On the Parity of the Generalized Frobenius Partition Functions

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Abstract

In his 1984 Memoir of the American Mathematical Society, George Andrews defined two families of functions, $\phi_k(n)$ and $c\phi_k(n)$, which enumerate two types of combinatorial objects which Andrews called generalized Frobenius partitions. These objects are generalizations of Frobenius partitions, which are bijective to ordinary partitions. Ordinary partitions and their functions have been popularized by Ramanujan in the 20^{th} century, which encouraged mathematicians to study them in the following years and to find what are called *Ramanujan–like congruences.* As part of his Memoir, Andrews proved a number of Ramanujan–like congruences satisfied by specific generalized Frobenius partition functions, and like Ramanujan, he paved the way for other authors to prove similar results for these functions, usually for a fixed parameter k. In this thesis, we gently introduce the reader to the ordinary partition and partition function, as well as the two families of generalized Frobenius partition functions that Andrews introduced. Our goal is to identify an **infinite** family of values of k such that $\phi_k(n)$ is even for all n in a specific arithmetic progression; in particular, our primary goal in this work is to prove that, for all positive integers ℓ, all primes $p \geq 5$, and all values r, $1 \leq r \leq p-1$, such that $24r+1$ is a quadratic nonresidue modulo p,

$$\phi_{p\ell-1}(pn+r) \equiv 0 \pmod{2}$$

for all $n \geq 0$. Such a result, which holds for $\phi_k(n)$ for infinitely many values of k, is rare in the study of arithmetic properties satisfied by generalized Frobenius partitions, primarily because of the unwieldy nature of the generating functions in question. We have also found and proven other results that pertain to generalized Frobenius partitions, which we will also be sharing in this thesis.

Contents

1 Introduction

In his 1984 AMS Memoir, George Andrews [1] defined two families of combinatorial objects known as *generalized Frobenius partitions.* To understand these objects, some preliminary concepts and ideas must be introduced. We start with the ordinary partition, which happens to be a special case of the generalized Frobenius partition.

Definition 1.1. *A partition λ of a positive integer n is a nonincreasingly ordered sum of positive integers $\lambda_1 + \cdots + \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i's are called the parts of the partition.*

For example, the integer 5 can be written as $4+1$, so the sum $4+1$ is a partition of 5. Note that by definition, the sum has to be written in a **nonincreasing** order, so $1+4$ is the same as $4+1$, but both of them will be represented by the latter since they are in the same equivalence class. To avoid any confusion, we will write the 7 partitions of 5:

$$5$$

$$4+1$$

$$3+2$$

$$3+1+1$$

$$2+2+1$$

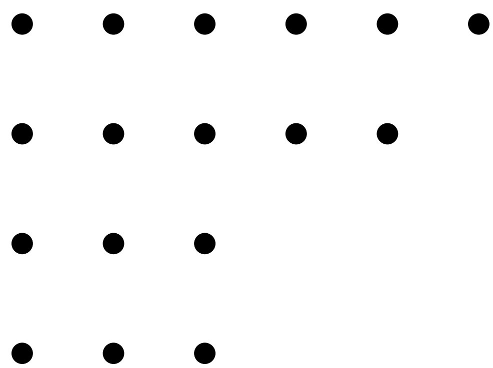
$$2+1+1+1$$

$$1+1+1+1+1$$

As we have seen, you can just represent a partition of a positive integer n by writing the parts as a sum. Another way of representing a partition is using what is called a *Ferrers graph*.

Definition 1.2. *Let $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ be a partition of n. The Ferrers graph G_λ corresponding to the partition is a set of rows made up of nodes where the i^{th} row has λ_i nodes for $1 \leq i \leq r$, and the nodes are left justified.*

We can explain it using some examples. The sum $6 + 5 + 3 + 3$ is a partition of 17, and can be represented by:



So we start with the first part $\lambda_1 = 6$ in the sum and place λ_1 many nodes in the first row of the graph, and then $\lambda_2 = 5$ many nodes in the second row of the graph, and so on until we reach the last part in the partition. Note that since each partition can be represented uniquely with a sequence of nonincreasing numbers, then it can also be represented uniquely with a Ferrers graph.

So far, we have introduced ordinary integer partitions and represented them in two ways. Using what we know so far, we can introduce a very important function in number theory, the ordinary integer partition function.

Definition 1.3. *Let n be a positive integer. The partition function $p(n)$ is the function that counts the number of ordinary partitions of n.*

For example, we previously wrote all the partitions of the integer 5, and noted that there were 7 of them, therefore $p(5) = 7$. This counting function is very important in number theory and has been studied extensively by many mathematicians, including Srinivasa Ramanujan. Ramanujan set out to find properties of this function by proving what are now called Ramanujan type congruences. He looked at infinite sequences of positive integers $p(an+b)$ under a specific modulus, laying down the foundation for other mathematicians to explore and find similar congruences. One of his results is the following theorem:

Theorem 1.4. *For all* $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod{5}.$$

This means that the numbers $\{p(4), p(9), p(14), p(19), p(24), p(29), \cdots\}$ are all congruent to $0 \pmod{5}$ (they are all multiples of 5). To be able to study the sequence of integers $p(n)$, mathematicians do not calculate $p(n)$ for every positive integer n. Instead, they use what is called the generating function of the sequence $p(n)$ due to Euler.

Definition 1.5. *Let* a_n *be a sequence of numbers. The generating function of the sequence is defined as the power series* $G(a_n, q)$ *defined by*

$$G(a_n, q) = \sum_{n=0}^{\infty} a_n q^n$$

where the element of the sequence a_i *is the coefficient of* q^i. *Note that* $|q| < 1$ *must be true for this series to converge, and so we will always assume that.*

The form of the generating function of $p(n)$ that we will use is:

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{\prod_{m=1}^{\infty} (1-q^m)}. \tag{1.1}$$

A cleaner way to write this is using what are called Pochhammer symbols. These symbols are going to play an important role in this paper, especially in our proofs. They are defined as follows:

Definition 1.6 (Pochhammer symbol)**.** *For* $n \geq 1$,

$$(A;q)_n = (A)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}),$$

$$(A;q)_\infty = (A)_\infty = \lim_{n\to\infty}(A;q)_n.$$

We also assume that $(A)_0 = 1$.

This means that the generating function in (1.1) for the sequence $p(n)$ is

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

A proof of this equality can be found in chapter 5 of [2]. We will be using Pochhammer symbols throughout this paper whenever we can, and so the reader should keep in mind what they represent throughout. A lemma that will be useful related to Pochhammer symbols is shown below.

Lemma 1.7. *For all primes* p,

$$(q;q)_\infty^p \equiv (q^p;q^p)_\infty \pmod{p}.$$

Proof. Recall that

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots,$$

and this means that

$$(q;q)_\infty^p = (1-q)^p(1-q^2)^p(1-q^3)^p\cdots \tag{1.2}$$

Now if we look at the binomial expansion of $(1-q^n)^p$, we can write

$$\begin{aligned}(1-q^n)^p &= \sum_{k=0}^{n} \binom{p}{k} (-q^n)^k \\ &= \binom{p}{0}(-q^n)^0 + \binom{p}{1}(-q^n)^1 + \cdots + \binom{p}{p}(-q^n)^p\end{aligned}$$

Note that all coefficients in the expansion above have a factor of p except for $\binom{p}{0} = \binom{p}{p} = 1$, which means that

$$(1-q^n)^p \equiv (1-q^{np}) \pmod{p},$$

proving the assertion. □

2 Generalized Frobenius Partitions

Now that we are familiar with some concepts from partition theory, we move along to Frobenius symbols, which were introduced by Frobenius in the year 1900 while he was studying the representation theory of the symmetric group. He wanted to represent integer partitions in a different way, so he found a bijective mapping between his symbols and Ferrers graphs.

