup_{\substack{T \cap T' \ni m \\ T \neq T'}} \{\{f_T^m, g_{T'}^m\}\} \right).$$

Obviously, $G_{\mathcal{M}}$ is a k-regular graph. Therefore the proof of Theorem 7 will be concluded by the following lemma

LEMMA 7.2. *$G_{\mathcal{M}}$ contains a perfect code iff there is a partition of $\mathcal{M}$ into k-tuples, i.e there exists $\mathcal{T}' \subset \mathcal{T}$ such that card($\{T; m \in t \in \mathcal{T}'\}) = 1, \forall m \in M$.*

Proof. Let C be a perfect code in $G_{\mathcal{M}}$. Then $C \cap V(G_T)$ is a code in G_T that covers all vertices of degree k for every k-tuple T. Using Lemma 7.1 we obtain that

$$\mathcal{T}' = \{T; C \cap V(G_T) \quad \text{is a code of type (i) from Lemma 7.1}\}.$$

is a partition of the system $\mathcal{M}$.

Conversely, let there is a partition $\mathcal{T}'$ of $\mathcal{M}$. Let C_T ($\overline{C_T}$, resp.) be a code of type (i) (type (ii), resp.) covering all vertices of degree k in G_T. Then

$$C = \bigcup_{T \in \mathcal{T}'} C_T \cup \bigcup_{T \notin \mathcal{T}'} \overline{C_T} \cup \bigcup_{m \in T \in \mathcal{T}'} \{f_T^m, g_T^m\}$$

is a perfect code in $G_{\mathcal{M}}$. □

By virtue of Lemma 7.2 the polynomial transformation **kRkP** $\propto$ **PC** in k-regular graphs is established. Both Theorem 7 and Theorem 1 are proved. □

IV. 3-regular planar graphs. It is easy to see that the result of the previous section holds also for 3-regular graphs. However our aim is to go one step further. In particular we place the requirement of planarity on input instances. After a lot of technical difficulties we are able to prove :

THEOREM 8. *The problem* **PC** *is* NP*-complete in 3-regular planar graphs.*

Proof. As in the proof of Theorem 7 the proof will be divided into several steps. We shall need several special graphs.

Two graphs $H_{x,k}; \overline{H}_{x,k}$ are visualized on Figures 1 and 2, where $k_1(x), ..., k_n(x)$, $(n = n(x))$, denote all clauses containing a variable x such that $k_j(x)$ preserves the counter-clockwise orientation determined by the planar representation of H.

Similarly, $x_1(k)$, $x_2(k)$, $x_3(k)$ denote variables occuring in a clause k under counter-clockwise orientation determined by the planar representation of H.

We have the following lemma

LEMMA 8.1. *Let* C *be a 1-code that covers all vertices of degree 3 in* $H_{x,k}(\overline{H}_{x,k}$, *resp.). Then* $\Phi_C(L_x^{k_i(x)}), \Phi_C(P_x^{k_i(x)}) \in \{\bullet, \otimes\}$ *and moreover provided* $\Phi_C(L_x^{k_i(x)}) \neq \Phi_C(P_x^{k_i(x)})$ *it holds either*

$$(1)\ \Phi_C(L_x^{k_i(x)}) = \bullet, \quad \Phi_C(P_x^{k_i(x)}) = \otimes, \quad \Phi_C(P_k^{x_i(k)}) = \bullet, \quad \Phi_C(L_k^{x_i(k)}) = \otimes$$
$$(\Phi_C(L_x^{k_i(x)}) = \bullet, \quad \Phi_C(P_x^{k_i(x)}) = \otimes, \quad \Phi_C(P_k^{x_i(k)}) = \Phi_C(L_k^{x_i(k)}) = \circ, \text{ resp.})$$

or

$$(2)\ \Phi_C(L_x^{k_i(x)}) = \otimes, \quad \Phi_C(P_x^{k_i(x)}) = \bullet, \quad \Phi_C(P_k^{x_i(k)}) = \Phi_C(L_k^{x_i(k)}) = \circ$$
$$(\Phi_C(L_x^{k_i(x)}) = \otimes, \quad \Phi_C(P_x^{k_i(x)}) = \Phi_C(P_k^{x_i(k)}) = \bullet, \quad \Phi_C(L_k^{x_i(k)}) = \otimes, \text{ resp.}) \ \Box$$

