Two studies of entanglement in the SYK

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Two parts

- Part 1: Eigenstate entanglement in the SYK.
  [Yichen Huang and YG, arXiv 1709.09160]

- Part 2: Entanglement growth in quenched SYK chain.
  [YG, Andrew Lucas and Xiao-Liang Qi, arXiv:1708.00871, JHEP]

Motivation

Thermalization in quantum many-body system.

- ▶ Quantum mechanics: a pure state keeps pure under time evolution; but can "look thermal" for any small sub-region:

$$|\Psi\rangle = \underbrace{\qquad\qquad}_{A} \quad \text{highly excited state}$$

looks thermal ↖ ↗ environment

$$\rho_A = \mathsf{Tr}_{A^c}\left(|\Psi\rangle\langle\Psi|\right) \approx \frac{e^{-\beta H_A}}{Z_\beta} \quad \text{for small subregion } A$$

$\beta = \frac{1}{T}$: determined by energy density.

- ▶ Part 1: entanglement entropy of subregions in the SYK model.

SYK

$N \gg 1$ number of Majorana fermions. Hilbert space dimension $d = 2^{N/2}$:

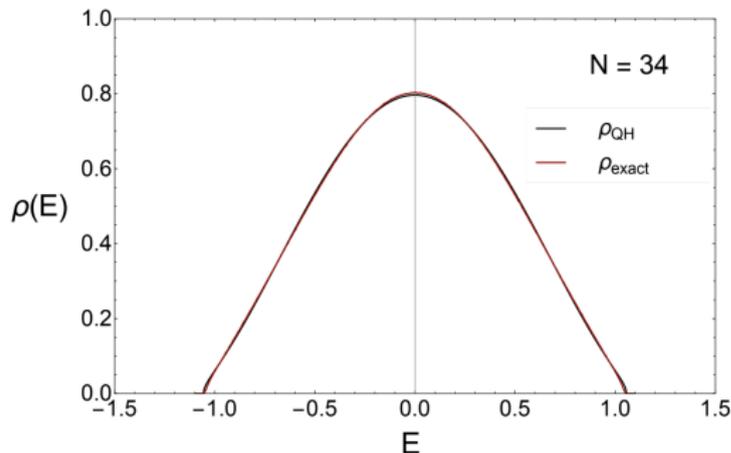$$H = \sum_{i<j<k<l} J_{ijkl} \chi_i \chi_j \chi_k \chi_l, \quad \overline{J_{ijkl}^2} = \frac{3!}{N^3} J^2.$$

Figure from [García-García, Verbaarschot 2017] $J = 2/3$, $\rho(E)$ normalized to 1.

Spectral properties

The spectral properties:

1. The mean energy of H is $\operatorname{Tr} H/d = 0$.

2. The ground-state energy is $-NE_0$ (extensive), where E_0 is a known constant.

3. To leading order, the density of states at energy $-NE$ is given by

$$D(E, E_0, N) = e^{\left[\frac{1}{2}\log 2 - \frac{1}{16}\arcsin^2\left(\frac{E}{E_0}\right)\right]N}.$$

[García-García, Verbaarschot 2017]

This formula is empirically valid except very close to the edge.

Arguments

- Divide the N fermions into two group $N = N_A + N_B$, $N_A \leqslant N/2$

$$H = H_A + H_B + H_\partial$$

- Self-similarity: H_A itself is a SYK hamiltonian with $\widetilde{J} = \left(\frac{N_A}{N}\right)^{3/2} J$
  $\Rightarrow$ GS energy density $-E_0^A = -\left(\frac{N_A}{N}\right)^{3/2} E_0$;

- Pick an arbitrary eigenstate $H|\psi\rangle = NE|\psi\rangle$,

Arguments

- Every interacting term in H get equal expectation value:

$$\langle\psi|H_A|\psi\rangle = \left(\frac{N_A}{N}\right)^4 NE = \text{Tr}\left(\rho_A H_A\right) \Rightarrow E^A = \left(\frac{N_A}{N}\right)^3 E$$