Definition 2.1. *A Frobenius symbol or Frobenius partition of* n *is a two rowed array of non-negative integers*

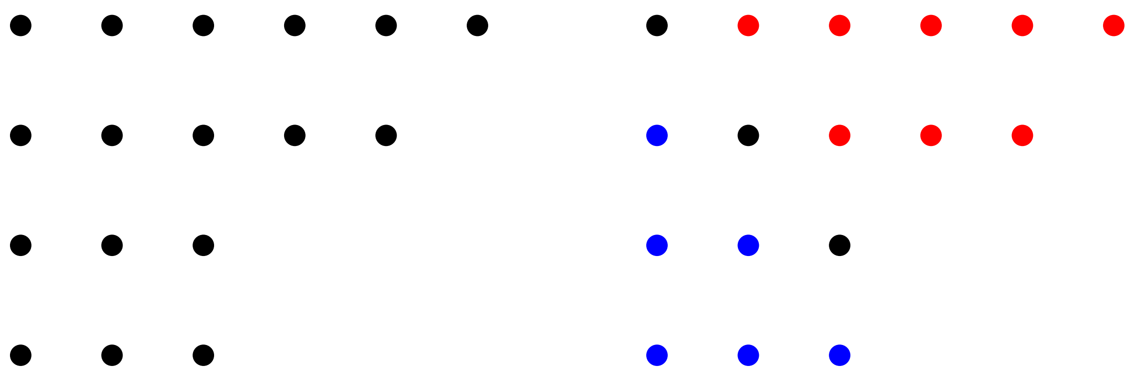
$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

wherein each row is of the same length $a_1 > a_2 > \cdots a_r \geq 0, b_1 > b_2 > \cdots > b_r \geq 0$ *and*

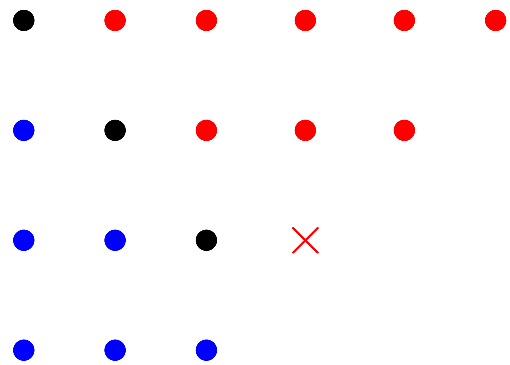
$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

Frobenius arrived at these symbols by starting with a Ferrers graph of a partition

π. He deleted the main diagonal which possesses say r dots. The remaining rows of dots to the right of the diagonal are enumerated to provide one strictly decreasing sequence of r nonnegative integers (the r^{th} such row might be empty thus producing 0, which is why our a_i's and b_i's are allowed to equal zero). The remaining dots below the diagonal are enumerated by columns to provide a second strictly decreasing sequence of r nonnegative integers. The resulting two sequences, which are both of length r, are then presented in the Frobenius notation given in the definition above. We will show by example how each F-partition corresponds to a unique Ferrers graph. We have already seen the Ferrers graph of the partition $6 + 5 + 3 + 3$ of 17.



We start with the graph on the left. We use some color coding to make a distinction between the different numbers involved in the Frobenius partition. Note that the three black dots (r dots) are the diagonal of this Ferrers graph. This means that there must be three a_i's as well as three b_i's. However, we notice that there are now three blue columns to the left of the diagonal (representing the b_i's), but only two red rows to the right of the diagonal (representing the a_i's). Now the number of dots in each one of the rows/columns represents what the corresponding a/b is going to be. Recall that in Definition 2.1, each row in the matrix had r elements, so we adjust the Ferrers graph as shown below to emphasize that one of the elements in the matrix must be a zero.



Now looking at this third graph, we note that we already saw that $r = 3$, because of the number of dots in the diagonal, and so there should be three b_i's as well as three a_i's. The b_i's are 3, 2, and 1 because of the three columns below the diagonal that have 3, 2, and 1 dots respectively. As for the a_i's, the sequence is 5, 3, and 0 because of the three red rows to the right of the diagonal (including the third empty row, which corresponds to 0 in the sequence). This means that the corresponding Frobenius symbol is:

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

George Andrews was inspired by these Frobenius symbols to generalize them and introduce new combinatorial objects which he called generalized Frobenius partitions in his 1984 Memoir of the AMS [1]. In this thesis, we focus our attention on two general classes of Frobenius partitions that he introduced:

- Generalized Frobenius partitions with up to k repetitions
- Generalized Frobenius partitions with k colors

2.1 Generalized Frobenius Partitions With Up To k Repetitions

Recall that Definition 2.1 restricted the numbers in each row of the Frobenius symbol to be strictly decreasing. A generalized Frobenius partition with up to k repetitions is similar to the ordinary Frobenius partition, except that each row is allowed up to k repetitions for each integer. This means that the numbers in each row are allowed to be nonincreasing (as opposed to being strictly decreasing). For example, the generalized Frobenius partitions of the integer 4 with up to 3 repetitions are:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

We first note that there are 11 of them, and that now the sequences in the rows are nonincreasing due to allowing the repetitions, so the only difference in this generalization is that

$$a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$$

$$b_1 \geq b_2 \geq \cdots \geq b_r \geq 0.$$

Each one of these partitions still represent the integer 4. For example, the first partition only has one nonzero element, which is 1. It also has 3 elements in each row, and if we sum the number of elements in a row with the sum of all elements in the matrix, we get $3 + 1 = 4$.

2.2 Generalized Frobenius Partitions With k Colors

Now we introduce generalized Frobenius partitions with k colors. Instead of colors, we will be indexing the integers with the "colors" $1, 2, \cdots, k$. For example, $1_1, 1_2, \cdots, 1_k$ are all different elements with k different colors. The elements in each row will still be arranged in strictly decreasing order, but the relationship will be defined differently. We say that two elements $h_i > \ell_j$ precisely when $h > \ell$, or when $h = \ell$ and $i > j$. For example,

$$2_3 > 1_3, \qquad 2_1 > 1_3, \qquad \text{and} \qquad 2_4 > 2_3.$$

Let us look at the generalized Frobenius partitions of the integer 2 with 2 colors:

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix} \quad \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix} \quad \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix} \quad \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix} \quad \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix} \quad \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix} \quad \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix} \quad \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix} \quad \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}$$

Note that there are nine of these partitions in total. It is worth mentioning that if we allow up to one repetition for the generalized Frobenius partition or one color for the colored generalized Frobenius partition, we are left with the ordinary Frobenius partition, which we have shown is isomorphic to the ordinary partition. Similar to the ordinary partition function $p(n)$, the generalized Frobenius partitions also have functions that count how many generalized Frobenius partition each positive integer has. They will be the focus of this paper, and we will be looking at specific Ramanujan type congruences that they satisfy.