Now, our aim is to present a polynomial transformation from **13p3S** to **PC** considered for 3-regular planar graphs. Let F constitute an instance of **13p3S**. Let H be a planar representation of this instance. For each variable x we put

$$V(H_x) = \bigcup_{i=1}^{n(x)} \{L_x^{k_i(x)}; P_x^{k_i(x)}; S_i; Z_i\},$$

$$E(H_x) = \bigcup_{i=1}^{n(x)} \{\{P_x^{k_i(x)}, S_i\}, \{P_x^{k_i(x)}, Z_{i-1}\}, \{L_x^{k_i(x)}, S_i\}, \{L_x^{k_i(x)}, Z_i\}, \{S_i, Z_{n+1-i}\}\}.$$

Further, for every clause we construct a graph H_k, see Figure 3. Finally we put

$$V(H_F) = \bigcup_x V(H_x) \cup \bigcup_k V(H_k) \cup \bigcup_{x \in k} V(H_{x,k}) \cup \bigcup_{\neg x \in k} V(\overline{H}_{x,k}),$$
$$E(H_F) = \bigcup_x E(H_x) \cup \bigcup_k E(H_k) \cup \bigcup_{x \in k} E(H_{x,k}) \cup \bigcup_{\neg x \in k} E(\overline{H}_{x,k}).$$

Obviously, graph H_F is planar and 3-regular.

To finish the proof of the Theorem 8 we are to prove :

LEMMA 8.2. *Graph H_F contains a perfect code iff the clause F is one-in-three satisfiable (i.e., there exists a* **true/false** *valuation of variables such that in each clause exactly one variable receives the value* **true**).

Proof. Let F be one-in-three satisfiable and let A(B, resp.) be the set of variables which receive the value **true** (**false,** resp.). We shall use the following notation (c.f. Lemma 8.1) :

$C_i(x)$ is a code of type (i) in $H_{x,k}$ and
$\overline{C}_i(x)$ is a code of type (i) in $\overline{H}_{x,k}$ $(i = 1,2)$;
$C(x,k)$ is a code in H_k containing the vertex $P_k^{x(k)}$ and covering all vertices of H_k except of $L_k^{x(k)}$ (such code is unique).

Then

$$C = \bigcup_{x\in A}\left(\bigcup_{x\in k} C_1(x,k) \cup \bigcup_{\neg x\in k}\overline{C}_1(x,k)\right) \cup \bigcup_{x\in B}\left(\bigcup_{x\in k} C_2(x,k)\cup\bigcup_{\neg x\in k}\overline{C}_2(x,k)\right)\cup$$

$$\cup\bigcup_{k}\left(\bigcup_{x\in k, x\in A} C(x,k)\cup\bigcup_{\neg x\in k, x\in B} C(x,k)\right)$$

is a perfect code in H_F.

On the other hand let H_F contain a perfect code C. Take a variable x and consider $C\cap V(H_x)$. Since C has to cover vertices $L_x^{k_i(x)}$ and $P_x^{k_i(x)}$ $(i = 1,...,n(x))$ it holds that

$$C\cap V(H_x)\cap\{S_i, Z_i; i = 1,...,n(x)\} = \emptyset.$$

Thus we obtain either

$$(1) \qquad C\cap V(H_x) = \{L_x^{k_i(x)}; i = 1,2,...,n(x)\}$$

or

$$(2) \qquad C\cap V(H_x) = \{P_x^{k_i(x)}; i = 1,2,...,n(x)\}.$$

Let A denote the set of variables such that (1) holds. These variables receive the value **true** and the remaining ones the value **false**. Considering a clause k with variables x_1, x_2, x_3 and using Lemma 8.1 we obtain that x_i receives the value **true** iff $P_k^{x_i(k)}\in C$ and $L_k^{x_i(k)}$ is covered by a "code"-vertex from $C\cap V(H_{x,k})$. Due to the construction of a graph H_k there is at most one **true**-variable in this clause. So it remains to examine the case when in k there is no **true**-variable. In this case Lemma 8.1 says that all vertices $L_k^{x_i(k)}, P_k^{x_i(k)}, i = 1,2,3$, have to be covered by $C\cap V(H_k)$. But H_k does not contain a code satisfying $\Phi_C(L_k^{x_i(k)}) = \Phi_C(P_k^{x_i(k)}) = \otimes$, as can be observed from Figure 3. □

Hence we have proved that **13p3S** $\propto$ **PC** in planar 3-regular graphs. Theorem 8 follows. □

As a concluding remark of this section we conjecture that our NP-completeness result could be strengthened to 3-regular planar bipartite graphs.

