A PARABOLIC SYSTEM ARISING IN FILM DEVELOPMENT

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IMA Preprint Series # 577
August 1989

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Key words and phrases: Existence and uniqueness, edge enhancement, large time behavior

§1 Introduction

In this paper we shall study the Cauchy problem:

$$P_t - P_{xx} = E(x)F(Q) - P + Q \qquad \text{in } R^1 \times [0,\infty), \tag{1.1}$$

$$Q_t = P - Q \qquad \text{in } R^1 \times [0,\infty), \tag{1.2}$$

$$P(x,0) = Q(x,0) = 0 \qquad \text{for } x \in R^1, \tag{1.3}$$

$$\max_x |P(x,t)| < \infty, \quad \max_x |Q(x,t)| < \infty, \qquad \text{for any } t < \infty. \tag{1.4}$$

where $E(x)$ is a step function:

$$E(x) = \begin{cases} C_1 & x < 0 \\ C_2 & x > 0 \end{cases}, \qquad \text{where } C_1 > C_2 > 0;$$

$F(u)$ is a C^2 positive decreasing function, $-\infty < u < \infty$. System (1.1) arises in film development; for background, see [1].

Define *dye* $D(x,T)$ at time T by

$$D(x,T) = E(x) \int_0^T F(Q(X,t))dt \tag{1.5}$$

where (P, Q) is the solution of (1.1)-(1.4). Then one expects the profile of D-curve to be as indicated in Figure 1.

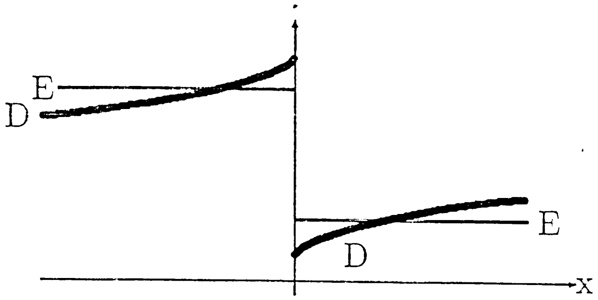

Figure 1

If it does, then we say that *edge enhancement* occurs (see [1]). In section 3, we shall prove that if T is sufficiently small, then this phenomenon does occur.

We shall be mainly interested however in the large time behavior of solutions. It turns out that this depends on the behavior of $F(u)$ as $u \to \infty$. If $F(u)$ reaches zero at a finite point $u = M > 0$, then both P and Q converge to M as $t \to \infty$ under some additioal assmmption on F (Theorem 4.3). If $F(u)$ approaches a positive limit as $u \to \infty$, then both P and Q increase linearly in t as $t \to \infty$ (Theorem 4.10).

The most interesting case is when $F(u)$ is strictly positive but converges to zero as $u \to \infty$; then P and Q still converge to ∞; more specifically, if $F(u)/u^{-\gamma} \to \beta$, $-F'(u)/u^{-\gamma-1} \to \gamma\beta$ as $u \to \infty$ for some $\beta > 0$ and $\gamma > 0$, then

$$\left.\begin{array}{l} P(x,t)/t^{\frac{1}{1+\gamma}} \longrightarrow \frac{1}{2}g(0) \\ Q(x,t)/t^{\frac{1}{1+\gamma}} \longrightarrow \frac{1}{2}g(0) \end{array}\right. \qquad \text{as } t \to \infty.$$

Here $g(x)$ is the solution of

$$\begin{array}{l} g''(x) + \frac{x}{2}g'(x) = \frac{1}{1+\gamma}g - 2^{\gamma}\beta E(x)/g^{\gamma}, \\ g(+\infty) = (2^{\gamma}C_2)^{\frac{1}{1+\gamma}}, \quad g(-\infty) = (2^{\gamma}C_1)^{\frac{1}{1+\gamma}}. \end{array} \tag{1.6}$$

§2 Preliminary Results

Let $W_p^{1,2}(K) = \{u : u, D_t u, D_x u \text{ and } D_{xx}u \text{ are all in } L^p(K)\}$, where $K \subset R^1 \times [0, \infty)$. By a solution (P,Q) of (1.1)-(1.4), we mean that $P \in W_p^{1,2}(K)$, $Q \in W_p^{1,2}(K)$ for any $p > 1$ and any compact subset K, and (P,Q) satisfies (1.1)-(1.4) ((1.1), (1.2) in a.e. sense).

In the sequel, various constants will be denoted by C. We shall first establish the existence and uniqueness of solutions of (1.1)-(1.4) for small time.

Lemma 2.1 There exists a $\delta > 0$, such that the system (1.1)-(1.4) has a unique solution (P,Q) in $R^1 \times [0, \delta)$; furthermore, $P > 0,\ Q > 0$.

Proof. From (1.2), we get

$$Q(x,t) = \int_0^t e^{-(t-s)} P(x,s)ds, \tag{2.1}$$

so that (1.1) can be rewritten in the form

$$(P_t - P_{xx})(x,t) = E(x)F\left(\int_0^t e^{-(t-s)}P(x,s)ds\right) - P + \int_0^t e^{-(t-s)}P(x,s)ds. \tag{2.2}$$

We first solve (2.2) with $P(x,0) = 0$. Let

$$B = C^0(R^1 \times [0,\delta]) \cap \{f : f \text{ is bounded in every strip } R^1 \times [0,t] \ \text{ for any } \ t < \delta\}.$$

For any $\bar{P} \in B$, let P be the unique solution of

$$\begin{aligned}&(P_t - P_{xx})(x,t) = E(x)F\left(\int_0^t e^{-(t-s)}\bar{P}(x,s)ds\right) - \bar{P} + \int_0^t e^{-(t-s)}\bar{P}(x,s)ds,\\ &P(x,0) = 0.\end{aligned}$$

Define an operator L by $L\bar{P} = P$. It is easy to check that L maps B into B, and L is a contraction provided δ is small. Therefore, there exists a unique fixed point P of L. Defining Q by (2.1), it is clear (P,Q) forms the unique solution of (1.1)-(1.4). By standard L^p-estimates (see [2]), P_t and P_{xx} are in $L^p(K)$ for any $p > 1$ and any compact subset $K \subset R^1 \times [0,\delta)$.

