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Invariant Subspaces for LPV Systems and Their Applications

Gary Balas, József Bokor, and Zoltán Szabó

Abstract—The aim of this note is to extend the notion of invariant subspaces known in the geometric control theory of the linear time invariant systems to the linear parameter-varying (LPV) systems by introducing the concept of parameter-varying invariant subspaces. For LPV systems affine in their parameters, algorithms are given to compute many parameter varying subspaces relevant in the solution of state feedback and observer design problems.

Index Terms—Distributions, input affine systems, invariant subspaces, linear parameter-varying (LPV) systems.

I. INTRODUCTION

Important engineering processes involve time-varying linear and nonlinear models. A general theory for the robust control of nonlinear systems is not computationally tractable and useful progress requires an intermediate level of complexity. Linear parameter-varying (LPV) modeling techniques have gained a lot of interest as they provide a systematic means of computing gain-scheduled controllers, especially those related to vehicle and aerospace control, [2], [4], [11], [18], [23].

Many of the control system design techniques using LPV models can be cast or recast as convex feasibility problem with infinite constraints that involve linear matrix inequalities (LMIs). This problem can be addressed by using affine LPV modeling that reduces the infinite constraints imposed on the LMI formulation to a finite number, [1], [30].

The pure LPV model is not quite matched for practical problems, e.g., to the flight control problem, where the scheduling variables are in fact system states (e.g. airspeed and angle of attack), rather than bounded external variables. An approach to this problem is to generate so-called quasi-LPV models, which are applicable when the scheduling variables are measured states, the dynamics are linear in the inputs and other states, and there exist inputs to regulate the scheduling variables to arbitrary equilibrium values.

The mathematically dual concepts of (A,B)(or controlled)-invariance and (C,A) (or conditioned)-invariance play an important role in the geometric theory of linear time-invariant (LTI) systems, [6], [32]. These concepts were used to study some fundamental problems of LTI control theory, such as disturbance decoupling (DDP), unknown input observer design, fault detection (FPRG), [19], [20], [32]. The nonlinear version of this geometrical approach is much more complex and deals with certain locally controlled or conditioned invariant distributions and codistributions, [14], [15], [24], [25].

The aim of this note is to extend these notions for the parameter-varying systems by introducing the notion of *parameter-varying* (A, B)-*invariant, parameter-varying* (C, A)-*invariant*, controllability and unobservability subspaces, and to give some algorithms to compute these subspaces if certain conditions are fulfilled.

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This note deals with the class of LPV systems of m inputs and p outputs that can be described as

$$\dot{x}(t) = A\left(\rho(t)\right) x(t) + B\left(\rho(t)\right) u(t)$$
$$y(t) = C\left(\rho(t)\right) x(t) \tag{1}$$

where

$$A\left(\rho(t)\right) = A_0 + \rho_1(t) A_1 + \cdots + \rho_N(t) A_N \tag{2}$$
$$B\left(\rho(t)\right) = B_0 + \rho_1(t) B_1 + \cdots + \rho_N(t) B_N \tag{3}$$
$$C\left(\rho(t)\right) = C_0 + \rho_1(t) C_1 + \cdots + \rho_N(t) C_N \tag{4}$$

and the dimension of the state space is supposed to be n.

It is assumed that each parameter ρ_i ranges between known external values $\rho_i(t) \in [\underline{\rho}_i, \overline{\rho}_i]$ and the parameter set, that contains all the vectors $\rho := [\rho_1(t), \ldots, \rho_N(t)]^T$ where $t \in [0, T]$, will be denoted by $\mathcal{P}$. For the sake of notational simplicity the time dependency of the matrices will be omitted ($A(\rho) := A(\rho(t))$) where it is possible.

In introducing the various parameter-varying invariant subspaces, an important goal was to set notions that lead to computationally tractable algorithms. In general, it is a hard task to give an exhaustive characterization for the solution of the fundamental problems such as the DDP or the FPRG problem even in the LPV case. However, since the main ingredient in the solution of these problems are certain local decomposition theorems—in observable and unobservable subsystems, for example—using suitable invariant subspaces instead of the distributions or codistributions one can get sufficient conditions for the solvability that can be useful for practical engineering problems.

