THE STRONG ϕ TOPOLOGY ON SYMMETRIC SEQUENCE SPACES

William H. Ruckle
Clemson University
Clemson, South Carolina 29631

1. Introduction - The Strong ϕ Topology

Let S be a linear space of real sequences written in functional notation

$$s = (s(j)) = (s(1), s(2), \dots) \quad .$$

There is a natural duality between S and the space ϕ of sequences which are eventually 0 given by the equation

$$<s,t> = \sum_j s(j)t(j) \qquad s \in S, \; t \in \phi \quad .$$

The series has only a finite number of nonzero terms since t is in ϕ.

A subset B of ϕ is called S-bounded if

$$p_B(s) = \sup\left\{ \left| \sum_j s(j)t(j) \right| \; : t \in B \right\} < \infty$$

for each s in S.

The strong ϕ ($\beta\phi$-) topology on S is the locally convex topology determined by all seminorms of the form p_B as B ranges over all S-bounded subsets of ϕ. Most familiar sequence spaces bear the $\beta\phi$ topology, e.g., ϕ with the strongest locally convex topology, the ℓ^p-spaces $1 \leqslant p \leqslant \infty$ with the BK topology, and ω (all sequences) with the topology of coordinate-wise convergence, but not ℓ^p $(0 < p < 1)$ with the FK topology. The concept of $\beta\phi$ topology is related to the concept of norming biorthogonal sequence; see, e.g., [11]. A biorthogonal sequence $\{e_n, E_n\}$ in a Banach space X is called norming if there is a bounded subset B of $[E_n]$ the linear span $\{E_n\}$ in the dual space X' of X such that

$$\|x\| \leqslant \sup\{|f(x)| : f \in B\} \quad .$$

It is easy to conclude from the relevant definitions that a total biorthogonal sequence $\{e_n, E_n\}$ is norming if and only if the BK space S consisting of all sequences $(E_n(x))$ as x ranges over X bears its $\beta\phi$ topology. Here the BK topology induced upon S is given by the norm

$$\left\|\left(E_n(x)\right)\right\| = \|x\| \qquad x \in X \quad .$$

A space S of sequences is called symmetric if the sequence s_π is in S for every s is in S and every permutation π on the set of indices. Here s_π is the sequence given by

$$s_\pi = \left(s(\pi(1)), s(\pi(2)), \dots\right) \quad .$$

Symmetric sequence spaces are considered in the 1934 paper of Köthe and Toeplitz [3], in the three papers of Garling [4 - 6] and two papers of the author [8, 9]. Besides ϕ, ω and the ℓ^p-spaces, two additional types of symmetric sequence spaces, Lorentz sequence spaces and Orlicz sequence spaces have been the object of intense investigation. See, for example, [7].

The purpose of this paper is to consider the relation between the combinatorial structure of a sequence space and its $\beta\phi$ topology. The departure from previous work is our greater emphasis upon the combinatorial properties of the space. In particular, we study the following target problem: Is every symmetric sequence space S barrelled in its $\beta\phi$ topology? We shall prove that the answer to this problem is "yes" for three of the four categories of symmetric spaces, but the answer in general is "no." The main positive result in this paper is Theorem 5.3 which asserts that every symmetric space of bounded sequences which contains a nonconvergent sequence is barrelled in its $\beta\phi$ topology, and the $\beta\phi$ topology coincides with the relative topology of the BK-space m of bounded sequences. This is a generalization of a result of Seever [13] for the particular space of finitely valued sequences. The main negative result is an example of a nonseparable symmetric BK-space which is not barrelled in its $\beta\phi$ topology.

2. Symmetry and the Classification of Symmetric Sequence Spaces

For s any sequence and π any permutation of indices, the sequence s_π is given by the formula

$$s_\pi(j) = s(\pi(j)) \qquad j \in N.$$

A sequence space S is called symmetric if s_π is in S whenever s is in S.

