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The Analytic Geometry of the Plane based upon the straight-line as the fundamental element:

A thesis

submitted to the faculty of the Graduate School of the University of Minnesota

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Introduction.

The whole of analytic geometry as hitherto studied depends on the possibility of representing the position of a point in a plane by two coordinates. These two coordinates are regarded as the distances from the point to two selected lines. Other systems have occasionally been used, e.g. the polar system and the trilinear system. The underlying idea of coordinates in plane geometry is the existence of two independent quantities that serve to determine the position of a point in the plane. But there are other geometrical elements than the point whose position it may be desirable to specify. The point certainly presents itself to the mind as the element 'par excellence', probably because all drawing is done with the point. But the straight line is essentially as simple. As with the point, so with

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the line, its position depends upon two independent quantities, those most naturally suggested being the intercepts on two selected lines. The plane is then doubly infinite in lines as well as points, and is of the same nature whether the point or the line be regarded as the fundamental element. When the line is considered the fundamental element, the point becomes the secondary element, uniquely determined by two straight lines, and this secondary element suggests an infinity of straight lines passing thru it, just as following the more usual idea, the secondary element, the straight line determined by two points, suggests an infinity of points lying on it.

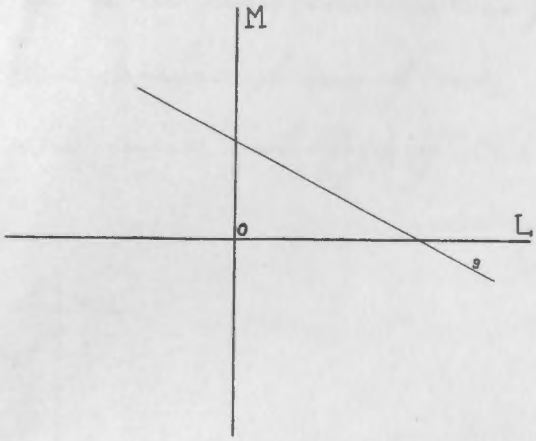
This relationship or duality between the line and the point in the plane makes it possible to build up an Analytic geometry using either the point or the line as the fundamental element. The theory of line geometry owes its origin to Plücker who suggested the idea of taking the straight line as the element of space; he, however did not apply it to the geometry of the plane. Prof. Cayley and

Dr. Booth have made some use of line coordinates in dealing with some particular phases of the subject but a systematic application has never been made in the development of the plane Analytic geometry without using the results of the ordinary Analytic geometry.

It is the purpose of the following pages to develop and prove some of the more important theorems of Analytic geometry of the plane, regarding the line as the fundamental element.

Line Coordinates.

The position of a line in a plane is uniquely determined if it is known to pass through two points; that is to say: 'two points determine a line.' In order to simplify matters consider these two points as lying on two fixed straight lines. These two lines may be any two lines in the plane but for convenience take two lines perpendicular to one another. A line is then determined as soon as it is known where the line cuts the two fixed line which are called lines or axes of reference, or simply axes; (i.e. as soon as two points on the line are determined.) In order to distinguish between the two lines call the horizontal one the L-axis and the vertical one the M-axis; call their intersection the origin, or O, and the two distances cut off on the axes by a line, the intercepts on the axes. Thus:



s is a line whose intercept on the L axis is twice as large as its intercept on the M axis.

The condition that a line shall move in some prescribed manner involves a functional relation between the intercepts on the axes, that is to say between the l and m coordinates. Any relation or equation between the l and m coordinates i.e., between l and m, will then represent not one line but <u>all</u> which satisfy the given condition

For example, suppose the lines are to satisfy the following condition:

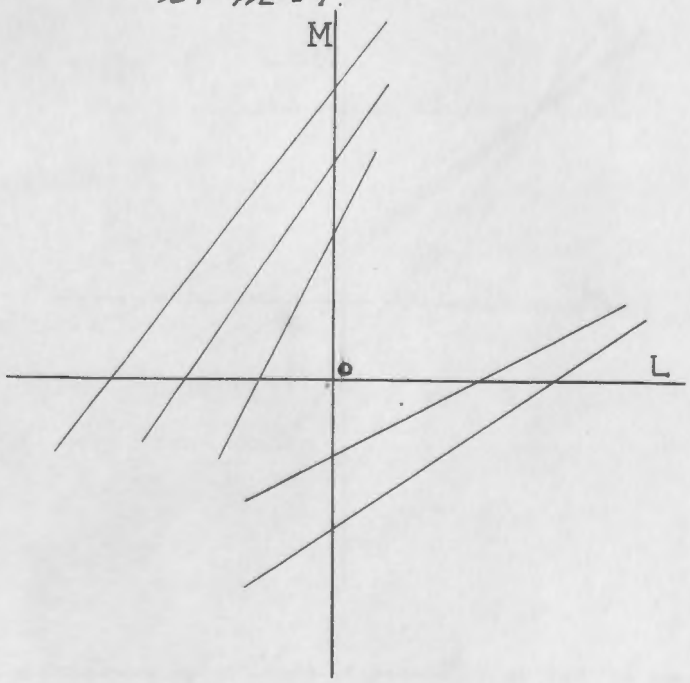
$$l + m = 1.$$

The question now naturally arises: What do l and m represent? Several possibilities present themselves. The first to suggest itself is that l

and m are the intercepts. Putting this interpretation upon l and m, the following is the plot of the lines satisfying the condition that

$$l + m = 1.$$

l	m
0	1
1	0
2	-1
3	-2
.....	.....
-1	2
-2	3
-3	4
....	....



A second possibility is to consider the l and m of the equation not as the intercepts but as the reciprocals of the intercepts.

As the following plot shows, this gives all lines passing thru a point, or a pencil of lines. With this interpretation then the equation

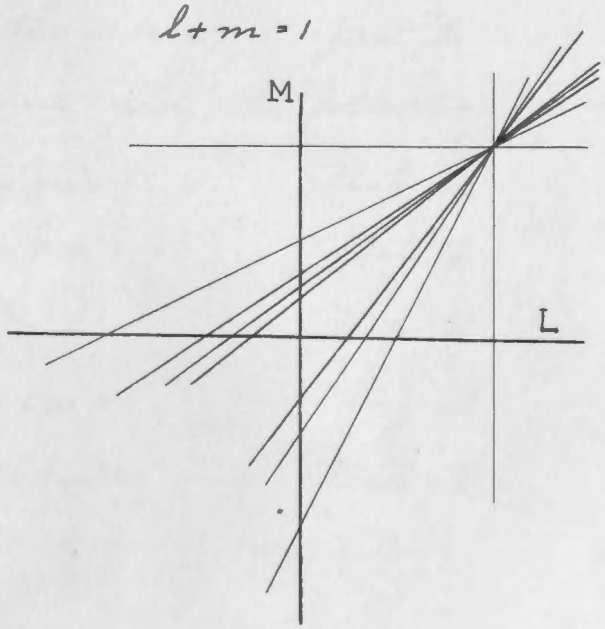
$$l + m = 1$$

is seen to represent all lines thru a point, the

exact dual of the theorem in point geometry, viz.- 'an equation of the first degree represents all points on a line'.

$l + m = 1$

$1/l$	$1/m$
∞	1
1	∞
$1/2$	-1
$1/3$	$-1/2$
$1/4$	$-1/3$
.....	.....
-1	$1/2$
$-1/2$	$1/3$
$-1/3$	$1/4$
$-1/4$	$1/5$



The plotting of other equations in l and m will bring out the fact that it is still more advantageous to consider not only the reciprocals but the negative reciprocals of l and m as the intercepts. This is of course an arbitrary scheme but not more so than any other would be; moreover results will show that this is convenient. In all the work that follows l and m, the coordinates of any line, will be considered as the negative reciprocals of the

intercepts on the L and M axes.

Plotting.

With the above interpretation of the coordinates of a line, plot the following equations:

(a) $l^2 + m^2 = 1$. Plate I.

(b) $3l + 4m + 1 = 0$. Plate II

(c) $m^2 = l$ Plate III.

(d) $120\,lm = 1$ Plate IV.

(e) $4l^2 + 9m^2 = 1$. Plate V.

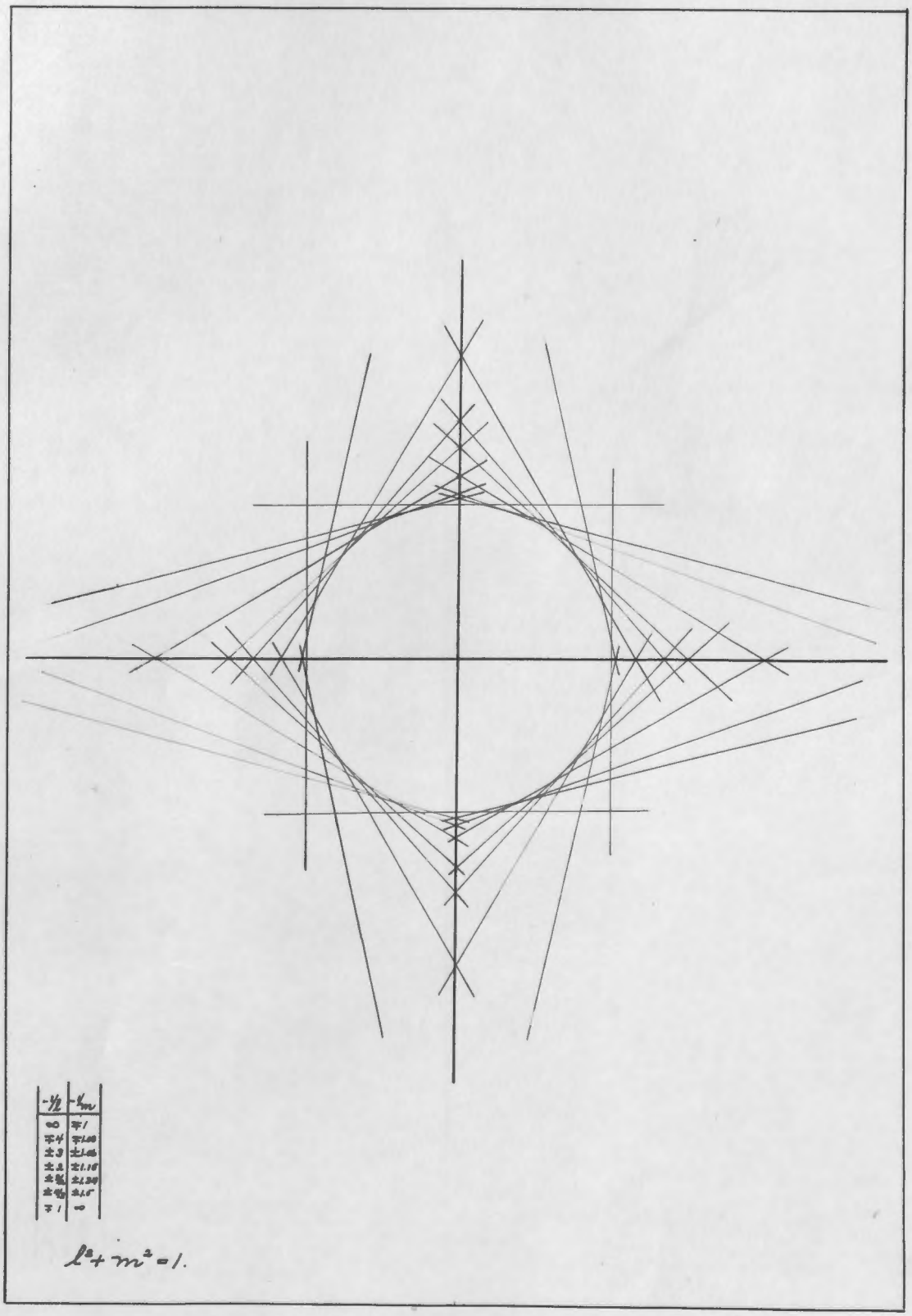
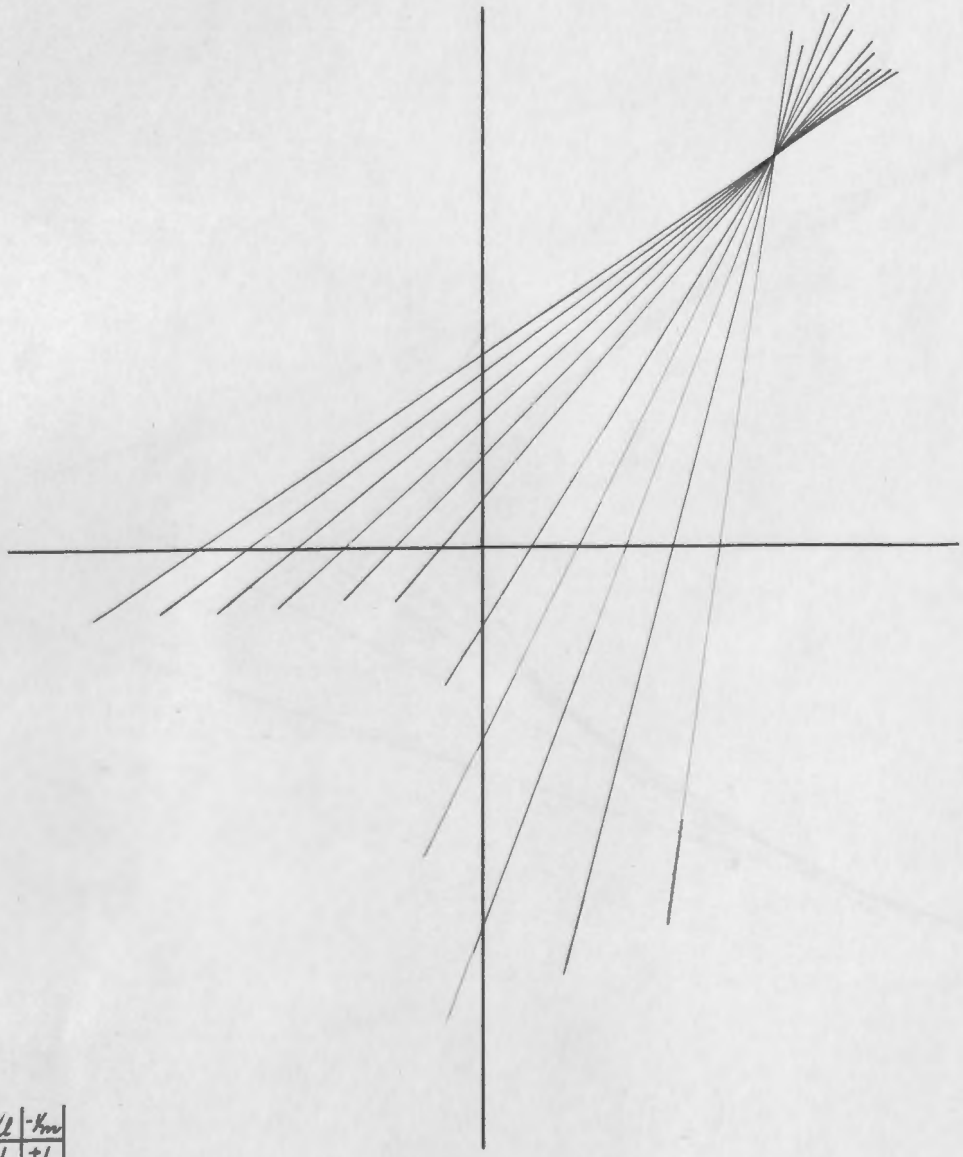

$-l/n$ | $-l/m$
∞ | ∓ 1
$\mp 4/3$ | ∓ 1.00
± 3 | ± 1.05
± 2 | ± 1.15
$\pm 9/4$ | ± 1.30
$\pm 9/5$ | ± 1.5
∓ 1 | ∞
$l^2 + m^2 = 1.$

PLATE I.



$-1/l$	$-1/m$
-1	$+1$
-2	$8/5$
$-9/2$	$4/7$
$-4/5$	$1/4$
1	-2
$1/2$	$-9/5$
$1/3$	$-1/2$

$3l + 4m + 1 = 0$

PLATE II.