To prove that $P(x,t) > 0$, we represent P by means of the fundamental solution $\Gamma(x,t;y,s) = \frac{1}{2\sqrt{\pi(t-s)}}e^{-\frac{(x-y)^2}{4(t-s)}}$:

$$P(x,t) = \int_0^t \int_{R^1} \Gamma(x,t;y,s)(E(y)F(Q) - P(y,s) + Q(y,s))dyds. \tag{2.3}$$

Then

$$\begin{aligned}|P(x,t)| &\le \int_0^t \int_{R^1} \Gamma(x,t;y,s)(E(y)F(Q) + C)dyds\\ &\le Ct.\end{aligned}$$

By (2.1), we also have the same estimate for $Q(x,t)$. Therefore,

$$\begin{aligned}P(x,t) &\ge \textstyle\int_0^t \int_{R^1} \Gamma(x,t;y,s)(E(y)F(Q) - 2Cs)dyds\\ &\ge Ct > 0,\end{aligned}$$

since $F(Q) > \frac{1}{2} > 0$ for $Q \le Ct \le C\delta$ if δ is small. By (2.1), also $Q(x,t) > 0$.

We next prove some a priori estimates (for global solution).

Lemma 2.2 If (P,Q) is a solution of (1.1)-(1.4) in $R^1 \times [0,T)$, then

$$P(x,t) > 0, \tag{2.4}$$

$$Q(x,t) > 0 \tag{2.5}$$

in $R^1 \times (0,T)$.

Proof. If the assertions are not true, then there exists a point (x_0, t_0) such that (2.4) and (2.5) hold for all $t < t_0$ but one of the inequalities becomes an equality at (x_0, t_0), i.e. either $P(x_0, t_0) = 0$, or $Q(x_0, t_0) = 0$. Note that $Q(x_0, t_0) = 0$ is impossible by (2.1), so that $P(x_0, t_0) = 0$. Since

$$P_t - P_{xx} + P = E(x)F(Q) + Q > 0$$

in the strip $\{0 \le t \le t_0\}$, the maximum principle then implies that $P \equiv 0$ in $\{0 \le t \le t_0\}$, a contradiction.

Lemma 2.3 Let (P,Q) be a solution of (1.1)-(1.3) in $R^1 \times [0,T)$. Then

$$P(x,t) \le CTe^T,$$

$$Q(x,t) \le CTe^T$$

in $R^1 \times [0,T)$.

Proof. From (2.3) and the fact that $P > 0$, we have

$$\begin{aligned} P(x,t) &\le \int_0^t \int_{R^1} \Gamma(x,t;y,s)(C_1F(0) + Q(y,s))dyds \\ &\le C_1F(0)t + \int_0^t \max_x Q(x,s)ds \\ &= C_1F(0)t + \int_0^t \int_0^s e^{-(t-s)} \max_x P(x,u)duds \qquad \text{(by (2.1))} \\ &\le C_1F(0)t + \int_0^t \max_x P(x,s)ds. \end{aligned}$$

Gronwall inequality then gives

$$\max_x P(x,t) \le C_1F(0)te^t \le CTe^T.$$

By (2.1), the same inequality holds for Q.

The a priori estimates of Lemma 2.3 enable us to apply the local existence and uniqueness result of Lemma 2.1, step by step, in order to obtain a unique global solution :

Theorem 2.4 There exists a unique solution of (1.1)-(1.4), for all $t > 0$; furthermore, $P > 0,\ Q > 0$.

We next establish additional regularities.

Theorem 2.5 The solution (P,Q) has the following regularity properties: P_{tt} and P_{txx} are in $L^p(K)$ for any $p > 1$; furthermore, P_t and P_x are continuous up to $\{t = 0\}$ except at (0,0). The same conclusions hold for Q.

Proof. Since (P,Q) is a solution of (1.1)-(1.3), for any $\xi \in C_0^\infty(R^1 \times (0,\infty))$, we have

$$-\int_0^\infty \int_{R^1} P(\xi_t + \xi_{xx}) = \int_0^\infty \int_{R^1} (E(x)F(Q) - P + Q)\xi.$$

Replacing ξ by ξ_t, and using integration by parts, we see that

$$\int_0^\infty \int_{R^1} P_t(\xi_t + \xi_{xx}) = -\int_0^\infty \int_{R^1} (E(x)F'(Q)Q_t - P_t + Q_t)\xi. \tag{2.6}$$

Equation (2.6) says that P_t is a weak solution of

$$P_{tt} - P_{txx} = (1 + E(x)F'(Q))Q_t - P_t. \tag{2.7}$$

Also

$$P_t(x,0) = E(x)F(0).$$

Since P_t, Q_t are in $L^p(K)$ for any compact subset $K \subset R^1 \times [0,\infty)$, standard L^p-estimates give us that P_{tt} and P_{txx} are in $L^p(K)$ for any $p > 1$. By Sobelev imbedding theorems, we see that P_t and P_x are continuous in $R^1 \times (0,\infty)$. Noting that $E(x)$ has a jump only at 0, we conclude that P_t and P_x are continuous up to $\{t = 0\}$ except at (0,0). By (2.1), all assertions are also true for Q.

§3 Edge Enhancement

In this section, we establish the edge enhancement phenomenon.

Theorem 3.1 Let $D(x,T)$ be the dye as defined in (1.6). Then, for any T sufficiently small, we have that

$$D(x,T) = E(x)g(x,T) + O(T^4)$$

where $g(x,T) \in C^\infty(R^1)$ and $\inf_{x,T} \frac{g(x,T)}{T} \geq c > 0$; furthermore,

$$g_x(x,T) \geq 0, \qquad x \in R^1$$

$$g_{xx}(x,T) \geq 0 \qquad \text{if } x < 0,$$

$$g_{xx} \leq 0 \qquad \text{if } x > 0,$$

$$\lim_{x\to\infty} g(x,T) = F(0)T(1+\frac{1}{6}C_2F(0)F'(0)T^2), \quad \lim_{x\to-\infty} g(x,T) = F(0)T(1+\frac{1}{6}C_1F(0)F'(0)T^2).$$

Proof. Let $f(x,t) = (1 + E(x)F'(Q))Q_t - P_t$. Then from (2.10)

$$\begin{aligned} P_t(x,t) &= \int_{R^1} \Gamma(x,t;y,0)E(y)F(0)dy + \int_0^t \int_{R^1} \Gamma(x,t;y,s)f(y,s)dys \\ &\triangleq I_1 + I_2. \end{aligned} \tag{3.1}$$

We compute

$$\begin{aligned} I_1 &= F(0)[C_1 \int_{-\infty}^{0} \Gamma(x,t;y,0)dy + C_2 \int_0^{\infty} \Gamma(x,t;y,0)dy] \\ &= F(0)[\frac{C_1+C_2}{2} + (C_2 - C_1) \int_{-\frac{x}{\sqrt{2t}}}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du], \end{aligned} \tag{3.2}$$

and

$$I_2 \leq \int_0^t |f|_{L^\infty(R^1 \times [0,\infty))} ds \leq Ct. \tag{3.3}$$