V. Trees. In this section we outline a recursive procedure DEF for computing the defect in (rooted) trees.

```
procedure $DEF(T : \text{tree}, t : \text{root}, x \in \{\bullet, \otimes, \circ\})$ : integer $\cup\ \infty$;
    case $x$ of
    $\bullet$ : $DEF :=$ if $V(T) = \{t\}$ then $0$ else $\sum_{u \in pre(t)}(DEF(T_u, u, \circ) - 1)$;
    $\circ$ : $DEF :=$ if $V(T) = \{t\}$ then $\infty$ else
            $\sum_{u \in pre(t)} \min\{(DEF(T_u, u, \circ), DEF(T_u, u, \otimes))\} + 1$;
    $\otimes$ : $DEF :=$ if $V(T) = \{t\}$ then $1$ else if
            $(\exists u \neq v \in pre(t)\ \&\ DEF(T_u, u, \circ) = DEF(T_u, u, \otimes) =$
                $= DEF(T_v, v, \circ) = DEF(T_v, v, \otimes) = \infty)$
    then $\infty$ else if
            $(\exists u \in pre(t) : DEF(T_u, u, \circ) = DEF(T_u, u, \otimes) = \infty)$
    then
        $DEF(T_u, u, \bullet) + \sum_{u \neq v \in pre(t)} \min\{DEF(T_v, v, \circ), DEF(T_v, v, \otimes)\}$
    else
            $\sum_{u \in pre(t)} \min\{DEF(T_u, u, \circ), DEF(T_u, u, \otimes)\} +$
        $+ \min_{u \in pre(t)}\{DEF(T_u, u, \bullet) - \min\{DEF(T_u, u, \circ), DEF(T_u, u, \otimes)\}\}$;

endprocedure.
```

Having this procedure the defect in a given tree T is given by

$$\min_{x \in \{\bullet, \otimes, \circ\}} DEF(T, t_0, x).$$

It remains to explain the notation used in the outlined procedure :

(1) t_0 is a root of a given tree T,

(2) $pre(t) \overset{def}{=} \{x; d(x, t_0) = d(t, t_0) + d(x, t) = d(t, t_0) + 1\}$,

(3) $T_u \overset{def}{=} T\big|\{x; d(x, t_0) = d(x, u) + d(u, t_0)\}$.

By a careful time and correctness analysis in amortized complexity fashion [12] we are able to prove the following

THEOREM 9. *Procedure DEF computes defect in trees and takes $\Theta(n)$ time.* $\square$

A similar result holds for graphs of bounded tree-width [10]. The details will appear elsewhere.

VI. Concluding remark. Given a k-regular graph G on n vertices, any dominating set in G has at least $\frac{n}{1+k}$ vertices. Moreover, there exists a dominating set with exactly $\frac{n}{1+k}$ vertices if and only if G contains a perfect code. Hence we have obtained a refinement of a well-known NP-complete problem on dominating sets in graphs (c.f.[4]) :

THEOREM 10. *The* **DOMINATING SET** *problem remains NP-complete even when restricted to planar 3-regular graphs.* $\square$

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Appendix.

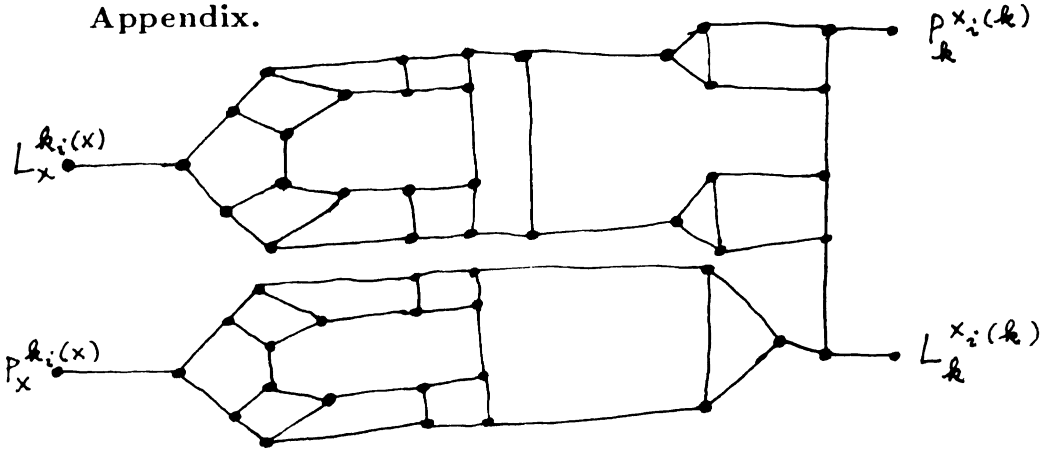

Figure 1.