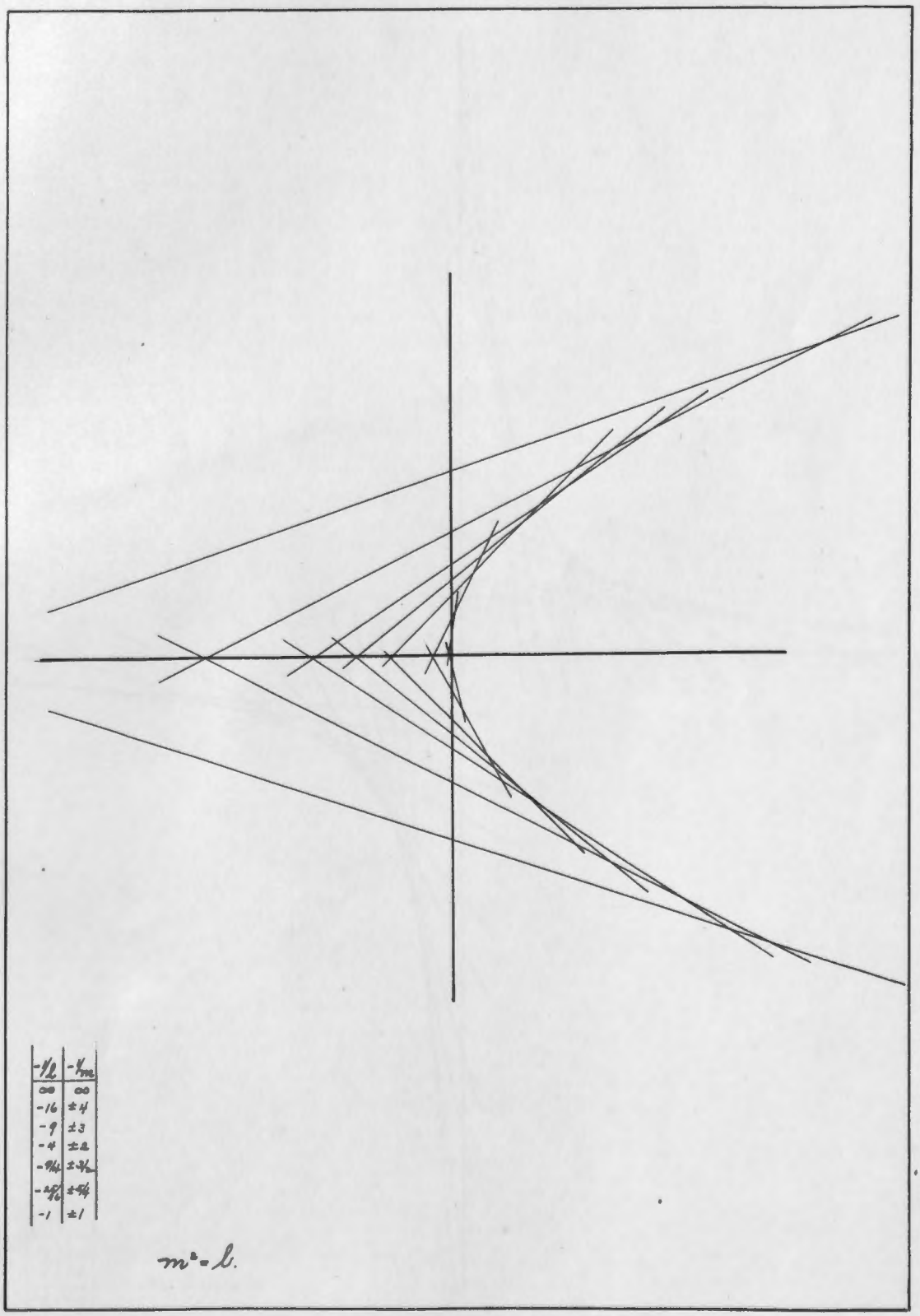


PLATE III.

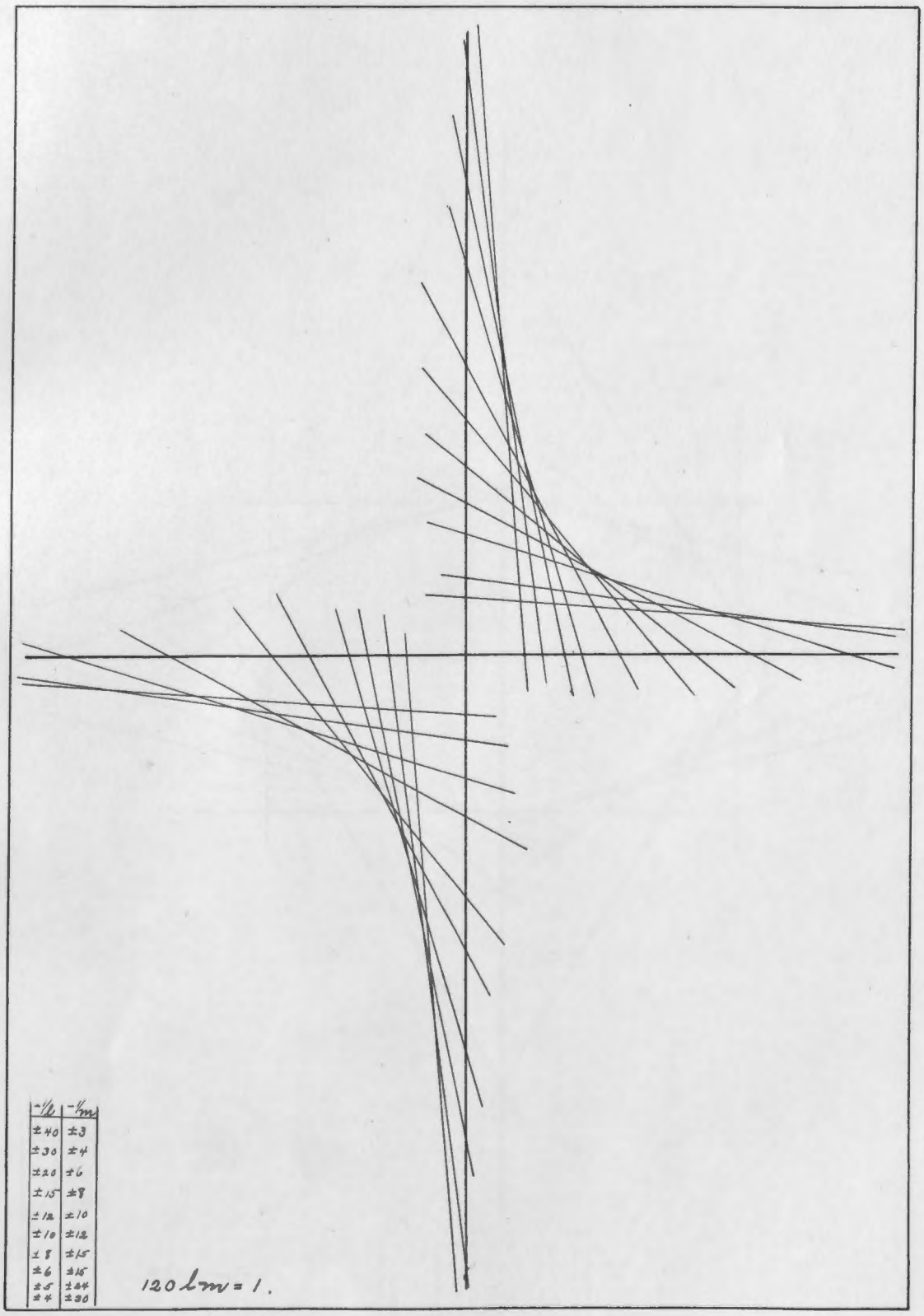


PLATE IV.

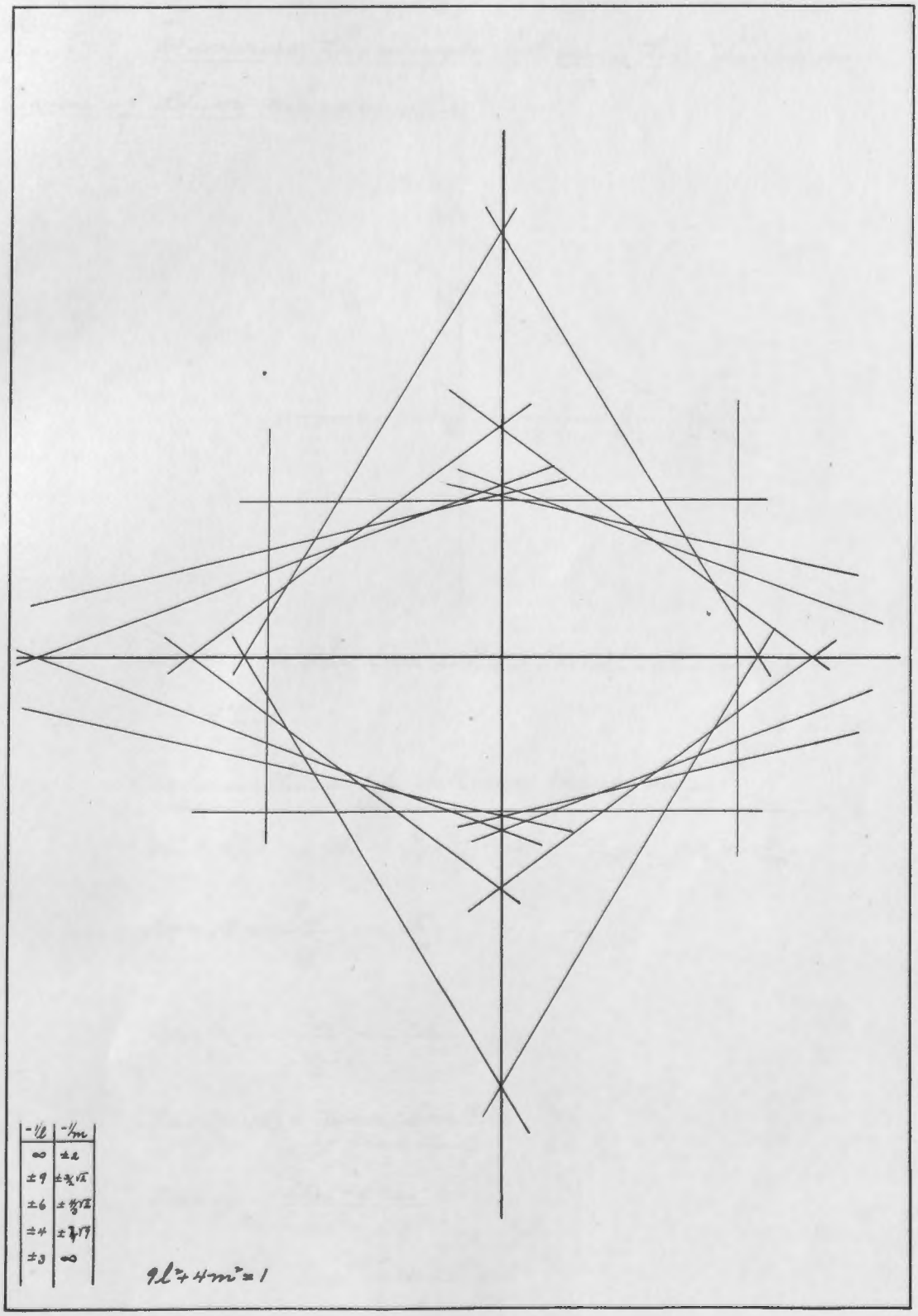
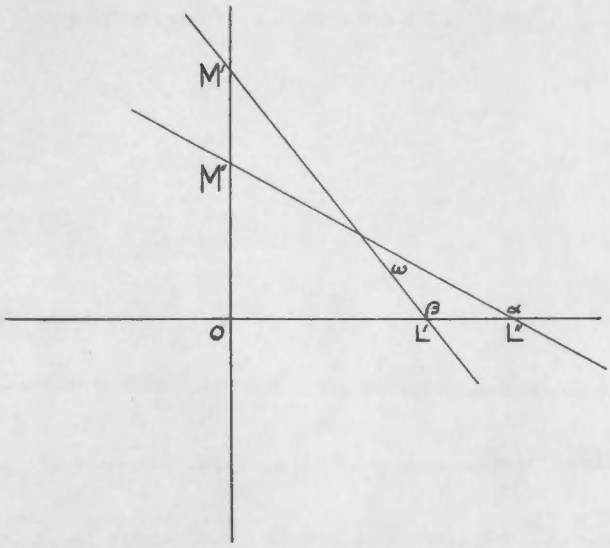

$-1/l$ | $-1/m$
∞ | ± 2
± 9 | $\pm \frac{2}{3}\sqrt{2}$
± 6 | $\pm \frac{4}{3}\sqrt{2}$
± 4 | $\pm \frac{7}{3}\sqrt{7}$
± 3 | ∞
$9l^2 + 4m^2 = 1$

PLATE V.

To express the angle between two lines in terms of their coordinates.



Let $M'L'$ be the line whose coordinates are $l'm'$.

" $M''L''$ " " " " " " $l''m''$.

To find the angle between them, or ω.

$OL' = -1/l'$, $OL'' = -1/l''$, $OM' = -1/m'$, $OM'' = -1/m''$.

$$\tan\beta = -\frac{-1/m'}{-1/l'} = -\frac{l'}{m'}$$

$$\tan\alpha = -\frac{-1/m''}{-1/l''} = -\frac{l''}{m''}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{l'/m' - l''/m''}{1 + l'/m' \cdot l''/m''}$$

$$\therefore \tan\omega = \frac{l'm'' - l''m'}{m'm'' + l'l''}$$

$$\text{or} \quad \omega = \tan^{-1}\frac{l'm'' - l''m'}{m'm'' + l'l''}.$$

Cor. I. Two lines are perpendicular to one another when, $m'm'' + l'l'' = 0$.

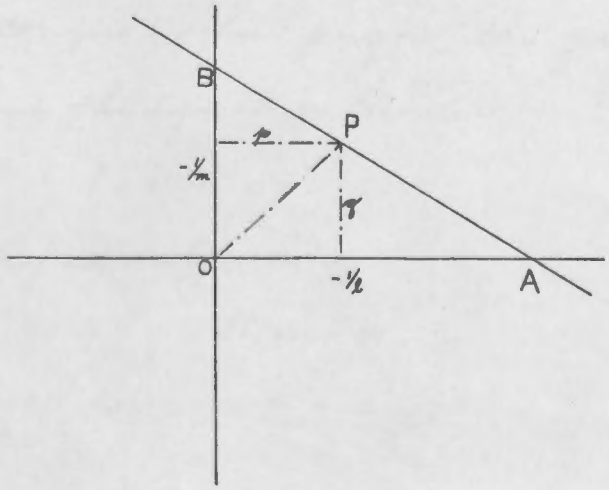
Cor. II. Two lines a parallel when

$$l'm'' = l''m'.$$

The Point.

In point-geometry the equation of a line is defined as an equation satisfied by the coordinates of all points on the line; so now in line geometry the equation of a point is defined as an equation satisfied by the coordinates of all straight lines passing thru the point.

To find the equation of any point, P.



In order to specify some definite point let p and q be the perpendicular distances of P from the L and M axes respectively. [This same plan will be followed hereafter in denoting any point.]

Through the point P draw any line, whose coordinates are l, m. It therefore cuts off on the axes the intercepts $-1/l$, $-1/m$.

Now

$$\triangle APO + \triangle BPO = \triangle BOA.$$

$$\therefore -1/l \cdot q - 1/m \cdot p = -1/m \cdot -1/l$$

or, $$lp + mq + 1 = 0.$$

When l and m assume all possible values this equation gives all lines thru the point P.

The origin: when $p = q = 0$, the given point P is the origin and the equation becomes

$$l \cdot 0 + m \cdot 0 + 1 = 0,$$

which is satisfied only when

$$l = m = \infty$$

or $$1/l = 1/m = 0.$$

Putting in these values the equation becomes

$$\infty \cdot 0 + \infty \cdot 0 + 1 = 0.$$

But $\infty \cdot 0$ is indeterminate and depends for its value, which is finite and different from 0, upon the way in which this limiting position has been reached. The equation therefore reduces to:

$$c = 0$$

where c means any constant whatever.

Moreover there is no inconsistency involved here, since a line thru the origin does not intersect the axes except at their own point of intersection, its only intercepts being, as noticed above

$$1/l = 1/m = 0.$$

It is therefore evident that <u>the equation of any point is of the first degree in l and m</u>

To prove the converse, viz - <u>Every equation of the first degree in l and m, is the equation of a point</u>

The most general equation of the first degree is :-

$$Al + Bm + C = 0.$$

Dividing by C, gives

$$\frac{A}{C}l + \frac{B}{C}m + 1 = 0$$

which differs from the general equation of a point only in the coefficients of l and m; since they are constants, replace $\frac{A}{C}$ by p and $\frac{B}{C}$ by q, and the equation becomes

$$pl + qm + 1 = 0.$$

∴ The general equation of the first degree represents a point. Moreover the position of the point in the plane can be at once determined by dividing by the absolute term, the resulting coefficients of l and m being the distances of the point from the L and M axes.

Suppose that in the general equation the absolute term is missing; it then takes the form:

$$Al + Bm = 0.$$

The question is: what point does the equation represent? If C were different from 0, it would represent a point whose distances from the axes were $\frac{A}{C}$ and $\frac{B}{C}$. But when $C = 0$, these distances become

infinite; that is to say

$$Al + Bm = 0$$

represents a point at infinity, or a system of lines all passing thru the same point at infinity; or in other words, it represents a system of parallel lines.

∴ Every equation of the first degree represents a point, or what is practically the same, the totality of lines thru a point.

Remark: In the work that follows

$$lp + mq + 1 = 0$$

will be used as the general equation of a point. Therefore unless otherwise specified the points at infinity are not considered.

<u>To find the condition that three lines pass thru the same point.</u>

Let the given point be:

$$lp + mq + 1 = 0$$

and let $l_1 m_1$; $l_2 m_2$; $l_3 m_3$; be the coordinates of the three lines.

If they are all to pass thru the given

point, their coordinates must satisfy the given equation

$$\therefore \quad l_1 p + m_1 q + 1 = 0$$

$$l_2 p + m_2 q + 1 = 0$$

$$l_3 p + m_3 q + 1 = 0$$

Or, eliminating p and q:

$$\begin{vmatrix} l_1 & m_1 & 1 \\ l_2 & m_2 & 1 \\ l_3 & m_3 & 1 \end{vmatrix} = 0.$$

<u>To find the equation of a point incident with two given lines.</u>

Let the coordinates of the lines be $l_1 m_1$ and $l_2 m_2$.

The general equation of any point (p, q) is:

$$lp + mq + 1 = 0. \quad (1)$$

Since the two lines are to be incident with the point their coordinates must satisfy the equation of the point.

$$\therefore \quad l_1 p + m_1 q + 1 = 0 \quad (2)$$

$$\text{and} \quad l_2 p + m_2 q + 1 = 0 \quad (3)$$

Eliminating p and q from (1), (2), and (3):

$$\begin{vmatrix} l & m & 1 \\ l_1 & m_1 & 1 \\ l_2 & m_2 & 1 \end{vmatrix} = 0$$

or

$$\frac{l - l_1}{l_2 - l_1} = \frac{m - m_1}{m_2 - m_1}.$$

To find the coordinates of a line incident with two given points.