For any sequence space S the α-dual or Köthe dual of S is the space S^α determined by the equation

$$S^\alpha = \left\{ t : \sum_j |s(j)t(j)| < \infty \ , \ \forall s \in S \right\} \quad .$$

A sequence space S is called perfect if $S^{\alpha\alpha} = S$. Köthe and Toeplitz [3] showed in 1934 that if S is perfect and symmetric then $S = \phi$, $S = \omega$, $S = m$ or $\ell^1 \subseteqq S \subsetneqq c_0$. This permits us to classify symmetric sequence spaces S which may not be perfect in terms of S^α (which is perfect).

S is very large if $S^\alpha = \phi$

S is large if $S^\alpha = \ell^1$

S is medium if $\ell^1 \subsetneqq S \subsetneqq c_0$

S is small if $S^\alpha = m$.

If S is a symmetric sequence space and $S^\alpha = \omega$ then $S \subset \phi$ so either $S = \phi$ or $S = \left\{ s \in \phi : \sum_j s(j) = 0 \right\}$. Henceforth we assume that all sequence spaces mentioned contain ϕ. Thus if $S^\alpha = \omega$ and $S \supset \phi$, $S = \phi$; the $\beta\phi$ topology on ϕ is the strongest locally convex topology whose various properties are well known.

We shall see that for small, large and very large symmetric sequence spaces the $\beta\phi$ topology is the relative FK topology of ℓ^1, m and ω respectively, and all of these spaces are barrelled. On the other hand, the collection of medium spaces admits a variety of topologies, some of which are not barrelled.

3. Very Large Symmetric Sequence Spaces ($S^{\alpha} = \phi$)

It is easy to see that a symmetric sequence space is very large if and only if it contains an unbounded sequence. The space ω is very large, and seems to be the only very large space mentioned in the literature. Here is an example which shows that many very large symmetric sequence spaces are possible.

3.1 Example of a very large sequence space distinct from ω. Let D consist of all finite linear combinations of rational sequences. Then D is a very large symmetric sequence space. If u is a sequence of real numbers which is linearly independent over the rationals, then u is not in D. To see this suppose

$$u = \sum_{n=1}^{k} a_n v_n$$

where each v_n is a sequence of rationals and each a_n is a real number $n = 1, 2, \ldots, k$. But this means that each u(j) is a finite rational combination of $\{a_1, \ldots, a_k\}$ contradicting the assumption that u is linearly independent over the rationals.

3.2 Theorem. If S is a symmetric sequence space which contains an unbounded sequence, then

(a) S is very large;

(b) the $\beta\phi$ topology on S is the relative topology of ω (the product topology);

(c) S is barrelled in the $\beta\phi$ topology.

Proof. We omit the straightforward proof of conclusion (a).

Conclusion (b) follows from the proof of Proposition 2 of [5] which does not use the fact that (e_n) forms a basis for the space.

(c) First we shall prove that if B is an S-bounded subset of ϕ then

(i) For each n

$$\sup\{|x(n)| : x \in B\} = M_n < \infty \ .$$

(ii) There is N such that for each x in B and $j > N$, $x(j) = 0$.

Assertion (i) is true since by our standing assumption e_n is in S for each n. To establish (ii), we assume the contrary, for the sake of obtaining a contradiction. This means we assume there is a sequence (x_n) in B and a sequence of indices (i_n) such that $i_n > i_{n-1} + 1$ and $x_n(i_n)$ is the last nonzero term in x_n for each n. Let v be an unbounded sequence in S. We define a permutation θ on the set of indices by induction. For $n < i_1$ let $\theta(n) = n$; let $\theta(i_1)$ be the smallest index h_1 such that

$$\left|v(h_1)x_1(i_1)\right| > 1 + \sum_{j<i_1} \left|v(\theta(j))\right| M_j \quad .$$

If $\theta(n)$ has been defined for $n < i_k$ let $\theta(i_k)$ be the smallest index h_k such that

$$\left|v(h_k)x_k(i_k)\right| > k + \sum_{j<i_k} \left|v(\theta(j))\right| M_j \quad .$$

Finally, let $\theta(i_k + 1)$ be the smallest index in the complement of $\{\theta(j) : j \leqslant i_k\}$ to ensure that θ is onto. If $\omega = v_\theta$ then

$$\left|\sum_j x_n(j)v_\theta(j)\right| \geqslant \left|x_n(i_n)v\,\theta(i_n)\right| - \left|\sum_{j<i_n} x_n(j)v_\theta(j)\right|$$

$$\geqslant \left|x_n(i_n)\,v\,\theta(i_n)\right| - \sum_{j<i_n} \left|x_n(j)\right| M_j$$

$$> n \ .$$

This shows ω is not bounded on B, contradicting the fact that B is S-bounded.

