

**OPERATOR MATRICES WITH CHORDAL  
INVERSE PATTERNS**

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# OPERATOR MATRICES WITH CHORDAL INVERSE PATTERNS\*

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**Abstract.** We consider invertible operator matrices whose conformally partitioned *inverses* have 0 blocks in positions corresponding to a chordal graph. In this event, we describe a) block entry formulae that express certain blocks (in particular, those corresponding to 0 blocks in the inverse) in terms of others, under a regularity condition, and b) in the Hermitian case, a formula for the inertia in terms of inertias of certain key blocks.

**1. Introduction.** For Hilbert spaces  $\mathcal{H}_i, i = 1, \dots, n$ , let  $\mathcal{H}$  be the Hilbert space defined by  $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$ . Suppose, further, that  $A : \mathcal{H} \rightarrow \mathcal{H}$  is a linear operator in matrix form, partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \\ \vdots & & & \vdots \\ A_{n1} & & \dots & A_{nn} \end{bmatrix},$$

in which  $A_{ij} : \mathcal{H}_j \rightarrow \mathcal{H}_i, i, j = 1, \dots, n$ . (We refer to such an  $A$  as an *operator matrix*.) We assume throughout that  $A$  is invertible and that  $A^{-1} = B = [B_{ij}]$  is partitioned conformably. We are interested in the situation in which some of the blocks  $B_{ij}$  happen to be zero. In this event we present (1) some formulaic relations among blocks of  $A$  (under a further regularity condition) and (2) a formula for the inertia of  $A$ , in terms of that of certain principal submatrices, when  $A$  is Hermitian. For this purpose we define an undirected graph  $G = G(B)$  on vertex set  $N \equiv \{1, \dots, n\}$  as follows: there is an edge  $\{i, j\}, i \neq j$ , in  $G(B)$  unless both  $B_{ij}$  and  $B_{ji}$  are 0.

An undirected graph  $G$  is called *chordal* if no subgraph induced by 4 or more vertices is a cycle. Note that if  $G(B)$  is not complete, then there is a chordal graph  $G$  (that is also not complete) such that  $G(B)$  is contained in  $G$ . Thus, if there is any symmetric sparsity in  $B$ , our results will apply (perhaps by ignoring the fact that some blocks are 0), even if  $G(B)$  is not chordal.

A *clique* in an undirected graph  $G$  is a set of vertices whose vertex induced subgraph in  $G$  is complete (i.e. contains all possible edges  $\{i, j\}, i \neq j$ ). A clique is *maximal* if it is not a proper subset of any other clique. Let  $\mathcal{C} = \mathcal{C}(G) = \{\alpha_1, \dots, \alpha_p\}$  be the collection

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of maximal cliques of the graph  $G$ . The intersection graph  $\mathcal{G}$  of the maximal cliques is an undirected graph with vertex set  $\mathcal{C}$  and an edge between  $\alpha_i$  and  $\alpha_j$ ,  $i \neq j$  if  $\alpha_i \cap \alpha_j \neq \phi$ . The graph  $G$  is connected and chordal if and only if  $\mathcal{G}$  has a *spanning tree*  $\mathcal{T}$  that satisfies the *intersection property*:  $\alpha_i \cap \alpha_j \subseteq \alpha_k$  whenever  $\alpha_k$  lies on the unique simple path in  $\mathcal{T}$  from  $\alpha_i$  to  $\alpha_j$ . Such a tree  $\mathcal{T}$  is called a *clique tree* for  $G$  and is generally not unique [BJL]. (See [Go] for general background facts about chordal graphs.) A clique tree is an important tool for understanding the structure of a chordal graph. For example, for a pair of nonadjacent vertices  $u, v$  in  $G$ , a  *$u, v$  separator* is a set of vertices of  $G$  whose removal (along with all edges incident with them) leaves  $u$  and  $v$  in different connected components of the result. A  $u, v$  separator is called *minimal* if no proper subset of it is a  $u, v$  separator. A set of vertices is called a *minimal vertex separator* if it is a minimal  $u, v$  separator for some pair of vertices  $u, v$ . (Note that it is possible for a proper subset of a minimal vertex separator to also be a minimal vertex separator.) If  $\alpha_i$  and  $\alpha_j$  are adjacent cliques in a clique tree for a chordal graph  $G$  then  $\alpha_i \cap \alpha_j$  is a minimal vertex separator for  $G$ . The collection of such intersections (including multiplicities) is independent of the clique tree and all minimal vertex separators for  $G$  occur among such intersections.

Given an  $n$ -by- $n$  operator matrix  $A = (A_{ij})$ , we denote the operator submatrix lying in block rows  $\alpha \subseteq N$  and block columns  $\beta \subseteq N$  by  $A[\alpha, \beta]$ . When the submatrix is principal (i.e.  $\beta = \alpha$ ), we abbreviate  $A[\alpha, \alpha]$  to  $A[\alpha]$ .

We define the *inertia* of an Hermitian operator  $B$  on a Hilbert space  $K$  as follows. The triple  $i(B) = (i_+(B), i_-(B), i_o(B))$  has components defined by

$i_+(B) \equiv$  the maximum dimension of an invariant subspace of  $B$  on which the quadratic form is positive.

$i_-(B) \equiv$  the maximum dimension of an invariant subspace of  $B$  on which the quadratic form is negative.

and

$i_o(B) \equiv$  the dimension of the kernel of  $B$  ( $\ker B$ ).