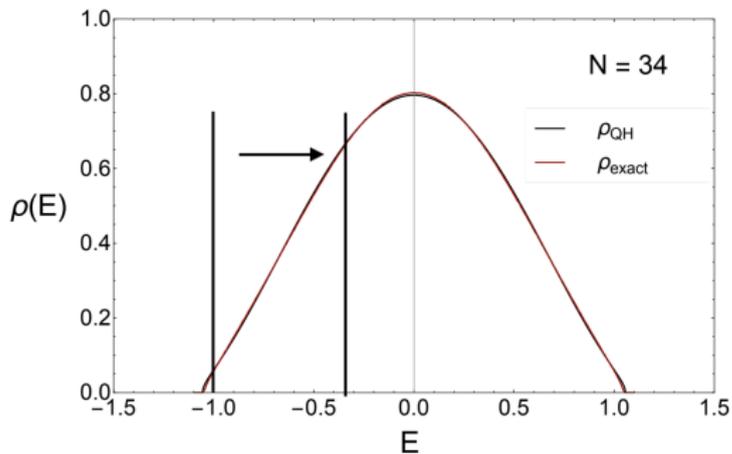- Given energy density E^A, thermal density matrix maximizes the entropy:

$$S_{A,vN} \leqslant \log D(E^A, E_0^A, N_A)$$

Result

Using the empirical formula for density of state, we have:

$$S_A \leqslant \left[\frac{1}{2} \log 2 - \frac{1}{16} \arcsin^2 \left(\frac{E}{E_0} \cdot \left(\frac{N_A}{N} \right)^{\frac{3}{2}} \right) \right] N_A$$

Assume SYK saturates the upper bound.

$$\frac{S_A}{N_A} = \left[\frac{1}{2} \log 2 - \frac{1}{16} \arcsin^2 \left(\left(\frac{N_A}{N} \right)^{\frac{3}{2}} \right) \right]$$

▶ Small subregion, $N_A \ll N \Rightarrow \frac{S_A}{N_A} = \frac{1}{2} \log 2$

▶ Ground state $E = E_0$, take half system $N_A = \frac{N}{2}$:

$$\frac{S_A}{N_A} = \left[\frac{1}{2} \log 2 - \frac{1}{16} \arcsin^2 \left(\frac{1}{2\sqrt{2}} \right) \right] \approx 0.3384$$

Numerical result ≈ 0.3375 [Liu, Chen, Balents arXiv:1709.06259]

Next

- We gain some understanding on the entanglement in the eigenstate of SYK (may be generalized to other all-to-all interacting systems)

- We want to understand how is the entanglement generated in SYK

- Part 2: work in a slightly different model, SYK chain.

When $|\Psi\rangle$ is not an eigenstate

Continue the general discussion on thermalization:

▶ Time evolution: $|\Psi\rangle \rightarrow |\Psi(t)\rangle = e^{-iHt}|\Psi\rangle$

$$\rho_A(t) = \mathsf{Tr}_{A^c}\left(|\Psi(t)\rangle\langle\Psi(t)|\right) \rightarrow \frac{e^{-\beta H_A}}{Z_\beta}$$

Reduced density matrix approaches thermal density matrix

▶ Check entanglement entropy

Operator independent, information scrambling, holographic interpretation ...

Entanglement growth

Common recipe:

- Choose a low entanglement excited state $|\Psi\rangle$;

- Time evolution $|\Psi(t)\rangle$;

- Entanglement entropy for subregions and see how fast it approaches the thermal value:

$$S_A(t) \to S_{A,\beta} \quad \text{thermal entropy}$$

SYK chain model

Maximally chaotic model with spatial locality [YG, XL Qi and D Stanford 2016] :

- One dimensional chain, each site $N \gg 1$
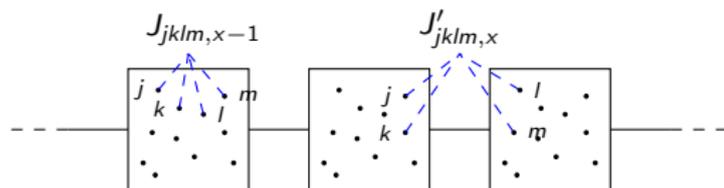
$$H = \sum_{x=1}^{M} \left[\sum_{j<k<l<m} \underbrace{J_{jklm,x} \, \chi_{j,x} \chi_{k,x} \chi_{l,x} \chi_{m,x}}_{\text{SYK term}} + \sum_{j<k;l<m} \underbrace{J'_{jklm,x} \, \chi_{j,x} \chi_{k,x} \chi_{l,x+1} \chi_{m,x+1}}_{\text{Nearest neighbour coupling}} \right]$$