Since the $\beta\phi$ topology on S is the relative ω topology, it follows that the dual space of S is the space ϕ with the natural duality. If B is an S-bounded subset of ϕ, then it satisfies (i) and (ii) so it is ω-bounded; see [3]. But the $\beta\phi$ topology on ω is an FK topology so that it is barrelled. Therefore, B is equicontinous. Since S-bounded implies S-equicontinuous it follows that S is barrelled. □

4. Small Symmetric Sequence Spaces ($S^{\alpha} = m$)

Examples of small symmetric sequence spaces are ℓ^p $(0 < p < 1)$. In [10] it is shown that the intersection of all small symmetric sequence spaces is ϕ. In other words, for each sequence u not in ϕ there is a small symmetric sequence space which does not contain u.

4.1 Lemma. Suppose S is a symmetric sequence space which properly contains ϕ but is contained in ℓ^1. If A is an unbounded subset of m, i.e., if

$$\sup\left\{\sup_j |x(j)| : x \in m\right\} = \infty ,$$

then there is s in S such that

$$\sup\left\{\left|\sum_j s(j)x(j)\right| : x \in A\right\} = \infty . \tag{4.1}$$

Proof. For each $k = 1, 2, \ldots,$ let

$$M(k) = \sup\left\{|x(k)| : x \in A\right\} .$$

If for any k, $M(k) = \infty$ then we may take s to be e and conclude the proof. Thus for the remainder of the proof we may assume $M(k) < \infty$ for each k. Of course, since A is unbounded in m it follows that $\sup_k M(k) = \infty$.

Let t be any sequence in S but not in ϕ. Let $h_1 < h_2 < \ldots$ be a sequence of positive integers such that for each j, $h_j - h_{j-1} > 1$ and $|t(h_j)| < |t(h_{j-1})|$. Let π be the permutation on the integers which interchanges h_{2n-1} and h_{2n} for all $n = 1, 2, \ldots$ and leaves other integers the same. If $v = t - t_{\pi}$ then v is in S, $v(j) = 0$ for $j \notin \{h_1, h_2, \ldots\}$ and $v(h_{2n-1}) = -v(h_{2n}) \neq 0$ for $n = 1, 2, \ldots$. Let $\{n_1, n_2, \ldots\}$ be a sequence of integers for which

$$\sum_{j>m} |v(h_{2n_j-1})| + |v(h_{2n_j})| < 2^{-(m+1)} |v(h_{n_m})| . \tag{4.2}$$

This is possible because $S \subset \ell^1$. Denote by θ the permutation which interchanges

h_{2n_j-1} and h_{2n_j} and leaves the remaining integers unchanged. If $u = \frac{1}{2}(v - v_\theta)$, then u is in S, $u(j) = 0$ for $j \notin \{h_{2n_j-1}, h_{2n_j}, j = 1, 2, \ldots\}$ and $u(h_{2n_j-1}) = -u(h_{2n_j}) \neq 0$. Denote $|u(h_{2n_k-1})|$ by a_k. Then because of (4.2)

$$\sum_{k>m} a_k < 2^{-m} a_m \quad .$$

We shall now define a sequence s which is a permutation of u and satisfies (4.1). Let x_1 be any sequence in A such that

$$\|x_1\| > \max\{1/a_1, M(1), M(2)\} + 1 \quad .$$

Here $\| \ \|$ denotes the norm in m, the sup-norm. Let m_1 be the smallest positive integer such that