Each component of  $i(B)$  may be a nonnegative integer, or  $\infty$  in case the relevant dimension is not finite. We say that two Hermitian operators  $B_1$  and  $B_2$  on  $K$  are *congruent* if there is an invertible operator  $C : K \rightarrow K$  such that

$$B_2 = C^* B_1 C.$$

According to the spectral theorem, if the bounded linear operator  $A : \mathcal{H} \rightarrow \mathcal{H}$  is Hermitian, then  $A$  is unitarily congruent (similar) to a direct sum:

$$U^* A U = \begin{bmatrix} A_+ & 0 & 0 \\ 0 & A_- & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

in which  $A_+$  is positive definite and  $A_-$  is negative definite. As  $i(A) = i(U^*AU)$ ,  $i_+(A)$  is the “dimension” of the direct summand  $A_+$ ,  $i_-(A)$  the dimension of  $A_-$ , and  $i_0(A)$  the dimension of the 0 direct summand, including the possibility of  $\infty$  in each case. It is easily checked that the following three statements are then equivalent:

- (i)  $A$  is congruent to  $\begin{bmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & 0 \end{bmatrix}$ , in which the sizes of the diagonal blocks are  $i_+(A)$ ,  $i_-(A)$  and  $i_0(A)$ , respectively;
- (ii) each of  $A_+$  and  $A_-$  is invertible;

and

- (iii)  $A$  has closed range.

We shall frequently need to make use of congruential representations of the form (i) and, so, assume throughout that each key principal submatrix (i.e. those corresponding to maximal cliques and minimal separators in the chordal graph  $G$  of the inverse of an invertible Hermitian matrix) has closed range. This may be a stronger assumption than is necessary for our formulae in section 3; so there is an open question here.

Chordal graphs have played a key role in the theory of positive definite completions of matrices and in determinantal formulae. For example, in [GJSW] it was shown that if the undirected graph of the specified entries of a partial positive definite matrix (with specified diagonal) is chordal, then a positive definite completion exists. (See e.g. [Jo] for definitions and background.) Furthermore, if the graph of the specified entries is not chordal, then there is partial positive definite data for which there is no positive definite completion. (These facts carry over in a natural way to operator matrices.) If there is a positive definite completion, then there is a unique determinant maximizing one that is characterized by having 0’s in the *inverse* in all positions corresponding to originally unspecified entries. Thus, if the graph of the specified entries is chordal, then the ordinary (undirected) graph of the inverse of the determinant maximizer is (generically) the same chordal graph. (In the partial positive definite operator matrix case such a zeros in-the-inverse completion still exists when the data is chordal and is an open question otherwise.) This was one of the initial motivations for studying matrices with chordal inverse (nonzero) patterns. Other motivation includes the structure of inverses of banded matrices, and this is background for section 2.

If an invertible matrix  $A$  has an inverse pattern contained in a chordal graph  $G$ , then  $\det A$  may be expressed in terms of certain key principal minors [BJ], as long as all relevant minors are nonzero:

$$\det A = \frac{\prod_{\alpha \in \mathcal{C}} \det A[\alpha]}{\prod_{\{\alpha, \beta\} \in \mathcal{E}} \det A[\alpha \cap \beta]}.$$

Here  $\mathcal{C}$  is the collections of maximal cliques of  $G$ , and  $\mathcal{T} = (\mathcal{C}, \mathcal{E})$  is a clique tree for  $G$ .

Thus, the numerator is the product of principal minors associated with maximal cliques, while the denominator has those associated with minimal vertex separators (with proper multiplicities). There is no natural analog of this determinantal formula in the operator case, but the inertia formula presented in section 3 has a logarithmic resemblance to it.

**2. Entry Formulae.** Let  $G = (N, E)$  be a chordal graph. We will say that an operator matrix  $A = [A_{ij}]$  is  $G$ -regular if  $A[\alpha]$  is invertible whenever  $\alpha \subseteq V$  is either a maximal clique of  $G$  or a minimal vertex separator of  $G$ . In this section we will establish explicit formulae for some of the block entries of  $A$  when  $G(A^{-1}) \subseteq G$ . Specifically, those entries are the ones corresponding to edges that are *absent* from  $E$  (see Theorem 3).

LEMMA 1. Let  $A = [A_{ij}]$  be a 3-by-3 operator matrix, and assume that

$$M_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad M_2 = \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \text{ and } A_{22} \text{ are each invertible.}$$

Then  $B = A^{-1}$  exists and satisfies  $B_{13} = 0$  if and only if  $A_{13} = A_{12}A_{22}^{-1}A_{23}$ .

*Proof.* Let us compute the Schur complement of  $A_{22}$  in  $A$ :

$$(1) \quad \begin{bmatrix} I & -A_{12}A_{22}^{-1} & 0 \\ 0 & I & 0 \\ 0 & -A_{32}A_{22}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -A_{22}^{-1}A_{21} & I & -A_{22}^{-1}A_{23} \\ 0 & 0 & I \end{bmatrix} \\ = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 & A_{13} - A_{12}A_{22}^{-1}A_{23} \\ 0 & A_{22} & 0 \\ A_{31} - A_{32}A_{22}^{-1}A_{21} & 0 & A_{33} - A_{32}A_{22}^{-1}A_{23} \end{bmatrix}.$$

If  $B = A^{-1}$  exists, then

$$(2) \quad \begin{bmatrix} B_{11} & B_{13} \\ B_{31} & B_{33} \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{13} - A_{12}A_{22}^{-1}A_{23} \\ A_{31} - A_{32}A_{22}^{-1}A_{21} & A_{33} - A_{32}A_{22}^{-1}A_{23} \end{bmatrix}^{-1},$$

and hence if  $B_{13} = 0$ , then necessarily we have  $A_{13} = A_{12}A_{22}^{-1}A_{23}$ . Conversely, if  $A_{13} = A_{12}A_{22}^{-1}A_{23}$ , then the (1,3) entry of the matrix on the right-hand side of (1) is zero. Note that  $A_{11} - A_{12}A_{22}^{-1}A_{21}$  and  $A_{33} - A_{32}A_{22}^{-1}A_{23}$  are invertible, because they are the Schur complements of  $A_{22}$  in  $M_1$  and  $M_2$ . Hence  $A$  is invertible, and by (2) we have  $B_{13} = 0$ .  $\square$