II. CONTROLLABILITY AND OBSERVABILITY OF LINEAR SYSTEMS

In what follows some standard results, see, e.g. [17] and [26], will be recalled about the controllability and observability of the LTV systems:

$$\dot{x}(t) = A(t) x(t) + B(t) u(t) \tag{5}$$
$$y(t) = C(t) x(t) \tag{6}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$, $x_0 = x(t_0)$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$.

A convenient way to study all solutions of a linear equation on the interval $[t_0, T]$, for all possible initial values simultaneously, is to introduce the matrix differential equation

$$\dot{X}(t) = A(t) X(t) \quad X(t_0) = \mathbb{I} \tag{7}$$

with $X(t) \in \mathbb{R}^{n \times n}$. Let us denote by $\Phi(t, t_0)$ the transition matrix, i.e., the solution of (7) at time t; it is known that $\Phi(t, t_0)$ is nonsingular for any t.

Definition 1: A state x_0 is said to be controllable at time t_0 if there exist a control function $u(t)$ depending on x_0 and t_0 and defined over some finite closed interval $[t_0, T]$ such that for the corresponding solution one has $x(T) = \phi_{x_0}(T, t_0) = 0$. If this is true for every state x and every t_0 then the system will be called (completely) controllable.

System (5) is called (completely) observable on an interval $[t_0, T]$ if any initial state x_0 at t_0 can be determined from knowledge of the system output $y(t)$ and input $u(t)$ over the interval $[t_0, T]$.

For a given system (5) let us denote by $\mathcal{R}$ the reachability subspace, i.e., the set of all states that can be reached from the origin in any finite time by means of control actions. The system is controllable if and only if $\mathcal{R} = \mathcal{X}$.

Analogously let us denote by $\mathcal{Q}$ the unobservability subspace, i.e., the set of all initial states that cannot be recognized from the output function. The system is observable if and only if $\mathcal{Q} = 0$.

One of the fundamental results, see [17], concerning controllability and observability properties of LTV systems uses the system solution in the formulation of its conditions. The solution of such a system is generally not available in closed form. It is also a standard result that one can derive a rank condition that guarantees controllability which

does not involve integration and can be obtain directly from the data $(A(t), B(t))$, see [26]:

Proposition 1: If (5) is analytic on an interval $\mathcal{I}$ and t is an arbitrary fixed element of $\mathcal{I}$, then (5) is completely controllable on every nontrivial subinterval of $\mathcal{I}$ if and only if

$$\text{rank}\left[B_0(t) \quad B_1(t) \quad \cdots \quad B_k(t)\right] = n \tag{8}$$

for some integer k, where

$$B_0(t) := B(t) \quad B_{i+1}(t) := A(t) B_i(t) - \frac{d}{dt} B_i(t). \tag{9}$$

If the analyticity condition is dropped, then the rank condition is only sufficient. An analogous result holds for observability, too.

It is also a standard way to study questions concerning controllability and observability using the Volterra expansions associated to the given differential equations. If the time-varying system has an affine structure then one can exploit this structure in deciding controllability and observability questions. Using the Peano–Baker formula for the transition matrix, one can prove the following result.

Lemma 1: For (1), the points attainable from the origin are those from the subspace spanned by the vectors

$$\mathcal{R}_{(\mathcal{A}, \mathcal{B})} := \text{span}\left\{\prod_{j=1}^{K} A_{l_j}^{i_j} B_k\right\} \tag{10}$$

where $K \geq 0$, $l_j, k \in \{0, \cdots, N\}$, $i_j \in \{0, \cdots, n-1\}$. Moreover, if one consider the finitely generated Lie-algebra $\mathcal{L}(A_0, \ldots, A_N)$ which contains $A_0, \ldots, A_N$, and a basis $\hat{A}_1, \ldots, \hat{A}_K$ of this algebra, then

$$\mathcal{R}_{(\mathcal{A}, \mathcal{B})} = \sum_{l=0}^{N} \sum_{n_1=0}^{n-1} \cdots \sum_{n_K=0}^{n-1} \text{Im}\left(\hat{A}_1^{n_1} \ldots \hat{A}_K^{n_K} B_l\right).$$