2.3 Generating Functions

The two main functions are $\phi_k(n)$ and $c\phi_k(n)$, which count the generalized Frobenius partitions of n with up to k repetitions, and the generalized Frobenius partitions of n with k colors respectively. Andrews [1] also introduced the generating functions for the functions $\phi_k(n)$ and $c\phi_k(n)$, which are going to be useful for us throughout this thesis. For the sequence $\phi_k(n)$, he proved the following theorem in his Memoir:

Theorem 2.2. *For* $|q| < 1$,

$$\Phi_k(q) = \frac{\sum_{m_1,m_2,\cdots,m_{k-1}=-\infty}^{\infty} \zeta^{(k-1)m_1+(k-2)m_2+\cdots+m_{k-1}} q^{Q(m_1,m_2,\cdots,m_{k-1})}}{(q;q)_\infty^k}$$

where $\zeta = e^{\frac{2\pi i}{k+1}}$, *and*

$$Q(m_1, m_2, \cdots, m_{k-1}) = m_1^2 + m_2^2 + \cdots + m_{k-1}^2 + \sum_{1\le i\le j\le k-1} m_i m_j.$$

He also showed in his Memoir that this generating function can be written in a different form shown in the theorem below.

Theorem 2.3. *Let* $|q| < 1$,

$$\Phi_k(q) = \frac{\sum_{\substack{j,r=-\infty \\ r\ge(k+1)|j|}}^{\infty} (-1)^{r+kj} q^{\binom{r+1}{2}-\binom{k+1}{2}j^2}}{(q;q)_\infty^2 (q^{k+1};q^{k+1})_\infty}.$$

As for $c\phi_k(n)$, Andrews starts with the lemma below, and uses it to show that the generating function of $c\phi_k(n)$ can be written in a form similar to the one for $\phi_k(n)$.

Lemma 2.4. *Let* $|q| < 1$, *then* $C\Phi_k(q)$ *is the constant term in*

$$CG_k(z) = \prod_{n=0}^{\infty} (1 + zq^{n+1})^k (1 + z^{-1}q^n)^k.$$

This means that after expanding this product and factoring out all coefficients of z^n, we will arrive at $a_0 z^0 + a_1 z + a_2 z^2 + \cdots$, where each a_i is implicitly a function of q. The function we are interested in is $C\Phi_k(q) = a_0$, the coeffecient of z^0. Andrews then performs some algebra and uses Jacobi's Triple Product Identity to arrive at the following:

Theorem 2.5. *For* $|q| < 1$,

$$C\Phi_k(q) = \frac{\sum_{m_1,m_2,\cdots,m_{k-1}=-\infty}^{\infty} q^{Q(m_1,m_2,\cdots,m_{k-1})}}{(q;q)_\infty^k}$$

where

$$Q(m_1, m_2, \cdots, m_{k-1}) = m_1^2 + m_2^2 + \cdots + m_{k-1}^2 + \sum_{1\le i\le j\le k-1} m_i m_j.$$

Note how the two generating functions for $\phi_k(n)$ and $c\phi_k(n)$ are closely related since the only difference is the root of unity, $\zeta = e^{\frac{2\pi i}{k+1}}$, in the generating function for the generalized Frobenius partition with up to k repetitions.

The object of this thesis is to prove congruences similar to the one in Theorem 1.4, but particular to ϕ_k and $c\phi_k$. Similar results have been found by other mathematicians, and we will mention a few of them below.

2.4 Previous Results on Generalized Frobenius Partitions

After introducing these generating functions in his Memoir, Andrews went on to find congruences, the first being

Theorem 2.6. *For all* $n \ge 0$,

$$\phi_2(5n + 3) \equiv c\phi_2(5n + 3) \equiv 0 \pmod 5.$$

He also proved the following result, which works for infinitely many indices and moduli:

Theorem 2.7. *For all integers* $n \ge 0$*, if* p *is a prime and* r *is an integer such that* $1 \le r \le p - 1$*, then*

$$c\phi_p(pn + r) \equiv 0 \pmod{p^2}.$$

After the Memoir was published, other mathematicians searched for different congruences related to generalized Frobenius partitions. Some focused on a specific index k, such as Kolitsch in his 1996 paper [15] where he proved the following result:

Theorem 2.8. *For all integers $\alpha \geq 0, n \geq 0$,*

$$c\phi_9(3^\alpha n + 3\lambda_{\alpha-1}) \equiv 0 \pmod{3^{2\alpha-1}}$$

when α is even, and

$$c\phi_9(3^\alpha n + 3\lambda_{\alpha-1}) \equiv 0 \pmod{3^{2\alpha}}$$

when α is odd. Here λ_α is the reciprocal of 8 modulo 3^α, and $\lambda_0 = 1$, and $\lambda_{-1} = \frac{1}{3}$.

Note that this theorem is true for infinitely many moduli because it depends on the value of α. Others have fixed both the modulus and the index, such as Hirschhorn and Sellers in their 2016 paper [12] where they proved:

Theorem 2.9. *Let $p \geq 5$ be a prime and let r be an integer, $1 \leq r \leq p-1$, such that $12r + 1$ is a quadratic nonresidue $\pmod{p}$, then for all $n \geq 0$,*

$$\phi_4(10pn + 10r + 1) \equiv 0 \pmod{5}.$$

A more general result that Garvan and Sellers proved in their 2014 paper [10] is:

Theorem 2.10. *Let p be a prime and r be an integer such that $1 \leq r \leq p-1$. If*

$$c\phi_k(pn + r) \equiv 0 \pmod{p}$$

for all $n \geq 0$, then

$$c\phi_{pN+k}(pn + r) \equiv 0 \pmod{p}$$

for all $N \geq 0, n \geq 0$.

There have been many more such results that have been proven by other mathematicians. The interested reader can see [3–9, 13, 14, 16, 18–25] for more congruence results.

As we have shown above, there exists an extensive literature on the subject of congruences satisfied by generalized Frobenius partition functions. Many of the authors we have cited above focus their work on a fixed value of the parameter k. Furthermore, very few congruences are known (mod 2), especially for the functions $\phi_k(n)$. In this thesis, we contribute to that literature and find congruences modulo 2 that are satisfied by generalized Frobenius partitions for infinitely many values of the parameter k. We highlight in the rest of this thesis how we looked for these congruences and proved them.

2.5 Producing the Sequences

We started the process of studying these partition functions by computing large lists of outputs for both ϕ_k and $c\phi_k$. You can imagine by looking at their generating functions that generating a large list of outputs is not an easy task to do by hand. This is why we resorted to compute these numbers with the help of Mathematica. We generated more than 1000 values for each of the first few values of k, but we will only show a few here below to illustrate the patterns that we noticed.

The function below was used to compute $\phi_k(n)$ for a fixed k for the first m integers.

```
Phi[k_, m_, q_] :=
 (
  mynumer = 0;
  Do[Do[mynumer =
     mynumer + (-1)^(r + (k*j)) q^(Binomial[r + 1, 2] -
          Binomial[k + 1, 2]*j^2), {r, (k + 1)*Abs[j],
     500}], {j, -500, 500}];
```

```
polyform = mynumer/
 Product[(1 - q^n)^2*(1 - q^((k + 1)*n)), {n, m}];
seriesform = Series[polyform, {q, 0, m}];
listform = CoefficientList[seriesform, q];
listform = Drop[listform, 1];
Return[listform]
)
```