Under the conditions of the preceding lemma, if we would like  $B_{31} = 0$  then we must also have  $A_{31} = A_{32}A_{22}^{-1}A_{21}$ . Notice that the graph of  $B$  in this case is a path:

$$G = \textcircled{1} - \textcircled{2} - \textcircled{3}.$$

Suppose now that  $A = [A_{ij}]$  is an invertible  $n \times n$  operator matrix, that  $1 < k \leq m < n$ , and that  $A^{-1} = [B_{ij}]$  satisfies  $B_{ij} = 0$  and  $B_{ji} = 0$  whenever  $i < k$  and  $j > m$ . In this case  $B$  has the block form

$$B = \begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} & 0 \\ \tilde{B}_{21} & \tilde{B}_{22} & \tilde{B}_{23} \\ 0 & \tilde{B}_{32} & \tilde{B}_{33} \end{bmatrix},$$

in which  $\tilde{B}_{11} = B[\{1, \dots, k-1\}]$ ,  $\tilde{B}_{22} = B[\{k, \dots, m\}]$  and  $\tilde{B}_{33} = B[\{m+1, \dots, n\}]$ . Let  $A = [\tilde{A}_{ij}]$  be partitioned conformably. If in addition to the above conditions we also have that  $A[\{1, \dots, m\}]$ ,  $A[\{k, \dots, n\}]$  and  $A[\{k, \dots, m\}]$  are invertible, then we simply have the case covered in the preceding Lemma, and we may deduce that

$$\begin{aligned} & A[\{1, \dots, k-1\}, \{m+1, \dots, n\}] \\ &= A[\{1, \dots, k-1\}, \{k, \dots, m\}] A[\{k, \dots, m\}]^{-1} A[\{k, \dots, m\}, \{m+1, \dots, n\}], \end{aligned}$$

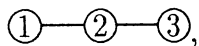
with a similar formula holding for  $A[\{m+1, \dots, n\}, \{1, \dots, k-1\}]$ . From this we may write explicit formulae for individual entries in  $A$ . For example, we may express any entry  $A_{ij}$  for which  $i < k$  and  $j > m$  as

$$(3) \quad A_{ij} = A[\{i\}, \{k, \dots, m\}] A[\{k, \dots, m\}]^{-1} A[\{k, \dots, m\}, \{j\}].$$

There is an obvious similarity between this situation and that covered in Lemma 1, which one sees simply by looking at the block structure of  $A^{-1}$ . But there are also some similarities which may be observed by looking at graphs. In the block case we just considered, the graph  $G(B)$  is a chordal graph consisting of exactly two maximal cliques, the sets  $\alpha_1 = \{1, \dots, m\}$  and  $\alpha_2 = \{k, \dots, n\}$ . The intersection  $\beta = \{k, \dots, m\}$  of  $\alpha_1$  and  $\alpha_2$  is a minimal vertex separator of  $G$  (in fact, the only minimal vertex separator in this graph). The formula (3) may then be written

$$(4) \quad A_{ij} = A[\{i\}, \beta] A[\beta]^{-1} A[\beta, \{j\}].$$

Note now in the 3-by-3 case that the equation  $A_{13} = A_{12} A_{22}^{-1} A_{23}$  has the same form as (4) when we let  $\beta = \{2\}$ . In fact, since  $\beta = \{2\}$  is a minimal separator of the vertices 1 and 3 in the graph



we see that  $\{2\}$  plays the same role in the 3-by-3 case as  $\{k, \dots, m\}$  does in the  $n \times n$  case.

In Theorem 3 we will encounter expressions of the form

$$(5) \quad A_{ij} = A[\{i\}, \beta_1] A[\beta_1]^{-1} A[\beta_1, \beta_2] A[\beta_2]^{-1} \cdots A[\beta_m]^{-1} A[\beta_m, \{j\}]$$

in which each  $\beta_k$  is a minimal vertex separator in a chordal graph. The sequence  $(\beta_1, \dots, \beta_m)$  is obtained by looking at a clique tree for the chordal graph, indentifying a path  $(\alpha_0, \alpha_1, \dots, \alpha_m)$  in the tree, and setting  $\beta_k = \alpha_{k-1} \cap \alpha_k$ .

These expressions turn out to be the natural generalization of (4) to cases in which the graph of  $B$  is any chordal graph. In addition, the results of this section generalize results of [JL2] from the scalar case to the operator case.

LEMMA 2. Let  $A : \mathcal{H} \rightarrow \mathcal{H}$  be an invertible operator matrix, with  $B = A^{-1}$ . Let  $G = (N, E)$  be the undirected graph of  $B$ . Let  $\{i, j\} \notin E$  and let  $\beta \subseteq N$  be any  $i, j$  separator for which  $A[\beta]$  is invertible. Then

$$A_{ij} = A[\{i\}, \beta] A[\beta]^{-1} A[\beta, \{j\}].$$

*Proof.* Without loss of generality we may assume that  $\beta = \{k, \dots, m\}$ , with  $k \leq m$ , and that  $\beta$  separates any vertices  $r$  and  $s$  for which  $r < k$  and  $s > m$ . Assuming then that  $i < k$  and  $j > m$ , we may write  $B$  as

$$\begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} & 0 \\ \tilde{B}_{21} & \tilde{B}_{22} & \tilde{B}_{23} \\ 0 & \tilde{B}_{22} & \tilde{B}_{33} \end{bmatrix}.$$

The result now follows from Lemma 1 and the remarks that follow it.  $\square$

If  $G$  is chordal and  $i$  and  $j$  are nonadjacent vertices then an  $i, j$  clique path will mean a path in any clique tree associated with  $G$  that joins a clique containing vertex  $i$  to a clique containing vertex  $j$ . One important property of any  $i, j$  clique path is that it will “contain” every minimal  $i, j$  separator in the following sense: If  $(\alpha_0, \dots, \alpha_m)$  is any  $i, j$  clique path, and if  $\beta$  is any minimal  $i, j$  separator then  $\beta = \alpha_{k-1} \cap \alpha_k$  for some  $k, 1 \leq k \leq m$ . Another important property of an  $i, j$  clique path is that every set  $\beta_k = \alpha_{k-1} \cap \alpha_k, 1 \leq k \leq m$ , is an  $i, j$  separator. It is *not* the case, however, that every  $\beta_k$  is a minimal  $i, j$  separator (see [JL2]).