Independent random coefficients: $\overline{J^2_{jklm,x}} = \frac{3! J_0^2}{N^3}, \quad \overline{J'^2_{jklm,x}} = \frac{J_1^2}{N^3}$

Thermofield double state
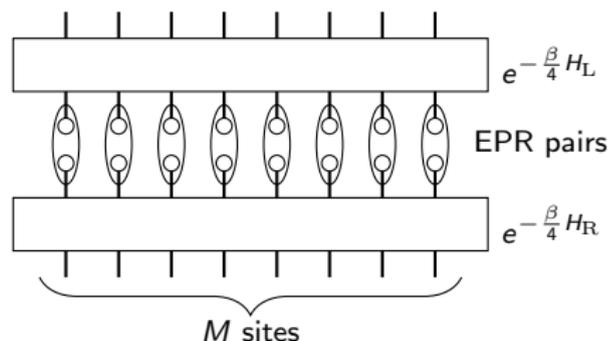
How to construct a convenient low entanglement state in SYK chain?
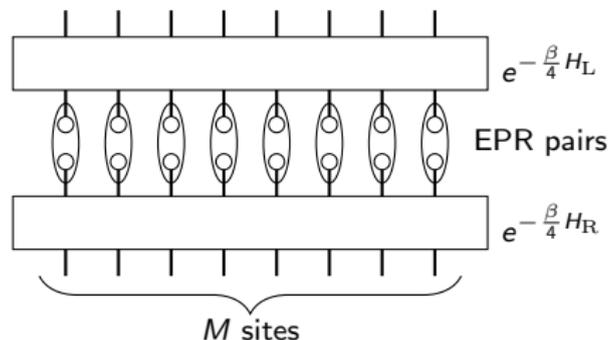Proposal: thermofield double state [Israel, Maldacena]

- Two copies of original Hilbert space $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R$;

$$|\text{TFD}\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta E_n}{2}} |n_L, n_R\rangle = \frac{1}{\sqrt{Z_\beta}} e^{-\frac{\beta}{4}(H_L + H_R)} |I\rangle, \quad H_R = H_L^T$$

- $|I\rangle = \prod_x |\text{EPR}\rangle_x$: no spatial entanglement.
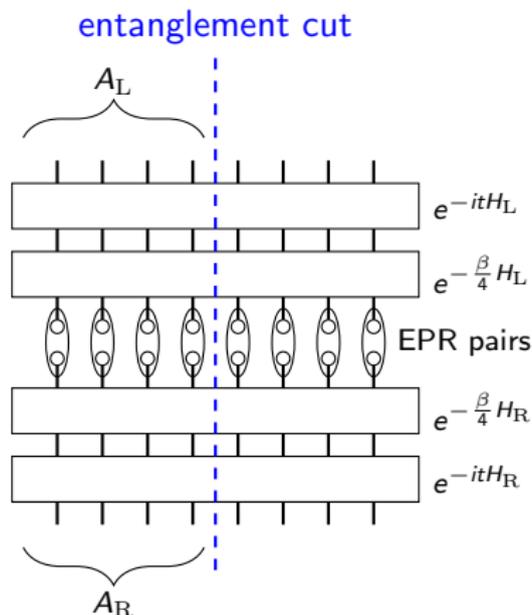
Thermofield double state cont'd

Two useful properties:

- If we cut horizontally: $\rho_{L/R} = \frac{1}{Z_\beta} e^{-\beta H} \Rightarrow$ TFD is an excited state.

- If we cut vertically $\Rightarrow$ short range entangled state.
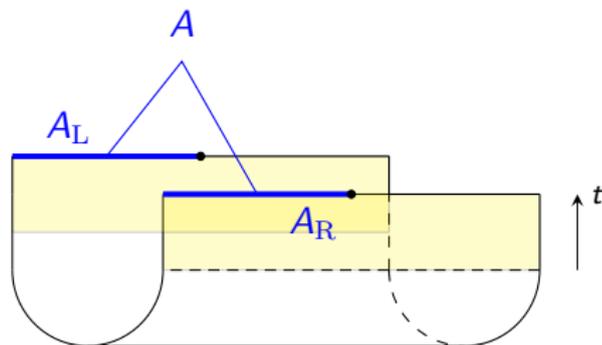
Time evolution and entanglement cut

Time evolution $U(t) = e^{-itH}$, $H = H_L + H_R$, TFD is not an eigenstate.