Now let us calculate $Q(x,t)$ for small t:

$$\begin{aligned} Q(x,t) &= \int_0^t e^{-(t-s)} P(x,s)ds \\ &= \int_0^t P(x,s)ds + \int_0^t P(x,s)(e^{-(t-s)} - 1)ds \\ &= \int_0^t P_t(x,s)(t-s)ds + O(t^3). \end{aligned} \tag{3.4}$$

Hence

$$\begin{aligned} \int_0^T Q(x,t)dt &= \frac{1}{2}\int_0^T P_t(x,s)(T-s)^2 ds + O(T^4) \\ &= \frac{1}{2}\int_0^T I_1(x,s)(T-s)^2 ds + \int_0^T I_2(x,s)(T-s)^2 ds + O(T^4) \quad \text{(by (3.1))} \\ &= \frac{1}{2}\int_0^T I_1(x,t)(T-t)^2 dt + O(T^4) \quad \text{(by (3.3))}. \end{aligned} \tag{3.5}$$

From (3.4), we know that $Q(x,t) = O(t^2)$. Therefore

$$F(Q(x,t)) = F(0) + F'(0)Q(x,t) + O(t^4).$$

Integrating in t and using (3.5) and (3.2), we get

$$\begin{aligned} \textstyle\int_0^T F(Q(x,t))dt &= F(0)T + F'(0) \textstyle\int_0^T Q(x,t)dt + O(T^4) \\ &\triangleq g(x,T) + O(T^4), \end{aligned}$$

where

$$g(x,T) = F(0)T + \frac{1}{2}F'(0)F(0)\int_0^T [\frac{C_1+C_2}{2} + (C_2 - C_1)\int_{-x/\sqrt{2t}}^{0} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du](T-t)^2 dt.$$

It is easy to check that $g(x,T)$ has all the properties stated in Theorem 3.1. It is also clear that D-curve has the profile indicated in Figure 1. Hence, if T is sufficiently small, edge enhancement occurs.

§4 Large Time Behavior

To study the large time behavior, we need to distinguish three cases:

$$\text{There exists an } M > 0, \text{ such that } F(M) = 0, \tag{4.1}$$

$$F(u) \to \beta > 0 \text{ as } u \to \infty, \tag{4.2}$$

$$F(u) \to 0 \text{ as } u \to \infty. \tag{4.3}$$

We shall begin with case 1.

Lemma 4.1 In addition to (4.1), we also assume that F satisfies

$$1 + \max_u C_1 F'(u) \geq 0. \tag{4.4}$$

Then $P_t > 0, \ Q_t > 0$ in $R^1 \times [0, \infty)$.

Proof. By (1.2) and Lemma 2.3, we see that

$$|Q_t|_{L^\infty(R^1 \times [0,1])} \leq C < \infty, \tag{4.5}$$

Representing P_t, the solution of (2.7) with $P_t(x, 0) = E(x)F(0)$, in terms of the fundamental solution, we get

$$P_t(x,t) = \int_{R^1} \Gamma(x,t;y,0)E(y)F(0)dy + \int_0^t \int_{R^1} \Gamma(x,t;y,s)[(1+E(y)F'(Q))Q_t - P_t]dyds \tag{4.6}$$

Hence

$$|P_t(x,t)| \leq C_1 F(0) + \tilde{C}t + \int_0^t \max_x |P_t(x,s)|ds, \quad (t \leq 1)$$

where $\tilde{C} = \max_{x, 0 \leq t \leq 1}(1 + E(x)F'(Q(x,s))|Q_t(x,t)| < \infty$ by (4.5). Gronwall's inequality then gives

$$\max_x |P_t(x,t)| \leq (C_1 + \tilde{C})e^t \leq C \quad (t \leq 1). \tag{4.7}$$

Using (4.5) and (4.7) in (4.6), we obtain

$$P_t(x,t) \geq C_2 F(0) - Ct > 0$$

in $R^1 \times [0, \delta]$ for some small $\delta > 0$. Since $Q_t = \int_0^t e^{-(t-s)} P_t(x,s)ds$, we also have $Q_t(x,t) > 0$. On the other hand, by the assumption (4.4):

$$P_{tt} - P_{txx} + P_t = (1 + E(x)F'(Q))Q_t \geq 0$$

as long as $Q_t \geq 0$. The argument used in the proof of Lemma 2.2 now shows that $P_t > 0$, and $Q_t > 0$ in $R^1 \times [0, \infty)$.

Lemma 4.2 Under the same assumptions of Lemma 4.1, we have

$$P(x,t) < M, \quad Q(x,t) < M \quad \text{in} \quad R^1 \times [0, \infty).$$

Proof. By Lemma 2.3, there is a $\delta > 0$, such that

$$P(x,t) < M \quad \text{in } R^1 \times [0,\delta).$$

Let $\bar{P} = M - P(x,t)$, $\bar{Q} = M - Q(x,t)$, then

$$\begin{matrix} \bar{P} > 0 \\ \bar{Q} > 0 \end{matrix} \quad \text{in } R^1 \times [0,\delta].$$

From (1.1)

$$\bar{P}_t - \bar{P}_{xx} = -E(x)F(Q) - \bar{P} + \bar{Q}.$$

Since $F(M) = 0$,

$$F(Q) = \int_0^1 F'(M + s(Q-M))ds(Q-M)$$

so that

$$\bar{P}_t - \bar{P}_{xx} + \bar{P} = [1 + E(x)\int_0^1 F'(M + s(Q-M))ds]\bar{Q} \geq 0$$

as long as $\bar{Q} \geq 0$. The same argument as before shows that $\bar{P} \geq 0, \bar{Q} \geq 0$.

Now we are in a position to state:

Theorem 4.3 If F satisfies (4.1) and (4.4), then for any $x \in R^1$

$$\begin{matrix} P(x,t) \to M \\ Q(x,t) \to M \end{matrix} \quad \text{as } t \to \infty.$$

Proof. By Lemma 4.1, the limits $\lim_{t\to\infty} P(x,t) \triangleq \bar{P}(x)$, $\lim_{t\to\infty} Q(x,t) \triangleq \bar{Q}(x)$ exist. By Lemma 4.2, $\bar{P}(x) \leq M, \bar{Q}(x) \leq M$.