$$|x_1(m_1)| > \|x_1\| - 1/2 \ .$$

Since $|x_1(m_1)|$ is larger than M(1) and M(2) it follows that $m_1 > 2$. Let

$$s(1) = -(\operatorname{sgn} x_1(m_1))a_1$$

$$s(j) = 0 \qquad 1 < j < m_1$$

$$s(m_1) = (\operatorname{sgn} x_1(m_1))a_1 \ .$$

Suppose we have defined x_h, m_h for $h < n$ and $s(j)$ for $j \leqslant m_{n-1}$. Let x_n in A be such that

$$\|x_n\| > 1 + \max\left\{a_n^{-1}(1 - 2^{-n})^{-1}\left(2^n + \sum_{j<m_{n-1}} |s(j)|\, M(j) + a_n M(m_{n-1} + 1)\right), M(1), \ldots, M(m_{n-1} + 2)\right\} \quad .$$

Let m_n be the smallest positive integer such that

$$|x_n(m_n)| > \|x_n\| - 2^{-n} \quad .$$

Note that $m_n > m_{n-1} + 2$. Let

$$s(m_{n-1}+1) = (-\operatorname{sgn} x_n(m_n))a_n$$

$$s(j) = 0 \qquad m_{n-1}+1 < j < m_n$$

$$s(m_n) = (\operatorname{sgn} x_n(m_n))a_n \quad .$$

Then s is a permutation of u since it exhausts the nonzero elements $\pm a_n$ and contains infinitely many 0's as well. This implies $s \in S$. For each n we have

$$\left|\sum_j s(j)x_n(j)\right| \geqslant \left|s(m_n)x_j(m_n)\right| - \sum_{j<m_n} |s(j)x_n(j)| - \sum_{j>m_n} |s(j)x_n(j)|$$

$$\geqslant a_n(\|x_n\| - 2^{-n}) - \sum_{j<m_{n-1}} |s(j)|M(j) - a_n M(m_{n-1}+1) - \|x_n\| \sum_{j>m_n} |s(j)|$$

$$\geqslant a_n(\|x_n\| - 2^{-n}) - \sum_{j<m_{n-1}} |s(j)|M(j) - a_n M(m_{n-1}+1) - \|x_n\| 2^{-n} a_n$$

$$\geqslant a_n\left((1-2^{-n})\|x_n\| - 2^{-n}\right) - \sum_{j<m_{n-1}} |s(j)|M(j) - a_n M(m_{n-1}+1)$$

$$\geqslant 2^n - 2^{-n} a_n \quad .$$

Consequently we conclude that (4.1) holds for s. □

4.2 Theorem. If S is a small symmetric sequence space then

(a) The $\beta\phi$ topology on S is the relative topology of the BK space ℓ^1.

(b) S is barrelled in the $\beta\phi$ topology.

Proof. If B is an S-bounded subset of ϕ then by Lemma 4.1 B is absorbed by the set $U_\infty = \{x \in \phi : \sup_n |x(n)| \leq 1\}$. Since S is contained in ℓ^1, U_∞ is S-bounded. Therefore, the $\beta\phi$ topology on S is determined by the norm

$$p_{U_\infty}(u) = \sup\left\{\sum_j u(j)x(j) : x \in U_\infty\right\}$$

$$= \sum_j |u(j)| \quad .$$

This confirms conclusion (a).

Conclusion (b) now follows from Lemma 4.1 just as (c) of Theorem 3.2 follows from (i) and (ii) in the proof of that Theorem. Since the $\beta\phi$ topology on S is the relative topology of ℓ^1, the dual space of S is m. If B is an S-bounded subset of m then by Lemma 4.1 it is uniformly bounded hence equicontinuous. Therefore, S is barrelled. □

5. Large Symmetric Sequence Spaces ($S^{\alpha} = \ell'$)

There are three classes of large symmetric sequence spaces

I $\quad S \subset c_0$

II $\quad S \subset c$ convergent sequences, but $S \not\subset c_0$

III $\quad S \subset m$, but S contains a nonconvergent sequence.

We first consider large symmetric sequence spaces of the first class. An example of such a space not c_0 is $c_0 \cap D$ where D is given by 3.1.