Finally we care the entanglement across the cut, i.e. entanglement entropy of $\rho_A(t) = \text{Tr}_{A^c}\left(|\text{TFD}\rangle\langle\text{TFD}|\right)$

Technicality: path integral representation
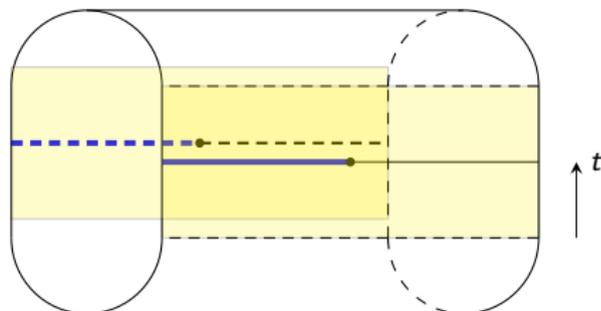
- TFD has a simple path integral representation: evolve Hamiltonian for period of $\frac{\beta}{2}$ (white tube)

- Real time evolution: grow the edges by real time t.

Technicality: reduced density matrix and replicas
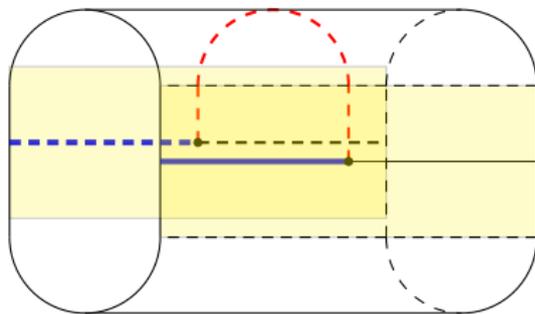
- Path integral representation of reduced density matrix:

- We are interested in n-Renyi entropy

$$S_{A,n} = \frac{\log \operatorname{Tr}\left(\rho_A^n\right)}{1 - n}$$

Technicality: replica and twist

- $Z_{A,n} = Z_\beta^n \operatorname{Tr}(\rho_A^n)$: partition function on a complicated manifold (similar to Calabrese-Cardy) $\Rightarrow$ partition function on the original manifold with twisted boundary conditions;

- Effective action (random average, introduce bilocal fields):

$$Z_{A,n} = \int \mathcal{D}G \mathcal{D}\Sigma\, e^{-N I_{A,n}}, \quad I_{A,n} = I_{n\ \text{replica}} + \Delta I_{\text{twist}}$$

- ΔI_{twist}: twisted interaction across the red line.

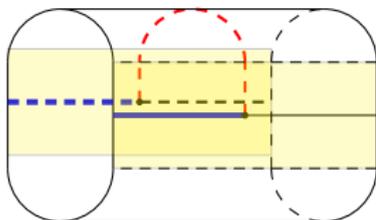# Replica diagonal ansatz

▶ Large N limit, saddle point of $I_{A,n}$.

$$\log Z_{A,n} = -N \min_{\{G,\Sigma\}} I_{A,n}[G,\Sigma]$$

▶ Replica diagonal ansatz $G^{\alpha\beta} = \delta^{\alpha\beta} G$: an upper bound of the minimization problem:

$$\min_{\{G,\Sigma\}} I_{A,n}[G,\Sigma] \leqslant \min_{\text{diag. } \{G,\Sigma\}} I_{A,n}[G,\Sigma]$$

(diagonal ansatz is perturbatively stable for small J_1/J)

▶ Twist term $\Delta I_{\text{twist}} = \frac{n}{8} J_1^2 \int_C d^2\tau \, G_x(\tau_1, \tau_2)^2 G_{x+1}(\tau_1, \tau_2)^2$:

Zero-th order effect: Linear growth

We consider the weak link limit where $J_1 \ll J$ (technically we need $J_1 \ll J/\sqrt{\beta J}$)

- To the zero-th order approximation, we take the saddle point of untwisted action, evaluate in imaginary time and analytic continue to real time;