Integrating $Q_t = P - Q$ over $(t-1,t)$, we get

$$Q(x,t)-Q(x,t-1) = P(x,t)-Q(x,t)+\int_{t-1}^{t}(P(x,s)-P(x,t))ds-\int_{t-1}^{t}(Q(x,s)-Q(x,t))ds.$$

Letting $t \to \infty$, we obtain

$$\bar{P}(x) - \bar{Q}(x) = 0. \tag{4.8}$$

Let $\xi(x) \in C_0^\infty(R^1)$. We multiply $P_t - P_{xx} = E(x)F(Q) - P + Q$ by ξ, then integrate over $R^1 \times (t-1,t)$, and finally let $t \to \infty$ to get

$$-\int_{R^1} \bar{P}(x)\xi''(x) = \int E(x)F(\bar{P}(x))dx. \tag{4.9}$$

This means that $\bar{P}(x)$ is a solution of

$$\bar{P}'' + E(x)F(\bar{P}) = 0. \tag{4.10}$$

and, in particular, $\bar{P}'' \leq 0$. But the function satisfying $0 \leq \bar{P} \leq M$, $\bar{P}'' \leq 0$ is clearly a constant and since $\bar{P}$ satisfies (4.10), this constant must be equal to M. This proves the assertions of the theorem.

Remark 4.4 We conjecture that Theorem 4.3 is not true without assumption (4.4), as the following example suggests. Let

$$F(u) = \begin{cases} M - u & \text{if } u \le M \\ 0 & \text{otherwise.} \end{cases}$$

Consider the system

$$\begin{aligned} &p' = KF(q) - p + q, \\ &q' = p - q, \\ &p(0) = q(0) = 0, \end{aligned} \tag{4.11}$$

where $K > 1$. This system corresponds to (1.1)-(1.4) with $E(x) \equiv K$, and certainly (4.4) is violated. Since $q'' = p' - q' = KF(q) - 2q'$ and $q'(0) = 0$, we see that $q'(t) = \int_0^t e^{-(t-s)} KF(q(s))ds > 0$ for any $t > 0$. Noting that $q'' = K(M - q) - 2q'$ for $q \le M$, we can solve this linear ODE with initial conditions: $q(0) = q'(0) = 0$. From the resulting expression for q, we see that $q(t_0) = M$ for some $t_0 > 0$. Since $q' > 0$, we conclude that $\lim_{t\to\infty} q(t) > M$. Hence the conclusions of Theorem 4.3 are not true in this case.

Now we turn to the remaining cases (4.2) and (4.3). We shall first consider the system

$$p' = KF(q) - p + q, \tag{4.12}$$

$$q' = p - q, \tag{4.13}$$

$$p(0) = q(0) = 0 \tag{4.14}$$

where K is a positive constant and F satisfies either (4.2) or (4.3). By Theorem 2.4, there exists a unique positive global solution of (4.11)-(4.13).

Lemma 4.5 If F satisfies either (4.2) or (4.3), then there exist positive constants c, C and t_0 such that

$$\begin{aligned} &0 < q' \le C \quad \text{for any } t \in R^+, \\ &|p'| \le C \quad \text{for any } t \in R^+. \end{aligned} \tag{4.15}$$

Moreover, if $F(u) \sim u^{-\gamma}$ at ∞ for $\gamma \ge 0$ (If $\gamma = 0$, we shall mean that F satisfies (4.2)), then

$$ct^{1/(1+\gamma)} \le p(t) \le Ct^{1/(1+\gamma)}, \quad ct^{1/(1+\gamma)} \le q(t) \le Ct^{1/(1+\gamma)} \tag{4.16}$$

for $t \ge t_0$.

Proof. Ther same argument of Remark 4.4 shows that $q' > 0$ in R^+. From this we deduce that

$$p' = KF(q) - q' \le KF(q) \le KF(0).$$

This, in turn, implies

$$q'(t) = \int_0^t e^{-(t-s)} p'(s)ds \le KF(0),$$

and $p' = KF(q) - q' \ge -q' \ge -KF(0)$. We have thus proved (4.15).

Next we show that $q(t) \to \infty$ as $t \to \infty$. Suppose this is not true, i.e.

$$0 < q(t) \le M < \infty \quad \text{for any } t. \tag{4.17}$$

Integrating $q'' = p' - q' = KF(q) - 2q'(t)$ over $(0,t)$, we get

$$q'(t) + 2q(t) = K\int_0^t F(q(s))ds. \tag{4.18}$$

Since $F(q(s)) \geq F(M)$, we see that the RHS of (4.18) is larger than $KF(M)t$, whereas the LHS of (4.18) is less than $KF(0) + 2M$ by (4.15) and (4.17). This is a contradiction if t is large.

Now integrating $q''/F(q) = K - 2q'/F(q)$ over $(0,t)$, and using integration by parts, we obtain

$$\begin{aligned}
\int_0^{q(t)} \frac{1}{F(s)}ds &= \frac{K}{2}t - \frac{q'(t)}{2F(q(t))} - \int_0^t \frac{F'(q)}{F^2(q)}q'^2 ds \\
&\geq \frac{K}{2}t - \frac{q'(t)}{2F(q(t))} \quad (\text{since } F' \leq 0) \\
&\geq \frac{K}{2}t - \frac{C}{F(q(t))} \quad (\text{since } 0 < q' \leq KF(0)).
\end{aligned}$$

Since $q(t) \to \infty$ as $t \to \infty$, we conclude from the above that

$$\begin{aligned}
\frac{K}{2}t &\leq \frac{C}{F(q(t))} + \int_0^{q(t)} \frac{1}{F(s)}ds \\
&\leq Cq(t)^\gamma + Cq(t)^{1+\gamma} \\
&\leq Cq(t)^{1+\gamma},
\end{aligned}$$

provided t is large. This gives $q(t) \geq ct^{1/(1+\gamma)}$. Next integrating $q'' = KF(q) - 2q'$ over $(0,t)$, and applying the result we just proved, we get

$$\begin{aligned}
q(t) &= \frac{K}{2}\int_0^t F(q(s))ds - \frac{q'(t)}{2} \\
&\leq C\int_0^t \frac{1}{q(s)^\gamma}ds + C \\
&\leq C\int_0^t s^{-\gamma/(1+\gamma)}ds + C \\
&\leq Ct^{\frac{1}{1+\gamma}}
\end{aligned}$$

provided t is large. Since $p(t)/q(t) = q'(t)/q(t) + 1$, and $q(t) \to \infty$ while q' remains bounded, we see that $p(t)/q(t) \to 1$ as $t \to \infty$. Hence all the estimates for q hold for p as well. The proof is complete.