A direct consequence of this fact is that if the inclusion $\mathcal{R}_{\mathcal{A}, \mathcal{B}} \subset \mathbb{R}^n$ is strict, i.e, if $\mathcal{R}_{\mathcal{A}, \mathcal{B}}$ is a proper subspace, then the system (5) cannot be completely controllable.

The main question is that under which condition is controllability guaranteed if the relation $\mathcal{R}_{\mathcal{A}, \mathcal{B}} = \mathbb{R}^n$, i.e., the multivariaiable Kalman rank condition, holds. In [28] and [29], an answer was given in terms of the solutions of a Wei–Normann equation, however, that solution is seldom accessible in practical situations. In [29], it was shown that if the parameter functions $\{1, \rho_1, \cdots, \rho_N\}$ are differential algebraically independent (relation that also involves the derivatives of ρ_i), then the multivariaiable Kalman rank condition is also sufficient for controllability. An analogous result holds for observability, as well.

III. PARAMETER-VARYING INVARIANT SUBSPACES

For LTI systems the concept of certain invariant subspaces and the corresponding global decompositions of the state equations induced by these invariant subspaces was one of the main thrusts for the development of geometric methods for solutions to problems of disturbance decoupling or noninteracting control; see [32]. Nonlinear systems can be studied using tools from differential geometry, when the central role is played by the concept of invariant distributions. From the geometric viewpoint results of the classical linear control can be seen as special cases of more general nonlinear results; for details, see [15] and [21].

Due to the computational complexity involved, these nonlinear methods have limited applicability in practice.

Linear time varying systems can be viewed as affine nonlinear systems [14], by augmenting the original state space to $\xi := [t, x]^T$. Restricting the investigations to linear subspaces, as special instances of distributions, then a subspace $\mathcal{V}$ of $\mathbb{R}^n$, will be an invariant distribution for system (5) if and only if $A(\rho(t))\mathcal{V} \subset \mathcal{V}$ for all $t \in \mathcal{I}$, where $\mathcal{I}$ is an interval on which the solutions are defined.

This fact motivates the introduction of the following notion for LPV systems.

Definition 2: A subspace $\mathcal{V}$ is called parameter-varying invariant subspace for the family of the linear maps $A(\rho)$ (or shortly $\mathcal{A}$-invariant subspace) if

$$A(\rho)\mathcal{V} \subset \mathcal{V} \quad \text{for all } \rho \in \mathcal{P}, \text{ i.e., for all } t \in \mathcal{I}. \qquad (11)$$

Let us observe, that if $\mathcal{V}$ is an $\mathcal{A}$-invariant subspace and $Im B(t) \subset \mathcal{V}$ for all t, then, considering the restrictions of the state matrices to this subspace, i.e.,

$$\bar{A}(\rho) := A(\rho)|_{\mathcal{V}} \qquad (12)$$

and using the invariancy property, (5) can be decomposed as

$$\dot{\bar{x}}(t) = \bar{A}(t)\bar{x}(t) + \tilde{A}_1(t)\tilde{x}(t) + \overline{B}(t)u(t) \qquad (13)$$
$$\dot{\tilde{x}}(t) = \tilde{A}_2(t)\tilde{x}(t). \qquad (14)$$

It is obvious that the subspace $\mathcal{R}_{(\mathcal{A},\mathcal{B})}$ in (10) is $A(\rho(t))$ invariant, i.e.,

$$A\left(\rho(t)\right)\mathcal{R}_{(\mathcal{A},\mathcal{B})} \subseteq \mathcal{R}_{(\mathcal{A},\mathcal{B})}, \qquad \text{for all } t \qquad (15)$$

moreover, one has that for the induced decomposition

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} = \mathcal{R}_{(\bar{\mathcal{A}},\bar{\mathcal{B}})}. \qquad (16)$$

These facts motivate the introduction of the following notion.