- Linear growth for n-th Renyi entropy $t \gg \beta$:

$$\frac{S_{A,n}(t)}{N} \approx \frac{n}{n-1} \gamma \log \cosh \frac{2\pi}{\beta} t + \text{const} \approx \frac{n}{n-1} \frac{2\pi\gamma}{\beta} t + \text{const}$$

parameter $\gamma = \frac{J_1^2}{8\pi J^2}$: relative strength of spatial coupling

Back reaction and soft modes
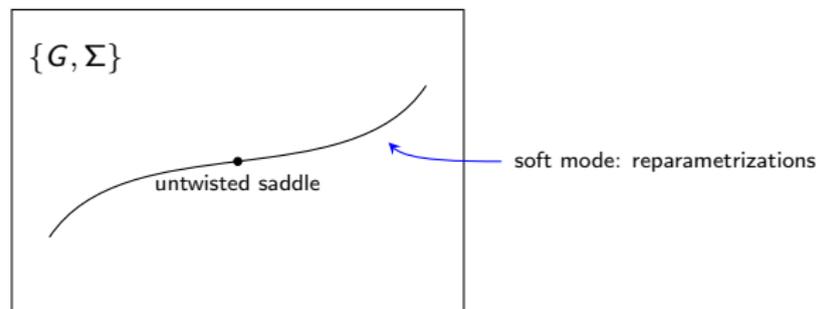
- Shift of the saddle caused by ΔI_{twist};

- In strong coupling $N \gg (\beta, \text{ time})J \gg 1$, the dynamics is dominated by reparametrization fields.

Long time saturation

Ansatz: uniform reparametrization $f_x(\tau) = f(\tau)$ — an upper bound calculation:

- Low energy sector of the untwisted action is described by Schwarzian;

$$I_{n \text{ replica}} = -nNM\frac{\alpha_S}{J} \int d\tau \, \text{Sch}\left(\tan\frac{\pi}{\beta}f(\tau), \tau\right)$$

- Twisted interaction ΔI_{twsit}

$$\Delta I_{\text{twist}} = -nN\frac{\gamma}{2}\log\eta_f, \quad \eta_f = \frac{\sin\frac{\pi}{\beta}\left(f(\tau_1) - f(\tau_2)\right)}{\left(\frac{\pi}{\beta}\right)^2 \epsilon_1\epsilon_2 f'(\tau_1)f'(\tau_2)}$$

ϵ_1, ϵ_2 cut-offs $\sim \frac{1}{J}$

Both terms are $SL(2, \mathbb{R})$ invariant $\Rightarrow$ Geometric interpretation.
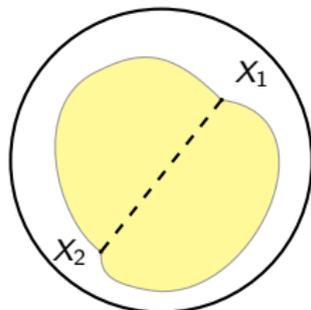
Geometric problem

- (Euclidean) Schwarzian action $\Rightarrow$ a curve with length $L = \beta J \gg 1$ in hyperbolic disk [Kitaev 2016] :

$$-\frac{\alpha_S}{J} \int d\tau \, \text{Sch}\left(\tan\frac{\pi}{\beta} f(\tau), \tau\right) \approx \alpha_S \left(L - A - 2\pi\right)$$

- Twisted interaction term:

$$\log \eta_f = \log \cosh D(X_1, X_2)$$
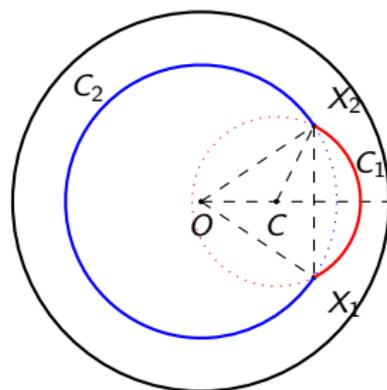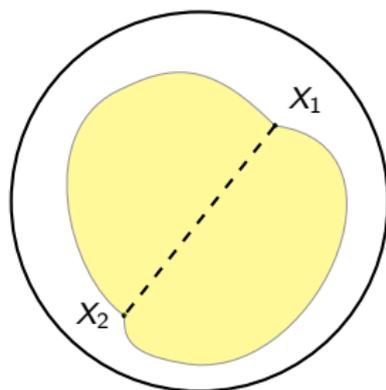
$X_{1,2}$ are positions of twist operators at time circle.