Lemma 4.6 If $F(u) \sim u^{-\gamma}$ at $\infty (\gamma \geq 0)$, then

$$p_1(t) \leq P(x,t) \leq p_2(t),$$

$$q_1(t) \leq Q(x,t) \leq q_2(t),$$

where (p_i, q_i) is a solution of (4.12)-(4.14) with $K = K_i$, and K_1 is sufficiently small while K_2 is sufficiently large.

Combining Lemma 4.5 and Lemma 4.6, we obtain

Corollary 4.7 Under the same assumptions of Lemma 4.6, we can find positive constants c, C and t_0, such that

$$ct^{1/(1+\gamma)} \le P(x,t) \le Ct^{1/(1+\gamma)},$$

$$ct^{1/(1+\gamma)} \le Q(x,t) \le Ct^{1/(1+\gamma)}$$

for all $x \in R^1$ and $t \ge t_0$.

Proof of Lemma 4.6. Let $u_i = P - p_i$, $v_i = Q - q_i$. It is easy to see that

$$u_{it} - u_{ixx} + u_i = E(x)F(Q) - K_iF(q_i) + v_i \triangleq J_i(x,t) \tag{4.19}$$

$$v_{it} = u_i - v_i$$

$$u_i(x,0) = v_i(x,0) = 0.$$

Representing the solution u_i of (4.19) with $u_i(x,0) = 0$, in terms of the fundamental solution, we get that for t small

$$\begin{aligned} u_1(x,t) &= \int_0^t \int_{R^1} \Gamma(x,t;y,s)[E(y)F(Q(y,s)) - K_1F(q_1(s)) + v_1(y,s) - u_1(y,s)]dyds \\ &\ge \int_0^t \int_{R^1} \Gamma(x,t;y,s)[\frac{E(y) - 2K_1}{2}F(0) - Cs]dyds \\ &\ge \int_0^t (\frac{C_2 - 2K_1}{2} - Cs)ds \\ &> 0 \quad (\text{if } K_1 < \tfrac{C_2}{2}\ ). \end{aligned}$$

Hence also

$$v_1(x,t) = \int_0^t e^{-(t-s)}u_1(x,s)ds > 0 \tag{4.20}$$

for t small. Using mean value theorem, we see that

$$J_1 = (E(x) - K_1)F(Q) + [1 + K_1F'(\lambda Q + (1-\lambda)q)](Q - q) \ge 0 \quad (\text{for some } \lambda, 0 < \lambda < 1)$$

as long as $Q - q = v_1 \ge 0$ and K_1 is sufficiently small. Now we use the maximum principle to argue as same as in the proof of Lemma 2.2 to deduce that $u_1(x,t) > 0,\ v_1(x,t) > 0$ in $R^1 \times (0,\infty)$.

Next we consider $u_2(x,t),\ v_2(x,t)$. The same argument as in proving (4.20) shows that $u_2(x,t) < 0,\ v_2(x,t) < 0$ in $R^1 \times (0,\delta]$ for some $\delta > 0$, provided K_2 is sufficiently large. We shall next show that

$$J_2(x,t) \le 0 \qquad \text{in } R^1 \times [0,\infty) \tag{4.21}$$

as long as $v_2(x,t) \le 0$. Hence the maximum principle gives that $u_2(x,t) < 0,\ v_2(x,t) < 0$ in $R^1 \times (0,\infty)$.

It remains to verify (4.21). By the assumptions on F, we have that $|F'(u)| \to 0$ as $u \to \infty$. We choose N so large such that $\frac{N}{2} \geq C_1F(0)$ and $1 + E(x)F'(u) \geq 0$ if $u \geq \frac{N}{2}$. If $q_2 \geq N$, $Q \leq \frac{N}{2}$, then

$$J_2(x,t) \leq E(x)F(0) - \frac{N}{2} \leq 0.$$

If $q_2 \geq N$, $Q \geq \frac{N}{2}$, then

$$\begin{aligned} J_2(x,t) &= (E(x) - K_2)F(q_2) + E(x)(F(Q) - F(q_2)) + Q - q_2 \\ &= (E(x) - K_2)F(q_2) + [1 + E(x)F'(\lambda q_2 + (1-\lambda)Q)](Q - q_2) \\ &\leq 0 \end{aligned}$$

as long as $v_2 = Q - q_2 \leq 0$, because $\lambda q_2 + (1-\lambda)Q \geq \frac{N}{2}$ implies that $1 + E(x)F'(\lambda q_2 + (1-\lambda)Q) \geq 0$.

Finally, if $q \leq N$, then

$$\begin{aligned} J_2(x,t) &\leq C_1F(0) - K_2F(N) + Q - q_2 \\ &\leq C_1F(0) - K_2F(N) \\ &\leq 0 \end{aligned}$$

as long as $v_2 = Q - q_2 \leq 0$ and K_2 is sufficiently large. The proof is complete.

Lemma 4.8 Under the same assumptions of Lemma 4.6, we have that

$$\begin{matrix} |P_t| \leq M \\ |Q_t| \leq M \end{matrix} \quad \text{in } R^1 \times [0,\infty).$$

From (1.1) and (1.2), we then get

Corollary 4.9 $|P_{xx}| \leq C < \infty$, $|Q_{xx}| \leq C < \infty$ in $R^1 \times [0,\infty)$.

Proof of Lemma 4.8. From (4.6), we obtain

$$\begin{aligned} |P_t(x,t)| &\leq C + C\int_0^t \int_{R^1} \Gamma(x,t;y,s)[|P - Q|(y,s) + |P_t|(y,s)]dyds \\ &\leq C + C\int_0^t se^{Cs}ds + \int_0^t \max_x |P_t(x,s)ds \quad \text{(by Lemma 2.3).} \end{aligned}$$

Gronwall inequality then gives

$$\max_x |P_t(x,t)| \leq g(t) \tag{4.22}$$

where $g(t)$ is a positive increasing function. Let $u = P_t - M$, $v = Q_t - M$. It is easy to see that

$$u_t - u_{xx} + u = (1 + E(x)F'(Q))v + ME(x)F'(Q) \tag{4.23}$$

$$v_t = u - v. \tag{4.24}$$

By Corollary 4.7 and the fact that $F'(u) \to 0$ as $u \to \infty$, we can find a $\tilde{t}$, such that

$$1 + E(x)F'(Q(x,t)) \geq 0 \quad \text{in } R^1 \times [\tilde{t}, \infty).$$

By (4.22), we can choose M so large such taht

$$\begin{matrix} u < 0 \\ v < 0 \end{matrix} \quad \text{in } R^1 \times [0, \tilde{t}].$$

We shall show that $u < 0$, $v < 0$ in $R^1 \times [0, \infty)$. If they are not true, then the argument as in the proof of Lemma 2.2, shows that there exists a point (x_0, t_0), such that

$$\begin{matrix} u < 0 \\ v < 0 \end{matrix} \quad \text{in } R^1 \times [0, t_0)$$

and $u(x_0, t_0) = 0$. Obviously, $\tilde{t} < t_0$. Hence we see that the RHS of (4.23) is nonpositive in $R^1 \times [\tilde{t}, t_0)$. The maximum principle implies that $u \equiv 0$ in the strip $R^1 \times [\bar{t}, t_0)$, which is a contradiction. This proves our assertions. Similarly, we can prove $P_t + M \geq 0$, $Q_t + M \geq 0$ in $R^1 \times [0, \infty)$. The proof is complete.