Definition 3: Let $\mathcal{B}(\rho)$ denote Im $B(\rho)$. Then a subspace $\mathcal{V}$ is called a parameter-varying (A,B)-invariant subspace (or shortly $(\mathcal{A}, \mathcal{B})$-invariant subspace) if for all $\rho \in \mathcal{P}$ any of the following equivalent conditions holds:

$$a.) \quad A(\rho)\mathcal{V} \subset \mathcal{V} + \mathcal{B}(\rho) \qquad (17)$$

and there exists a mapping $F \circ \rho: [0, T] \to \mathbb{R}^{m \times n}$ such that

$$b.) \quad (A(\rho) + B(\rho)F(\rho))\,\mathcal{V} \subset \mathcal{V}. \qquad (18)$$

Dealing with parametric uncertainties a similar concept was introduced in [7], called *robust controlled invariant* subspace. If one sets the gain matrix to be constant then the resulting subspace will be more restrictive, this approach was used in [9] and [22], and was termed as *generalized controllability* (A, B)-*invariant subspace*.

The dual notion of the previous definition is the following.

Definition 4: Let $\mathcal{C}(\rho)$ denote Ker $C(\rho)$. Then a subspace $\mathcal{W}$ is called a parameter-varying (C,A)-invariant subspace (or shortly $(\mathcal{C}, \mathcal{A})$-invariant subspace) if for all $\rho \in \mathcal{P}$ any of the following equivalent conditions holds:

$$a.) \quad A(\rho)\left(\mathcal{W} \cap \mathcal{C}(\rho)\right) \subset \mathcal{W} \qquad (19)$$

and there exists a mapping $G \circ \rho : [0, T] \to \mathbb{R}^{n \times p}$ such that:

$$b.) \quad (A(\rho) + G(\rho)C(\rho))\mathcal{W} \subset \mathcal{W}. \qquad (20)$$

The mathematically dual concepts of (A,B) (or controlled)-invariance and (C,A) (or conditioned)-invariance play an important role in the geometric theory of LTI systems. These concepts were used to study some fundamental problems of LTI control theory, e.g., [6], [19], [20], and [32].

In the time-varying case, one can deal with these problems using the much more complex mathematical objects given by the locally controlled or conditioned invariant distribution algorithms or codistribution algorithms, respectively, [15], [24]. The main problem that arises in practical situations is that either one cannot perform the computations or one cannot verify the conditions under the given algorithms provide the desired results.

If certain conditions are fulfilled, e.g., if the parameter functions are differential algebraically independent, then the parameter invariant

subspaces defined above coincide with the corresponding invariant distribution or codistribution, respectively. However, to give sufficient conditions for the solution of certain state feedback and observer filter design problems it is enough that some decompositions of the state equations could be performed. The parameter-varying versions of these invariant spaces are suitable objects to define the required decompositions, therefore they can play the same role in the solution of the fundamental problems, such as disturbance decoupling (DDP), see [31], unknown input observer design, fault detection (FPRG), see [3], [5], [27], as their counterparts in the time invariant context.

IV. INVARIANCE ALGORITHMS

In [10], an algorithm was given to determine the robust controlled invariant subspace, however since the number of conditions is not finite, the algorithm, in general, is quite complex. From a practical point of view, it is an important question to characterize these parameter-varying subspaces by a finite number of conditions. Assuming the special structure of the matrix $A(\rho)$ in (1), it is immediate that if the inclusions hold for all A_i, then they hold also for all $\rho \in \mathcal{P}$. It is not so straightforward under which conditions the reverse implication is true, too.

In what follows, a sufficient condition will be given that characterizes property (11) using only a finite number of constrains.

