ON BOUNDARY REGULARITY OF VORTEX PATCHES FOR 3D INCOMPRESSIBLE EULER SYSTEMS

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ON BOUNDARY REGULARITY OF VORTEX PATCHES FOR 3D INCOMPRESSIBLE EULER SYSTEMS

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Abstract. The vorticity equations of 3D incompressible inviscid flows with discontinuous initial vorticities $\boldsymbol{\omega}_0 = \boldsymbol{\varpi}_0 \chi_{\bar{\Omega}_0}$, where $\boldsymbol{\varpi}_0 \in C^\alpha$ and $\partial\Omega_0 \in C^{1+\alpha}$, are considered under formulations as integro-differential equations over time-dependent domains. We show that the solutions $\boldsymbol{\omega}$ will remain as $C^\alpha(\Omega_t)$ vortex patches for $0 < t < T$ for some $T > 0$, i.e., $\boldsymbol{\omega} = \boldsymbol{\varpi}\chi_{\bar{\Omega}_t}$ with $\boldsymbol{\varpi} \in C^\alpha$ and $\partial\Omega_t = \partial\boldsymbol{\Phi}(\Omega_0, t) \in C^{1+\alpha}$, where $\boldsymbol{\Phi} \in C^{1+\alpha}(\Omega_t)$ is the particle trajectories. Furthermore, we show that C^α singularities of $\boldsymbol{\varpi}$ in $\bar{\Omega}_t$ or $C^{1+\alpha}$ singularities of $\partial\Omega_t$ could occur at $t = T$ iff for some $0 < \beta \leq \alpha$, $\int_0^T \left(|\boldsymbol{\omega}(t)|_0 + \min\left(|\varphi(t)|_0, |\boldsymbol{\omega}(t)|_\beta \right) \right) dt = \infty$, where φ are related to Ω_t by $\Omega_t = \{\varphi < 0\}$. Consequently, we establish global existence for axisymmetric flows if $\alpha > 1/3$, without the assumption $|\boldsymbol{\omega}_0| \leq cr$ as required in [12], [13].

Key words. Incompressible flow, Euler system, vorticity equation, vortex patches, particle patches, Newtonian potentials, axisymmetric flows.

AMS subject classification. 35L45, 35Q35, 76

1. Introduction. In this paper we are concerned with the vortex motion of an inviscid incompressible flow in R^3. The vorticity equation is given by

$$\boldsymbol{\omega}_t + \mathbf{u} \cdot \nabla\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla\mathbf{u}, \tag{1.1}$$
$$\boldsymbol{\omega}(\mathbf{x}, 0) = \boldsymbol{\omega}_0(\mathbf{x}),$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{x}, t)$ is the vorticity field, $\mathbf{u}$ is the fluid velocity field that is solenoidal and satisfies $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. The initial vorticity $\boldsymbol{\omega}_0$ is assumed to be a solenoidal C^α 'patch', i.e., $\boldsymbol{\omega}_0(\mathbf{x}) = \boldsymbol{\varpi}_0(\mathbf{x})\chi_{\bar{\Omega}_0}$, where $\boldsymbol{\varpi}_0$ is a C^α vector-value function for some $0 < \alpha < 1$, Ω_0 is a bounded $C^{1+\alpha}$ domain, and $\chi_{\bar{\Omega}_0}$ is the characteristic function of $\bar{\Omega}_0$, the closure of Ω_0.

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It is known [9] [15] that the initial value problem (1.1) admits a unique weak solution with stratified regularity for a short time. In our particular setting, the initial vorticity is C^{α} continuous up to the boundary $\partial\Omega_0$ from inside the domain Ω_0. Since the vorticity is material, one sees that the jump discontinuity initiated along $\partial\Omega_0$ will propagate along with the fluid particle path $\mathbf{\Phi}(x,t)$. For any $t>0$, the vorticity $\boldsymbol{\omega}$ is only supported in the closure of $\Omega_t = \mathbf{\Phi}(\Omega_0,t)$. It is interesting to know whether the vorticity will still be C^{α} continuous in $\bar{\Omega}_t$ and $\partial\Omega_t \in C^{1+\alpha}$ for $t>0$. We call a weak solution $\boldsymbol{\omega}$ of (1.1) a C^{α} (or smooth) vortex patch solution if $\boldsymbol{\omega} \in \mathbf{C}^{\alpha}\left(\bar{\Omega}_t\right)$ and $\partial\Omega_t \in C^{1+\alpha}$, i.e., $\boldsymbol{\omega}$ has the form $\boldsymbol{\omega}(\mathbf{x},t) = \boldsymbol{\varpi}(\mathbf{x},t)\chi_{\bar{\Omega}_t}$ with $\boldsymbol{\varpi} \in C^{\alpha}(R^3)$.