THEOREM 3. Let  $G = (N, E)$  be a connected chordal graph, and let  $A : \mathcal{H} \rightarrow \mathcal{H}$  be a  $G$ -regular operator matrix. then the following assertions are equivalent:

- (i)  $A$  is invertible and  $G(A^{-1}) \subseteq G$ ;
- (ii) for every  $\{i, j\} \notin E$  there exists a minimal  $i, j$  separator  $\beta$  such that

$$A_{ij} = A[\{i\}, \beta] A[\beta]^{-1} A[\beta, \{j\}];$$

- (iii) for every  $\{i, j\} \notin E$  and every minimal  $i, j$  separator  $\beta$  we have

$$A_{ij} = A[\{i\}, \beta] A[\beta]^{-1} A[\beta, \{j\}];$$

- (iv) for every  $\{i, j\} \notin E$ , every  $i, j$  clique path  $(\alpha_0, \alpha_1, \dots, \alpha_m)$  and any  $k, 1 \leq k \leq m$  we have

$$A_{ij} = A[\{i\}, \beta_k] A[\beta_k]^{-1} A[\beta_k, \{j\}],$$

in which  $\beta_k = \alpha_{k-1} \cap \alpha_k$ ;

and

(v) for every  $\{i, j\} \notin E$  and every  $i, j$  clique path  $(\alpha_0, \alpha_1, \dots, \alpha_m)$  we have

$$A_{ij} = A[\{i\}, \beta_1] A[\beta_1]^{-1} A[\beta_1, \beta_2] \cdots A[\beta_m]^{-1} A[\beta_m, \{j\}],$$

in which  $\beta_k = \alpha_{k-1} \cap \alpha_k$ .

*Proof.* We will establish the following implications:

$$\begin{aligned} (iv) &\implies (iii) \implies (ii) \implies (iv); \\ (i) &\iff (iv) \iff (v). \end{aligned}$$

$(iv) \implies (iii)$  follows from the observation that every minimal  $i, j$  separator equals  $\beta_k$  for some  $k$ ,  $1 \leq k \leq m$ .

$(iii) \implies (ii)$  is immediate.

For  $(ii) \implies (iv)$ , let  $\{i, j\} \notin E$ , and let  $(\alpha_0, \alpha_1, \dots, \alpha_m)$  be a shortest  $i, j$  clique path. We will induct on  $m$ . For  $m = 1$  there is nothing to show, since in this case  $\beta_1 = \alpha_0 \cap \alpha_1$  is the only minimal  $i, j$  separator. Now let  $m \geq 2$ , and suppose that  $(iv)$  holds for all nonadjacent pairs of vertices for which the shortest clique path has length less than  $m$ . Since every minimal  $i, j$  separator equals  $\beta_k$  for some  $k$ , we have, by  $(ii)$ ,

$$A_{ij} = A[\{i\}, \beta_k] A[\beta_k]^{-1} A[\beta_k, \{j\}]$$

for some  $k$ ,  $1 \leq k \leq m$ . It will therefore suffice to show that for  $k = 1, 2, \dots, m - 1$  we have

$$(6) \quad A[\{i\}, \beta_k] A[\beta_k]^{-1} A[\beta_k, \{j\}] = A[\{i\}, \beta_{k+1}] A[\beta_{k+1}]^{-1} A[\beta_{k+1}, \{j\}]$$

Let us first observe that for  $k = 1, \dots, m - 1$ ,

$$(7) \quad A[\beta_k, \{j\}] = A[\beta_k, \beta_{k+1}] A[\beta_{k+1}]^{-1} A[\beta_{k+1}, \{j\}].$$

Indeed, suppose  $r \in \beta_k$ . Then  $(\alpha_k, \alpha_{k+1}, \dots, \alpha_m)$  is an  $r, j$  clique path of length  $m - k$ , and by the induction hypothesis we may write

$$A_{rj} = A[\{r\}, \beta_{k+1}] A[\beta_{k+1}]^{-1} A[\beta_{k+1}, \{j\}],$$

and equation (7) follows. A similar argument shows that for  $k = 2, \dots, m$  we have

$$(8) \quad A[\{i\}, \beta_{k+1}] = A[\{i\}, \beta_k] A[\beta_k]^{-1} A[\beta_k, \beta_{k+1}].$$

By (7) and (8), both sides of (6) are equal to

$$A[\{i\}, \beta_k] A[\beta_k]^{-1} A[\beta_k, \beta_{k+1}] A[\beta_{k+1}]^{-1} A[\beta_{k+1}, \{j\}],$$

and hence (6) holds, as required.

(i)  $\implies$  (iv) follows from Lemma 2.