Lemma 2: If the functions $1, \rho_1, \ldots, \rho_N$ are linearly independent over $\mathbb{R}$ then $A(\rho)\mathcal{V} \subset \mathcal{W} \forall \rho \in \mathcal{P}$ if and only if

$$A_i\mathcal{V} \subset \mathcal{W}, \qquad i = 0, \ldots, N.$$

We are interested in finding supremal $\mathcal{A}$-invariant subspaces in a given subspace $\mathcal{K}$ or containing a given subspace $\mathcal{L}$. As far as the first purpose is concerned the $\mathcal{A}$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm over $\mathcal{L}$ can be defined as:

$$\mathcal{AISAL}: \quad \mathcal{V}_0 = \mathcal{L}$$
$$\mathcal{V}_{k+1} = \mathcal{L} + \sum_{i=0}^{N} A_i \mathcal{V}_k, \qquad k \geq 0$$
$$\mathcal{V}^* = \lim_{k \to \infty} \mathcal{V}_k. \qquad (21)$$

Obviously, the algorithm will stop after a finite number of steps, i.e., $\mathcal{V}^* = \mathcal{V}_{n-1}$.

Theorem 1: The subspace $\mathcal{V}^*$ given by (21) is such that

$$\mathcal{L} \subset \mathcal{V}^* \qquad (22)$$
$$\mathcal{V}^* \text{ is } \mathcal{A} - \text{invariant} \qquad (23)$$

and assuming that conditions of Lemma 2. hold, it is minimal with these properties.

Similar to the linear case the subspace $\mathcal{V}^*$ will be denoted by $\langle \mathcal{A} | \mathcal{L} \rangle$. By duality, one has the $\mathcal{A}$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm in $\mathcal{K}$, i.e.,

$$\mathcal{AISAK}: \quad \mathcal{W}_0 = \mathcal{K}$$
$$\mathcal{W}_{k+1} = \mathcal{K} \cap \bigcap_{i=0}^{N} A_i^{-1} \mathcal{W}_k, \qquad k \geq 0$$
$$\mathcal{W}^* = \lim_{k \to \infty} \mathcal{W}_k \qquad (24)$$

where $A_i^{-1}\mathcal{W}_k$ denotes the inverse image of $\mathcal{W}_k$ under A_i^{-1}. The subspace $\mathcal{W}^*$ will be denoted by $\langle \mathcal{K} | \mathcal{A} \rangle$.

Theorem 2: The subspace $\mathcal{W}^*$ given by (24) is such that

$$\mathcal{W}^* \subset \mathcal{K}$$
$$\mathcal{W}^* \text{ is } \mathcal{A} - \text{invariant}$$

and assuming that that conditions of Lemma lem:pc. holds, it is maximal with these properties.

The direct application of nonlinear algorithms to LPV systems yield time-dependent distributions. In order to get rid of the time-dependency, we will continue with the following argument. The set of all $(\mathcal{A}, \mathcal{B})$-invariant subspaces contained in a given subspace $\mathcal{K}$, is an upper semilattice with respect to subspace addition. This semilattice admits a maximum which can be computed from the $(\mathcal{A}, \mathcal{B})$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm:

$$\mathcal{ABISA}: \quad \mathcal{V}_0 = \mathcal{K} \tag{25}$$

$$\mathcal{V}_{k+1} = \mathcal{K} \cap \bigcap_{i=0}^{N} A_i^{-1}(\mathcal{V}_k + \mathcal{B}). \tag{26}$$

The limit of this algorithm will be denoted by $\mathcal{V}^*$ and its calculation needs at most n steps.

The set of all $(\mathcal{C}, \mathcal{A})$-invariant subspaces—note that $\mathcal{C} = \operatorname{Ker} C$—containing a given subspace $\mathcal{L}$, is a lower semilattice with respect to subspace intersection. This semilattice admits a minimum which can be computed from the $(\mathcal{C}, \mathcal{A})$-$\mathcal{I}$nvariant $\mathcal{S}$ubspace $\mathcal{A}$lgorithm that can be obtained by duality from the $\mathcal{ABISA}$ algorithm, see [27].