Theorem 4.10 If F satisfies (4.2), then

$$\lim_{t \to \infty} \frac{P(x,t)}{t} = \frac{C_1 + C_2}{4}\beta, \quad \lim_{t \to \infty} \frac{Q(x,t)}{t} = \frac{C_1 + C_2}{4}\beta \quad \text{uniformly in } x \text{ as } t \to \infty.$$

Proof. From (1.1) and (1.2)

$$(P+Q)_t - \frac{1}{2}(P+Q)_{xx} = E(x)F(Q) + \frac{1}{2}(P-Q)_{xx}.$$

Making a change of variables: $y = \sqrt{2}x, w = P + Q$, we get

$$w_t - w_{yy} = E(y)F(Q) + (P-Q)_{yy}. \tag{4.25}$$

Also

$$w(y, 0) = 0.$$

Representing w in terms of the fundamental solution, we have

$$\begin{aligned} w(y,t) &= \textstyle\int_0^t \int_{R^1} \Gamma(y,t;z,s)E(z)F(Q(z,s))dzds + \int_0^t \int_{R^1} \Gamma(y,t;z,s)(P-Q)_{zz}(z,s)dzds \\ &\triangleq I_1 + I_2. \end{aligned} \tag{4.26}$$

Since by Corollary 4.7, $F(Q(y,t)) \to \beta$ uniformly in y as $t \to \infty$, we can get

$$\begin{aligned} \frac{I_1}{t} &= \frac{1}{t}\int_0^t \int_{R^1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} E(y + \sqrt{2(t-s)}u) F(Q(y + \sqrt{2(t-s)}u, s)dsdu \\ &= \int_{R^1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \int_0^1 E(y + \sqrt{2t(1-v)}u) F(Q(y + \sqrt{2t(1-v)}u, tv)dv \\ &\longrightarrow \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-u^2/2} C_1 \beta du + \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} C_2 \beta du \\ &= \frac{C_1 + C_2}{2}\beta \quad \text{uniformly in } y \text{ as } t \to \infty. \end{aligned} \tag{4.27}$$

We turn to the estimate of I_2. Write

$$\begin{aligned} I_2 &= \textstyle\int_0^{t-1}\int_{R^1}\Gamma(y,t;z,s)(P-Q)_{zz}(z,s)dzds + \int_{t-1}^{t}\int_{R^1}\Gamma(y,t;z,s)(P-Q)_{zz}(z,s)dzds \\ &\triangleq J_1 + J_2. \end{aligned}$$

By Corollary 4.9,

$$\frac{|J_2|}{t} \le \frac{C}{t}\int_{t-1}^{t} ds = \frac{C}{t}.$$

As to J_1, using integrations by parts, and noting that boundary terms disappear because $(P-Q)_y$ and $(P-Q)$ are bounded in y at ∞, we obtain

$$\begin{aligned} \frac{|J_1|}{t} &\le \frac{C}{t}\int_0^{t-1}\int_{R^1}|\Gamma_t(y,t;z,s)||P-Q|dzds \\ &\le \frac{C}{t}\int_0^{t-1}\int_{R^1}\left(\frac{1}{t-s}+\frac{(y-z)^2}{(t-s)^2}\right)\Gamma(y,t;z,s)|Q_t(z,s)|dzds \\ &\le \frac{C}{t}\int_0^{t-1}\frac{1}{t-s}ds \\ &= \frac{C\log t}{t} \longrightarrow 0, \quad \text{as } t \to \infty. \end{aligned}$$

From (4.26)-(4.27), and the estimates on the J_i, we conclude that

$$\frac{w(y,t)}{t} \to \frac{C_1+C_2}{2}\beta \quad \text{uniformly in } y \text{ as } t \to \infty. \tag{4.28}$$

Noting that $Q_t = P - Q$, and Q_t is bounded whereas $Q(x,t) \to \infty$ as $t \to \infty$, we get

$$\frac{P(x,t)}{Q(x,t)} = 1 + \frac{Q_t(x,t)}{Q(x,t)} \longrightarrow 1 \qquad \text{uniformly in } x \text{ as } t \to \infty. \tag{4.29}$$

Combining (4.28) and (4.29), we obtain the assertions of the theorem.

§5 Large Time Behavior(Continued)

In this section, we consider the case (4.3); more specifically, we shall assume that

$$\left\{ \begin{array}{l} F(u)u^\gamma \to \beta > 0 \\ -F'(u)u^{\gamma+1} \to \gamma\beta > 0 \end{array} \right. \quad \text{as } u \to \infty. \tag{5.1}$$

We shall study the large time behavior of $w \equiv P + Q$. It will be convenient to rewrite (4.25) as

$$\begin{aligned} w_t - w_{yy} &= 2^\gamma E(y)F(w) + E(y)F(w)\left(\tfrac{F(Q)}{F(w)} - 2^\gamma\right) + (P-Q)_{yy} \\ &\triangleq 2^\gamma E(y)F(w) + f_1(y,t) + f_2(y,t). \end{aligned}$$

Consider the solution of

$$\begin{cases} u_t - u_{yy} = 2^\gamma E(y)F(w) \\ u(y,0) = 0. \end{cases} \tag{5.2}$$

Lemma 5.1 The function u satisfies

$$\frac{|u-w|_{L^\infty(R^1)}(t)}{t^{1/(1+\gamma)}} \longrightarrow 0 \qquad \text{as } t \to \infty, \tag{5.3}$$

and consequently

$$ct^{\frac{1}{1+\gamma}} \le u(y,t) \le Ct^{\frac{1}{1+\gamma}} \tag{5.4}$$

for all $y \in R^1$ and $t \ge N$, N a large constant.