The problem on regularity for vortex patches was first proposed by Majda in [12]. A classical example of smooth vortex patches called Hill's stationary spherical vortex patch can be found in [2], [14]. The regularity problem was attacked by many authors in literatures. In the case that flows are planar and vorticities are constant, Chemin [6] and Bertozzi and Constantin [4] proved $C^{1+\alpha}$ regularity of Ω_t for $t>0$. In our previous paper [11], we established global existence for C^{α} vortex patches in 2D for any non-constant initial vorticity. For 3D flows, due to the vorticity stretching, the situation becomes much more complex. In [9] the authors showed existence of $C^{\alpha,\partial\Omega_t}$ solutions (for the definition, see [9, Def. 1.1]) for a small time interval. Note that $\boldsymbol{\omega} \in C^{\alpha,\partial\Omega_t}$ only implies that $\boldsymbol{\omega}$ is of C^{α} away from $\partial\Omega_t$. See [15] for some similar results. For smooth initial data $\boldsymbol{\omega}_0 \in C^{\alpha}(R^3)$, local existence was showed in [1] and [5].

There are also a few results on global existence for 3D problems. In the case that flows have axisymmetric structures, it has been showed in [13] that there exists a global $C^{\alpha,\partial\Omega_t}$ solution if the initial vorticity $\boldsymbol{\omega}_0$ (with x_3 as symmetric axis, for instance) satisfies the condition

$$|\omega_0| \leq cr, \tag{1.2}$$

where c is a constant, r is the distance to the symmetric axis. This assumption was introduced in [12] for $C_0^{\infty}(R^3)$ initial data. Notice that the condition (1.2) in general fails if $\boldsymbol{\omega}_0 \in C^\alpha$.

Some numerical evidences, however, suggest that vorticities could develop singularities in finite time [7],[12], even though they are initially smooth everywhere and supported in compact domains. In our particular problem, there are two kinds of singularities that could possibly be formed: $C^\alpha\left(\bar{\Omega}_t\right)$ singularities for ϖ and $C^{1+\alpha}$ singularities for Ω_t. That leads to the question of which quantities are essentially responsible for this breakdown phenomenon. In the case that initial vorticities $\omega_0 \in C^\alpha\left(R^3\right)$, the authors in [1] showed that any $C^\alpha\left(R^3\right)$ solutions will not blow up at $t = T$ as long as the vorticity strength does not accumulate fast enough, i.e.,

$$\int_0^T |\boldsymbol{\omega}(\cdot,t)|_0 \, dt < \infty. \tag{1.3}$$

This result was established for $C_0^\infty\left(R^3\right)$ initial data earlier in [3]. In [7], the author proposed the following condition

$$\int_0^T \left(\frac{|\boldsymbol{\omega}(\cdot,t)|_\beta}{\|\mathbf{u}(t)\|_{L^2}}\right)^{\frac{5}{2\beta+5}} dt < \infty, \; 0 < \beta \le 1 \tag{1.4}$$

that guarantees extension of smooth solutions beyond $t = T$.

The main purposes of the present paper are to study existence of a C^α vortex patch solution for the vorticity equation (1.1) with initial data being a C^α patch, and to establish conditions analogous to (1.3) and (1.4) that will guarantee required smoothness of solutions up to $t = T$.