5.1 Lemma. Suppose S is a large symmetric sequence space. A subset B of ℓ^1 is S-bounded if and only if

$$\sup\left\{\sum_j |y(j)| : y \in B\right\} < \infty \quad . \tag{5.1}$$

Proof. Condition (5.1) implies B is bounded in the BK topology of ℓ^1 hence m-bounded. Since $S^{\alpha} = \ell^1$, $S \subset S^{\alpha\alpha} = m$ so B is S-bounded. This shows (5.1) is sufficient.

If x is in S then the set $\langle x \rangle$ consisting of all sequences x_π where π ranges over all permutations is ℓ^1-bounded because $\langle x \rangle$ is uniformly bounded. Since $\langle x \rangle$ is bounded, by Satz 1, §5 of [3] it is completely bounded. Hence, if B is an S-bounded subset of ℓ^1, so is $\langle B \rangle = \{y_\pi : y \in B,\ \pi \text{ is a permutation}\}$. This is because for x in S

$$\sup\left\{\left|\sum_j x(j) y_\pi(j)\right| : y \in B\right\} = \sup\left\{\left|\sum_j u(j) y(j)\right| : u \in \langle x \rangle,\ y \in B\right\} < \infty.$$

Now assume, for the sake of obtaining a contradiction, that B is an S-bounded subset of ℓ^1 which does not satisfy (5.1). For each n let x_n in B satisfy

$$\sum_j |x_n(j)| > 4^n + 1$$

Since each x_n is in ℓ^1 we can find a sequence (π_n) of permutations and a sequence (M_n) of disjoint subsets of indices such that

$$\sum_{j \notin M_n} |(x_n)_{\pi_n}(j)| < 1.$$

It follows that

$$\sum_{j \in M_n} |(x_n)_{\pi_n}(j)| > 4^n$$

for each n. Each $(x_n)_{\pi_n}$ is a member of the set <B> which is S-bounded. Therefore, the partial sums of $\sum_n 2^{-n}(x_n)_{\pi_n}$ form a Cauchy sequence in the $\sigma(\ell^1, S)$ topology on ℓ^1. By Satz 2, §4 of [3] there is x in ℓ^1 such that $\sum_n 2^{-n}(x_n)_{\pi_n} = x$ in the $\sigma(\ell^1, S)$ topology. Since $\phi \subset S$, $\sum_n 2^{-n}(x_n)_{\pi_n}(j) = x(j)$ for each j, we have

$$\sum_{j \in M_n} |x(j)| \geqslant 2^{-n} \sum_{j \in M_n} |(x_n)_{\pi_n}(j)| - \sum_{m \neq n} 2^{-m} \sum_{j \in M_n} |(x_m)_{\pi_m}(j)|$$

$$\geqslant 2^n - 1$$

This contradicts the fact that x is in ℓ^1.□

The following theorem follows from Lemma 5.1 very much as Theorem 4.2 follows from 4.1. Therefore, we omit the proof.

5.2 Theorem. If S is a large symmetric sequence space which is contained in c_0 then

(a) The $\beta\phi$ topology on S is the relative topology of the BK space c_0;

(b) S is barrelled in the $\beta\phi$ topology.

If S is of class II then $S = [e]$ the span of $e = (1,1,\ldots)$ or $S = T \oplus [e]$ where T is a symmetric sequence space which is small, medium or large of class I. This S is barrelled if and only if T is. This essentially reduces the study of large symmetric sequence spaces of class II to those which are small, medium, or large, class I.

For symmetric sequence spaces of class III we have the following result.

5.3 Theorem. If S is a large symmetric sequence space which contains a divergent sequence then

(a) The $\beta\phi$ topology on S is the relative topology of m;

(b) S is dense in m;

(c) S is barrelled in the $\beta\phi$ topology.

The remainder of this section is devoted to the proof of 5.3. First we establish several lemmas.