Proof. It is clear that $u - w = I_1 + I_2$ where

$$I_i(y,t) = \int_0^t \int_{R^1} \Gamma(y,t;z,s) f_i(z,s)dzds, \qquad i = 1,2.$$

I_2 is the same in the proof of Theorem 4.10 so that $I_2(y,t)/t^{1/(1+\gamma)} \to 0$ uniformly in y as $t \to \infty$. To complete the proof of (5.3), it remains to show that $I_1(y,t)/t^{1/(1+\gamma)} \to 0$ uniformly in y as $t \to \infty$. Noting that

$$h(t) \triangleq \max_y |\frac{F(Q)}{F(w)} - 2^\gamma| \to 0 \qquad \text{as } t \to \infty$$

by (4.29) and (5.1), we get

$$\begin{aligned} |I_1(y,t)|/t^{1/(1+\gamma)} &\le \frac{1}{t^{1/(1+\gamma)}} \int_0^t \int_{R^1} \Gamma(y,t;z,s)F(w)w^\gamma \frac{1}{w^\gamma} \left| \frac{F(Q)}{F(w)} - 2^\gamma \right| dzds \\ &\le \frac{C}{t^{1/(1+\gamma)}} \int_0^t \frac{h(s)}{s^{\gamma/(1+\gamma)}} ds \qquad \text{(by Lemma 5.1)} \\ &= C \int_0^1 \frac{h(tv)}{v^{\gamma/(1+\gamma)}} dv \longrightarrow 0 \qquad \text{as } t \to \infty. \end{aligned}$$

From (5.3) and Corollary 4.7, (5.4) follows.

Next we prove

Lemma 5.3 There exist positive constants C and t_0 such that

$$|u_y(y,t)| \le Ct^{\frac{1}{1+\gamma} - \frac{1}{2}}$$

for all $y \in R^1$ and $t \ge t_0$.

Proof. Representing u in terms of the fundamental solution, we have

$$u(y,t) = \int_0^t \int_{R^1} \Gamma(y,t;z,s) 2^\gamma E(z) F(w(z,s)) dzds.$$

Differentiating the above equation in y and using Corollary 4.7, we see that

$$\begin{aligned}
|u_y(y,t)| &\leq C\int_0^t\int_{R^1}\Gamma(y,t;z,s)\frac{|x-y|}{(t-s)w^\gamma(z,s)}dzds \\
&\leq C\int_0^t\frac{1}{(t-s)^{1/2}s^{\gamma/(1+\gamma)}}ds \\
&\leq Ct^{\frac{1}{1+\gamma}-\frac{1}{2}}.
\end{aligned}$$

To study the large time behavior of u, we shall introduce the scaled function $u_\alpha(y,t) = u(a+\alpha y, \alpha^2 t)/\alpha^{\frac{2}{1+\gamma}}$ for any $a \in R^1$ and $\alpha > 0$. Then

$$\begin{aligned}
u_{\alpha t} - u_{\alpha yy} &= 2^\gamma\alpha^{\frac{2\gamma}{1+\gamma}}E(a+\alpha y)F(w) \\
&= \frac{2^\gamma E(a+\alpha y)}{u_\alpha^\gamma}F(u)u^\gamma + 2^\gamma E(a+\alpha y)\alpha^{\frac{2\gamma}{1+\gamma}}(F(w)-F(u)) \\
&\triangleq g_{1\alpha} + g_{2\alpha}. \qquad (5.5)
\end{aligned}$$

In the above expressions, all functions are evaluated at $(a+\alpha y, \alpha^2 t)$. Using the assumption (5.1) and (5.3), we obtain

$$\begin{aligned}
|g_{2\alpha}(y,t)| &\leq C\alpha^{\frac{2\gamma}{1+\gamma}}|F'(\lambda u + (1-\lambda)w)||u-w| \qquad (0<\lambda<1) \\
&\leq C\frac{1}{\alpha^{\frac{2}{1+\gamma}}}|u-w|_{L^\infty(R^1)}(\alpha^2 t) \longrightarrow 0 \quad \text{as } \alpha\to\infty, \text{ for any } t>0; \qquad (5.6)
\end{aligned}$$

here, as before, all functions are evaluated at $(a+\alpha y, \alpha^2 t)$. Note that, by (5.4),

$$ct^{\frac{1}{1+\gamma}} \leq u_\alpha(y,t) \leq Ct^{\frac{1}{1+\gamma}} \quad \text{for } \alpha \text{ sufficiently large.} \qquad (5.7)$$

We next estimate $g_{1\alpha}(y,t)$ as follows.

If $\alpha t^2 \leq N$, N is the constant in (5.4), then

$$g_{1\alpha}(y,t) \leq C\alpha^{\frac{2\gamma}{1+\gamma}} \leq C\frac{1}{t^{\gamma/(1+\gamma)}}.$$

If $\alpha^2 t \geq N$, then by (5.7)

$$\begin{aligned}
g_{1\alpha}(y,t) &\leq \frac{C}{u_\alpha^\gamma(y,t)} \\
&\leq \frac{C}{t^{\gamma/(1+\gamma)}}.
\end{aligned}$$

We conclude that

$$g_{1\alpha}(y,t) \leq \frac{C}{t^{\gamma/(1+\gamma)}} \quad \text{for any } \alpha, t. \qquad (5.8)$$

From (5.6) and (5.8), we see that the RHS of (5.5) is bounded in any compact subset $K \subset R^1 \times (0,\infty)$. Applying L^p-estimates, we get $|u_\alpha|_{W_p^{1,2}(K)} \leq C < \infty$. By Sobelev imbedding theorem, we can find a sequence $\alpha_n \to \infty$, such that $u_{\alpha_n}(y,t) \to \bar{u}(y,t)$ unifomly in K as $\alpha_n \to \infty$, for some function $\bar{u}$. By a diagonal argument, we may assume that

$$u_{\alpha_n}(y,t) \to \bar{u}(y,t) \quad \text{in} \quad R^1 \times (0,\infty). \tag{5.9}$$

By (5.7), we have

$$ct^{1/(1+\gamma)} \leq \bar{u}(y,t) \leq Ct^{1/(1+\gamma)} \qquad \text{for any } t > 0.$$

Since $u_{\alpha_n}(y,0) = 0$, we can represent u_{α_n} in terms of the fundamental solution to get

$$u_{\alpha_n}(y,t) = \int_0^t \int_{R^1} \Gamma(y,t;z,s) g_{1\alpha_n}(z,s)dzds + \int_0^t \int_{R^1} \Gamma(y,t;z,s) g_{2\alpha_n}(z,s)dzds.$$

Letting $\alpha_n \to \infty$, we see that (by (5.6) and (5.9))

$$\bar{u}(y,t) = \int_0^t \int_{R^1} \Gamma(y,t;z,s) 2^\gamma \beta \frac{E(z)}{\bar{u}^\gamma(z,s)} dzds; \tag{5.10}$$

i.e.

$$\begin{cases} \bar{u}_t - \bar{u}_{yy} = 2^\gamma \beta E(y)/\bar{u}^\gamma \quad \text{in } R^1 \times (0,\infty) \\ \bar{u}(y,0) = 0 \\ ct^{1/(1+\gamma)} \leq \bar{u}(y,t) \leq Ct^{1/(1+\gamma)}. \end{cases} \tag{5.11}$$

Lemma 5.3 The solution of (5.11) is unique.