Let the equations of the points be:

$$lp' + mq' + 1 = 0 \qquad (1)$$

$$lp'' + mq'' + 1 = 0 \qquad (2)$$

The coordinates of a line incident with both points will satisfy both equations, (1) and (2).

Solving (1) and (2) simultaneously for l and m, gives

$$l = \frac{q' - q''}{p'q'' - p''q'}.$$

$$m = \frac{p'' - p'}{p'q'' - p''q'}$$

To find the condition that three points may lie on the same straight line.

Let the given points be:

$$lp' + mq' + 1 = 0$$

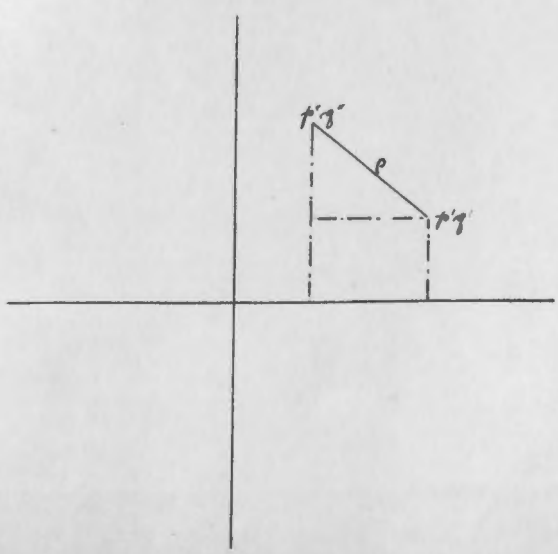
$$lp'' + mq'' + 1 = 0$$

$$lp''' + mq''' + 1 = 0$$

If these three points are to lie on the same straight line, the coordinates of the line must satisfy the three equations. But if l and m are to satisfy these three equations:

$$\begin{vmatrix} p' & q' & 1 \\ p'' & q'' & 1 \\ p''' & q''' & 1 \end{vmatrix} = 0.$$

To find the distance between two points whose equations are given.



Let the equation of the point $p'q'$ be:

$$lp' + mq' + 1 = 0 \qquad (1)$$

and of $p''q''$:

$$lp'' + mq'' + 1 = 0. \qquad (2)$$

From the figure it is evident that:

$$(p' - p'')^2 + (q' - q'')^2 = \rho^2 \qquad (3.)$$

where ρ represents the distance from $p'q'$ to $p''q''$.

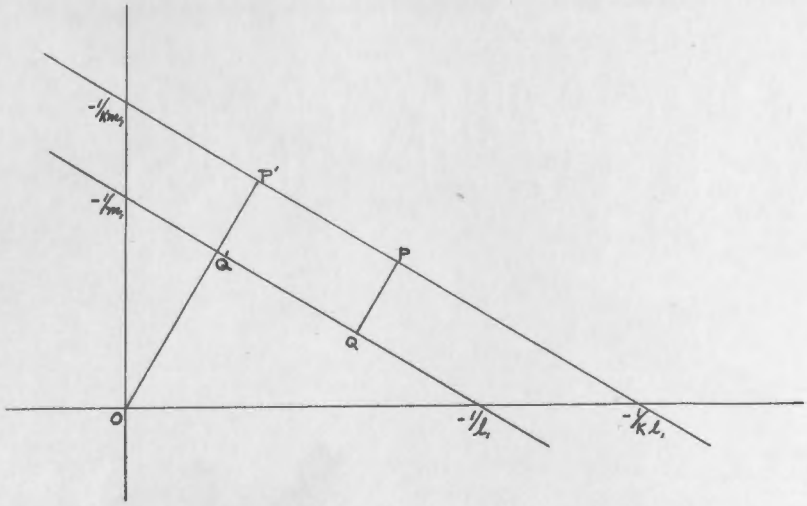
Since p', q' and p'', q'' are known when the equations of the points are given, (i.e. (1) and (2)) equation (3) involves no unknown quantities except the required distance ρ. Therefore (3). will give the distance between any two points whose equations are given.

To find the perpendicular distance from a given straight line to a given point.

Let the equation of the given point P be:

$$lp + mq + 1 = 0$$

and let $l, m,$ be the coordinates of the given line.



Draw PQ, thru P and perpendicular to the given line. PQ is then the distance to be found.

Thru P draw a second line parallel to the given line. If l_1, m_1 are the coordinates of the given line, Kl_1, Km_1 are the coordinates of any line parallel to it.

Since this line passes thru the given point its coordinates must satisfy the equation of the given point:

$$\therefore \quad Kl_1 p + Km_1 q + 1 = 0$$

$$\text{or } K = \frac{-1}{l_1 p + m_1 q}$$

Through O draw OP' perpendicular to the given line. Then

$$OQ' = \frac{1}{\sqrt{l_1^2 + m_1^2}} \quad \text{and} \quad OP' = \frac{1}{K\sqrt{l_1^2 + m_1^2}}$$

Substituting the value previously found for K

$$OP' = -\frac{l_1 p + m_1 q}{\sqrt{l_1^2 + m_1^2}}.$$

From the figure:

$$QP = Q'P' = OP' - OQ'$$

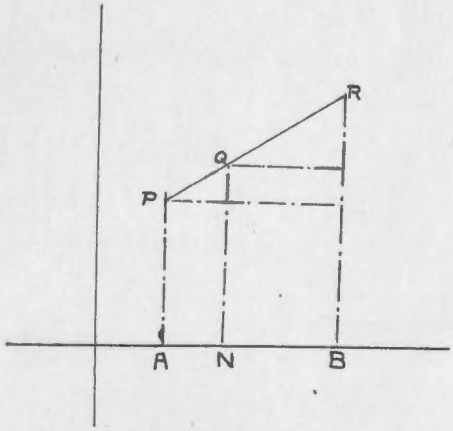
$$= -\frac{l_1 p + m_1 q}{\sqrt{l_1^2 + m_1^2}} - \frac{1}{\sqrt{l_1^2 + m_1^2}}$$

$$-QP = PQ = \frac{l_1 p + m_1 q + 1}{\sqrt{l_1^2 + m_1^2}}.$$

Hence, the perpendicular distance from a given straight line to a given point is obtained by substituting the coordinates of the line in the equation of the point and dividing by the square root of the sum of the squares of the coordinates of the line.

If the value of the radical $\sqrt{l_1^2 + m_1^2}$ be always considered positive, PQ will be positive or negative according as the distance from P to the origin is greater or less than the distance of the line from the origin.

To find the equation of a point that divides in a given ratio the distance between two given points.



Let the equations of the two given points, P and R, be respectively:

$$lp' + mq' + 1 = 0$$

$$lp'' + mq'' + 1 = 0.$$

and let $\mu : \nu$ be the given ratio in which the required point, Q is to divide the distance PR.

From the figure:

$$AN : BN :: PQ : QR :: \mu : \nu$$

$$\therefore \quad AN \cdot \nu - BN \cdot \mu = 0.$$

$$(p - p')\nu - (p'' - p)\mu = 0$$

$$\text{or} \quad p = \frac{\mu p'' + \nu p'}{\mu + \nu}.$$

Similarly:

$$q = \frac{\mu q'' + \nu q'}{\mu + \nu}.$$

But p and q are the distances of the required point Q, therefore its equation is:

$$l \cdot \frac{\mu p'' + \nu p'}{\mu + \nu} + m \frac{\mu q'' + \nu q'}{\mu + \nu} + 1 = 0.$$

The most useful case is when Q bisects the distance PR. Then the distances of the point of bisection from the axes are:

$$p = \tfrac{1}{2}(p' + p'') \qquad q = \tfrac{1}{2}(q' + q'')$$

and hence the equation of the point:

$$l(p' + p'') + m(q' + q'') + 2 = 0.$$

<u>To find the equation of a point on a line incident with two given points.</u>

Let the equations of the two given points be:

$$lp' + mq' + 1 = 0 \qquad (1)$$

$$\text{and} \quad lp'' + mq'' + 1 = 0. \qquad (2)$$

Consider the equation:

$$lp' + mq' + 1 + \lambda(lp'' + mq'' + 1) = 0. \qquad (3)$$

It is the equation of a point since it is of the first degree; and if $l, m,$ be the coordinates of the line incident with the two given points

$$l_1 p' + m_1 q' + 1 = 0$$

$$l_1 p'' + m_1 q'' + 1 = 0$$

and therefore:

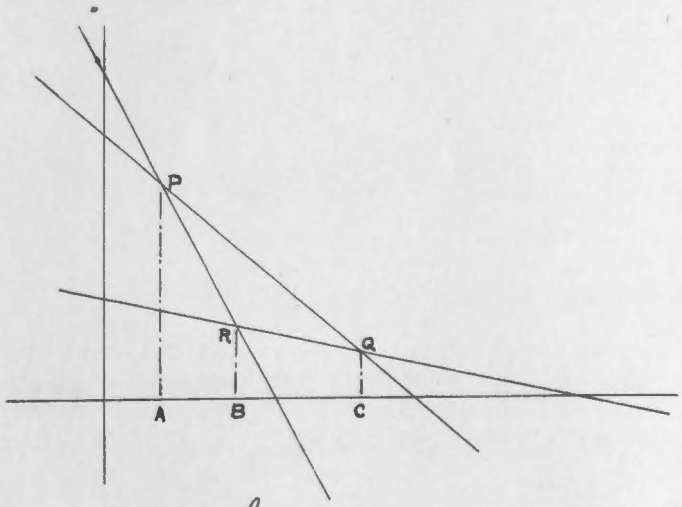
$$l_1 p' + m_1 q' + 1 + \lambda(l_1 p'' + m_1 q'' + 1) = 0$$

But this equation shows that the line $l_1 m_1$ is also incident with the point given by (3).

Hence (3) is the equation of a point on a line incident with the two given points. Since there are an infinite number of such points (3) represents for all possible values of λ, all points on a line or a range of points.

<u>To find the area of a triangle in terms of the coordinates of its sides.</u>

Let $l_1 m_1$; $l_2 m_2$; $l_3 m_3$; be the coordinates of the three lines forming the triangle PQR.



The equation of the point P is given by:

$$\begin{vmatrix} l & m & 1 \\ l_1 & m_1 & 1 \\ l_2 & m_2 & 1 \end{vmatrix} = 0.$$

or $l(m_1 - m_2) + m(l_2 - l_1) + l_1 m_2 - l_2 m_1 = 0.$

Hence for the point P:

$$p_1 = \frac{m_1 - m_2}{l_1 m_2 - l_2 m_1} \qquad q_1 = \frac{l_2 - l_1}{l_1 m_2 - l_2 m_1}.$$

Similarly for Q:

$$p_2 = \frac{m_2 - m_3}{l_2 m_3 - l_3 m_2} \qquad q_2 = \frac{l_3 - l_2}{l_2 m_3 - l_3 m_2}.$$

And for R:

$$p_3 = \frac{m_3 - m_1}{l_3 m_1 - l_1 m_3} \qquad q_3 = \frac{l_1 - l_3}{l_3 m_1 - l_1 m_3}.$$

The position of each vertex of the triangle is now known. With these determined it will be possible to find the area of the triangle.

From the figure:

$$\triangle PQR = APQC - BRQC - APRB.$$

$$APQC = \triangle APC + \triangle PAC$$

$$= \tfrac{1}{2} \cdot PA \cdot AC + \tfrac{1}{2} \cdot QC \cdot AC = \tfrac{1}{2} AC\,(PA + QC)$$

$$= \tfrac{1}{2}(p_2 - p_1)(q_1 + q_2)$$

Similarly:

$$BQRC = \tfrac{1}{2}(p_2 - p_3)(q_2 + q_3)$$

And:

$$APRB = \tfrac{1}{2}(p_3 - p_1)(q_3 + q_2)$$

$$\therefore \triangle PQR = \tfrac{1}{2}\left[(p_2 - p_1)(q_1 + q_2) - (p_2 - p_3)(q_2 + q_3) - (p_3 - p_1)(q_3 + q_1)\right]$$

$$= \tfrac{1}{2}\left[p_2 q_1 - p_1 q_2 + p_3 q_2 - p_2 q_3 + p_1 q_3 - p_3 q_1\right].$$

or in determinant form:

$$2\,\triangle PQR = \begin{vmatrix} p_1 & q_1 & 1 \\ p_2 & q_2 & 1 \\ p_3 & q_3 & 1 \end{vmatrix}$$

Substituting in this the values previously found for p_1, q_1; p_2, q_2; p_3, q_3; gives:-

$$2\,\Delta\,PQR = \begin{vmatrix} \dfrac{m_1 - m_2}{\Delta_1} & \dfrac{l_2 - l_1}{\Delta_1} & 1 \\ \dfrac{m_2 - m_3}{\Delta_2} & \dfrac{l_3 - l_2}{\Delta_2} & 1 \\ \dfrac{m_3 - m_1}{\Delta_3} & \dfrac{l_1 - l_3}{\Delta_3} & 1 \end{vmatrix}$$

where

$$\Delta_1 = l_1 m_2 - l_2 m_1$$

$$\Delta_2 = l_2 m_3 - l_3 m_2$$

$$\Delta_3 = l_3 m_1 - l_1 m_3 .$$

$$= \frac{1}{\Delta_1 \Delta_2 \Delta_3} \begin{vmatrix} m_1 - m_2 & l_2 - l_1 & \Delta_1 \\ m_2 - m_3 & l_3 - l_2 & \Delta_2 \\ m_3 - m_1 & l_1 - l_3 & \Delta_3 \end{vmatrix}$$

$$= \frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta_1 \Delta_2 \Delta_3} \begin{vmatrix} m_1 - m_2 & l_2 - l_1 \\ m_2 - m_3 & l_3 - l_2 \end{vmatrix}$$

$$= \frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta_1 \Delta_2 \Delta_3} \cdot \left[\Delta_1 + \Delta_2 + \Delta_3\right] - \frac{\left[\Delta_1 + \Delta_2 + \Delta_3\right]^2}{\Delta_1 \Delta_2 \Delta_3} .$$

$$= \frac{1}{\Delta_1 \Delta_2 \Delta_3} \begin{vmatrix} l_1 & m_1 & 1 \\ l_2 & m_2 & 1 \\ l_3 & m_3 & 1 \end{vmatrix}^2$$

$$\therefore \triangle PQR = \frac{1}{2\Delta_1 \Delta_2 \Delta_3} \begin{vmatrix} l_1 & m_1 & 1 \\ l_2 & m_2 & 1 \\ l_3 & m_3 & 1 \end{vmatrix}^2$$

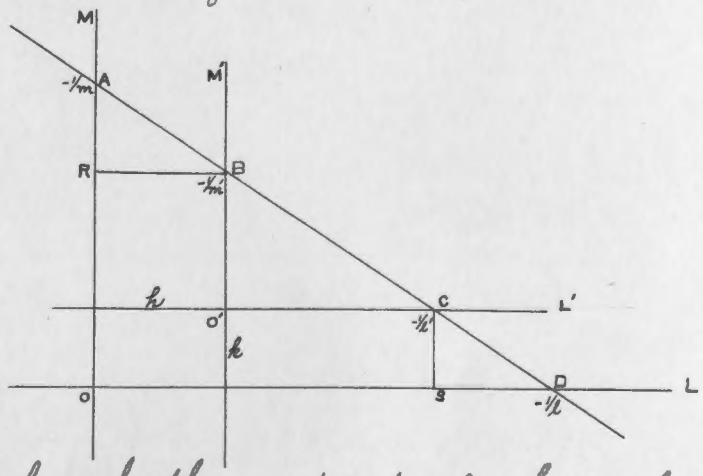
or

$$\triangle PQR = \frac{1}{2 \cdot \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} \cdot \begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} \cdot \begin{vmatrix} l_3 & m_3 \\ l_1 & m_1 \end{vmatrix}} \begin{vmatrix} l_1 & m_1 & 1 \\ l_2 & m_2 & 1 \\ l_3 & m_3 & 1 \end{vmatrix}^2,$$

a result involving only the given coordinates of the sides of the triangle.

Transformation of Coordinates.

To change the origin of coordinates without changing the direction of the axes.



Let l, m be the coordinates of a line referred to the axes OL and OM.

and let l', m' be the coordinates of the same line referred to the axes O'L' and O'M', where O' is at a distance h from OM and k from OL.

Through C draw CS parallel to OM.

" B " BR " " OL.

In the similar triangles BO'C and CSD,

$$\frac{O'B}{O'C} = \frac{CS}{SD} \quad \text{or} \quad \frac{-1/m'}{-1/l'} = \frac{h}{SD}$$

$$\therefore \; SD = h \cdot \frac{m'}{l'}.$$

In the similar triangles BO'C and ARB

$$\frac{O'B}{O'C} = \frac{AR}{RB} \quad \text{or} \quad \frac{l'}{m'} = \frac{AR}{h}$$

$$\therefore \quad AR = h \cdot \frac{l'}{m'}.$$

Now:

$$OD = \left|\frac{1}{l}\right| = h + \frac{1}{l'} + k \cdot \frac{m'}{l'}$$

$$\therefore \quad \frac{1}{l} = \frac{hl' + km' + 1}{l'} \quad \text{or} \quad l = \frac{l'}{hl' + km' + 1}$$

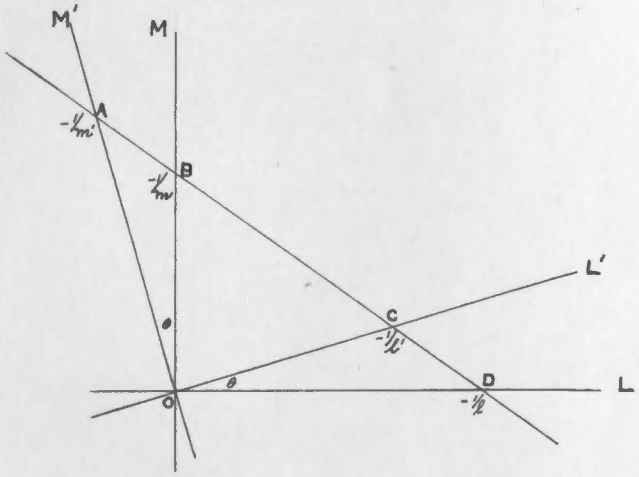
And

$$OA = \left|\frac{1}{m}\right| = k + \frac{1}{m'} + h \cdot \frac{l'}{m'}$$

$$\therefore \quad \frac{1}{m} = \frac{hl' + km' + 1}{m'} \quad \text{or} \quad m = \frac{m'}{hl' + km' + 1}$$

Hence to find what a given equation becomes when the origin is moved to the point h, k, the new axes being parallel to the old substitute $\frac{l'}{hl' + km' + 1}$ for l and $\frac{m'}{hl' + km' + 1}$ for m. After the substitution is made write l and m instead of l' and m'; so that practically this transformation is effected by writing $\frac{l}{hl + km + 1}$ in the place of l, and $\frac{m}{hl + km + 1}$ in the place of m.