5.4 Lemma. Suppose S is a symmetric space of sequences which contains ϕ and also contains a divergent sequence. For each subset M of indices and $\epsilon > 0$ there is a sequence $e_{M,\epsilon}$ in S such that (a) $|e_{M,\epsilon}(j)| < \epsilon$ for $j \notin M$; (b) $|e_{M,\epsilon}(j) - 1| < \epsilon$ for $j \in M$; (c) $e_M - e_{M,\epsilon} \in c_0$ where $e_M(j) = 1$ for $j \in M$ and $e_M(j) = 0$ for $j \notin M$.

Proof. We first prove the lemma under the assumption that M is infinite and has an infinite complement. Let s be a bounded nonconvergent sequence in S. Let $h_1 < h_2 < \ldots$ and $k_1 < k_2 < \ldots$ be two sequences of indices such that $h_n < k_n < h_{n+1}$ for each n while $\lim_n s(h_n) = a$ and $\lim_n s(k_n) = b$ exist and are distinct. Let π be the permutation which interchanges h(n) and k(n) for $n = 1, 2, \ldots$ and leaves the other integers the same. Let $t = s - s_\pi$; then $t(h_n) = -t(k_n)$, $\lim_n t(h_n) = a - b$, $\lim_n t(k_n) = b - a$. If $u = (a-b)^{-1} t$ we have $\lim_n u(h_n) = 1$ and $\lim_n u(k_n) = -1$. Given $\epsilon > 0$, let N be such that $|u(h_n) - 1| < \epsilon/2$ and $|u(k_n) + 1| < \epsilon/2$ if h_n or $k_n > N$. Let v be the sequence for which $v(j) = 0$ for $j \leqslant N$ and $v(j) = u(j)$ for $j > N$. Since S contains ϕ, v is in S. For simplicity we shall assume that h_1 and k_1 are greater than N. Let θ be a permutation on the indices which (a) leaves each index in $\sim(\{h_n\} \cup \{k_n\})$, the complement of $\{h_n\} \cup \{k_n\}$, the same; (b) maps h_{2n} onto k_n for each $n = 1, 2, \ldots$; (c) maps k_n onto h_{2n} for each $n = 1, 2, \ldots$; (d) leaves each h_{2n-1} unchanged. If $w = (v + v_\theta)/2$ then $w(j) = 0$ for j in $\sim(\{h_n\} \cup \{k_n\})$; $|w(j)| = |v(k_n) + v(h_{2n})|/2 \leqslant |v(k_n) + 1|/2 + |v(h_{2n}) - 1|/2 < \epsilon/2$ for $j = k_n$; $|w(j)| = |u(h_{2n}) + u(k_n)|/2 < \epsilon/2$ for $j = h_{2n}$; while for $j = h_{2n-1}$, $|v(j) - 1| = |v(h_{2n-1}) - 1| < \epsilon/2$.

Since $\{h_{2n-1}\}$ is an infinite subset of indices with an infinite complement, there is a permutation ρ which takes $\{h_{2n-1}\}$ onto M and the complement of $\{h_{2n-1}\}$ onto the complement of M. If $e_{M,\epsilon} = w_\rho$, $e_{M,\epsilon}$ satisfies (a), (b) and (c).

If M is a finite set of indices then $e_M \in \phi \subset S$. If M has a finite complement let σ be the permutation of indices which maps $\{h_{2n-1}\}$ onto $\sim\{h_{2n-1}\}$ and let $e_{M,\epsilon} = w + w_\sigma - e_{\sim K}$. □

The following lemma is found on p. 108 of [12] as well as in the book of Köthe [2] and the works of Bourbaki [1].

5.5 Lemma. Let (f_n) be a sequence of continuous linear functions on m and let (M_j) be a sequence of finite subsets of indices. There is a set M of indices which is a union of a subsequence of (M_j) such that whenever s is a member of m with support on M we have

$$f_n(s) = \sum_{j \in M} s(j) f_n(e_j) \quad .$$

Conclusion of the proof of Theorem 5.3.

Let m_0 be the space of finitely valued sequences. It is well known (see, e.g., [15]) that m_0 is dense in m. It is clear that m_0 is the linear span of all sequences e_M as M ranges over all sets of indices. By Lemma 5.4 if S is a