For (iv)  $\implies$  (i), let the maximal cliques of  $G$  be  $\alpha_1, \alpha_2, \dots, \alpha_p$ ,  $p \geq 2$ . We will induct on  $p$ . In case  $p = 2$  then the result follows from Lemma 1, so let  $p > 2$  and suppose that the implication holds whenever the maximal cliques number fewer than  $p$ . Let  $\mathcal{T}$  be a clique tree associated with  $G$ , let  $\{\alpha_k, \alpha_{k+1}\}$  be any edge of  $\mathcal{T}$ , and suppose the vertex sets of the two connected components of  $\mathcal{T} - \{\alpha_k, \alpha_{k+1}\}$  are  $\mathcal{C}_1 = \{\alpha_1, \dots, \alpha_k\}$  and  $\mathcal{C}_2 = \{\alpha_{k+1}, \dots, \alpha_p\}$ . Set  $V_1 = \cup_{i=1}^k \alpha_i$  and  $V_2 = \cup_{i=k+1}^p \alpha_i$ . (Let  $G_V$  be the subgraph of  $G$  induced by the subset  $V$  of vertices.) Since induced subgraphs of a chordal graph are necessarily chordal,  $G_{V_1}$  and  $G_{V_2}$  are chordal graphs, and since (iv) holds for the matrix  $A$ , (iv) holds as well for  $A[V_1]$  and  $A[V_2]$ . By the induction hypothesis,  $A[V_1]$  and  $A[V_2]$  are invertible. Note also that  $V_1 \cap V_2 = \alpha_k \cap \alpha_{k+1}$ , which follows from the intersection property. Since  $A[V_1 \cap V_2]$  is invertible, we may now apply Lemma 1 to the matrix  $A$  (in which  $A_{11}$  is replaced by  $A[V_1 \setminus V_2]$ ,  $A_{22}$  by  $A[V_1 \cap V_2]$  and  $A_{33}$  by  $A[V_2 \setminus V_1]$ ), and conclude that  $A^{-1}[V_1 \setminus V_2, V_2 \setminus V_1] = 0$  and  $A^{-1}[V_2 \setminus V_1, V_1 \setminus V_2] = 0$ . In other words, if we set  $B = A^{-1}$  then  $B_{ij} = 0$  and  $B_{ji} = 0$  whenever  $i \in V_1 \setminus V_2$  and  $j \in V_2 \setminus V_1$ . Now if  $\{i, j\} \notin E$  then  $\alpha_k$  and  $\alpha_{k+1}$  may be chosen (renumbering the  $\alpha$ 's if necessary) so that  $i \in V_1 \setminus V_2$  and  $j \in V_2 \setminus V_1$ . Hence it must be that  $B_{ij} = 0$  and  $B_{ji} = 0$  whenever  $\{i, j\} \notin E$ .

For (iv)  $\implies$  (v), let  $\{i, j\} \notin E$ , and let  $(\alpha_0, \alpha_1, \dots, \alpha_m)$  be any  $i, j$  clique path. First, we must observe that for any  $k, 1 \leq k \leq m$ ,

$$(9) \quad A[\beta_k, \{j\}] = A[\beta_k, \beta_{k+1}] A[\beta_{k+1}]^{-1} A[\beta_{k+1}, \{j\}].$$

Indeed, by assumption, for any  $r \in \beta_k$  we have

$$A_{rj} = A[\{r\}, \beta_{k+1}] A[\beta_{k+1}]^{-1} A[\beta_{k+1}, \{j\}],$$

and (9) follows from this. By successively applying (9) we obtain

$$\begin{aligned} A_{ij} &= A[\{i\}, \beta_1] A[\beta_1]^{-1} A[\beta_1, \{j\}] \\ &= A[\{i\}, \beta_1] A[\beta_1]^{-1} A[\beta_1, \beta_2] A[\beta_2]^{-1} A[\beta_2, \{j\}] \\ &\quad \vdots \\ &= A[\{i\}, \beta_1] A[\beta_1]^{-1} A[\beta_1, \beta_2] \cdots A[\beta_{m-1}, \beta_m] A[\beta_m]^{-1} A[\beta_m, \{j\}], \end{aligned}$$

as required.

For  $(v) \implies (iv)$ , let  $\{i, j\} \notin E$ , and let  $(\alpha_0, \dots, \alpha_m)$  be an  $i, j$  clique path. Let  $r \in \alpha_{k-1}$ ,  $1 < k \leq m$ . We may write [because of assumption  $(v)$ ]

$$A_{ir} = A[\{i\}, \beta_1] A[\beta_1]^{-1} \cdots A[\beta_{k-1}]^{-1} A[\beta_{k-1}, \{r\}],$$

and because  $r \in \beta_k \implies r \in \alpha_k$  we thus have

$$(10) \quad A[\{i\}, \beta_k] = A[\{i\}, \beta_1] A[\beta_1]^{-1} \cdots A[\beta_{k-1}]^{-1} A[\beta_{k-1}, \beta_k].$$

It may be similarly shown that

$$(11) \quad A[\beta_k, \{j\}] = A[\beta_k, \beta_{k+1}] A[\beta_{k+1}]^{-1} \cdots A[\beta_m]^{-1} A[\beta_m, \{j\}].$$

By using (10) and (11) we therefore obtain

$$\begin{aligned} A_{ij} &= A[\{i\}, \beta_1] \cdots A[\beta_{k-1}, \beta_k] A[\beta_k]^{-1} A[\beta_k, \beta_{k+1}] \cdots A[\beta_m, \{j\}] \\ &= A[\{i\}, \beta_k] A[\beta_k]^{-1} A[\beta_k, \{j\}], \end{aligned}$$

as required.  $\square$

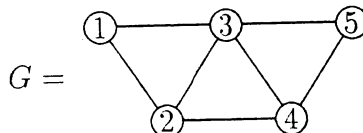
**3. Inertia Formula.** In [JL1], it was shown that if  $A \in M_n(\mathbb{C})$  is an invertible Hermitian matrix and if  $G = G(A^{-1})$  is a chordal graph, then the inertia of  $A$  may be expressed in terms of the inertias of certain principal submatrices of  $A$ . Precisely, let  $\mathcal{C}$  denote the collection of maximal cliques of  $G$ , and let  $\mathcal{T} = (\mathcal{C}, \mathcal{E})$  be a clique tree associated with  $G$ . If  $G(A^{-1}) = G$ , then it turns out that

$$(11) \quad i(A) = \sum_{\alpha \in \mathcal{C}} i(A[\alpha]) - \sum_{\{\alpha, \beta\} \in \mathcal{E}} i(A[\alpha \cap \beta]).$$

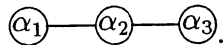
It is helpful to think of (11) as a generalization of the fact that if  $A^{-1}$  is block diagonal (meaning, of course, that  $A$  is block diagonal) then the inertia of  $A$  is simply the sum of the inertias of the diagonal blocks of  $A$ . To see what (11) tells us in a specific case, suppose that  $A^{-1}$  has a pentadiagonal nonzero-pattern, as in

$$A^{-1} \sim \begin{bmatrix} X & X & X & & \\ X & X & X & X & \\ X & X & X & X & X \\ & X & X & X & X \\ & & X & X & X \end{bmatrix}$$