To change the direction of the axes without changing the origin, both systems being rectangular.



Let l and m be the coordinates of a line referred to the axes OL and OM.

and let l' and m' be the coordinates of the same line referred to the axes OL' and OM', making an angle θ with the original axes.

From the figure:

$$\triangle AOD = \triangle AOC + \triangle COD$$

$$\therefore \frac{1}{l} \cdot \frac{1}{m'} \sin(90^\circ + \theta) = \frac{1}{l'} \cdot \frac{1}{m'} + \frac{1}{l} \cdot \frac{1}{l'} \cdot \sin\theta$$

$$\text{or} \quad \frac{1}{l m'} \cos\theta = \frac{1}{l' m'} + \frac{1}{l l'} \sin\theta$$

$$\therefore \quad l = l' \cos\theta - m' \sin\theta$$

Likewise: $\triangle AOC = \triangle AOB + \triangle BOC$

$$\therefore \frac{1}{m'} \cdot \frac{1}{l'} = \frac{1}{m'} \cdot \frac{1}{m} \sin\theta + \frac{1}{m'} \cdot \frac{1}{l'} \sin(90^\circ - \theta)$$

$$\text{or} \quad m = l' \sin\theta + m' \cos\theta.$$

Some General Properties of a Sheaf of Lines.

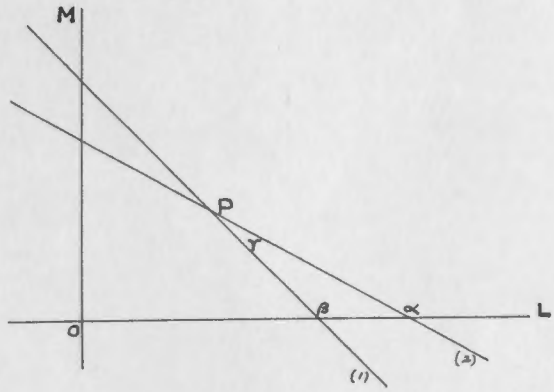
The general equation of the first degree in l and m has been shown to represent a system of lines incident with a point or as it is generally called 'a pencil of lines'.

There are other kinds of systems of lines defined by functional relations between l and m; in general $l = f(m)$ defines some system of lines, a particular kind of function defining a particular kind of system. Any system of lines defined by some such functional relation is called a sheaf of lines or simply a sheaf.

A point of incidence of a sheaf is the point of intersection of any two lines of the sheaf.

A point of contact of a sheaf is the limiting position of a point of incidence as the two lines approach coincidence.

To find the direction of a line of the sheaf $l = f(m)$, through a point of contact (which is not the origin.)



Let (1) and (2) be two nearly coincident lines of the sheaf, $l = f(m)$, and P their point of incidence.

Let l' and m' be the coordinates of (1)

and $l' + \Delta l$ " $m' + \Delta m$ " " " (2)

where Δl and Δm denote small increments in l and m.

Then

$$\tan\beta = -\frac{l'}{m'}, \quad \tan\alpha = -\frac{l' + \Delta l}{m' + \Delta m}$$

and

$$\tan\gamma = \frac{l'(m' + \Delta m) - m'(l' + \Delta l)}{m'(m' + \Delta m) + l'(l' + \Delta l)} = \frac{l'\Delta m - m'\Delta l}{m'(m' + \Delta m) + l'(l' + \Delta l)}$$

Now let (1) and (2) approach coincidence. The point of incidence P, then becomes a point of contact and moreover $\gamma = 0$.

Therefore, as the two lines approach coincidence

$$\frac{l'\Delta m - m'\Delta l}{m'(m'+\Delta m) + l'(l'+\Delta l)} \doteq 0.$$

In order that this equation be true the numerator must approach 0 or the denominator ∞, but the latter is impossible unless l or m become ∞ which cannot happen as long as the point of contact is not the origin.

Therefore $l'\Delta m - m'\Delta l \doteq 0$ as Δl and Δm approach 0.

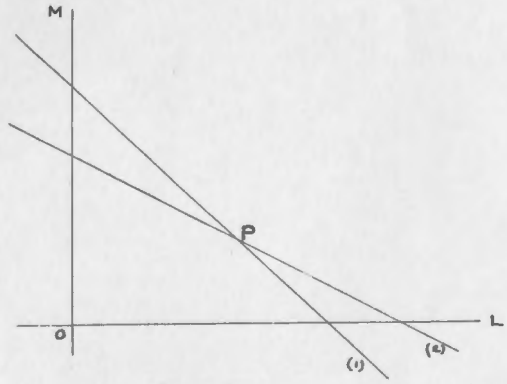
and in the limit

$$\frac{l'}{m'} = \frac{dl}{dm} \quad [\text{for } l = l' \text{ and } m = m']$$

But $\frac{l'}{m'}$ is the tangent of the angle made by the two coincident lines thru the point of contact.

∴ The first derivative of the function represents the direction of a line thru a point of contact, the coordinates of the line being substituted in the derivative.

To find the equation of a point of contact of any sheaf $l = f(m)$.



Let (1) and (2) be two nearly coincident lines of the sheaf

Let l', m' be the coordinates of (1)

and $l' + \Delta l,\ m' + \Delta m$ " " " (2)

The equation of the point P, the intersection of (1) and (2) is :

$$\frac{l - l'}{(\Delta l + l') - l'} = \frac{m - m'}{(\Delta m + m') - m'}$$

or

$$\frac{l - l'}{m - m'} = \frac{\Delta l}{\Delta m}.$$

Let Δl and Δm approach 0 then in the limit

$$\frac{l - l'}{m - m'} = \frac{dl}{dm}$$ where $\frac{dl}{dm}$ represents the value of the derivative for $l'm'$.

or $l - l' = \frac{dl}{dm}(m - m')$

which is the equation of the point P as the two lines approach coincidence : the equation of the point of contact of $l'm'$.

The Sheaf of the Second Class.

When an algebraic equation in l and m is of the second degree it means that for every value of l there are two values of m, and for every value of m, two values of l; – that is to say, thru every point of the plane pass two lines of the sheaf. Such a sheaf is said to be of the <u>second class</u>. Likewise, when the equation is of the third degree, there are three lines of the sheaf thru every point of the plane, and the sheaf is said to be of the third class. In general :– <u>The class of a sheaf is equal to the degree of its equation</u>.

The most general equation of the second degree, and therefore the most <u>general equation of a sheaf of the second class is</u> :

$$al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0.$$

where a, h, b, f, g and c, denote constants.

Some General Properties of the Sheaf of the Second Class.

To find the equation of the point of contact of the sheaf of the second class whose equation is:

$$al^2+2hlm+bm^2+2gl+2fm+c=0.$$

I. The equation of the point of contact of any sheaf is given by: $l-l'=\frac{dl}{dm}(m-m')$

Differentiating the given equation

$$al\,dl+bm\,dm+hm\,dl+hl\,dm+g\,dl+f\,dm=0$$

or $$\frac{dl}{dm}=-\frac{bm+hl+f}{al+hm+g}$$

Putting in this value in the general equation of the point of contact and at the same time substituting the coordinates of the line $l'm'$ whose point of contact is to be determined, gives:

$$l-l'=-\frac{bm'+hl'+f}{al'+hm'+g}(m-m')$$

or: $al^2+hlm'+gl-al'^2-hl'm'-l'g=-bmm'-hl'm-fm+bm'^2+hl'm'+fm'$

or: $al^2+h(lm'+l'm)+bmm'+gl+fm=al'^2+2hl'm'+bm'^2+gl'+fm'$

Adding $gl'+fm'+c$ to both members, the second

number will vanish, since $l'm'$ is a line of the sheaf, and the equation of the point of contact becomes:

$$all' + h(lm' + l'm) + bmm' + g(l+l') + f(m+m') + c = 0.$$

II. Another method of obtaining the equation of the point of contact.

Let $l'm'$ and $l''m''$ be the coordinates of any two lines of the sheaf

$$al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0.$$

The equation of the point of incidence of these two lines is given by:

$$a(l-l')(l-l'') + 2h(l-l')(m-m'') + b(m-m')(m-m'') =$$
$$al^2 + 2hlm + bm^2 + 2gl + 2fm + c,$$

for the equation represents a point, the terms above the first degree destroying each other, and it is evidently satisfied by the two lines of the sheaf $l'm'$ and $l''m''$.

Putting $l'' = l'$ and $m'' = m'$, gives the equation of the point of contact:

$$a(l-l')^2 + 2h(l-l')(m-m') + b(m-m')^2 = al^2 + 2hlm + bm^2 + 2gl + 2fm + c.$$

Or expanding:

$$2all' + 2h(l'm + m'l) + 2bmm' + 2gl + 2fm + c = al'^2 + 2hl'm' + bm'^2$$

Add to both members $2gl'+2fm'+c$; the right hand member will vanish since $l'm'$ is a line of the sheaf, and the equation of the point of contact becomes:

$$all'+h(l'm+lm')+bmm'+g(l+l')+f(m+m')+c=0,$$

a result which is identical with that obtained by the first method.

It is perhaps worth while noticing that the equation of the contact point of $l'm'$ is obtained from the equation of the sheaf by writing ll' for l^2, $l'm+lm'$ for $2lm$, mm' for m^2, $l+l'$ for $2l$ and $m+m'$ for $2m$.

Pole and Polar

The equation of a point of contact of the sheaf whose equation is:

(1) $al^2+2hlm+bm^2+2gl+2fm+c=0,$

is given by:

(2) $all'+h(l'm+lm')+bmm'+g(l+l')+f(m+m')+c=0$

when $l'm'$ is a line of the sheaf (1)

If $l'm'$ is not a line of the sheaf, or in

other words if (2) is not a point of contact, then thru it there are two lines belonging to the sheaf (1)

Let them be $(l_1 m_1)$ and $(l_2 m_2)$.

The equations of the points of contact of these lines are respectively:

(3) $$a l l_1 + h(l_1 m + l m_1) + b m m_1 + g(l + l_1) + f(m + m_1) + c = 0$$

(4) $$a l l_2 + h(l_2 m + l m_2) + b m m_2 + g(l + l_2) + f(m + m_2) + c = 0$$

The conditions that (3) and (4) shall lie on the line $l'm'$ are:

(5) $$a l' l_1 + h(l_1 m' + l' m_1) + b m' m_1 + g(l' + l_1) + f(m' + m_1) + c = 0$$

(6) $$a l' l_2 + h(l_2 m' + l' m_2) + b m' m_2 + g(l' + l_2) + f(m' + m_2) + c = 0$$

But (5) and (6) are also the conditions that the point (2) shall lie on the lines $l_1 m_1$ and $l_2 m_2$

$\therefore$ (2) is a point incident with the lines $l_1 m_1$ and $l_2 m_2$, and the line $l'm'$ is a line incident with the points of contact of these two lines.

<u>The point (2) and the line $l'm'$ are called pole and polar with respect to (1)</u>

The equation of the pole of a line $l'm'$ is identical in form with that of the point of contact of $l'm'$, but in the latter $l'm'$ is a line of the

sheaf while in the latter it is not.

If $l'm'$ is a line of the sheaf, the two points of contact of l, m, and of $l_2\, m_2$ coincide and each of the lines l, m, and $l_2\, m_2$ coincides with $l'm'$. Therefore the point of contact is a particular case of a pole which lies on its own polar.

<u>If the pole of a line $l'm'$ lies on $l''m''$, then will the pole of $l''m''$ lie on $l'm'$</u>

The equations of the poles of $l'm'$ and $l''m''$ with respect to the sheaf of the second class are:

(1) $a l l' + h(l'm + lm') + b m m' + g(l + l') + f(m + m') + c = 0$

(2) $a l l'' + h(l''m + lm'') + b m m'' + g(l + l'') + f(m + m'') + c = 0$

The point (1) will lie on $l''m''$ if

$$a l'' l' + h(l'm'' + l''m') + b m'' m' + g(l'' + l') + f(m'' + m') + c = 0$$

But this is also the condition that (2) shall lie on $l'm'$; which proves the proposition.

Cor. I The locus of the polars of all points lying on a fixed line is a point; viz. the pole of the fixed line.

Cor II. The point of intersection of the polars of two given points P and Q, is the pole of the line PQ.

The Reduction of the General Equation of the Second Degree to Certain Standard Forms.

As before let the general equation of the second degree be:

(1) $al^2 + 2hlm + bm^2 + 2gl + 2fm + c = 0.$

With the equation in this form very little can be inferred about the nature and properties of the sheaf it represents. The purpose of the next few pages is to show that by a transfer of the origin and a rotation of the axes this equation can be simplified and reduced to certain standard forms.

The terms of the first degree can be made to vanish by changing the origin to some new point say p, q. In order to do this, make use of the following substitutions

$$l = \frac{l}{lp + mq + 1}, \quad m = \frac{m}{lp + mq + 1}$$

The equation then becomes:

(2) $l^2(a+cp^2+2gp)+m^2(b+2fq+q^2c)+2lm(h+gq+fp+cpq)$
$+2l(g+pc)+2m(f+qc)+c=0.$

If the terms of the first degree are to vanish: $g+pc=0$ and $f+qc=0$

or: $p=-g/c$, $q=-f/c$,

a transformation which is always possible provided $c\neq 0$. If $c=0$, p and q become infinite and the transformation cannot be effected in this way. The case of $c=0$ will therefore be taken up separately.

Substituting the above values of p and q in (2) gives:

$$l^2(ac^2-cg^2)+2lm(hc^2-fgc)+m^2(bc^2-f^2c)+c^3=0$$

or, dividing by c:

(3) $$l^2(ac-g^2)+2lm(hc-fg)+m^2(bc-f^2)+c^2=0$$

To free (3) of the term in lm, rotate the axes thru the angle θ, making use of the following substitutions

$$l=l\cos\theta-m\sin\theta,\quad m=l\sin\theta+m\cos\theta.$$

(3) then becomes:

$$(ac-g^2)(l\cos\theta-m\sin\theta)^2+2(hc-fg)(l\cos\theta-m\sin\theta)(l\sin\theta+m\cos\theta)$$
$$+(bc-f^2)(l\sin\theta+m\cos\theta)^2+c^2=0.$$

or, expanding:

(4) $[(ac-g^2)\cos^2\theta + 2(hc-fg)\cos\theta\sin\theta + (bc-f^2)\sin^2\theta]\,l^2 + [(ac-g^2)\cdot\sin^2\theta - 2(hc-fg)\sin\theta\cos\theta + (bc-f^2)\cos^2\theta]\,m^2 + [-2(ac-g^2)\cdot\cos\theta\sin\theta + 2(hc-fg)(\cos^2\theta-\sin^2\theta) + 2(bc-f^2)\sin\theta\cos\theta]\,lm + c^2 = 0.$

If the coefficient of lm is to vanish

$$2\sin\theta\cos\theta\,(bc-ac+g^2-f^2) + 2(hc-fg)(\cos^2\theta-\sin^2\theta) = 0$$

(5) $$\text{or} \quad \tan 2\theta = \frac{2(fg-hc)}{c(b-a)+g^2-f^2}$$

a transformation which is always possible since the tangent can assume all values between $+\infty$ and $-\infty$.

From (5) the values of $\sin\theta$ and $\cos\theta$ can be computed and these values when substituted in (4) give an equation of the form:

$$Al^2 + Bm^2 = C$$

$$\text{or} \quad \underline{\frac{A}{C}l^2 + \frac{B}{C}m^2 = 1}$$

where

$A = (ac-g^2)\cos^2\theta + 2(hc-fg)\cos\theta\sin\theta + (bc-f^2)\sin^2\theta$

$B = (ac-g^2)\sin^2\theta + 2(hc-fg)\cos\theta\sin\theta + (bc-f^2)\cos^2\theta$

$C = -c^2$

When $c = 0$, the general equation becomes

(1) $$al^2 + 2hlm + bm^2 + 2gl + 2fm = 0$$

This is satisfied for $l = m = 0$

i.e. the line at infinity is a line of the sheaf.

The equation of the point of contact of a line $l'm'$ is in general:

$$all' + h(l'm + lm') + bmm' + g(l + l') + f(m + m') = 0$$

If $l' = m' = 0$, this becomes

$gl + fm = 0$, the equation of the contact-point of the line at ∞. This contact point is the point at ∞ defined by the system of parallel lines that satisfy the equation $gl + fm = 0$.

[Note: It is readily obtained by placing the two terms of the first degree of the equation equal to 0.]