Optimization

- Total action = Schwarzian + twisted interaction

$$I_{A,n} = nN \min_{\{f\}} \left(M\alpha_S(L - A - 2\pi) + \frac{\gamma}{2}\log\cosh D(X_1, X_2) \right)$$

- Look for the curve (with fixed length L) in hyperbolic space minimizes the action $I_{A,n}$

Late time saturation

We can solve the geometric problem, and find (after analytic continuation) the Renyi entropy in the late time:

$$\lim_{t \to \infty} S_{A,n}(t) = \frac{n}{n-1} NM \left[\underbrace{\frac{2\pi^2 \alpha_S}{\beta J}}_{\frac{1}{4} c_V T} + \underbrace{\gamma \left(\frac{1}{M} + \log \frac{4\sqrt{2}\alpha_S}{\gamma} \right)}_{\to 0, \text{ at } \gamma \to 0} \right]$$

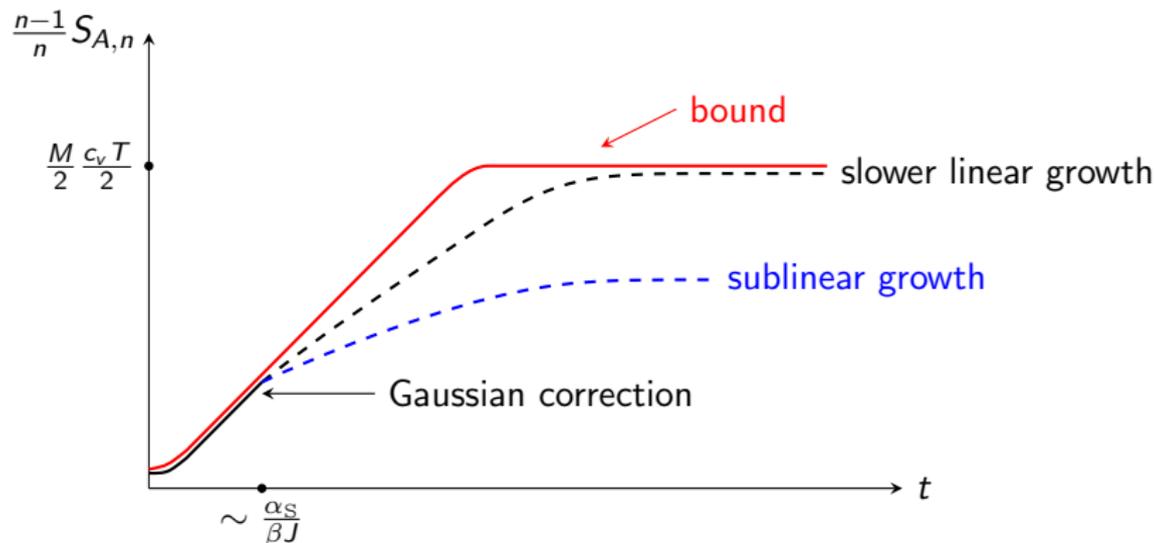There are $\frac{M}{2}$ sites in A, the Renyi entropy persite is:

$$s_{A,n}(\infty) = N \left[\left(1 + \frac{1}{n-1} \right) \frac{1}{2} c_V T + \frac{1}{2} \gamma \left(\frac{1}{M} + \log \frac{4\sqrt{2}\alpha_S}{\gamma} \right) \right]$$

Comparing to thermal Renyi entropy:

$$s_{A,n}^{\text{th}} = N \left[S_0 + \left(1 + \frac{1}{n} \right) \frac{1}{2} c_V T \right]$$

Conclusion

We compute an upper bound for Renyi entanglement growth.
Significantly less than the thermal value.

Discussions

Physical implication:

- ▶ soft modes/reparametrization modes (corresponds to the specific heat) thermalize quickly;

- ▶ "localized" degree of freedoms thermalize very slowly: thermalization time is a function of N;

  (zero temperature entropy is missing in our setting, protected subspace? [Kitaev])

Open questions:

- ▶ What is the nature of the zero temperature entropy?

- ▶ Will these modes finally thermalize?