Now if we run the above function for $\phi_4, \phi_6, \phi_9, \phi_{13}$, the output, when entered into a table, looks like this:

n	ϕ_4	ϕ_6	ϕ_9	ϕ_{13}
1	1	1	1	1
2	3	3	3	3
3	6	6	6	6
4	12	12	12	12
5	20	21	21	21
6	35	38	38	38
7	56	62	63	63
8	92	103	106	106
9	142	163	170	170
10	221	258	271	272
11	330	394	419	422
12	496	602	646	653
13	724	894	972	986
14	1056	1325	1454	1481
15	1512	1927	2140	2188
16	2155	2788	3126	3211
17	3028	3978	4509	4652
18	4240	5648	6462	6698
19	5866	7925	9162	9541
20	8085	11066	12912	13510
21	11035	15311	18040	18965
22	15001	21075	25056	26469
23	20234	28789	34550	36672
24	27184	39146	47388	50542
25	36276	52872	64581	69210
26	48220	71102	87581	94312
27	63736	95072	118118	127802
28	83932	126598	158574	172398
29	109974	167730	211842	231399
30	143601	221383	281825	309298

n	ϕ_4	ϕ_6	ϕ_9	ϕ_{13}
31	186670	290880	373261	411568
32	241871	380836	492462	545536
33	312154	496605	647132	720184
34	401616	645402	847332	947320
35	514872	835725	1105422	1241487
36	658155	1078808	1437345	1621565
37	838568	1387939	1862657	2110798
38	1065536	1780464	2406376	2739076
39	1349894	2276998	3099196	3543212
40	1705760	2904051	3979978	4570070
41	2149497	3693296	5096415	5877362
42	2702155	4685019	6508454	7538003
43	3388234	5927381	8289498	9641619
44	4238852	7481141	10531260	12300719
45	5290436	9419044	13345873	15653388
46	6588695	11831994	16872536	19871858
47	8187360	14828996	21281205	25167169
48	10153256	18545300	26781776	31801158
49	12564986	23142970	33629868	40093904
50	15519658	28821931	42139824	50440694
51	19131620	35821654	52693626	63323693
52	23541131	44435790	65758899	79335672
53	28913518	55015884	81902734	99198100
54	35450320	67990925	101816692	123794454
55	43389128	83873772	126337376	154197432
56	53018040	103287626	156481531	191715877
57	64676696	126976603	193476356	237935305
58	78774896	155841101	238808196	294783585
59	95795522	190954902	294267174	364590846
60	116318829	233612506	362014791	450181418

n	ϕ_4	ϕ_6	ϕ_9	ϕ_{13}
61	141027890	285355211	444648266	554961041
62	170740520	348035058	545294847	683045692
63	206418640	423853783	667703980	839387635
64	249209612	515448485	816377748	1029955021
65	300461836	625949462	996702687	1261913079
66	361778460	759095160	1215130065	1543874244
67	435041352	919315656	1479362078	1886156142
68	522481400	1111887752	1798601300	2301129584
69	626714160	1343054304	2183812358	2803583944
70	750831761	1620232208	2648065211	3411209424
71	898455050	1952182229	3206901839	4145112143
72	1073853840	2349287997	3878807777	5030481149
73	1282021436	2823794280	4685722044	6097309922
74	1528832096	3390181818	5653684352	7381317196
75	1821146752	4065499566	6813541353	8924953225
76	2167018355	4869866316	8201831147	10778666622
77	2575840444	5826931979	9861758496	13002302277
78	3058615504	6964548465	11844401052	15666840245
79	3628160212	8315404523	14210048610	18856328416
80	4299457122	9917923284	17029845486	22670263976
81	5089936210	11817131454	20387623135	27226255492
82	6019934395	14065856559	24382128626	32663274539
83	7113079988	16725912938	29129525236	39145309982
84	8396892428	19869696525	34766408906	46865796782
85	9903303452	23581793518	41453206238	56052613080
86	11669439995	27961103320	49378249591	66974098141
87	13738326800	33123011984	58762380028	79945863598
88	16159906544	39202210548	69864436068	95338947998
89	18991982436	46355622760	82987472052	113589072737
90	22301547573	54766140758	98486133685	135207665044
91	26166041880	64646551171	116775030359	160794395443
92	30675083219	76244480773	138338625156	191052043002
93	35932137690	89847644354	163742475972	226803429101
94	42056768624	105790376928	193646461800	269011413102
95	49186849832	124460628774	228819837867	318801701572
96	57481484195	146308578624	270158883355	377489681929
97	67123923392	171855925266	318707006731	446611059584
98	78325352720	201707227242	375678243270	527957785887
99	91328729438	236562210600	442484037190	623619103357
100	106413683242	277230666307	520764452663	736029536138

After compiling the list of outputs from the ϕ_k function, we set out to look for arithmetic progressions $an + b$, that when plugged into the function ϕ_k, produce an output that is congruent to 0 (mod 2), for all integers $n \geq 0$. Some were not as easy to identify as others, so we used the function below to test the first few values of a progression for divisibility by 2. Most of the time we looked for arithmetic progressions $an + b$ where $a < 50$.

```
  NModCheck[list_, m_, s_] :=
(proglist = List[];
 deadcongs = List[];
 l = Length[list];
 breaker = 1;
 Do[
  Do[
   Do[If[Mod[list[[i*r + j]], m] != 0, breaker = 0; Break[]];
    , {r, 1, (l - j)/i}];
   If[breaker == 1, AppendTo[proglist, List[i, j]], breaker = 1]
   , {j, 0, i - 1}]
  , {i, s}];
 If[Length[proglist] == 0, Print["No check"]; Return[proglist],
  Print["Entering check"]];
 Do[Do[If[
    Mod[proglist[[i]][[1]], proglist[[j]][[1]]] == 0 &&
     Mod[proglist[[i]][[2]], proglist[[j]][[1]]] ==
      Mod[proglist[[j]][[2]], proglist[[j]][[1]]],
    AppendTo[deadcongs,
     List[proglist[[i]][[1]], proglist[[i]][[2]]]]; Break[]]
   , {j, 1, i - 1}]
```

```
  , {i, 2, Length[proglist]}];
newlist = proglist;
newlist = Complement[newlist, deadcongs];
Print["The conguences are"];
Do[Print[i,
  StringForm["     ``n+`` \[Congruent] `` (mod ``) ",
   newlist[[i]][[1]], newlist[[i]][[2]], 0, m]], {i,
  Length[newlist]}]
)
```

An example output after testing the first 1000 values of each of the previous partition functions would be the following:

ϕ_4	ϕ_6	ϕ_9	ϕ_{13}
$4n+3$	$7n+3$	$5n+3$	$7n+3$
$5n+3$	$7n+4$	$5n+4$	$7n+4$
$5n+4$	$7n+6$		$7n+6$
$10n+5$			
$10n+7$			
$14n+5$			
$14n+7$			
$14n+9$			
$14n+11$			

This means that our code hypothesizes that each one of the progressions above are congruent to 0 (mod 2) for all integers $n \geq 0$ when plugged into the corresponding ϕ_k function. For example, the code predicts that

$$\phi_4(4n+3) \equiv 0 \pmod 2$$

for all integers $n \geq 0$ since $4n+3$ is shown under the ϕ_4 column. Our code has the nice quality of not reporting any arithmetic progressions that are subprogressions of ones that it has already reported. This means, for example, that if Mathematica thinks that $2n$ (all even numbers) might be a progression where a congruence is satisfied, then it will not report $4n$ (all multiples of 4), since it is already covered by $2n$.