Now in order to simplify the general equation when $c = 0$, instead of transforming the origin to some new point, rotate the axes until the L axis is parallel to the above mentioned system of parallel lines. That is to say, let

$$\tan\theta = \frac{l}{m} = -f/g$$

then $\sin\theta = \dfrac{f}{\sqrt{f^2+g^2}}$ and $\cos\theta = \dfrac{g}{\sqrt{f^2+g^2}}$

Substituting in equation (1) by means of the equations:

$$l = l\cos\theta - m\sin\theta$$

$$m = l\sin\theta + m\cos\theta$$

making use of the values just found for $\sin\theta$ and $\cos\theta$,

$$a\left[\frac{lg}{\sqrt{f^2+g^2}} - \frac{mf}{\sqrt{f^2+g^2}}\right]^2 + 2h\left[\frac{lg}{\sqrt{f^2+g^2}} - \frac{mf}{\sqrt{f^2+g^2}}\right]\left[\frac{lf}{\sqrt{f^2+g^2}} + \frac{mg}{\sqrt{f^2+g^2}}\right]$$

$$+ b\left[\frac{lf}{\sqrt{f^2+g^2}} + \frac{mg}{\sqrt{f^2+g^2}}\right] + 2g\left[\frac{lg}{\sqrt{f^2+g^2}} - \frac{mf}{\sqrt{f^2+g^2}}\right] - 2f\left[\frac{lf}{\sqrt{f^2+g^2}} - \frac{mg}{\sqrt{f^2+g^2}}\right] = 0.$$

or

$$l^2\left[ag^2+2hfg+bf^2\right] + lm\left[-2agf-2hf^2+2hg^2+2bfg\right] + m^2\left[af^2-2fgh+bg^2\right]$$
$$+ 2l\left[g^2+f^2\right]^{3/2} = 0.$$

in which the term in m has vanished.

Now transfer the origin to the point:

$$-\frac{ag^2+2hfg+bf^2}{2(g^2+f^2)^{3/2}}, \quad \frac{2agf+2hf^2-2hg^2-2bfg}{2(g^2+f^2)^{3/2}}$$

[Note:- these coordinates are obtained by dividing the coefficient of l^2 and lm by the coefficient of l.]

The coefficients of l^2 and lm will then vanish and the equation becomes

$$m^2(af^2-2fgh+bg^2) + 2(g^2+f^2)^{3/2}l = 0$$

or $\underline{m^2 = 2Kl}$, where $K = \dfrac{2(g^2+f^2)^{3/2}}{2fgh-af^2-bg^2}$.

The general equation of the second degree can therefore always be reduced to one of the two standard forms

I, when $c \neq 0$ $\qquad \frac{A}{C} l^2 + \frac{B}{C} m^2 = 1$

II when $c = 0$ $\qquad m^2 = 2Kl.$

The next step will be to investigate in some detail the nature and properties of the sheaf represented by each of these two types of equations.

The Parabolic Sheaf.

As has been previously noticed the line at infinity is one of the lines of the sheaf defined by the equation:

$$al^2 + 2hlm + bm^2 + 2gl + 2fm = 0$$

since it is satisfied for

$$l = m = 0$$

Such a sheaf is called a <u>parabolic</u> <u>sheaf</u>

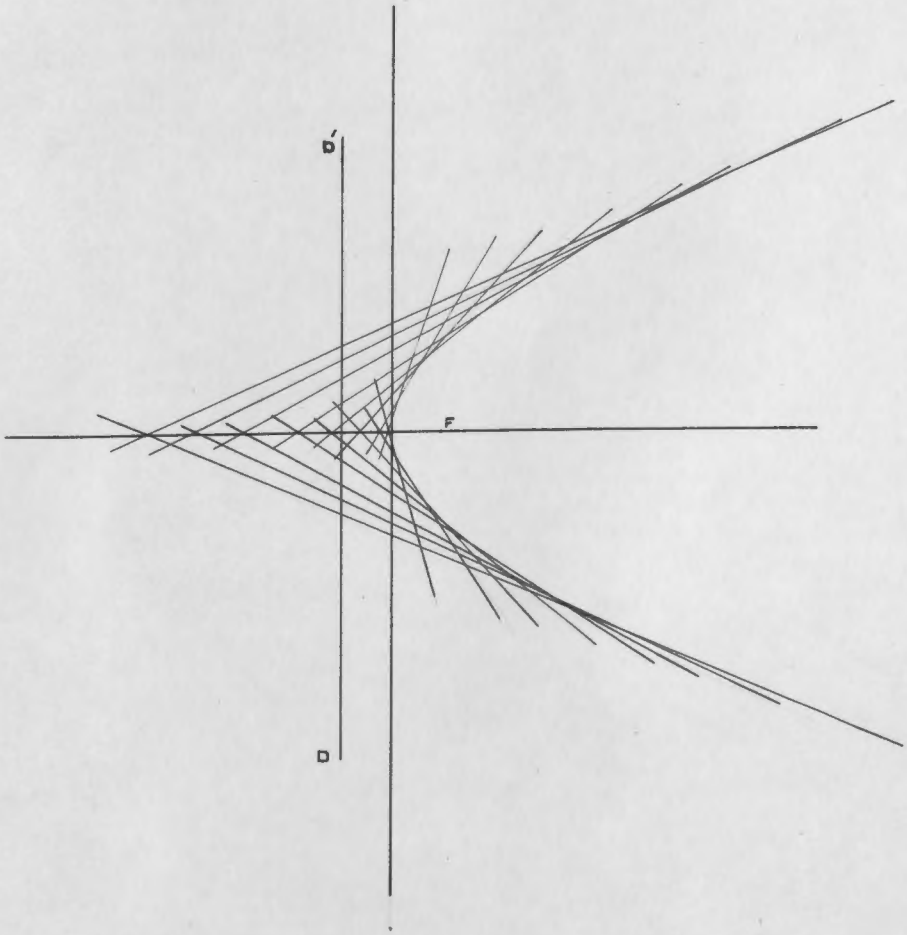


PLATE VI

Since such an equation can, by a proper choice of the origin and axes, always be transformed to the simpler standard form

$$m^2 = 2Kl,$$

this will always be used hereafter in discussing the parabolic sheaf

The equation of the point of contact of any line $l'm'$ of the sheaf is

$$mm' = K(l+l').$$

This is satisfied for

$$l' = m' = 0$$

$\therefore$ there is a point of contact at infinity, as might have been expected inasmuch as the line at infinity is a part of the sheaf. Substituting $l' = m' = 0$, in the equation of the point of contact gives

$$Kl = 0$$

which means that the point of contact of the line at infinity is situated on the L axis at an infinite distance from the origin.

When $\quad l' = m' = \infty$

$$Kl = \infty, \text{ or } l = \infty$$

∴ the origin is the point of contact of the M axis. This point is called the vertex of the sheaf.

The equation of the pole of any line $l, m,$ in the plane is:

$$m m_1 = k(l + l_1)$$

When $l_1 = m_1 = 0$

this becomes $kl = 0$

which means that the pole of the line at infinity is situated on the L axis. Moreover this is the point of contact of the line at infinity, that is to say the pole of the line at infinity, the point at infinity lies on its own polar. For reasons that will be evident in dealing with the other sheafs of the second class this point i.e. the pole of the line at infinity is called the centre of the sheaf. The centre of the parabolic sheaf, then, lies at infinity.

Conjugate lines are lines such that each passes thru the pole of the other

The focus of a sheaf of the second class

is a point such that all conjugate lines thru it are at right angles to one another.

Focal radii are lines drawn from the foci to a point of contact.

To find the equation of the focus of the parabolic sheaf, $m^2 = 4kl$.

Let

(1) $\alpha l + \beta m + 1 = 0$, define the point

where α and β are to be determined.

Suppose $l_1 m_1$ is any line thru this point.

Its pole will be of the form

$$m m_1 = k(l + l_1)$$

(2) or $$k l - m_1 m + k l_1 = 0$$

The line thru the points (1) and (2) will be conjugate to the line $l_1 m_1$, since it passes thru the pole of $l_1 m_1$; its coordinates will be found by solving (1) and (2) simultaneously, giving

$$l = \frac{-m_1 - \beta k l_1}{\beta k + \alpha m_1}, \quad m = \frac{\alpha k l_1 - k}{\beta k + \alpha m_1}$$

The line $l_1 m_1$ and the line whose coordin

-ates have just been determined will be at right angles provided

$$ll_1 + mm_1 = 0$$

or $$l_1 \cdot \frac{-m_1 - \beta K l_1}{\beta K + \alpha m_1} + m_1 \cdot \frac{\alpha K l_1 - K}{\beta K + \alpha m_1} = 0$$

(3) or $$l_1(-m_1 - \beta K l_1) + m_1(\alpha K l_1 - K) = 0$$

where α and β are to be determined.

Since the line $l_1 m_1$ passes thru the point

(1) $$\alpha l_1 + \beta m_1 + 1 = 0$$

$$\therefore \quad l_1 = \frac{-\beta m_1 - 1}{\alpha}$$

Substituting this value of l_1 in (3) gives:

$$\left(\frac{-1-\beta m_1}{\alpha}\right)\left(-m_1 + \beta K \cdot \frac{1+\beta m_1}{\alpha}\right) + m_1 \cdot \left(\alpha K \cdot \frac{-\beta m_1 - 1}{\alpha} - K\right) = 0$$

(4) or $$m_1^2(\alpha\beta - \beta^3 K - \alpha^2 \beta K) + m_1(\alpha - 2\beta^2 K - 2\alpha^2 K) - \beta K = 0$$

Moreover since the focus is such a point that <u>all</u> conjugate lines thru it are at right angles equation (4) must be true for <u>all</u> values of m_1. Hence equation (4) is an identity, and in order that it be such

(a) $(\alpha\beta - \beta^3 K - \alpha^2 \beta K) = \beta(\alpha - \beta^2 K - \alpha^2 K) = 0$

(b) $(\alpha - 2\beta^2 K - 2\alpha^2 K) = 0$ or $\alpha - 2\alpha^2 K = 2\beta^2 K$

(c) $\beta K = 0$

From (c), since $K \neq 0$, $\beta = 0$.

Substituting this value in (b)

$$\alpha = 2\alpha^2 K \quad \therefore \alpha = \frac{1}{2K}$$

The focus therefore lies on the L axis at a distance $\frac{1}{2K}$ from the origin and its equation is:

$$\frac{1}{2K} l + 1 = 0 \quad \text{or} \quad l + 2K = 0.$$

[Note: If in equation (3) m_1 had been eliminated instead of l_1, by substituting for m_1 $\frac{-\alpha l_1 - 1}{\beta}$ the resulting equation would have been:

$$l_1\left(\frac{\alpha l_1 + 1}{\beta} - \beta K l_1\right) - \frac{\alpha l_1 + 1}{\beta}(\alpha K l_1 - K) = 0$$

or $\quad l_1^2(\alpha - \beta^2 K - \alpha^2 K) + l_1 + K = 0$

But since $K \neq 0$, this equation is not true for all values of l_1, and therefore is not an identity. It is however satisfied for the values already found, viz. $\alpha = \frac{1}{2K}$, $\beta = 0$, $l = -2K$. Taking account of both equations the one in m_1 is seen to be an identity but not so the one in l_1. This could be true only when the focus lies on the L axis, a condition which agrees with the values already found for α and β.]

The directrix of the parabola is defined as the polar of the focus, and its coordinates are seen by inspection of the equation of the focus to be $(-2K, 0)$.

To find the ratio between the distance from a point of contact to the focus and the distance from that same point of contact to the directrix.

Let $l'm'$ be any line of the sheaf $m^2 = 2K.l$.

Its point of contact is:

$$mm' = K(l+l')$$

or $$\frac{1}{l'}.l - \frac{m'}{Kl'}.m + 1 = 0$$

The distance from this point to the focus whose coordinates are $(\frac{1}{2K}, 0)$ is:

$$\sqrt{\left\{\left(\frac{1}{2K} - \frac{1}{l'}\right)^2 + \left(0 - \frac{m'}{Kl'}\right)^2\right\}} = d$$

or $$\sqrt{\left\{\frac{1}{4K^2} - \frac{1}{Kl'} + \frac{1}{l'^2} + \frac{m'^2}{Kl'^2}\right\}} = \sqrt{\left\{\frac{1}{4K^2} + \frac{1}{Kl'} + \frac{1}{l'^2}\right\}}$$

$$= \frac{1}{2K} + \frac{1}{l'} = d.$$

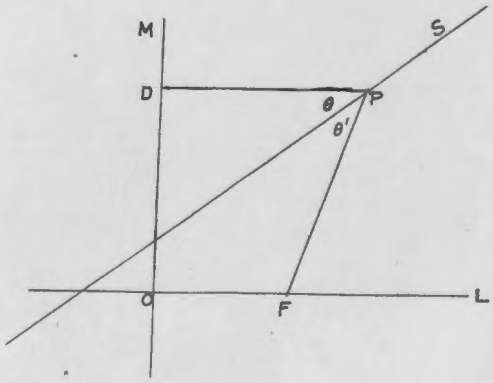
The distance from this same point to the directrix whose coordinates are $(2K, 0)$ is:

$$\frac{\frac{2K}{l'} - \frac{m'}{Kl'}\cdot 0 + 1}{\sqrt{4K^2+0}} = \frac{1}{l'} + \frac{1}{2K} = d'$$

Therefore, the distance from any point of contact to the focus is equal to the distance from that same point of contact to the directrix.

This property identifies the 'parabolic sheaf' with the system of tangents to a parabola; or in other words the locus of the contact points of a parabolic sheaf is a parabola.

To show that the angle between any line of the sheaf and the line thru its point of contact parallel to the axis is equal to the angle between that same line and the focal radius through its point of contact.



Let: S be any line of the sheaf.

θ, the angle between S and the focal radius FP

θ', the angle between S and DP, parallel to OL.

The equation of the point of contact of any line $l_1 m_1$ of the sheaf is:

$$m m_1 = K(l + l_1) \quad \text{or} \quad \frac{1}{l_1} \cdot l - \frac{m_1}{K l_1} \cdot m + 1 = 0$$

The coordinates of a line thru this point and parallel to the L axis are: $0, -\frac{m_1}{K l_1}$

The equation of the focus is: $l + 2K = 0$.

Therefore the coordinates of a focal radius thru the point of contact given above are:

$$-2K, \quad \frac{-K(2K - l_1)}{m_1}$$

The angle between $l_1 m_1$ and $0, \frac{-m}{K l_1}$ is given by:

$$\tan\theta = \frac{l_1 \frac{-m_1}{K l_1} - m_1 \cdot 0}{-m_1 \cdot \frac{m_1}{K l_1}} = \frac{l_1}{m_1}.$$

The angle between $-2K, \frac{-K(2K - l_1)}{m_1}$ and $l_1 m_1$ is:

$$\tan\theta' = \frac{l_1 \cdot \frac{K(2K - l_1)}{m_1} - m_1 \cdot 2K}{-2K \cdot l_1 + m_1 \cdot \frac{-K(2K - l_1)}{m_1}} = \frac{l_1}{m_1}.$$

$\therefore \theta = \theta'$

<u>This proves the well known property of a parabolic reflector</u>.

It has been shown that the general equation of the sheaf of the second class,

$$al^2+2hlm+bm^2+2gl+2fm+c=0$$

when $c \neq 0$

can always be reduced to the simpler form:

$$(1) \qquad \frac{A}{C}l^2+\frac{B}{C}m^2=1$$

Case I. When A and B are both positive (1) may be written:

$$\underline{a^2l^2+b^2m^2=1}$$

where $a^2=\frac{A}{C}$, and $b^2=\frac{B}{C}$.

Case II. When A is positive and B negative (1) may be written:

$$\underline{a^2l^2-b^2m^2=1}$$

where $a^2=\frac{A}{C}$, and $-b^2=\frac{B}{C}$.

Case III. If $A=B$, equation (1) takes the form:

$$\underline{l^2+m^2=\frac{1}{r^2}}$$

where $\frac{1}{r^2}=\frac{C}{A}=\frac{C}{B}$.

These three will be considered the standard forms of the equation of the second degree when $c \neq 0$. By using these standard forms it will not prove difficult to determine some of the characteristic properties of the sheaf represented and according to these properties to name them.

The Elliptic Sheaf.

The equation

$$a^2 l^2 + b^2 m^2 = 1$$

is not satisfied for

$$l = m = 0$$

$\therefore$ the line at infinity is no part of the sheaf.

The point of contact of any line $l'm'$ is:

$$a^2 l l' + b^2 m m' = 1$$

Since this is not satisfied for

$$l' = m' = 0$$

no point of contact lies at infinity; or all points of contact lie in the finite part of the plane. Moreover the original equation is not satisfied for $l = m = \infty$

therefore no line of the sheaf passes thru the origin.