These algorithms can be obtained in the same way as those for the LTI case; see [32]. The minimality (maximality) of these subspaces follows from the fact that if one assumes that the parameter functions are linearly independent, then any relation involving subspace inclusions of the type $A(\rho)\mathcal{V} \subset \mathcal{W}$ will be equivalent with the same type of inclusions with the matrices A_i, i.e., $A_i\mathcal{V} \subset \mathcal{W}$ for all i.

For the LPV case (with constant B matrix) one can get the following definition for the controllability subspace.

Definition 5: A subspace $\mathcal{R}$ is called parameter-varying controllability subspace if there exists a constant matrix K and a parameter varying matrix $F : [0, T] \to \mathbb{R}^{m \times n}$ such that

$$\mathcal{R} = \langle \mathcal{A} + B\mathcal{F} | \operatorname{Im} BK \rangle \tag{27}$$

where the notation $\mathcal{A} + B\mathcal{F}$ stems for the system $A(\rho) + BF(\rho)$.

As in the classical case, it can be seen that the family of controllability subspaces contained in a given subspace $\mathcal{K}$ is closed under subspace addition. Hence, this family has a maximal element which can be computed from the parameter-varying $\mathcal{C}$ontrollability $\mathcal{S}$ubspace $\mathcal{A}$lgorithm:

$$\mathcal{CSA}: \quad \mathcal{R}_0 = 0 \tag{28}$$

$$\mathcal{R}_{k+1} = \mathcal{V}^* \cap \left(\sum_{i=0}^{N} A_i \mathcal{R}_k + \mathcal{B} \right) \tag{29}$$

$$\mathcal{R}^* = \lim_{k \to \infty} \mathcal{R}_k \tag{30}$$

where $\mathcal{V}^*$ is computed by $\mathcal{ABISA}$.

Theorem 3: The subspace $\mathcal{R}^*$ is the largest parameter—varying controllability subspace in $\mathcal{C}$.

This subspace plays a central role in solving the DDP problem in the affine LPV case; for details, see [31].

The $\mathcal{CSA}$ algorithm can be obtained by dualizing the observability codistribution algorithm, see [24], and applying it for subspaces.

In what follows, a few properties of the controllability subspaces, known in the LTI case, [32], will be recast in the parameter varying context.

Theorem 4: If $\hat{\mathcal{B}} \subset \mathcal{B}$ and $\mathcal{R} = \langle \mathcal{A} | \mathcal{B} \rangle$ then $\mathcal{R} = \langle \mathcal{A} | \mathcal{B} \cap \mathcal{R} \rangle$. Conversely, if $\mathcal{R} = \langle \mathcal{A} | \mathcal{B} \cap \mathcal{R} \rangle$ then there exists $K : \mathcal{U} \to \mathcal{U}$ for which $\mathcal{R} = \langle \mathcal{A} | \operatorname{Im} BK \rangle$.

Theorem 5: A subspace $\mathcal{R}$ is parameter-varying controllability subspaces if and only if there exists $F : [0, T] \to \mathbb{R}^{m \times n}$ such that $\mathcal{R} = \langle \mathcal{A} + B\mathcal{F} | \mathcal{B} \cap \mathcal{R} \rangle$.

Let $\mathcal{R} \subset \mathcal{X}$ a fixed subspace and define $\Gamma = \{ \mathcal{Z} : \mathcal{Z} = \mathcal{R} \cap (\sum_{i=0}^{N} A_i \mathcal{Z} + \mathcal{B}) \}$.

Lemma 3: There exists a unique minimal element of Γ.

The minimal element $\mathcal{Z}^*$ can be computed from the following algorithm:

$$\mathcal{Z}_0 = 0 \quad \mathcal{Z}_{k+1} = \mathcal{R} \cap \left(\sum_{i=0}^{N} A_i \mathcal{S}_k + \mathcal{B} \right). \tag{31}$$

Theorem 6: A subspace $\mathcal{R} \subset \mathcal{X}$ is parameter-varying controllability subspace if and only if it is $(\mathcal{A}, \mathcal{B})$-invariant and $\mathcal{R} = \mathcal{Z}^*$.