Now we will look at the code used to generate the $c\phi_k(n)$ sequences. Below is the code used to generate the first m integers in the sequence $c\phi_4(n)$. This function was broken down into subfunctions that, when combined, produce the desired generating function that output the sequence as a list. The first two functions combined form the Q function we have seen in the definition of the generating function of $c\phi_k(n)$ in Theorem 2.5, and the third one is the actual generating function.

```
  Qfunction1[intlist_] :=
 (
 sum = 0;
 Do[sum = sum + intlist[[i]]^2, {i, Length[intlist]}];
 Return[sum]
 )
 Qfunction[intlist_] :=
(
 sum = Qfunction1[intlist];
 Do[a = intlist[[i]]; minilist = intlist[[i + 1 ;; Length[intlist]]];
   sum = sum + a*Total[minilist], {i, Length[intlist] - 1}];
 Return[sum]
 )
   Cphi4[n_, m_, q_] :=
(
 If[m > n, Return["Are you sure? m>n"]];
 \[Zeta] = Exp[(2 \[Pi]*I)/5];
 coeffs =
```

```
  FullSimplify[
   CoefficientList[
    Series[Sum[
       Sum[Sum[\[Zeta]^(3*m1 + 2*m2 + m3)*q^
          Qfunction[List[m1, m2, m3]], {m1, -n, n}], {m2, -n,
        n}], {m3, -n, n}]/Product[(1 - q^i)^4, {i, n}], {q, 0, m}],
    q]];
 coeffs = Drop[coeffs, 1];
 Return[coeffs]
 )
```

Now that we have provided the reader with an example of the sequences we generated, we are in good shape to move on to prove our main results.

3 New Congruence Results For The Generalized Frobenius Partition

3.1 Preliminaries

Before we tackle the main results in this thesis, a few preliminary results that are used in our proofs are to be shown. Note that the proofs in this section for Theorems 3.1, 3.2, 3.3, and 3.4 are all from Hirschhorn's *The Power of q* [11].

Theorem 3.1. *For* $|q| < 1$,

$$(1+a)(1+aq)(1+aq^2)\cdots = 1 + \sum_{k\geq 1} a^k \frac{q^{\frac{(k^2-k)}{2}}}{(1-q)\cdots(1-q^k)}.$$

Proof. We will provide a proof from Hirschhorn [11]. Assume $|q| < 1$, and

$$(1+a)(1+aq)(1+aq^2)\cdots = \sum_{k\geq 1} a^k c_k \tag{3.3}$$

where $c_k = c_k(q)$. (The condition $|q| < 1$ ensures that the product converges.) If in (3.3) we put aq for a, we find

$$(1+aq)(1+aq^2)(1+aq^3)\cdots = \sum_{k\geq 0} a^k q^k c_k. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\sum_{k\geq 0} a^k c_k(q) = (1+a)\sum_{k\geq 0} a^k q^k c_k(q) = c_0 + \sum_{k\geq 1} a^k (q^k c_k + q^{k-1} c_{k-1}). \tag{3.5}$$

It follows from (3.3) that $c_0 = 1$, and from (3.5) that, for $k \geq 1$,

$$c_k = q^k c_k + q^{k-1} c_{k-1},$$

or,

$$c_k = \frac{q^{k-1}}{1-q^k} c_{k-1}. \tag{3.6}$$

Thus, we find that by applying (3.6) recursively, and by using the fact that $c_0 = 1$, we get that

$$c_k = \frac{q^{\frac{k^2-k}{2}}}{(1-q)\cdots(1-q^k)},$$

and (3.3) becomes

$$(1+a)(1+aq)(1+aq^2)\cdots = 1 + \sum_{k\geq 1} a^k \frac{q^{\frac{k^2-k}{2}}}{(1-q)\cdots(1-q^k)},$$

which proves our assertion. □

Note that we can use Pochhamer notation to rewrite the statement of Theorem 3.1 as

$$(-a;q)_\infty = \sum_{k\geq 0} a^k \frac{q^{\frac{k^2-k}{2}}}{(q;q)_k}. \tag{3.7}$$

Theorem 3.2 (Jacobi's Triple Product Identity)**.** *For* $|q| < 1$,

$$(-a^{-1}q;q^2)_\infty(-aq;q^2)_\infty(q^2;q^2) = \sum_{k=-\infty}^{\infty} a^k q^{k^2}$$

Proof. If in (3.7) we replace q by q^2 and a by aq, we find

$$(-aq;q^2)_\infty = \sum_{k\geq 0} a^k \frac{q^{k^2}}{(q^2;q^2)_k}.$$

That is,

$$(1+aq)(1+aq^3)(1+aq^5)\cdots = \sum_{k\geq 0} a^k \frac{q^{k^2}}{(q^2;q^2)_k}. \tag{3.8}$$

We will now perform a three-step process that Hirschhorn calls "a pull-back of degree n". If in (3.8) we suppose $q \neq 0$ and replace a by aq^{-2n}, we obtain

$$(1+aq^{1-2n})(1+aq^{3-2n})\cdots(1+aq^{-1})(1+aq)(1+aq^3)\cdots = \sum_{k\geq 0} a^k \frac{q^{k^2-2nk}}{(q^2;q^2)_k}. \tag{3.9}$$

Let us multiply (3.9) by q^{n^2}, and recall that $n^2 = 1+3+\cdots+(2n-1)$. We obtain

$$(q^{2n-1}+a)(q^{2n-3}+a)\cdots(q+a)(1+aq)(1+aq^3)\cdots = \sum_{k\geq 0} a^k \frac{q^{k^2-2nk+n^2}}{(q^2;q^2)_k},$$

or

$$(a+q)(a+q^3)\cdots(a+q^{2n-1})(1+aq)(1+aq^3)\cdots = \sum_{k\geq 0} a^k \frac{q^{(k-n)^2}}{(q^2;q^2)_k}. \tag{3.10}$$

Now suppose $a \neq 0$, and divide (3.10) by a^n, and we obtain

$$(1+a^{-1}q)(1+a^{-1}q^3)\cdots(1+a^{-1}q^{2n-1})(1+aq)(1+aq^3)\cdots = \sum_{k\geq 0} a^{k-n}\frac{q^{(k-n)^2}}{(q^2;q^2)_k} = \sum_{k=-n}^{\infty} a^k \frac{q^{k^2}}{(q^2;q^2)_{n+k}}.$$

We can rewrite this equality as

$$(-a^{-1}q;q^2)_n(-aq;q^2)_\infty = \sum_{k\geq -n} a^k \frac{q^{k^2}}{(q^2;q^2)_{n+k}}. \tag{3.11}$$

Incidentally, although we assumed $q \neq 0$ on the way to proving (3.11), we note that it is true for $q = 0$ also. So for $|q| < 1, a \neq 0$, and taking $n \to \infty$, for each fixed k, we get

$$(q^2;q^2)_{n+k} \to (q^2;q^2)_\infty$$

And so combining this with (3.11), we get

$$(-a^{-1}q;q^2)_\infty(-aq;q^2)_\infty = \sum_{k=-\infty}^{\infty} a^k \frac{q^{k^2}}{(q^2;q^2)_\infty}$$

or,

$$(-a^{-1}q;q^2)_\infty(-aq;q^2)_\infty(q^2;q^2)_\infty = \sum_{k=-\infty}^{\infty} a^k q^{k^2}.$$

This is Jacobi's Triple Product Identity, completing the proof. □

Theorem 3.3 (Euler's Pentagonal Number Theorem). *Let* $|q| < 1$, *then*

$$(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2-k}{2}}.$$

Proof. We can prove this theorem using Theorem 3.2. We have

$$(-a^{-1}q;q^2)_\infty(-aq;q^2)_\infty(q^2;q^2)_\infty = \sum_{k=-\infty}^{\infty} a^k q^{k^2}.$$

If we replace q by $q^{\frac{3}{2}}$ then set $a = -q^{\frac{1}{2}}$, we find that

$$(q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2-k}{2}},$$

which since the left side can be seen to be $(q;q)_\infty$, is the required result. □

Theorem 3.4 (Jacobi)**.** *Let* $|q| < 1$, *then*

$$(q;q)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)q^{\frac{k^2+k}{2}}$$