The graph of  $A^{-1}$  is then



which is chordal. The maximal cliques of  $G$  are  $\alpha_1 = \{1, 2, 3\}$ ,  $\alpha_2 = \{2, 3, 4\}$  and  $\alpha_3 = \{3, 4, 5\}$ , and the clique tree associated with the graph  $G$  is



Equation (11) now tells us that the inertia of  $A$  is given by

$$i(A) = i(A[\{1, 2, 3\}]) + i(A[\{2, 3, 4\}]) + i(A[\{3, 4, 5\}]) \\ - i(A[\{2, 3\}]) - i(A[\{3, 4\}]).$$

Thus, we may compute the inertia of  $A$  by adding the inertia of these submatrices:

$$\begin{bmatrix} X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \end{bmatrix}$$

and subtracting the inertias of these:

$$\begin{bmatrix} X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \\ X & X & X & X & X \end{bmatrix}$$

Our goal in this section is to generalize formula (11) to the case in which  $A = [A_{ij}]$  is an invertible  $n$ -by- $n$  Hermitian operator matrix. We will be concerned with the case in which one of the components of inertia is finite, so that in (11) we will replace  $i$  by  $i_+$ ,  $i_-$  or  $i_o$ .

For a chordal graph  $G = (N, E)$ , we will say that an invertible  $n$ -by- $n$  operator matrix  $A$  is *weakly  $G$ -regular* (or simply *weakly regular*) if for every maximal clique or minimal vertex separator  $\alpha$  both  $A[\alpha]$  and  $A^{-1}[\alpha^c]$  have closed range.

LEMMA 4. Let  $M : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$  be represented by the 2-by-2 matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Suppose that  $A$  is invertible, and that

$$M^{-1} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}.$$

Then  $\dim \ker A = \dim \ker S$ .

*Proof.* Let  $x_1, x_2, \dots, x_n$  be linearly independent elements of  $\ker A$ . Then for  $1 \leq k \leq n$  we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_k \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ y_k \end{bmatrix},$$

in which  $y_k = Cx_k$ ,  $k = 1, \dots, n$ . Since  $M$  is invertible it follows that  $y_1, y_2, \dots, y_n$  are linearly independent. Observe now that

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} 0 \\ y_k \end{bmatrix} = \begin{bmatrix} x_k \\ 0 \end{bmatrix},$$

from which it follows that  $y_k \in \ker S$ . It follows now that  $\dim \ker S \geq \dim \ker A$ ; by reversing the argument we find that  $\dim \ker A \geq \dim \ker S$ . Thus  $\dim \ker A = \dim \ker S$ .  $\square$

LEMMA 5. Let  $M : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$  be Hermitian and invertible, and suppose that

$$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}.$$

If  $i_+(M) < \infty$ , then  $i_o(A) < \infty$ .

*Proof.* Clearly  $i_+(A) < \infty$ , so let  $n = i_+(A)$ . Let  $H$  be an invertible operator for which  $H^*AH = I_n \oplus -I \oplus O$ , in which  $I_n$  denotes the identity operator on an  $n$ -dimensional subspace, and  $-I$  and  $O$  are operators on spaces of respective dimensions  $i_-(A)$  and  $i_o(A)$ . Then

$$\begin{bmatrix} H^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} I_n & & B_1 \\ & -I & B_2 \\ & & O & B_3 \\ B_1^* & B_2^* & B_3^* & C \end{bmatrix}.$$

We may reduce this further by another congruence:

$$\begin{aligned} & \begin{bmatrix} I_n & & & \\ & I & & \\ & & I & \\ -B_1^* & B_2^* & O & I \end{bmatrix} \begin{bmatrix} I_n & & B_1 \\ & -I & B_2 \\ & & O & B_3 \\ B_1^* & B_2^* & B_3^* & C \end{bmatrix} \begin{bmatrix} I_n & & -B_1 \\ & I & B_2 \\ & & I & O \\ & & & I \end{bmatrix} \\ &= \begin{bmatrix} I_n & & O \\ & -I & O \\ & & O & B_3 \\ O & O & B_3 & S \end{bmatrix}, \end{aligned}$$

in which  $S = C - B_1^*B_1 + B_2^*B_2$ . Hence

$$i_+(M) = n + i_+ \left( \begin{bmatrix} O & B_3 \\ B_3^* & S \end{bmatrix} \right),$$

and thus

$$i_+ \left( \begin{bmatrix} O & B_3 \\ B_3^* & S \end{bmatrix} \right) < \infty.$$

But this implies that the zero block in this matrix must act on a space of finite dimension. Recalling that this dimension equals  $i_o(A)$ , we obtain the desired conclusion.  $\square$

The following Lemma generalizes a result of [Ha] for finite-dimensional matrices (see also [JL1]).

LEMMA 6. *Let  $M : \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow \mathcal{H}_1 \oplus \mathcal{H}_2$  be Hermitian and invertible, with*

$$M = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \text{ and } M^{-1} = \begin{bmatrix} P & Q \\ Q^* & R \end{bmatrix}.$$

*If  $i_+(M) < \infty$ , and if  $A$  and  $R$  both have closed range, then*

$$i_+(M) = i_+(A) + i_o(A) + i_+(R).$$

*Proof.* If  $i_o(A) = 0$  then  $A$  is invertible and the result follows from the fact that  $R$  is the inverse of the Schur complement  $C - B^*A^{-1}B$  and that  $i_+(M) = i_+(A) + i_+(C - B^*A^{-1}B)$ . Hence, suppose that  $i_o(A) > 0$ . Since  $i_+(M) < \infty$  we have as well from Lemma 5 that  $i_o(A) < \infty$ .