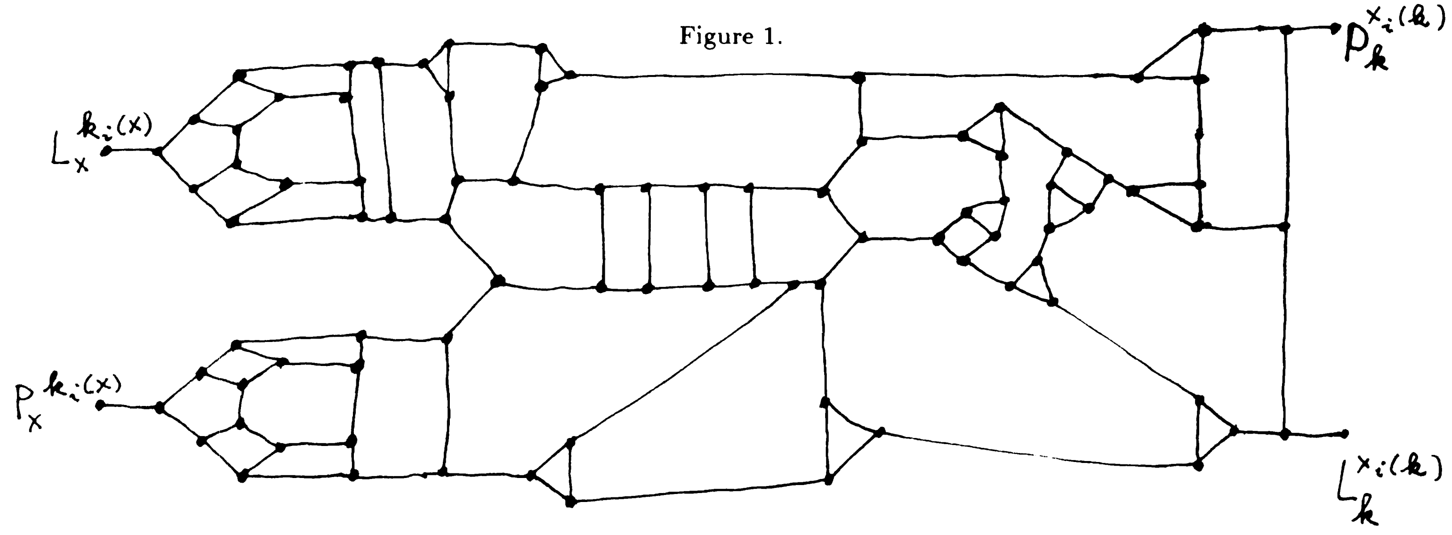

Figure 2.

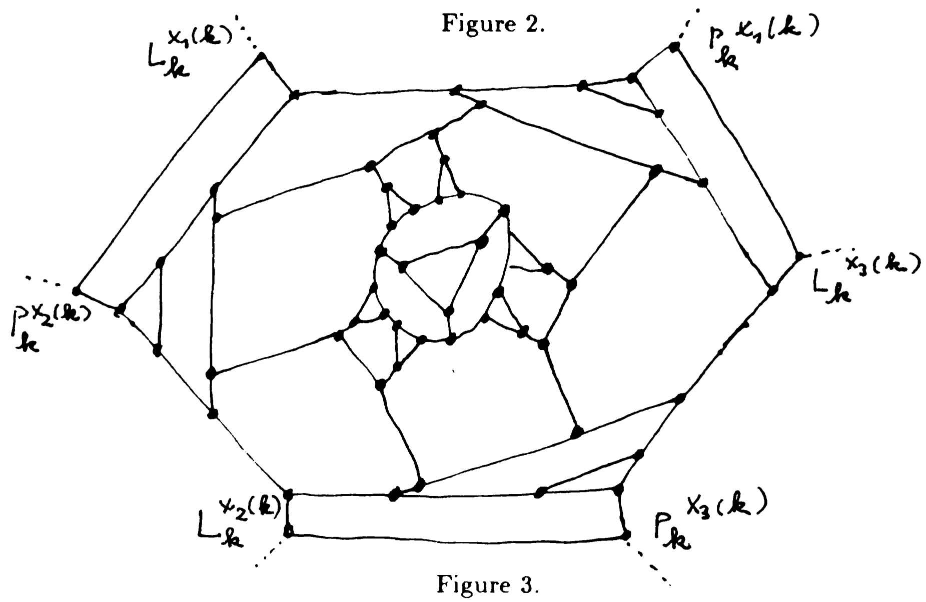

Figure 3.

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