Proof. We start with Jacobi's Triple Product Identity,

$$(-a^{-1}q;q^2)_\infty(-aq;q^2)_\infty(q^2;q^2)_\infty = \sum_{k=-\infty}^{\infty} a^k q^{k^2}.$$

If we replace q by $q^{\frac{1}{2}}$ and replace a by $-aq^{\frac{1}{2}}$, we obtain

$$(a^{-1};q)_\infty(aq;q)_\infty(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k a^k q^{\frac{k^2+k}{2}}.$$

This can be written as

$$(1-a^{-1})(a^{-1}q;q)_\infty(aq;q)_\infty(q;q)_\infty = \sum_{k\geq 0}(-1)^k(a^k - a^{-k-1})q^{\frac{k^2+k}{2}}.$$

Next, multiply by $a^{\frac{1}{2}}$ to obtain the identity

$$(a^{\frac{1}{2}} - a^{-\frac{1}{2}})(a^{-1}q;q)_\infty(aq;q)_\infty(q;q)_\infty = \sum_{k\geq 0}(-1)^k\left(a^{k+\frac{1}{2}} - a^{-k-\frac{1}{2}}\right)q^{\frac{k^2+k}{2}}.$$

If we suppose $a \neq 1$ and divide by $a^{\frac{1}{2}} - a^{-\frac{1}{2}}$, we obtain

$$\begin{aligned}(a^{-1}q;q)_\infty(aq;q)_\infty(q;q)_\infty &= 1 + \sum_{k\geq 1}(-1)^k\left(\frac{a^{k+\frac{1}{2}} - a^{-k-\frac{1}{2}}}{a^{\frac{1}{2}} - a^{-\frac{1}{2}}}\right)q^{\frac{k^2+k}{2}} \\ &= 1 + \sum_{k\geq 1}(-1)^k(a^k + a^{k-1} + \cdots + a^{-k})q^{\frac{k^2+k}{2}}.\end{aligned}$$

Now if we let $a \to 1$, we obtain

$$(q;q)_\infty^3 = \sum_{k\geq 0}(-1)^k(2k+1)q^{\frac{k^2+k}{2}},$$

which is the desired result. □

We are now done with the preliminary results that will be needed to prove our main results, and we move on to proving them.

3.2 Infinitely Many Parity Results for Generalized Frobenius Partitions.

Our first result pertains to generalized Frobenius partitions with up to k colors.

Theorem 3.5. *For all $k \geq 1$, and $n \geq 0$,*

$$c\phi_{2k}(2n+1) \equiv 0 \pmod{2}.$$

Proof. As we mentioned earlier in Lemma 2.4, Andrews showed that the generating function for $c\phi_{2k}(n)$ is the constant term in $CG_{2k}(z)$. We then have

$$\begin{aligned} CG_{2k}(z) &= \prod_{n=0}^{\infty}(1+zq^{n+1})^{2k}(1+z^{-1}q^n)^{2k} \\ &\equiv \prod_{n=0}^{\infty}(1^2+(zq^{n+1})^2)^k(1^2+(z^{-1}q^n)^2)^k \pmod 2 \text{ by Lemma 1.7} \\ &= \prod_{n=0}^{\infty}(1+(zq^{n+1})^2)^k(1+(z^{-1}q^n)^2)^k \\ &= \prod_{n=0}^{\infty}(1+z^2q^{2n+2})^k(1+z^{-2}q^{2n})^k. \end{aligned}$$

Note that when expanding $CG_{2k} \pmod 2$, we can only get q's with even powers since both q^{2n+2} and q^{2n} have even powers. This means that the terms of the form $a_{2N+1}z^0q^{2N+1}$ must have $a_{2N+1} \equiv 0 \pmod 2$ for all $N \geq 0$, thus $c\phi_{2k}(2n+1) \equiv 0 \pmod 2$ for all $n \geq 0$ as desired. □

We note that this result was already proven by Baruah and Sarmah in [4] by employing Theorem 2.10 that Garvan and Sellers proved in [10].

We also note that the fact that 2 is a prime allowed us to say that

$$\prod_{n=0}^{\infty}(1+zq^{n+1})^{2k}(1+z^{-1}q^n)^{2k} \equiv \prod_{n=0}^{\infty}(1+z^2q^{2n+2})^k(1+z^{-2}q^{2n})^k \pmod 2,$$

using Lemma 1.7, and we did not use any other properties of 2, so an analogous statement is actually still true for a general prime p, not just 2. This leads us to the following corollary:

Corollary 3.6. *For all $k \geq 1, n \geq 0$, primes $p \geq 2$, and $1 \leq r \leq p-1$,*

$$c\phi_{pk}(pn+r) \equiv 0 \pmod p.$$

Proof. Following the reasoning of the previous proof, we can write

$$CG_{pk}(z) = \prod_{n=0}^{\infty}(1 + zq^{n+1})^{pk}(1 + z^{-1}q^{n})^{pk}$$
$$\equiv \prod_{n=0}^{\infty}(1 + (zq^{n+1})^{p})^{k}(1 + (z^{-1}q^{n})^{p})^{k} \pmod{p}.$$

Note that all powers of q in this product will be multiples of p, so the coefficient of $z^0q^{pn+r}, 1 \leq r \leq p-1$ will be congruent to $0 \pmod{p}$. □

Before we talk about our next result, an important idea is required to understand the statement and the proof.

Definition 3.7. *An integer m is called a quadratic residue* (mod n) *if it is congruent to a perfect square* (mod n). *In other words, m is a quadratic residue* (mod n) *if there exists an integer x such that*

$$x^2 \equiv m \pmod{n}.$$

Otherwise, m is a quadratic nonresidue (mod n).

We now look at our main result for ϕ_k. This result is significant because it applies to infinitely many values of k and infinitely many progressions as well. Not many results appear in the literature on the parity of ϕ_k for infinitely many values of k, and so we are particularly proud of the Theorem below.

Theorem 3.8. *Let $p \geq 5$ be a prime and let r be an integer, $1 \leq r \leq p-1$, such that $24r+1$ is a quadratic nonresidue* (mod p). *Then for all integers $n \geq 0$, and $\ell \geq 1$,*

$$\phi_{p\ell-1}(pn+r) \equiv 0 \pmod{2}.$$

Proof. We know from Theorem 2.3 that one way of writing the generating function for ϕ_k is

$$\Phi_k(q) = \frac{\sum\limits_{\substack{j,r=-\infty \\ r \geq (k+1)|j|}}^{\infty} (-1)^{r+kj} q^{\binom{r+1}{2} - \binom{k+1}{2} j^2}}{(q;q)_\infty^2 (q^{k+1};q^{k+1})_\infty}.$$

Therefore, we have

$$\begin{aligned}
\Phi_k(q) &= \frac{\sum\limits_{\substack{j,r=-\infty \\ r \geq (k+1)|j|}}^{\infty} (-1)^{r+kj} q^{\binom{r+1}{2} - \binom{k+1}{2} j^2}}{(q;q)_\infty^2 (q^{k+1};q^{k+1})_\infty} \\
&\equiv \frac{\sum\limits_{\substack{j,r=-\infty \\ r \geq (k+1)|j|}}^{\infty} q^{\binom{r+1}{2} - \binom{k+1}{2} j^2}}{(q;q)_\infty^2 (q^{k+1};q^{k+1})_\infty} \pmod 2 \\
&= \frac{\sum\limits_{\substack{r=-\infty \\ r \geq (k+1)|j|}}^{\infty} \sum\limits_{j=-\infty}^{\infty} q^{\binom{r+1}{2} - \binom{k+1}{2} j^2}}{(q;q)_\infty^2 (q^{k+1};q^{k+1})_\infty}.
\end{aligned}$$