The dual notion of parameter-varying controllability subspace is the following:

Definition 6: A subspace $\mathcal{S}$ is called parameter-varying unobservability subspace if there exists a constant matrix H and a parameter varying matrix $G : [0, T] \to \mathbb{R}^{n \times p}$ such that

$$\mathcal{S} = \langle \operatorname{Ker} HC | \mathcal{A} + \mathcal{G}C \rangle \tag{32}$$

where $\mathcal{A} + \mathcal{G}C$ denotes the system $A(\rho) + G(\rho)C$.

By dualization one can show that the family of parameter-varying unobservability subspaces containing a given subspace $\mathcal{L}$ is closed under subspace intersection. The minimal element of this family can be computed as the result of the parameter-varying $\mathcal{U}$nobservability $\mathcal{S}$ubspace $\mathcal{A}$lgorithm that can be obtained by duality from $\mathcal{CSA}$, for details see [27]. These subspaces were used in the solution of fault detection filter design problems. Stabilization issues were treated using LMI techniques; for details see [3] and [27]. Moreover, using these parameter varying unobservability subspaces it is also possible to solve fault detection filter design problems for qLPV systems; for details, see [5].

In [13], exactly the same algorithms were obtained in the context of bilinear systems.

V. CONCLUSION

This note investigated the extension of invariant subspace concepts used in the geometric theory of LTI systems to LPV and qLPV systems. The basic concept was the construction of subspaces invariant for a family of linear mappings. These were called parameter varying invariant subspaces. For LPV systems affine in their parameters algorithms were given to compute the relevant invariant subspaces used in relation with state feedback and observer design problems like DDP, decoupling, inversion, or detection filter design. These algorithms use only linear algebraic tools and can offer a computational alternative to some nonlinear problems reformulated in LPV or qLPV terminology.

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Robust PI Controller Design Satisfying Sensitivity and Uncertainty Specifications

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Abstract—This note presents a control design method for determining proportional–integral-type controllers satisfying specifications on gain margin, phase margin, and an upper bound on the (complementary) sensitivity for a finite set of plants. The approach can be applied to plants that are stable or unstable, plants given by a model or measured data, and plants of any order, including plants with delays. The algorithm is efficient and fast, and as such can be used in near real-time to determine controller parameters (for online modification of the plant model including its uncertainty and/or the specifications). The method gives an optimal controller for a practical definition of optimality. Furthermore, it enables the graphical portrayal of design tradeoffs in a single plot, highlighting the effects of the gain margin, complementary sensitivity bound, low frequency sensitivity and high frequency sensor noise amplification.

Index Terms—Design, gain and phase margin, linear systems, proportional–integral (PI) control, robustness.

I. INTRODUCTION

Although many methods for tuning proportional–integral (PI) and proportional–integral-derivative (PID) controllers exist, extensive research in design techniques continues, driven by the strong use of such controllers in industry. Depending on the types of specifications that the design must satisfy, the tuning methods reported in the literature can be summarized as falling into one of two categories.

One class of methods considers gain and phase margin specifications. Ho *et al.* [1], [2] developed simple analytical formulae to tune PI and PID controllers for commonly used first-order and second-order plus dead-time plant models to meet gain and phase margin specifications. Ho *et al.* [3], [4] reported tuning formulae for the design of PID controllers that satisfy both robustness and performance requirements. Crowe and Johnson [5] presented an automatic PI control design algorithm to satisfy gain and phase margin based on a converging algorithm. Suchomoski [6] developed a tuning method for PI and PID controllers that can shape the nominal stability, transient performance, and control signal to meet gain and phase margins.

A second class of design methods focuses on sensitivity specifications, and is based on the premise that gain and phase margin specifications may fail to guarantee a reasonable bound on the sensitivity. Ogawa [7] used the QFT-framework to propose a PI design technique that satisfies a bound on the sensitivity for an uncertain plant. Poulin and Pomerleau [8] developed a PI design methodology for integrating processes that bounds the maximum peak resonance of the closed loop. The peak resonance constraint is equivalent to bounding the complementary sensitivity, which can be converted to bounding the sensitivity. Cavicchi [9] described a design method for