<u>Such a sheaf is called an elliptic sheaf.</u>

The pole of any line $l_1 m_1$ is given by:

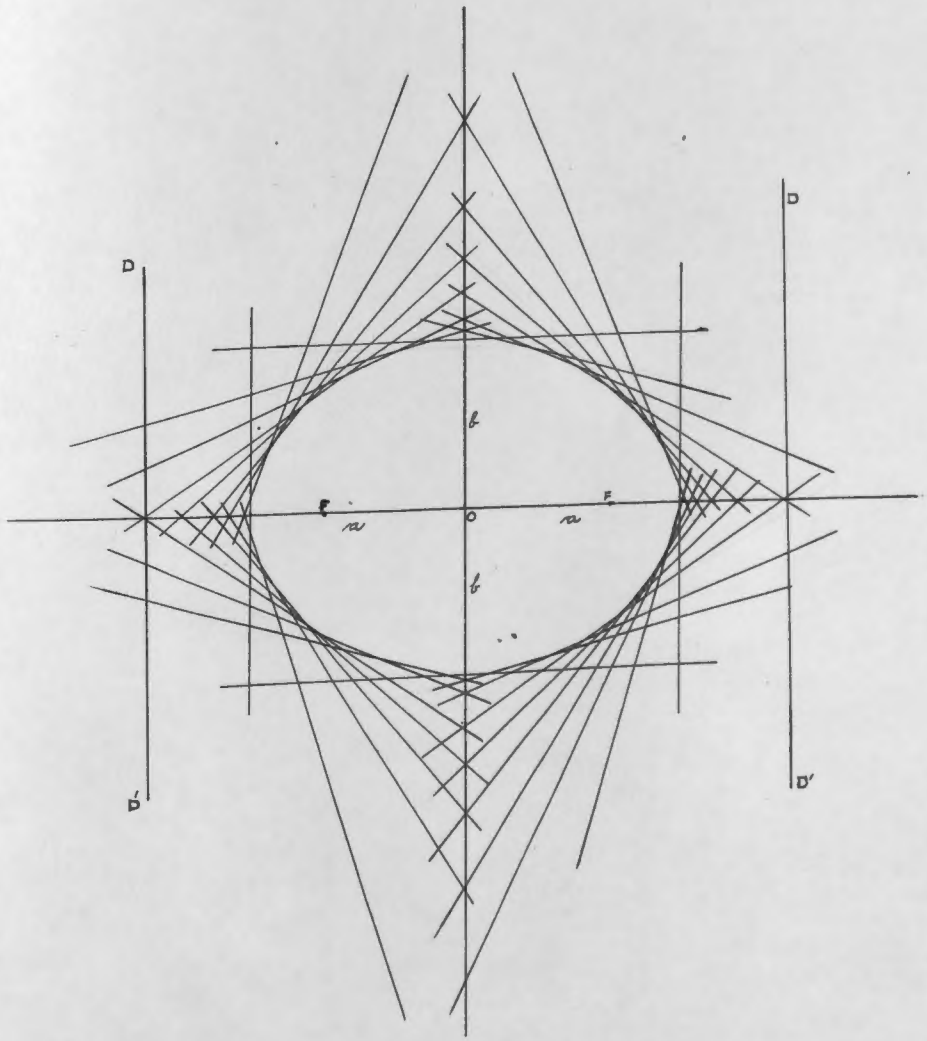


PLATE VII.

$$a^2 l l_1 + b^2 m m_1 = 1.$$

The pole of the line at infinity for which

$$l_1 = m_1 = 0$$

is then

$$a^2 l \cdot 0 + b^2 m \cdot 0 = 1$$

But this is the equation of the origin ∴ in accordance with a previous definition (viz. 'the pole of the line at infinity is the centre of the sheaf') <u>the origin is the centre of the elliptic sheaf.</u>

When $l = 0$, $m = \pm 1/b$

∴ there are two lines of the sheaf parallel to the L axis, at a distance b above and below the axis.

When $m = 0$, $l = \pm \frac{1}{a}$

∴ there are two lines of the sheaf parallel to the M axis, at a distance a to the right and left of the axis.

The points of contact of the lines $(\pm \frac{1}{a}, 0)$ are:

$$a^2 \cdot l \cdot \frac{\pm 1}{a} + b^2 m \cdot 0 = 1$$

$$\text{or } l = \pm \frac{1}{a}$$

$\therefore$ the points of contact of the lines parallel to the M-axis, lie on the L axis.

These two points are called the vertices of the elliptic sheaf. The distance between the two vertices is $2a$; this length is called the <u>major axis</u> of the elliptic sheaf.

Similarly the points of contact of the lines $(0, \pm \frac{1}{b})$ are

$$a^2 l \cdot 0 + b^2 m \cdot \pm \frac{1}{b} = 1$$

$$\text{or} \quad m = \pm \frac{1}{b}$$

$\therefore$ the points of contact of the lines of the sheaf parallel to the L axis lie on the M-axis. The distance between these two points is called the <u>minor axis</u> of the elliptic sheaf.

To find the equation of the focus of the elliptic sheaf.

Let the equation of the sheaf be:

$$a^2 l^2 + b^2 m^2 = 1.$$

Let the equation of the focus be

(1) $$\alpha l + \beta m + 1 = 0$$

where α and β are to be determined.

Let $l_1 m_1$ be any line through the point (1)

Its pole will be

(2) $$a^2 l l_1 + b^2 m m_1 = 1$$

The line through the points (1) and (2) will be conjugate to the line $l_1 m_1$; its coordinates will be found by solving (1) and (2) simultaneously giving:

$$l = \frac{-\beta - b^2 m_1}{\alpha b^2 m_1 - a^2 \beta l_1}, \quad m = \frac{\alpha + a^2 l_1}{\alpha b^2 m_1 - a^2 \beta l_1}.$$

The line $l_1 m_1$ and the line whose coordinates have just been determined will be at right angles to one another provided

$$l l_1 + m m_1 = 0$$

(3) or $$l_1(-\beta - b^2 m_1) + m_1(\alpha + a^2 l_1) = 0$$

Since the line $l_1 m_1$ passes thru the point (1), $\alpha l_1 + \beta m_1 + 1 = 0$

or $l_1 = \frac{-1-\beta m_1}{\alpha}$

Substituting this value of l_1 in equation (3)

$$\left(\frac{1+\beta m_1}{\alpha}\right)\left(\beta + b^2 m_1\right) + m_1\left(\alpha + a^2 \cdot \frac{-1-\beta m_1}{\alpha}\right) = 0$$

or

(4) $$m_1^2(\beta b^2 - \beta a^2) + m_1(\alpha^2 + \beta^2 - a^2 + b^2) + \beta = 0,$$

from which α and β are to be determined.

Now since the focus is such a point that <u>all</u> conjugate lines thru it are at right angles equation (4) must be true for <u>all</u> values of m_1, and must therefore be an identity

$\therefore$ (a) $\beta b^2 - \beta a^2 = \beta(b^2 - a^2) = 0$

(b) $\alpha^2 + \beta^2 - a^2 + b^2 = 0$

(c) $\beta = 0$

From (c) $\beta = 0$

and from (b) $\alpha^2 = a^2 - b^2$

or $\alpha = \pm\sqrt{a^2 - b^2}$

Substituting these values in the assumed equation of the focus gives as the true equation

$$\pm\sqrt{a^2 - b^2}\, l + 1 = 0$$

Therefore the elliptic sheaf has two foci both of which lie on the major axis at a distance $\sqrt{a^2-b^2}$ to the right and left of the centre.

[Note: As in the parabolic sheaf so here, if m_1 had been eliminated from equation (3), the result would have been an equation in l_1,

$$l_1^2(\alpha l_2 - a^2\alpha) + l_1(b^2-a^2-\alpha^2-\beta^2) - \alpha = 0$$

If this is to be an identity

$$\alpha = 0 \quad \text{and} \quad \beta = \pm\sqrt{b^2-a^2}$$

But from the equation in m_1,

$$\alpha = \pm\sqrt{a^2-b^2} \quad \text{and} \quad \beta = 0$$

Both are sufficient conditions but one or the other will always give imaginary points.

When $a > b$, the condition obtained from the equation in m_1 gives two real foci while the equation in l_1 then gives imaginary points. When $a < b$, the conditions are reversed. The conclusion must therefore be that there are two real foci and these always lie on the major axis while the imaginary ones lie on the minor axis.]

To show that the sum of the distances from any point of contact to the two foci is constant and equal to $2a$.

Let the equation of the sheaf be:

$$a^2 l^2 + b^2 m^2 = 1$$

The equation of the point of contact of the line $l'm'$ is:

$$a^2 l l' + b^2 m m' = 1$$

The coordinates of the point are:

$$-a^2 l', \; -b^2 m'$$

The coordinates of one focus are

$$\sqrt{a^2-b^2}, \; 0$$

Let the distance from the point of contact to this focus be d; then

$$d^2 = \left[\sqrt{a^2-b^2} + a^2 l_1\right]^2 + (-b^2 m_1)^2$$

$$= a^2 - b^2 + 2a^2 l_1 \sqrt{a^2-b^2} + a^4 l_1^2 + b^4 m_1^2$$

$$= a^2(1 + a^2 l_1^2) + b^2(b^2 m_1^2 - 1) + 2a^2 l_1 \sqrt{a^2-b^2}$$

$$= a^2(1 + a^2 l_1^2) - a^2 b^2 l_1^2 + 2a^2 l_1 \sqrt{a^2-b^2}$$

$$= a^2\left(1 + 2l_1\sqrt{a^2-b^2} + l_1^2(a^2-b^2)\right)$$

$$\therefore \; d = a\left(1 + l_1\sqrt{a^2-b^2}\right)$$

The coordinates of the other focus are:

$-\sqrt{a^2-b^2}, 0.$

Let the distance from the point of contact be d', then

$$d'^2 = (-\sqrt{a^2-b^2} + a^2 l_1)^2 + (-b^2 m_1)^2$$

$$= a^2 - b^2 - 2a^2 l_1 \sqrt{a^2-b^2} + a^4 l_1^2 + b^4 m_1^2$$

$$= a^2(1 - 2 l_1 \sqrt{a^2-b^2} + l_1^2 (a^2-b^2))$$

$$\therefore d' = a(1 - l_1 \sqrt{a^2-b^2})$$

Therefore $d + d' = 2a$ $\therefore$ constant.

This identifies the 'elliptic' sheaf with the totality of tangents to an ellipse; i.e. the locus of the contact points is an ellipse.

A normal is a line perpendicular to a line of the sheaf at its point of contact.

To show that the product of the l-intercept of any line of the sheaf and its normal is constant and equal to the difference of the squares of the semi-major and semi-minor axes.

Let $l'm'$ be the coordinates of any line of the sheaf: $a^2 l^2 + b^2 m^2 = 1$

Its point of contact will be:

(1) $$a^2 l l' + b^2 m m' = 1.$$

The coordinates of a line perpendicular to $l'm'$ must satisfy the condition

(2) $$l_1 l' + m_1 m' = 0$$

where $l_1 m_1$ are the coordinates of the line perpendicular to $l'm'$.

If the line $l_1 m_1$ is to pass thru the point of contact of $l'm'$, its coordinates must satisfy (1)

(3) $$\therefore \quad a^2 l_1 l' + b^2 m_1 m' = 1$$

Solving (2) for m_1 gives $m_1 = -\frac{l'}{m'} \cdot l_1$

Substituting this value in (3)

$$a^2 l_1 l' + b^2 m' \cdot -\frac{l_1 l'}{m'} = 1$$

or $$l_1 (a^2 l' - b^2 l') = 1$$

$$\therefore \quad l_1 = \frac{1}{l'(a^2 - b^2)}$$

therefore the l-intercept of the normal is $-l'(a^2 - b^2)$

and " " " " " line $l'm'$ is $\frac{-1}{l'}$

$$\therefore \quad \frac{-1}{l'} \cdot -l'(a^2 - b^2) = a^2 - b^2.$$

Q. E. D.

The equations of the foci of the elliptic sheaf have been shown to be:

(1) $$\pm\sqrt{a^2-b^2}\,l+1=0$$

The equation of the pole of any line $l_1 m_1$ is:

(2) $$a^2 l_1 l + b^2 m_1 m = 1$$

If (1) and (2) are to be the same point

$$a^2 l_1 \cdot \frac{-1}{\pm\sqrt{a^2-b^2}} + b^2 m_1 \cdot 0 = 1$$

$$\therefore \quad l_1 = \frac{\pm\sqrt{a^2-b^2}}{a^2}, \quad m_1 = 0.$$

These then are the coordinates of the polars of the foci; as in the parabolic sheaf they are called <u>directrices</u>. The elliptic sheaf has then two directrices as well as two foci.

<u>To find the ratio between the distance from a point of contact to the focus and to the directrix.</u>

Let $a^2 l l_1 + b^2 m m_1 = 1$ be a point of contact

If $\sqrt{a^2-b^2}\,l+1=0$ is the focus, then the coordinates of the corresponding directrix are:

$$\frac{-\sqrt{a^2-b^2}}{a^2}, \quad 0$$

Let the distance from the point of contact to the focus be d, then as has already been shown

$$d = a(1 + l_1\sqrt{a^2-b^2})$$

Let the distance from the point of contact to the directrix be d', then

$$d' = \frac{a^2 l_1 \cdot \frac{-\sqrt{a^2-b^2}}{a^2} + b^2 m_1 \cdot 0 + 1}{-\frac{\sqrt{a^2-b^2}}{a^2}}$$

$$= \frac{a^2(1 + l_1\sqrt{a^2-b^2})}{\sqrt{a^2-b^2}}$$

$$\therefore \quad \varepsilon = \frac{d}{d'} = \frac{\sqrt{a^2-b^2}}{a}.$$

ε is called the <u>eccentricity</u>; for the elliptic sheaf it is always less than unity.

<u>To show that any line of the sheaf makes equal angles with the focal radii drawn to its point of contact.</u>

Let $l'm'$ be any line of the elliptic sheaf

$$a^2l'^2 + b^2m'^2 = 1$$

and the equation of its point of contact

(1) $$a^2ll' + b^2mm'$$

The equations of the foci are:

(2) $$\pm\sqrt{a^2-b^2}\,l + 1 = 0.$$

The coordinates of any line through the point of contact and the focus can be found by solving (1) and (2) simultaneously, giving:

$$l = \frac{-1}{\pm\sqrt{a^2-b^2}} \qquad m = \frac{\pm\sqrt{a^2-b^2} + a^2 l'}{\pm b^2 m'\sqrt{a^2-b^2}}$$

Using the upper signs the angle θ, between this line and the line $l'm'$ is given by:

$$\tan\theta = \frac{l'm - lm'}{ll' + mm'}$$

$$= \frac{l' \cdot \frac{\sqrt{a^2-b^2} + a^2 l'}{b^2 m'\sqrt{a^2-b^2}} + m' \frac{1}{\sqrt{a^2-b^2}}}{\frac{-l'}{\sqrt{a^2-b^2}} + m' \cdot \frac{\sqrt{a^2-b^2} + a^2 l'}{b^2 m'\sqrt{a^2-b^2}}}$$

$$= \frac{l'\sqrt{a^2-b^2} + a^2 l'^2 + b^2 m'^2}{-b^2 l' m' + m'\sqrt{a^2-b^2} + a^2 l' m'}$$

$$= \frac{l'\sqrt{a^2-b^2} + 1}{m'\sqrt{a^2-b^2}\left(\sqrt{a^2-b^2}\,l' + 1\right)} = \frac{1}{m'\sqrt{a^2-b^2}}$$

Using the lower signs the angle θ' is given by:

$$\tan\theta' = \frac{\frac{m'}{\sqrt{a^2-b^2}} + l' \frac{\sqrt{a^2-b^2} - a^2 l'}{-b^2 m'\sqrt{a^2-b^2}}}{\frac{l'}{\sqrt{a^2-b^2}} + m' \frac{-\sqrt{a^2-b^2} + a^2 l'}{-b^2 m'\sqrt{a^2-b^2}}}$$

$$= \frac{1}{m'\sqrt{a^2-b^2}}$$

$$\therefore \quad \theta = \theta'$$

Or the line $l'm'$ makes equal angles with the focal radii thru its point of contact. Since

the normal is perpendicular to the line of the sheaf it also makes equal angles with the focal radii. Therefore a line of the sheaf and its normal bisect the angles between the focal radii drawn to the corresponding point of contact.

The Hyperbolic Sheaf.

The equation

$$a^2 l^2 - b^2 m^2 = 1$$

is not satisfied for

$$l = m = 0$$

$\therefore$ the line at infinity is not a line of the sheaf

When $l = \infty$, $m = \pm\infty$

$\therefore$ thru the origin pass two lines of the sheaf.

The point of contact of any line $l'm'$ of the sheaf is

$$a^2 l l' - b^2 m m' = 1$$

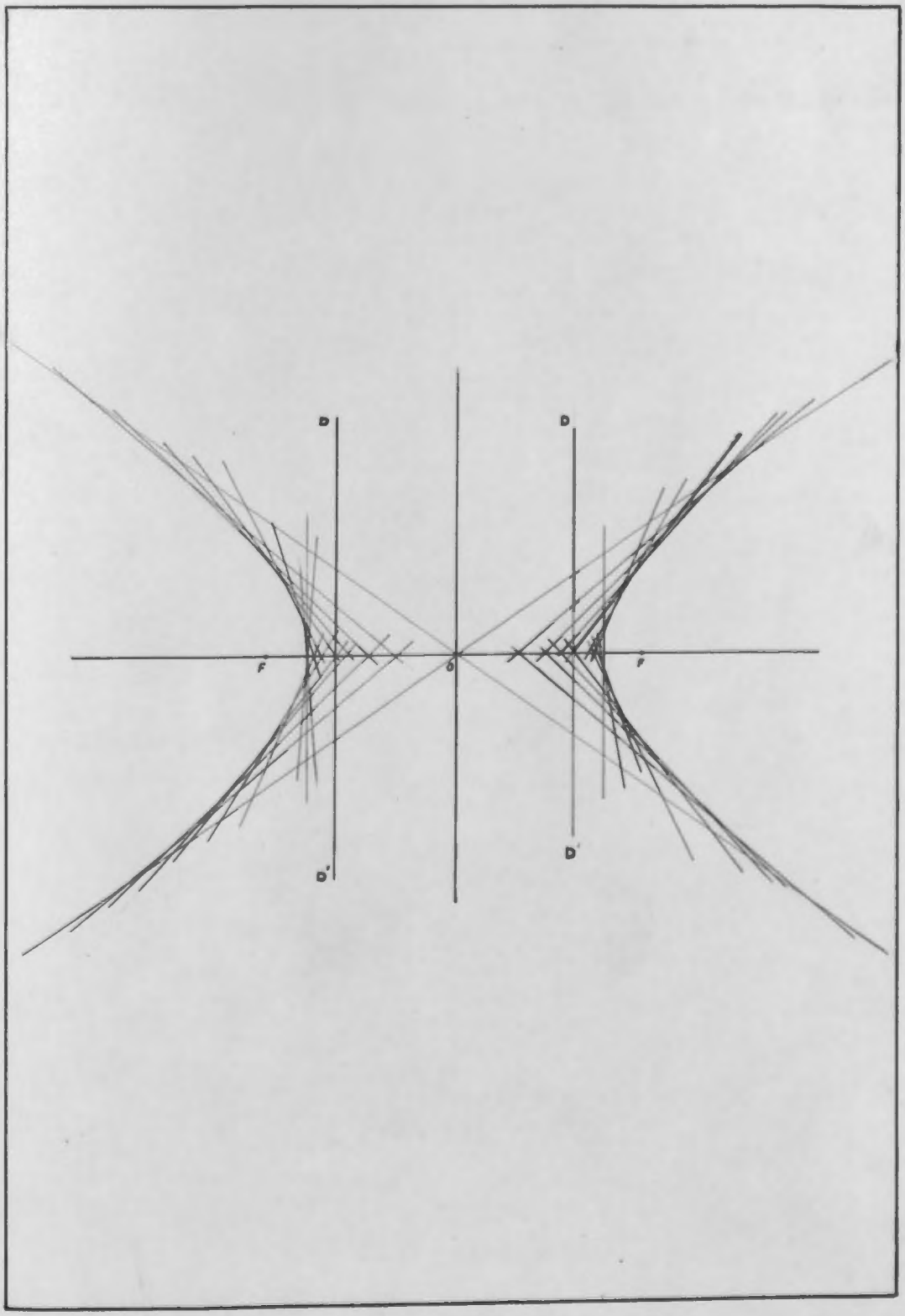


PLATE VIII

For $l' = m' = \infty$, i.e. a line through the origin this becomes

$$a^2 l \cdot \infty - b^2 m \cdot \infty = 1$$

which is true only for the point at infinity. Since there are two lines of the sheaf thru the origin there are two points of contact at infinity. These two lines which lie in the finite part of the plane but whose contact points lie at infinity are called <u>asymptotic lines</u>. The sheaf is called a <u>hyperbolic sheaf</u>.