Note that expanding the inner sum in the expression above would give us:

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} q^{\binom{r+1}{2} - \binom{k+1}{2} j^2} &= q^{\binom{r+1}{2} - 0} + q^{\binom{r+1}{2} - \binom{k+1}{2}(-1)^2} + q^{\binom{r+1}{2} - \binom{k+1}{2}(1)^2} + q^{\binom{r+1}{2} - \binom{k+1}{2}(-2)^2} \\
&\quad + q^{\binom{r+1}{2} - \binom{k+1}{2}(2)^2} + q^{\binom{r+1}{2} - \binom{k+1}{2}(-3)^2} + q^{\binom{r+1}{2} - \binom{k+1}{2}(3)^2} \cdots \\
&= q^{\binom{r+1}{2} - 0} + 2q^{\binom{r+1}{2} - \binom{k+1}{2}(1)} + 2q^{\binom{r+1}{2} - \binom{k+1}{2}4} + 2q^{\binom{r+1}{2} - \binom{k+1}{2}9} + \cdots \\
&\equiv q^{\binom{r+1}{2}} \pmod 2.
\end{aligned}$$

This means that $r \geq (k+1)|0| = 0$, and we can now write

$$\begin{aligned}
\Phi_k(q) &\equiv \frac{\sum\limits_{\substack{r=-\infty \\ r\geq(k+1)|j|}}^{\infty} q^{\binom{r+1}{2}}}{(q;q)_\infty^2 (q^{k+1};q^{k+1})_\infty} \pmod 2 \\
&= \frac{\sum\limits_{r=0}^{\infty} q^{\binom{r+1}{2}}}{(q;q)_\infty^2 (q^{k+1};q^{k+1})_\infty} \\
&\equiv \frac{(q;q)_\infty^3}{(q;q)_\infty^2 (q^{k+1};q^{k+1})_\infty} \pmod 2 \text{ using Theorem 3.4} \\
&= \frac{(q;q)_\infty}{(q^{k+1};q^{k+1})_\infty}.
\end{aligned}$$

Replacing k by $p\ell - 1$,

$$\begin{aligned}
\Phi_{p\ell-1}(q) &\equiv \frac{(q;q)_\infty}{(q^{p\ell};q^{p\ell})_\infty} \pmod 2 \\
&= (q;q)_\infty \sum_{j=0}^{\infty} p(j) q^{p\ell j} \\
&\equiv \sum_{m\in\mathbb{Z}} q^{\frac{3}{2}m^2 - \frac{1}{2}m} \sum_{j=0}^{\infty} p(j) q^{p\ell j} \pmod 2 \text{ using Theorem 3.3.}
\end{aligned}$$

Now assume that

$$pn + r = \frac{3}{2}m^2 - \frac{1}{2}m + p\ell j$$

for some integers m, j, ℓ. Then

$$r \equiv \frac{3}{2}m^2 - \frac{1}{2}m \pmod p.$$

Hence

$$\begin{aligned}
24r + 1 &\equiv 36m^2 - 12m + 1 \pmod p \\
&\equiv (6m-1)^2 \pmod p
\end{aligned}$$

which contradicts the assumption that $24r+1$ is a quadratic nonresidue (mod p). Therefore, $pn+r$ can never be represented as $\frac{3}{2}m^2 - \frac{1}{2}m + p\ell j$ for integers m, ℓ, j. This proves that

$$\phi_{p\ell-1}(pn+r) \equiv 0 \pmod{p}$$

for all $n \geq 0$. □

3.3 Generalized Frobenius Partition With Up To 4 Repetitions

Recall that when we looked at Mathematica's output for possible arithmetic progressions $an+b$ that satisfy $\phi_k(an+b) \equiv 0 \pmod{2}$, the case where $k=4$ had several progressions that satisfied that result. However, a lot of those arithmetic progressions did not fit Theorem 3.8. After further investigation, we found that these arithmetic progressions fall under different criteria, and in fact, are summarized by the three theorems in this section. Our main focus for the rest of this thesis is going to be on ϕ_4 only. First, we will start with an equality proved by Sarmah and Gayan which we obtained through personal communication [17].

Theorem 3.9.

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n = \frac{(q^2;q^2)_\infty^7 (q^{10};q^{10})_\infty^2}{(q;q)_\infty^6 (q^4;q^4)_\infty^2 (q^5;q^5)_\infty^2}$$

Note that if we consider Theorem 3.9 (mod 2), then we end up with the following:

Corollary 3.10.

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n \equiv (q^{10};q^{10})_\infty \pmod{2}$$

Proof. Note that because we are looking at the result (mod 2), we can use Lemma

1.7 as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \phi_4(2n+1)q^n &= \frac{(q^2;q^2)_\infty^7 (q^{10};q^{10})_\infty^2}{(q;q)_\infty^6 (q^4;q^4)_\infty^2 (q^5;q^5)_\infty^2} \\
&\equiv \frac{(q^2;q^2)_\infty^7 (q^{10};q^{10})_\infty^2}{(q^2;q^2)_\infty^3 (q^4;q^4)_\infty^2 (q^{10};q^{10})_\infty} \pmod 2 \\
&= \frac{(q^2;q^2)_\infty^4 (q^{10};q^{10})_\infty}{(q^4;q^4)_\infty^2} \\
&\equiv \frac{(q^2;q^2)_\infty^4 (q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^4} \pmod 2 \\
&= (q^{10};q^{10})_\infty .
\end{aligned}$$

The result follows. □

The first arithmetic progression in our list for $\phi_4(n)$ was $4n+3$, and so the following theorem addresses why that progression was on the list.

Theorem 3.11. *For all* $n \geq 0$,

$$\phi_4(4n+3) \equiv 0 \pmod 2.$$

Proof. We first start by noting that $4n+3$ is an odd number for any $n \geq 0$, which means it is a subset of the progression $2n+1$ since it covers all odd integers. This means that we can use the result of Sarmah and Gayan in Theorem 3.9. Combining their result with Theorem 3.3 yields

$$\sum_{n=0}^{\infty} \phi_4(2n+1)q^n \equiv (q^{10};q^{10})_\infty \equiv \sum_{m=-\infty}^{\infty} q^{10(\frac{3}{2}m^2+\frac{1}{2}m)} \pmod 2. \tag{3.12}$$

If we want to track what each element of $\phi_4(4n+3)$ is going to be $\pmod 2$, we can look at (3.12) above, and substitute $2k+1$ for each n to get

$$\sum_{k=0}^{\infty} \phi_4(2(2k+1)+1)q^{2k+1} = \sum_{k=0}^{\infty} \phi_4(4k+3)q^{2k+1}.$$

This means that the values $\phi_4(4k+3)$ that we are interested in are the coefficients of q^{2k+1}. Equation (3.12) implies that all the powers in the series can be expressed as pentagonal numbers, and if $q^{2k+1} = q^{10\left(\frac{3}{2}m^2+\frac{1}{2}m\right)}$, then

$$2k+1 = 10\left(\frac{3}{2}m^2 + \frac{1}{2}m\right)$$

must be true. However, we know that $\frac{3}{2}m^2+\frac{1}{2}m$ is an integer because it is a pentagonal number, therefore $10\left(\frac{3}{2}m^2 + \frac{1}{2}m\right)$ must be divisible by 2 because of the factor of 10. On the other hand, $2k+1$ is always odd. This means that the coefficient for every q^{2k+1} is $0 \pmod 2$. Therefore

$$\phi_4(4n+3) \equiv 0 \pmod 2$$

for all $n \geq 0$, proving the assertion. □

We also noticed that our algorithm conjectured that $5n+3, 5n+4, 10n+5, 10n+7$ are also arithmetic progressions that give an output congruent to $0 \pmod 2$ when plugged into ϕ_4. The progressions $5n+3, 5n+4$ are already covered by Theorem 3.8. However, $10n+5, 10n+7$ are not. Note that $10n+3$ and $10n+9$ are not on the list because they are subprogressions of $5n+3, 5n+4$, which already fall under Theorem 3.8. We will add them to the following theorem for the sake of completeness, and to provide another proof.