Proof. Suppose u_1, u_2 are the two solutions of (5.11). Setting $v = u_1 - u_2$, we have

$$v_t - v_{yy} + g(y,t)v = 0 \qquad \text{in } R^1 \times (0,\infty)$$

where $g(y,t) = 2^\gamma \beta E(y) \gamma [\lambda u_1(y,t) + (1-\lambda)u_2(y,t)]^{\gamma-1}/u_1^\gamma(y,t)u_2^\gamma(y,t) \geq 0$ (for some λ, $0 < \lambda < 1$). Set $v_\epsilon = v + \epsilon$. From the third equation of (5.13), we see that $v_\epsilon > 0$ in $R^1 \times [0,\delta]$ for some $\delta > 0$. Since

$$v_{\epsilon t} - v_{\epsilon yy} + g(y,t)v_\epsilon = \epsilon g(y,t) \geq 0,$$

the maximum principle gives us that $v_\epsilon > 0$ in $R^1 \times [0,\infty)$. Letting $\epsilon \to 0$, we deduce that $v \geq 0$. Similarly, we can prove that $v \leq 0$. Hence $v \equiv 0$.

By Lemma 5.3, we conclude that

$$u_\alpha(y,t) \longrightarrow \bar{u}(y,t) \qquad \text{as } \alpha \to \infty.$$

For $\alpha > 0$, $\bar{\alpha} > 0$, we have that

$$u_{\alpha\bar{\alpha}}(y,t) \longrightarrow \bar{u}(y,t) \qquad \text{as } \alpha \to \infty.$$

On the other hand

$$u_{\alpha\bar{\alpha}}(y,t) = \frac{1}{\bar{\alpha}^{\frac{2}{1+\gamma}}} u_\alpha(\bar{\alpha}y, \bar{\alpha}^2 t) \longrightarrow \frac{1}{\bar{\alpha}^{\frac{2}{1+\gamma}}} \bar{u}(\bar{\alpha}y, \bar{\alpha}^2 t) \qquad \text{as } \alpha \to \infty.$$

Therefore,

$$\bar{u}(y,t) = \frac{1}{\bar{\alpha}^{\frac{2}{1+\gamma}}}\bar{u}(\bar{\alpha}y, \bar{\alpha}^2 t).$$

Let $\bar{\alpha} = \frac{1}{\sqrt{t}}$, we see that $\bar{u}(y,t)$ must have the form:

$$\bar{u}(y,t) = t^{\frac{1}{1+\gamma}} g(\frac{y}{\sqrt{t}}). \tag{5.12}$$

It is easy to see that g must satisfy:

$$g''(x) + \frac{x}{2}g'(x) = \frac{1}{1+\gamma}g - 2^\gamma \beta \frac{E(x)}{g^\gamma}. \tag{5.13}$$

Set $f(t) = \lim_{y\to\infty} \bar{u}(y,t)$. Then rewrite (5.10) as

$$\bar{u}(y,t) = \int_0^t \int_{R^1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} 2^\gamma \beta \frac{E(y+\sqrt{2(t-s)}u)}{\bar{u}(y+\sqrt{2(t-s)}u, s)} duds.$$

Letting $y \to \infty$, we see that

$$f(t) = \int_0^t \frac{2^\gamma \beta C_2}{f^\gamma(s)} ds;$$

i.e.

$$f' = \frac{2^\gamma \beta C_2}{f^\gamma}, \quad f(0) = 0.$$

Hence $f(t) = (2^\gamma \beta C_2)^{\frac{1}{1+\gamma}} t^{\frac{1}{1+\gamma}}$. Comparing with (5.12), we see that

$$\lim_{y\to\infty} g(y) = (2^\gamma \beta C_2)^{\frac{1}{1+\gamma}}. \tag{5.14}$$

Similarly,

$$\lim_{y\to-\infty} g(y) = (2^\gamma \beta C_1)^{\frac{1}{1+\gamma}}. \tag{5.15}$$

Since $\bar{u}$ is unique, the solution of (5.13)-(5.15) is also unique. From the relation $u_\alpha(0,t) \to t^{\frac{1}{1+\gamma}} g(0)$, we see that

$$\frac{u(a,\alpha^2 t)}{\alpha^{2/(1+\gamma)}} \longrightarrow t^{\frac{1}{1+\gamma}} g(0) \qquad \text{as} \quad \alpha \to \infty.$$

In particular,

$$\frac{u(0,t)}{t^{1/(1+\gamma)}} \longrightarrow g(0) \qquad \text{as } t \to \infty.$$

Using Lemma 5.2, we get that for any $y \in [-M, M]$

$$\begin{aligned} \frac{|u(y,t) - u(0,t)|}{t^{1/(1+\gamma)}} &\leq \frac{|u_y(\lambda y, t)||y|}{t^{1/(1+\gamma)}} \\ &\leq \frac{CM}{\sqrt{t}}. \end{aligned}$$

We conclude that for any $M > 0$

$$\frac{u(y,t)}{t^{1/(1+\gamma)}} \longrightarrow g(0) \quad \text{uniformly in } y \in [-M, M] \text{ as } t \to \infty.$$

Using (5.3) and the fact that $w = P + Q$, we have established

Theorem 5.4 If F satisfies the assuption (5.1), then for any $M > 0$

$$\lim_{t\to\infty} \frac{P(x,t)}{t^{\frac{1}{1+\gamma}}} = \lim_{t\to\infty} \frac{Q(x,t)}{t^{\frac{1}{1+\gamma}}} = \frac{1}{2}g(0),$$

uniformly in $x \in [-M, M]$; here g is the unique solution of (5.13)-(5.15).

Acknowledgement. I am grateful to Dr. David S. Ross from Eastman Kodak company who introduce the problem to us at an IMA industrial problem seminar; I would also thank Professor Avner Friedman for several stimulating conversations.

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