Hence, let  $n = i_o(A)$ , and let us consider the special case in which  $R = 0$ . Since we require, by Lemma 4, that  $i_o(R) = i_o(A) = n$ ,  $R$  must act on an  $n$ -dimensional space. Hence we have

$$M^{-1} = \begin{bmatrix} P & Q \\ Q^* & O_n \end{bmatrix}$$

where  $O_n$  denotes the zero operator on  $n$ -dimensional Hilbert space. By an appropriately chosen congruence of the form  $T_1 = H \oplus I$ , we may reduce  $M$  to the form

$$M_1 = T_1^* M T_1 = \begin{bmatrix} I_k & & B_1 \\ & -I & B_2 \\ & & O_n & B_3 \\ B_1^* & B_2^* & B_3^* & C \end{bmatrix},$$

where  $k = i_+(A)$ . With

$$T_2 = \begin{bmatrix} I_k & & -B_1 \\ & -I & B_2 \\ & & I_n & O \\ & & & I \end{bmatrix},$$

we then have

$$M_2 = T_2^* M_1 T_2 = \begin{bmatrix} I_k & & O \\ & -I & O \\ & & O_n & B_3 \\ O & O & B_3^* & S \end{bmatrix},$$

in which  $S = C - B_1^* B_1 + B_2^* B_2$ . The matrix

$$\begin{bmatrix} O_n & B_3 \\ B_3^* & S \end{bmatrix}$$

is an invertible operator on a  $2n$ -by- $2n$  Hilbert space, and in this case its inertia must be  $(n, n, 0)$ . From the form of  $M_2$  we see that we must have

$$\begin{aligned} i_+(M) &= k + i_+ \left( \begin{bmatrix} O_k & B_3 \\ B_3^* & S \end{bmatrix} \right) \\ &= k + n \\ &= i_+(A) + i_o(A). \end{aligned}$$

Since  $i_+(R) = 0$ , this last expression equals  $i_+(A) + i_o(A) + i_+(R)$ .

Now let us consider the general case, in which we make no assumption concerning the dimension of the space on which  $R$  acts. Choose an invertible matrix of the form  $T_1 = I \oplus H$  so that  $T_1^* M^{-1} T_1$  has the form

$$M_1^{-1} = T_1^* M^{-1} T_1 = \begin{bmatrix} P & Q_1 & Q_2 & Q_3 \\ Q_1^* & I_\ell & & \\ Q_2^* & & -I & \\ Q_3^* & & & O_n \end{bmatrix},$$

in which  $\ell = i_+(R)$  and  $n = i_o(R) [= i_o(A)]$ . Then with

$$T_2 = \begin{bmatrix} I & & & \\ -Q_1^* & I_\ell & & \\ Q_2^* & & I & \\ O & & & I_n \end{bmatrix},$$

we obtain

$$M_2^{-1} = T_2^* M_1^{-1} T_2 = \begin{bmatrix} S & O & O & Q_3 \\ O & I_\ell & & \\ O & & -I & \\ Q_3^* & & & O_n \end{bmatrix}.$$

From the form of  $M_2^{-1}$ , and by simple calculations, we find that  $M_2 = T_2^{-1} T_1^{-1} M (T_1^{-1})^* (T_2^{-1})^*$  has the form

$$M_2 = \begin{bmatrix} A & O & O & B_2 \\ O & I_\ell & & \\ O & & -I & \\ B_2^* & & & C_2 \end{bmatrix}$$

for some operators  $B_2$  and  $C_2$ . Hence we have

$$(12) \quad i_+(M) = i_+(R) + i_+ \left( \begin{bmatrix} A & B_2 \\ B_2^* & C_2 \end{bmatrix} \right).$$

Observe that

$$\begin{bmatrix} A & B_2 \\ B_2^* & C_2 \end{bmatrix}^{-1} = \begin{bmatrix} S & Q_3 \\ Q_3^* & O_n \end{bmatrix},$$

and thus by the special case we considered previously,

$$(13) \quad i_+ \left( \begin{bmatrix} A & B_2 \\ B_2^* & C_2 \end{bmatrix} \right) = i_+(A) + i_o(A).$$

Thus combining (12) and (13) we obtain

$$i_+(M) = i_+(A) + i_o(A) + i_+(R),$$

as required.  $\square$

The following lemma will be used in the proof of the main result of this section. First, let  $G = (V, E)$  be any connected chordal graph, and let  $\mathcal{T} = (\mathcal{C}, \mathcal{E})$  be any clique tree associated with  $G$ . For any pair of maximal cliques  $\alpha$  and  $\beta$  that are adjacent in  $\mathcal{T}$ , let  $\mathcal{T}_\alpha$  and  $\mathcal{T}_\beta$  be the subtrees of  $\mathcal{T} - \{\alpha, \beta\}$  that contain, respectively,  $\alpha$  and  $\beta$ , and let  $\mathcal{C}_\alpha$  and  $\mathcal{C}_\beta$  be the vertex sets of  $\mathcal{T}_\alpha$  and  $\mathcal{T}_\beta$ . Define

$$V_{\alpha \setminus \beta} = \left( \bigcup_{\gamma \in \mathcal{C}_\alpha} \gamma \right) \setminus \beta,$$

with  $V_{\beta \setminus \alpha}$  defined similarly.

LEMMA 7. [BJL] *Under the assumptions of the preceding paragraph, the following hold:*

- (i)  $V_{\alpha \setminus \beta} \cap V_{\beta \setminus \alpha} = \emptyset$ ;
- (ii)  $(\alpha \cap \beta)^c = V_{\alpha \setminus \beta} \cup V_{\beta \setminus \alpha}$ ;

and

- (iii)  $\alpha^c$  is the disjoint union

$$\alpha^c = \bigcup_{\beta \in \text{adj } \alpha} V_{\beta \setminus \alpha},$$

in which  $\text{adj } \alpha = \{\beta \in \mathcal{C} : \{\alpha, \beta\} \in \mathcal{E}\}$ .