Theorem 3.12. *For all $n \geq 0$,*

$$\phi_4(10n+r) \equiv 0 \pmod 2$$

where $r = 3, 5, 7,$ or 9.

Proof. First note that since r can only be one of the odd integers above, then $\frac{r-1}{2}$ is an integer. Similar to the proof of Theorem 3.11, we start by substituting $5k + \frac{r-1}{2}$

for every n in (3.12) to get

$$\sum_{k=0}^{\infty} \phi_4\left(2\left(5k+\frac{r-1}{2}\right)+1\right) q^{5k+\frac{r-1}{2}} = \sum_{k=0}^{\infty} \phi_4(10n+r) q^{5k+\frac{r-1}{2}}. \tag{3.13}$$

Again, this means that the values of $\phi_4(10n+r)$ that we are interested in are the coefficients of $q^{5k+\frac{r-1}{2}}$. By (3.12), all powers in the series in (3.13) can be expressed as pentagonal numbers, and if $q^{5k+\frac{r-1}{2}} = q^{10\left(\frac{3}{2}m^2+\frac{1}{2}m\right)}$, then

$$5k+\frac{r-1}{2} = 10\left(\frac{3}{2}m^2+\frac{1}{2}m\right).$$

However, we know that $\frac{3}{2}m^2+\frac{1}{2}m$ is an integer because it is a pentagonal number, therefore $10\left(\frac{3}{2}m^2+\frac{1}{2}m\right)$ is divisible by 5 because of the factor of 10. On the other hand, $5k+\frac{r-1}{2}$ can only be one of $5k+1, 5k+2, 5k+3$, or $5k+4$. None of these are divisible by 5. This means that the coefficient for every $q^{5k+\frac{r-1}{2}}$ is 0 (mod 2). Therefore

$$\phi_4(10n+r) \equiv 0 \pmod 2$$

for all $n \geq 0$, when $r = 3, 5, 7$, or 9, proving our assertion. □

Our code found other progressions $an+b$ that satisfied $\phi_4(an+b) \equiv 0 \pmod 2$. In fact, there are infinitely many of them. They are summarized below, but first, a final lemma is in order. We will use Wilson's Theorem to prove this lemma, and so we will start with that theorem.

Theorem 3.13 (Wilson's Theorem). *$p > 1$ is a prime number if and only if*

$$(p-1)! \equiv -1 \pmod p.$$

Lemma 3.14. *Let $p \neq r$ both be primes with p being congruent to $a \pmod r$ and $1 \leq r \leq p-1$. The multiplicative inverse of $r, r^{-1} \pmod p$ can be written as*

$$r^{-1} \equiv \frac{\frac{(r-1)!}{a}p+1}{r} \pmod p.$$

Proof. We first start by noting that since $p \equiv a \pmod{r}$, and $p \neq r$, then a is one of the values $1, 2, \cdots, r-1$. This means that $\frac{(r-1)!}{a}$ is an integer. Furthermore,

$$\begin{aligned}\frac{(r-1)!}{a}p + 1 &\equiv \frac{(r-1)!}{a}a + 1 \pmod{r} \\ &= (r-1)! + 1 \\ &\equiv -1 + 1 \pmod{r} \text{ by Theorem 3.13} \\ &= 0.\end{aligned}$$

This means that the numerator in $\frac{\frac{(r-1)!}{a}p+1}{r}$ is some multiple of r, so the whole fraction is an integer. Now if we look at this fraction $\pmod{p}$, we note that

$$r \cdot \frac{\frac{(r-1)!}{a}p + 1}{r} \equiv \frac{(r-1)!}{a}p + 1 \equiv 0 + 1 \equiv 1 \pmod{p}.$$

□

In particular, this means that $5^{-1} \equiv \frac{\frac{24}{a}p+1}{5} \pmod{p}$ when $p \equiv a \pmod{5}$ and $p \geq 7$.

We are now in a position to prove the existence of infinitely many congruences (mod 2) satisfied by ϕ_4 which are summarized by the following theorem.

Theorem 3.15. *Let $p \geq 7$ be prime and let r be an odd integer, $1 \leq r \leq p-1$, such that $(6r-1)\frac{\frac{24}{a}p+1}{5}$ is a quadratic nonresidue $\pmod{p}$. Then for all integers $n \geq 0$,*

$$\phi_4(2pn + r) \equiv 0 \pmod{2}$$

where $p \equiv a \pmod{5}$.

Proof. First note that since r is always an odd integer, then $\frac{r-1}{2}$ is an integer. We start by substituting $pn + \frac{r-1}{2}$ for every n in (3.12) to get:

$$\sum_{k=0}^{\infty} \phi_4(2(pk + \frac{r-1}{2}) + 1)q^{pk+\frac{r-1}{2}} = \sum_{k=0}^{\infty} \phi_4(2pk + r)q^{pk+\frac{r-1}{2}}. \qquad (3.14)$$

Similar to the proofs of Theorems 3.11 and 3.12, the values of $\phi_4(2pk + r)$ that we are interested in are the coefficients of $q^{pk+\frac{r-1}{2}}$. By (3.12), all powers in the series in (3.14) can be expressed as pentagonal numbers, and if $q^{pk+\frac{r-1}{2}} = q^{10\left(\frac{3}{2}m^2+\frac{1}{2}m\right)}$, then

$$pk + \frac{r-1}{2} = 10\left(\frac{3}{2}m^2 + \frac{1}{2}m\right).$$

Looking at this equation $\pmod p$ yields:

$$\frac{r-1}{2} \equiv 10(\frac{3}{2}m^2 + \frac{1}{2}m) \pmod p$$

$$\Rightarrow r - 1 \equiv 10(3m^2 + m) \pmod p$$

$$\Rightarrow 12(r-1) \equiv 10(36m^2 - 12m) \pmod p$$

$$\Rightarrow 12(r-1) + 10 \equiv 10(36m^2 - 12m + 1) \pmod p$$

$$\Rightarrow 12r - 2 \equiv 10(6m-1)^2 \pmod p$$

$$\Rightarrow 6r - 1 \equiv 5(6m-1)^2 \pmod p$$

$$\Rightarrow (6r-1)5^{-1} \equiv (6m-1)^2 \pmod p.$$

However, by Lemma 3.14,

$$(6r-1)\frac{\frac{24}{a}p+1}{5} \equiv (6r-1)5^{-1} \pmod p$$

which means

$$(6r-1)\frac{\frac{24}{a}p+1}{5} \equiv (6m-1)^2 \pmod p$$

which contradicts the assumption that $6(r-1)\frac{\frac{24}{a}p+1}{5}$ is a quadratic nonresidue (mod p). Therefore, $pn + \frac{r-1}{2}$ can never be represented as $10\left(\frac{3}{2}m^2 + \frac{1}{2}m\right)$ for any integer m. So the coefficients corresponding to $q^{pn+\frac{r-1}{2}}$ are zero $\pmod 2$, and therefore,

$$\phi_4(2pn + r) \equiv 0 \pmod 2$$

for all $n \geq 0$. □

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