The pole of any line $l_1 m_1$ is:

$$a^2 l l_1 - b^2 m m_1 = 1$$

For $l_1 = m_1 = 0$ (the line at infinity) this becomes

$$a^2 l \cdot 0 - b^2 m \cdot 0 = 1,$$

the equation of the origin. As in the elliptic sheaf the pole of the line at infinity is called the centre of the sheaf.

Going back to the original equation

$$a^2 l^2 - b^2 m^2 = 1$$

When $l = 0$, m is imaginary $\therefore$ there are no lines of the sheaf parallel to the L axis.

When $m = 0$, $l = \pm \frac{1}{a}$

$\therefore$ there are two lines of the sheaf parallel to the M-axis, at a distance $\underline{a}$ to the right and left of the axis.

The points of contact of the lines $(\pm \frac{1}{a}, 0)$ are:

$$a^2 l \cdot \pm \frac{1}{a} + b^2 m \cdot 0 = 1$$

$$\text{or } l = \pm \frac{1}{a}.$$

$\therefore$ the points of contact of the lines parallel to the M axis lie on the L axis. These two points are called the <u>vertices</u> of the sheaf. The distance between them is $2a$; this length is called the <u>transverse axis</u> of the hyperbolic sheaf. If B and B' be two points on the M axis such that $B'O = OB = b$, the line BB' is called the conjugate axis.

Since the only difference in the equation of the elliptic and hyperbolic sheaf is the sign of b^2, the results obtained there apply in general equally well here by a simple change of sign. When such is the case the result will be indicated without giving the details of the work.

By analogy from the elliptic sheaf, the equations of the foci of the hyperbolic sheaf are:

$$\pm\sqrt{a^2+b^2}\,l+1=0.$$

The hyperbolic sheaf has thus two foci both of which lie on the transverse axis.

To show that the difference of the distances from any point of contact to the two foci is constant and equal to $2a$.

Let the equation of the sheaf be:

$$a^2l^2-b^2m^2=1$$

The equation of the point of contact of any line $l'm'$ is:

$$a^2ll'-b^2mm'=1$$

The equations of the foci are:

(1) $$\sqrt{a^2+b^2}\,l+1=0$$

(2) $$\text{and } -\sqrt{a^2+b^2}\,l+1=0$$

Let the distance from the point of contact to the focus given by (1) be d; then

$$d^2=\left[\sqrt{a^2+b^2}+a^2l'\right]^2+(b^2m')^2$$

$$=a^2+b^2+2a^2l'\sqrt{a^2+b^2}+a^4l'^2+b^4m'^2$$

$$d^2 = a^2(1+a^2l_1^2) + b^2(1+b^2m_1^2) + 2a^2l_1\sqrt{a^2+b^2}$$

$$= a^2(1+a^2l_1^2) + a^2b^2l_1^2 + 2a^2l_1\sqrt{a^2+b^2}$$

$$= a^2\left[1+2l_1\sqrt{a^2+b^2} + l_1^2(a^2+b^2)\right]$$

$$\therefore\ d = a\left(1+l_1\sqrt{a^2+b^2}\right)$$

Let the distance from the point of contact to the focus given by (2) be d'; then

$$d'^2 = \left[-\sqrt{a^2+b^2} + a^2l_1\right]^2 + (b^2m_1)^2$$

$$= a^2\left[l_1^2(a^2+b^2) - 2l_1\sqrt{a^2+b^2} + 1\right]$$

$$\therefore\ d' = a\left(l_1\sqrt{a^2+b^2} - 1\right)$$

Therefore

$$d - d' = 2a \quad \therefore \text{ constant.}$$

This identifies the hyperbolic sheaf with the totality of tangents to a hyperbola; i.e. the locus of the contact points is a hyperbola.

In a manner exactly analogous to that used for the elliptic sheaf it can be shown that the product of the l-intercepts of any line of the sheaf and of its normal, is constant and equal to the sum of the squares of the semi-transverse and semi-conjugate axes.

Changing the sign of b^2 in the coordinates of the directrices of the elliptic sheaf gives:

$$\pm \frac{\sqrt{a^2+b^2}}{a^2}, 0$$

as the coordinates of the directrices of the hyperbolic sheaf.

As in the elliptic sheaf, it can be shown that the ratio (ε) between the distance from any point of contact to the focus and to the directrix is constant; and also that:

$$\varepsilon = \frac{\sqrt{a^2+b^2}}{a}.$$

For the hyperbolic sheaf this ratio is always greater than unity.

By a method analogous to that used for the elliptic sheaf it can be shown that any line of the sheaf makes equal angles with the focal radii drawn to its point of contact.

A direct result of this property of the elliptic and hyperbolic sheaf is the following:

If an elliptic and hyperbolic sheaf have

the same foci, the line and the normal to one of the sheaves at any one of their four common contact points, are respectively the normal and the line to the other. That is: the two sheaves intersect orthogonally.

The Circular Sheaf.

The equation

$$l^2 + m^2 = \frac{1}{r^2}$$

is a special case of the equation of the elliptic sheaf where $a = b = r$. The properties given for the elliptic sheaf will therefore be true for this special form also; viz.

The line at infinity is not a line of the sheaf.

All points of contact lie in the finite part of the plane.

There are two lines of the sheaf parallel to the L axis at a distance r above and below the axis.

There are two lines of the sheaf parallel to the M axis, also at a distance r to the right and left of the axis.

The distance from any line $(l_1 m_1)$ of the sheaf, $\quad l^2 + m^2 = \frac{1}{r^2}$

to the origin is:

$$\frac{0 \cdot l_1 + 0 \cdot m_1 + 1}{\sqrt{l_1^2 + m_1^2}} = d$$

$\therefore \quad d = \pm r$, a constant and therefore independent of $l_1 m_1$ $\therefore$ the distance from the origin to any line of the sheaf is constant and equal to r.

The distance from any point of contact

$$l l' + m m' = \frac{1}{r^2}$$

or $\quad r^2 l l' + r^2 m m' - 1 = 0$

to the origin is given by:

$$(-r^2 l')^2 + (-r^2 m')^2 = d^2$$

$$\therefore \quad d^2 = r^4 (l'^2 + m'^2) = r^2$$

$$\therefore \quad d = \pm r.$$

Therefore the distance from any point of contact to the origin is also r.

Such a sheaf is called a <u>circular sheaf</u>. The property just proved <u>identifies the circular sheaf with the totality of tangents to a circle</u>, i.e. <u>the locus of the contact points is a circle</u>.

In the elliptic sheaf the coordinates of the focus are $(\pm\sqrt{a^2-b^2}, 0)$. Since for the circular sheaf $a = b$, these coordinates become $(0,0)$, and the focus coincides with the origin or the centre of the sheaf.

A means of determining from the equation the nature of a sheaf of the second class without transforming the equation to a standard form.

Let the general equation of the sheaf of the second class be:

$$al^2+2hlm+bm^2+2gl+2fm+c=0.$$

The pole of the line

$$l'=m'=0 \quad \text{(i.e. the line at } \infty\text{)}$$

is:

$$gl+fm+c=0$$

which by previous definition is the centre of the sheaf.

Let $l_1 m_1$ be any line of the sheaf

If this line is incident with the centre of the sheaf

$$gl_1+fm_1+c=0$$

$$\text{or} \quad l_1=\frac{-c-fm_1}{g}$$

Substituting this value in the original equation:

$$a\frac{(c+fm_1)^2}{g^2}+2hm_1\frac{-c-fm_1}{g}+bm_1^2+2(-c-fm_1)+2fm_1+c=0$$

or

$$m_1^2(af^2-2fgh+bg^2)+2m_1(acf-ghc)+ac^2-cg^2=0$$

Solving for m_1

$$m_1=\frac{acf-ghc}{af^2-2fgh+bg^2}\pm\frac{g}{af^2-2fgh+bg^2}\cdot\sqrt{c(ch^2-abc-2fgh+bg^2+af^2)}$$

Let $c(ch^2-abc-2fgh+bg^2+af^2)=c\begin{vmatrix} g & f & c \\ h & b & f \\ a & h & g \end{vmatrix}=c\Delta.$

Now if $c\Delta$ is positive there will be two lines of the sheaf thru the centre $\therefore$ a hyperbolic sheaf.

If $c\Delta$ is negative m_1 is imaginary $\therefore$ no lines of the sheaf pass thru the centre, and consequently the sheaf is elliptic.

If $c\Delta=0$, the two values of m_1 are equal. However $c\Delta=0$ includes two possibilities

Either $c=0$ and then the sheaf is parabolic;

Or $\Delta=0$.

i.e. $ch^2-abc-2fgh+bg^2+af^2=0$

(1) or $(h^2-ab)(g^2-ac)=(hg-af)^2$.

The meaning of this is most easily seen by

solving the original equation for l. This gives:

$$(2) \qquad l = \frac{hm+g}{a} \pm \frac{1}{a}\sqrt{(h^2-ab)m^2+a(hg-af)m+g^2-ac}$$

Now (1) is the condition that the quantity under the radical in (2) shall be a perfect square

Therefore when (1) is true

$$l = \frac{hm+g}{a} \pm \frac{\sqrt{h^2-ab}\,m+\sqrt{g^2-ac}}{a}$$

or $\quad al - (h \pm \sqrt{h^2-ab})m - (g \pm \sqrt{g^2-ac}) = 0$

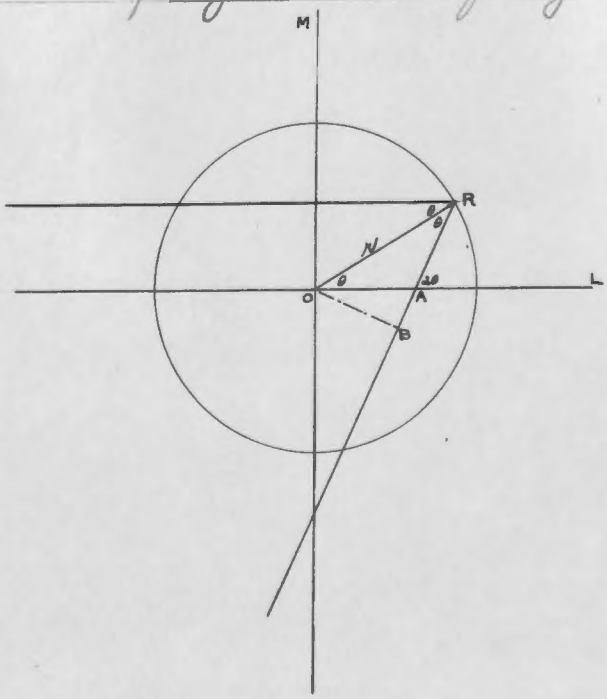
That is to say: the equation of the second degree has degenerated into two equations of the first degree, each of which represents a point. Therefore:

If $\Delta = 0$, the sheaf of the second class degenerates into two sheafs of the first class or two pencils of lines.

Though it is beyond the scope of this paper to go into the theory of sheafs of higher class than the second, it will be interesting to carry out in detail a few problems which involve equations of higher degree than the second, as well as a few others for the solution of which line coordinates are particularly adapted.

The Caustic, formed by the reflection of rays of light upon a circular mirror.

First: To find the equation of the caustic when the source of light is at infinity.



Let the radius of the circle be r

and RA, a reflected ray

θ, the angle of reflection

From the figure:

$$\sin\theta = \frac{OB}{r} = \frac{1}{r\sqrt{l^2+m^2}}$$

where l, m are the coordinates of the reflected ray.

Also: $OA = \frac{1}{2} r \cdot \sec\theta$

$$\text{or} \quad -\frac{1}{l} = \frac{r}{2\cos\theta} = \frac{r}{2\sqrt{1-\sin^2\theta}}$$

$$\therefore \quad l = -\frac{2}{r}\sqrt{1-\sin^2\theta}$$

Substituting the value of $\sin\theta$, found above:

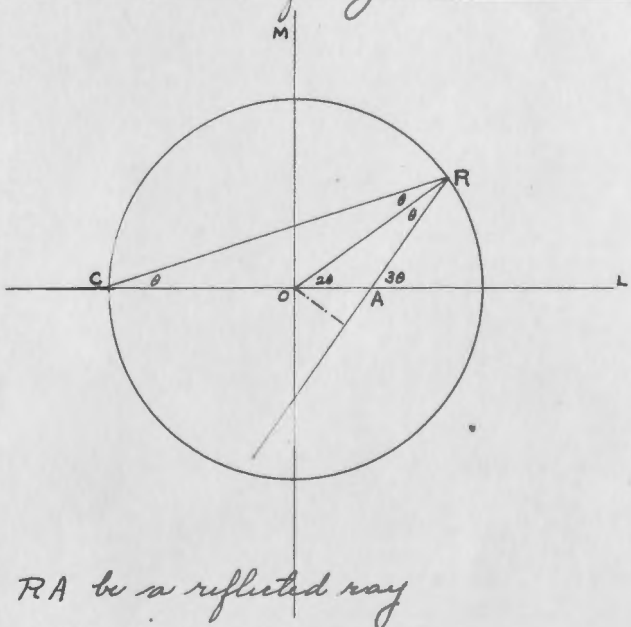
$$l = -\frac{2}{r}\sqrt{\frac{r^2(l^2+m^2)-1}{r^2(l^2+m^2)}}$$

$$\therefore \quad l^2 r^4 (l^2+m^2) = 4r^2(l^2+m^2) - 4$$

$$\text{or} \quad r^2(l^2+m^2)(l^2r^2-4) + 4 = 0.$$

The envelope (i.e. the locus of contact points) is found by methods of calculus to be an epicycloid which is of the sixth degree. This shows that it is of the fourth class.

Second: To find the equation of the caustic when the source of light is on the circumference.



Let RA be a reflected ray

θ, the angle of reflection

C, the source of light

From the figure

$\angle CRO = \theta$

$\therefore \angle RAL = 3\theta$

In the triangle ORA

$$\frac{OA}{OR} = \frac{\sin\theta}{\sin(180-3\theta)}$$

(1) or $-kr\sin\theta = \sin 3\theta$

But $\sin\theta = \dfrac{1}{r\sqrt{l^2+m^2}}$

and

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta = \sin\theta(3-4\sin^2\theta)$$

Substituting these values in (1)

$$lr\sin\theta + \sin\theta(3 - 4\sin^2\theta) = 0$$

or $$lr + (3 - 4\sin^2\theta) = 0$$

$$\therefore \quad lr + 3 - 4\cdot\frac{1}{r^2(l^2+m^2)} = 0$$

$$lr^3(l^2+m^2) + 3r^2(l^2+m^2) - 4 = 0$$

$$r^2(lr+3)(l^2+m^2) - 4 = 0.$$

By the methods of calculus the envelope (i.e. the locus of contact points) has been shown to be a cardioid which is of the fourth degree.

This shows that the curve is of the third class.

When the source of light is at a finite distance, greater or less than r, the equation of the caustic is not so simply obtained and inasmuch as an exhaustive treatment of this phase of the subject would take considerable time, the development of the equations will not be taken up, but some drawings will be given to show the different forms of the caustic as the source of light varies from infinity to zero.

The Caustic

In the drawings that follow only the reflected rays are drawn so as not to confuse the lines of the caustic with the incident rays.

Five cases have been considered:

I. When the source of light is at infinity Plate IX.

II. When the source of light is at a distance greater than r. Plate X.

III. When the source of light is on the circumference. Plate XI

IV. When the source of light is at a distance equal to r/2. Plate XII.

V. When the source of light is at a distance less than r/2. Plate XIII.

In cases I and II only a semicircular mirror is used.