We should note the following consequences of Lemma 7. Suppose  $B = [B_{ij}]$  is a matrix satisfying  $G(B) \subseteq G$ , in which  $G$  is a chordal graph, and let  $\mathcal{T}$  be a clique tree associated with  $G$ . If  $\{\alpha, \beta\}$  is an edge of  $\mathcal{T}$ , then  $B[(\alpha \cap \beta)^c]$  is essentially a direct sum of the matrices  $B[V_{\alpha \setminus \beta}]$  and  $B[V_{\beta \setminus \alpha}]$ . The reason for this is that there are no edges between vertices in  $V_{\alpha \setminus \beta}$  and vertices in  $V_{\beta \setminus \alpha}$ , and hence  $B_{ij} = 0$  whenever  $i \in V_{\alpha \setminus \beta}$  and  $j \in V_{\beta \setminus \alpha}$ . Similarly, if  $\alpha$  is any maximal clique of  $G$  then  $B[\alpha^c]$  is essentially a direct sum matrices of the form  $B[V_{\beta \setminus \alpha}]$  as  $\beta$  runs through all cliques that are adjacent in  $\mathcal{T}$  to  $\alpha$ .

LEMMA 8. Let  $\mathcal{H} = \mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_n$ , let  $A: \mathcal{H} \rightarrow \mathcal{H}$  be an invertible operator matrix, and let  $G = G(A^{-1})$  be a connected chordal graph. If  $\mathcal{T} = (\mathcal{C}, \mathcal{E})$  is any clique tree associated with  $G$ , then

$$(14) \quad \sum_{\alpha \in \mathcal{C}} \dim \ker A[\alpha] = \sum_{\{\alpha, \beta\} \in \mathcal{E}} \dim \ker A[\alpha \cap \beta].$$

*Proof.* Let us look first at the left-hand side of (14). by Lemma 4 and by Lemma 7 we have

$$(15) \quad \begin{aligned} \sum_{\alpha \in \mathcal{C}} \dim \ker A[\alpha] &= \sum_{\alpha \in \mathcal{C}} \dim \ker A^{-1}[\alpha^c] \\ &= \sum_{\alpha \in \mathcal{C}} \sum_{\beta \in \text{adj } \alpha} \dim \ker A^{-1}[V_{\beta \setminus \alpha}]. \end{aligned}$$

On the other hand, by applying Lemmas 4 and 7 we may see that the right-hand side of (14) is

$$(16) \quad \begin{aligned} \sum_{\{\alpha, \beta\} \in \mathcal{E}} \dim \ker A[\alpha \cap \beta] &= \sum_{\{\alpha, \beta\} \in \mathcal{E}} \dim \ker A^{-1}[(\alpha \cap \beta)^c] \\ &= \sum_{\{\alpha, \beta\} \in \mathcal{E}} (\dim \ker A^{-1}[V_{\beta \setminus \alpha}] + \dim \ker A^{-1}[V_{\alpha \setminus \beta}]). \end{aligned}$$

Observe that with every edge  $\{\alpha, \beta\}$  of  $\mathcal{T}$  we may associate exactly two terms in the right-most expression of (15), namely  $\dim \ker A^{-1}[V_{\beta \setminus \alpha}]$  and  $\dim \ker A^{-1}[V_{\alpha \setminus \beta}]$ . But this just means that (15) and (16) contain all the same terms, and hence (14) is established.  $\square$

THEOREM 9. Let  $G = (N, E)$  be a connected chordal graph, let  $A = [A_{ij}]$  be an  $n$ -by- $n$  weakly  $G$ -regular Hermitian operator matrix, and suppose that  $G(A^{-1}) \subseteq G$ . If  $i_+(A) < \infty$ , then for any clique tree  $\mathcal{T} = (\mathcal{C}, \mathcal{E})$  associated with  $G$  we have

$$i_+(A) = \sum_{\alpha \in \mathcal{C}} i_+(A[\alpha]) - \sum_{\{\alpha, \beta\} \in \mathcal{E}} i_+(A[\alpha \cap \beta]).$$

*Proof.* Since  $i_+(A) < \infty$ , we must have  $i_+(A[\alpha]) < \infty$  for any  $\alpha \subseteq N$ , and by Lemma 5 we know that  $i_o(A[\alpha]) < \infty$  for any  $\alpha \subseteq N$ . By Lemma 6 we may write

$$(17) \quad \begin{aligned} &\sum_{\alpha \in \mathcal{C}} i_+(A[\alpha]) - \sum_{\{\alpha, \beta\} \in \mathcal{E}} i_+(A[\alpha \cap \beta]) \\ &= \sum_{\alpha \in \mathcal{C}} [i_+(A) - i_+(A^{-1}[\alpha^c]) - i_o(A[\alpha])] \\ &\quad - \sum_{\{\alpha, \beta\} \in \mathcal{E}} [i_+(A) - i_+(A^{-1}[(\alpha \cap \beta)^c]) - i_o(A[\alpha \cap \beta])] \\ &= \sum_{\alpha \in \mathcal{C}} i_+(A) - \sum_{\{\alpha, \beta\} \in \mathcal{E}} i_+(A) - \sum_{\alpha \in \mathcal{C}} i_+(A^{-1}[\alpha^c]) \\ &\quad + \sum_{\{\alpha, \beta\} \in \mathcal{E}} i_+(A^{-1}[(\alpha \cap \beta)^c]) - \sum_{\alpha \in \mathcal{C}} i_o(A[\alpha]) + \sum_{\{\alpha, \beta\} \in \mathcal{E}} i_o(A[\alpha \cap \beta]). \end{aligned}$$

The last two terms of the last expression in (17) cancel by Lemma 8, and the two middle terms cancel by an argument similar to that used in the proof of Lemma 8. Finally, since  $\mathcal{T}$  has exactly one more vertex than the number of edges, the right-hand side of (17) equals  $i_+(A)$ . This proves the theorem.  $\square$

Of course, a similar statement is true for  $i_-(A)$ , and the corresponding statement for  $i_o(A)$  is already contained in Lemma 8.

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