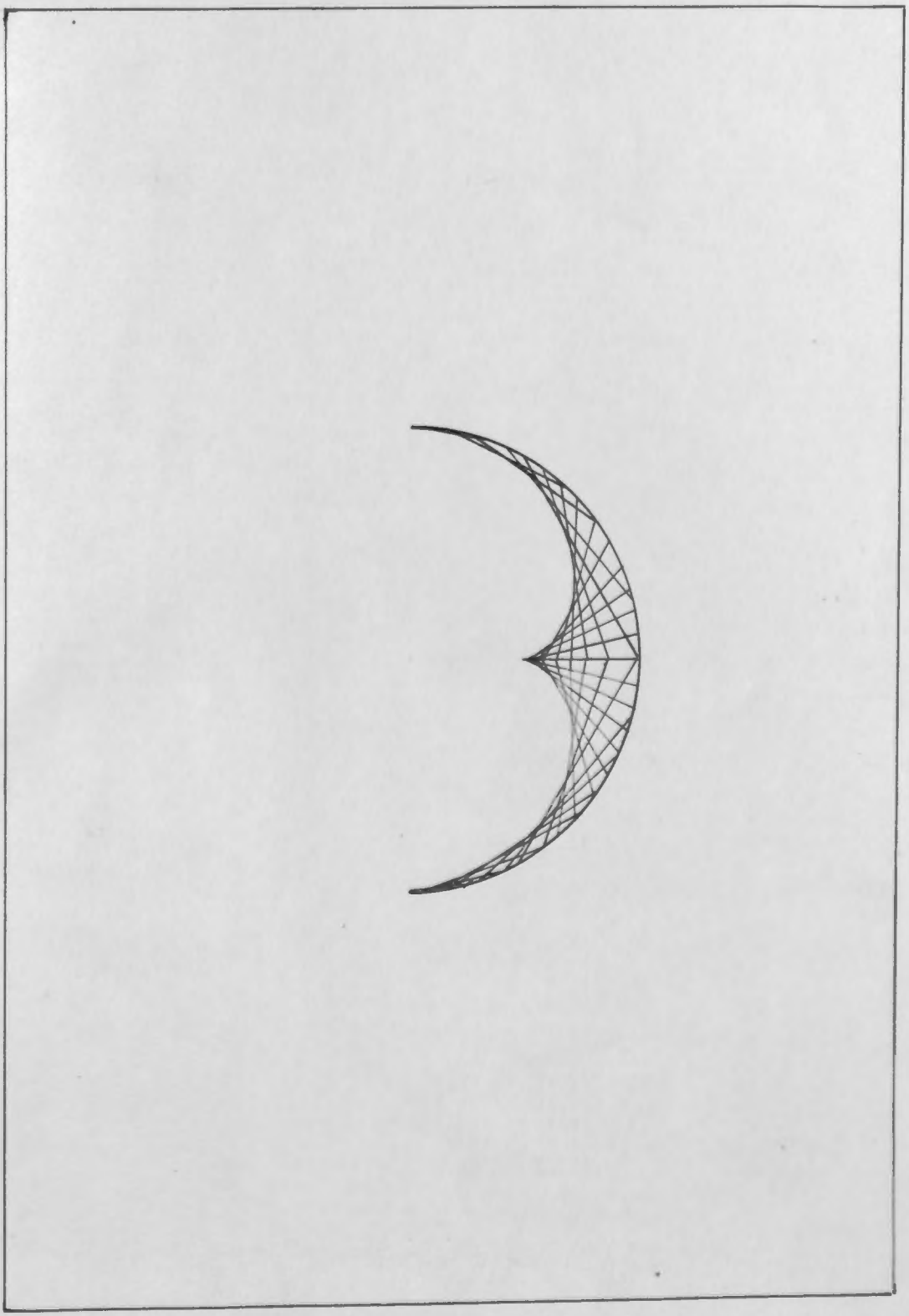


PLATE IX

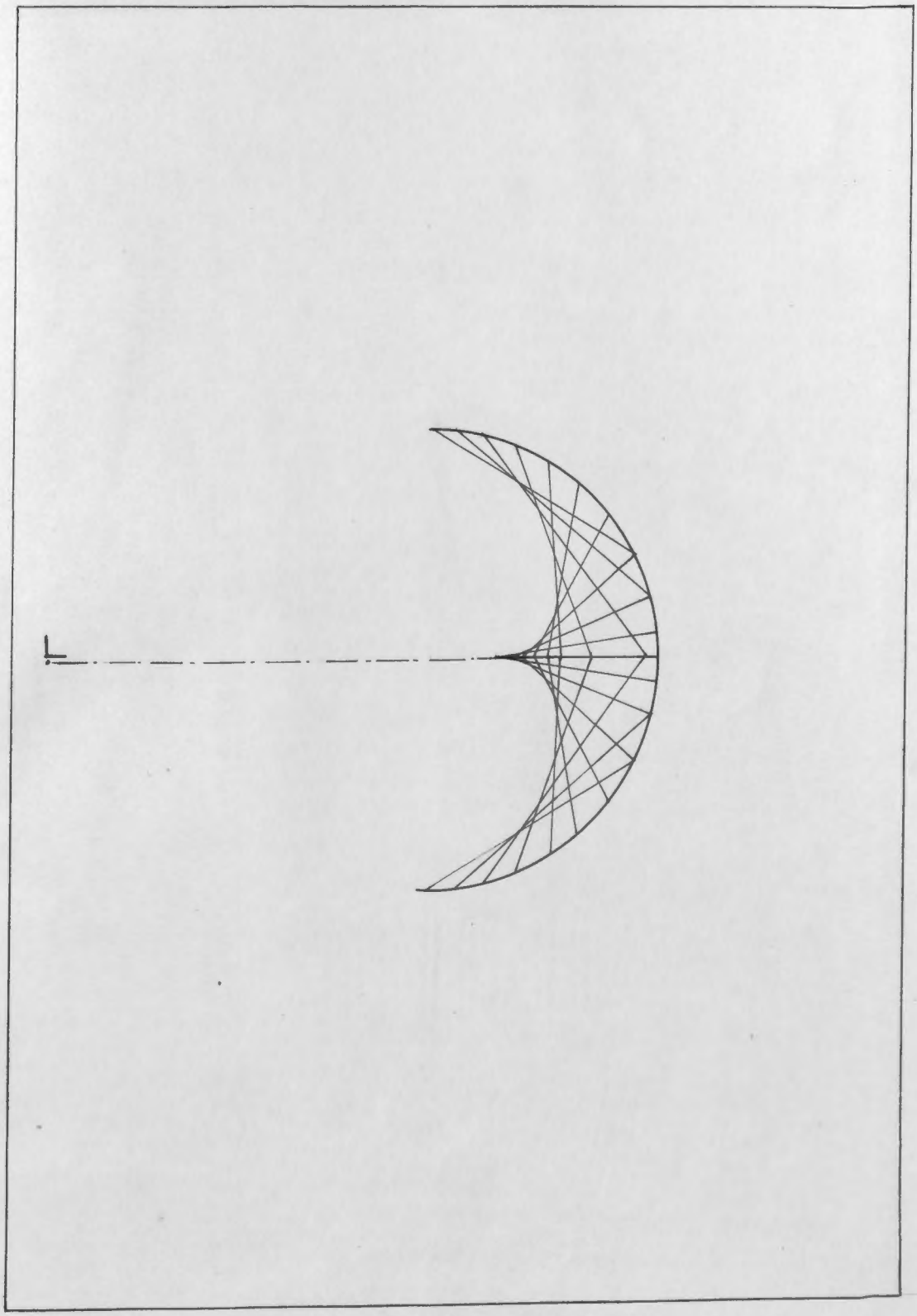


PLATE X

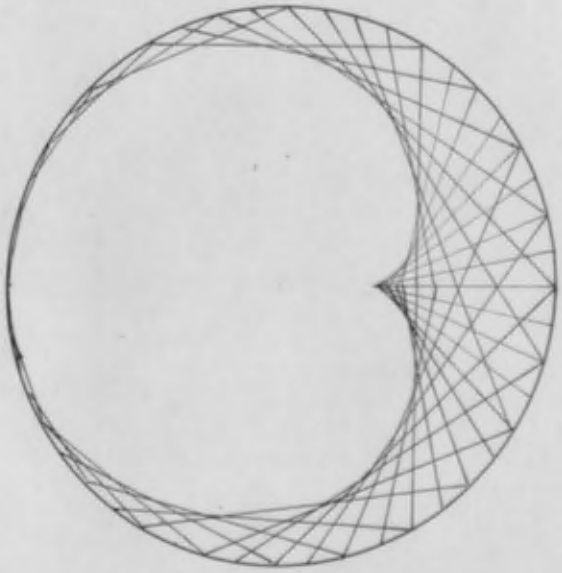


PLATE XI

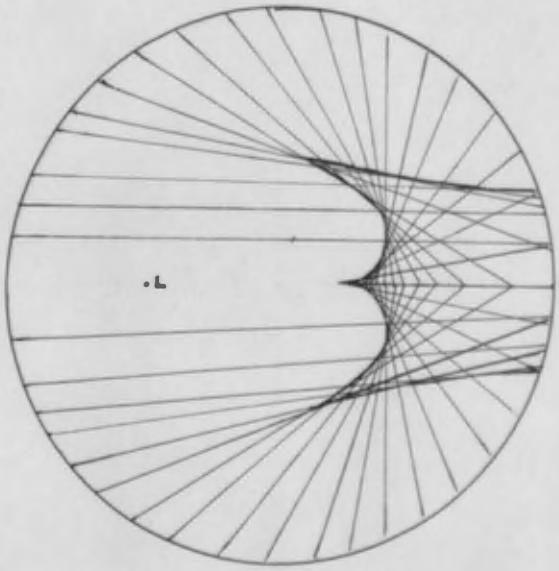


PLATE XII

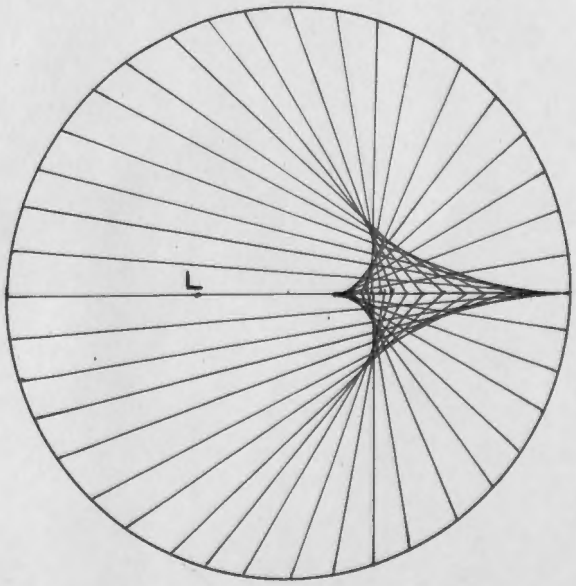


PLATE XIII.

A few simpler problems will illustrate the applicability of line coordinates as well or better than would the more complicated equations of the caustic.

I. Find the equation of a sheaf such that the sum or difference of the intercepts of the lines is constant ($= a$).

Since $-\frac{1}{l}, -\frac{1}{m}$ are the intercepts of a line whose coordinates are l, m, the stated condition gives directly

$$\mp \frac{1}{l} - \frac{1}{m} = a$$

or $$l \pm m + a\,lm = 0$$

Since this is an equation of the second degree with the absolute term missing it is known at once to be a parabolic sheaf.

II. Find the equation of the sheaf such that the product of the intercepts is constant ($= a^2$).

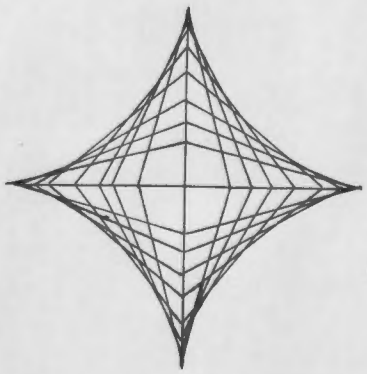
$$-\frac{1}{l} \cdot -\frac{1}{m} = a^2$$

or $$a^2 lm = 1$$

$$\therefore lm = \frac{1}{a^2}$$

which since it is an equation of the second degree and $c\Delta > 0$ is the equation of a hyperbolic sheaf.

III. Find the equation of the sheaf such that the part of the line intercepted by the axes is constant and equal to K.



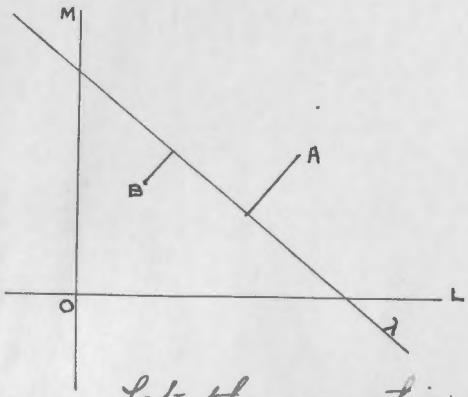
Since the part of the line intercepted by the coordinate axes is equal to the square root of the sum of the squares of the intercepts

$$\frac{1}{l^2} + \frac{1}{m^2} = K^2$$

or $$l^2 + m^2 - K^2 l^2 m^2 = 0.$$

which is the equation of a hypocycloid of which the condition given above is a well known property.

IV. Find the equation of the sheaf such that the sum of the squares of the distances from two fixed points is constant $(= K^2)$.



Let A, B, be the fixed points and λ the line whose coordinates are $l_1 m_1$.

Let the equations of A and B respectively be:

$$pl + qm + 1 = 0$$
$$rl + sm + 1 = 0$$

The perpendicular distances from the points A and B to the line λ are:

$$\frac{pl_1 + qm_1 + 1}{\sqrt{l_1^2 + m_1^2}}, \quad \frac{rl_1 + sm_1 + 1}{\sqrt{l_1^2 + m_1^2}}$$

Squaring and adding:

$$\frac{(pl_1 + qm_1 + 1)^2}{l_1^2 + m_1^2} + \frac{(rl_1 + sm_1 + 1)^2}{l_1^2 + m_1^2} = K^2$$

$$\therefore (p^2 + r^2 - K^2) l_1^2 + (q^2 + s^2 - K^2) m_1^2 + 2(pq + rs) l_1 m_1 + 2(p + r) l_1 + 2(q + s) m_1 + 2 = 0$$

which is the equation of either an elliptic or hyperbolic sheaf - depending on the values of the coefficients of l_1^2 and m_1^2.

V Find the equation of the sheaf such that the difference of the squares of the perpendiculars from two fixed points is constant ($=K^2$)

As in the previous problem let the squares of the distances be

$$\frac{(pl_1+qm_1+1)^2}{l_1^2+m_1^2}, \text{ and } \frac{(rl_1+sm_1+1)^2}{l_1^2+m_1^2}$$

Subtracting gives

$$(p^2+r^2-K^2)l_1^2+(q^2+s^2-K^2)m_1^2+2(pq+rs)l_1m_1+2(p+r)l_1+2(q+s)m_1=0$$

Since this contains no absolute term it is the equation of a parabolic sheaf

VI Find the equation of a sheaf such that the product of the distances from two fixed points is constant ($=K^2$)

As before let the perpendicular distances be

$$\frac{pl_1+qm_1+1}{\sqrt{l_1^2+m_1^2}}, \quad \frac{rl_1+sm_1+1}{\sqrt{l_1^2+m_1^2}}$$

Then their product is:

$$(pr-K^2)l_1^2+(qs-K^2)m_1^2+(ps+qr)l_1m_1+(r+p)l_1+(q+s)m_1+1=0$$

which is an elliptic or hyperbolic sheaf depending upon the coefficients of l^2 and m^2

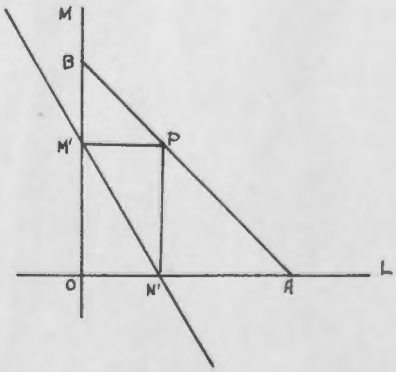
VII. Find the equation of a sheaf such that the sum of one intercept and K times the other is constant (= c).

$$-\frac{1}{l} + K \cdot -\frac{1}{m} = c$$

or $Kl + m + clm = 0$

a parabolic sheaf.

VIII. From any point in the hypotenuse of a right triangle perpendiculars PM' and PN' are drawn to the legs. Find the equation of the sheaf made up of the lines M'N'.



Let the sides of the right triangle be the coordinate axes.

Let OA = a, OB = b

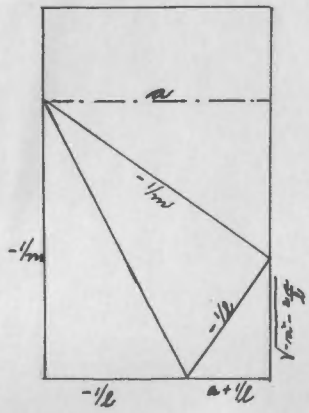
From similar triangles:

$$\frac{a}{-1/l} = \frac{b}{b + 1/m}$$

$\therefore \quad ablm + al + bm = 0$

a parabolic sheaf.

IX. A leaf of a book is folded so that one corner moves along an opposite side. Find the equation of the sheaf of lines determined by the crease.



From similar triangles:

$$\frac{a}{-1/m} = \frac{\sqrt{-a^2-\frac{2a}{l}}}{-1/l}$$

or $am = l\sqrt{-a^2-\frac{2a}{l}}$

$$a^2m^2 = l^2\left(-a^2-\frac{2a}{l}\right)$$

$$\therefore\ a^2m^2+a^2l^2+2al=0.$$

a parabolic sheaf.