We shall show that there exists a unique C^α vortex patch solution $\boldsymbol{\omega}(\mathbf{x},t)$ for $t < T$ for some $T > 0$. This regularity result enables us to establish some conditions that ensure large time existence. We shall show that the solution will not be singular at $t = T$ provided that for some $0 < \beta \le \alpha$, $\theta(t)$ defined by

$$\theta(t) = \int_0^t \left(|\omega(s)|_{0,\Omega_s} + \min\left(|\nabla\varphi(s)|_{0,\partial\Omega_s}, |\omega(s)|_{\beta,\Omega_s}\right)\right) ds \tag{1.5}$$

is bounded for $t < T$, where $\varphi(\mathbf{x},t)$ is related to Ω_t by $\Omega_t = \{\mathbf{x} : \varphi(\mathbf{x},t) < 0\}$ (see the next section for further explanation of the notations). It will also be proved that in that case the solution can be extended to $t < T + T_0$ for some $T_0 > 0$ depending only on $\theta(T)$. Note that in (1.5), β can be arbitrarily small. In particular, we observe from (1.5) that the quantity $\|\omega(t)\|_{\beta,\Omega_t}$ alone for any small β could control the breakdown of vorticities. In the case of axisymmetry, we shall show that if $\alpha > 1/3$, then for any $T > 0$, there exists $\beta(T) > 0$ such that $\|\omega(t)\|_{\beta,\Omega_t}$ is bounded for $t < T$. Therefore, the above obversation leads to global existence of C^α vortex patches for axisymmetric flows for $\alpha > 1/3$, without the assumption (1.2).

We point out that unlike the problem (1.1) with $C^\alpha(R^3)$ initial data, where singularities occur only when the vorticity becomes singular, for the vortex patch problem, singularities could be caused also by lack of $C^{1+\alpha}$ smoothness of the domain $\bar{\Omega}_t$. Comparing (1.5) to (1.3) and (1.4), it seems that (1.5) is a reasonable condition to avoid developing singularities for vorticities or the boundaries of supports $\bar{\Omega}_t$ in finite time. Notice also that for axisymmetric flows, due to the axisymmetry, condition (1.2) holds automatically if $\boldsymbol{\omega}_0 \in C^1$. However, this condition in general fails for $\boldsymbol{\omega}_0 \in C^\alpha$. Hence our result of global existence for axisymmetric flows is new even for the problem with compactly supported smooth initial data.

Technically, the approach in the present paper to the problem (1.1) is completely different from those in [9] and [15] where the authors worked with the Euler system and used the tools of pseudo-differential operators. The vorticity equation (1.1) will first be formulated as a non-local integro-differential equation (2.6) for the fluid particle trajectory over the time-dependent domain $\bar{\Omega}_t$. We shall then directly work with problem (2.6). The main idea for this treatment comes from our previous paper [8] concerning motion of charged particle patches. Some techniques from potential theory and the theory of singular integrals will be used to derive necessary estimates for related Newtonian potentials over $C^{1+\alpha}$ domains. For axisymmetric flows, we shall establish L^∞ bounds for vorticities by using L^p estimates.

The paper is organized as follows. The main results and necessary notations will be presented in the next section. In §3, we shall first study the integro-differential equation (2.6) for general initial data. The results will then be applied to the problem (1.1) to establish existence and regularity of vortex patches. The non-singularity condition (1.5) will be verified in §4. In §5 we shall establish global existence for axisymmetric flows.

2. Preliminaries and Main Results. Throughout the paper, we assume that the velocity vanishes at infinity. It is well-known that the velocity $\mathbf{u}$ can be recovered from the vorticity field through the Biot-Savart law:

$$\mathbf{u}(\mathbf{x},t) = \int_{R^3} \boldsymbol{\omega}(\mathbf{y},t) \times \nabla \left(\frac{1}{4\pi |\mathbf{x}-\mathbf{y}|} \right) d\mathbf{y} = \int_{R^3} \mathbf{K}(\mathbf{x}-\mathbf{y})\, \boldsymbol{\omega}(\mathbf{y},t)\, d\mathbf{y}, \tag{2.1}$$

where

$$\mathbf{K}(\mathbf{x}) = \frac{1}{4\pi |\mathbf{x}|^3} \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}. \tag{2.2}$$

Let $\boldsymbol{\Phi}(x,t)$ be the fluid particle trajectory, i.e., $\boldsymbol{\Phi}$ solves

$$\frac{\partial \boldsymbol{\Phi}}{\partial t} = \mathbf{u}(\boldsymbol{\Phi},t), \qquad \boldsymbol{\Phi}(\mathbf{x},0) = \mathbf{x}. \tag{2.3}$$

Set

$$\Omega_t = \boldsymbol{\Phi}(\Omega_0,t) = \{\mathbf{y} = \boldsymbol{\Phi}(\mathbf{x},t) : \ \mathbf{x} \in \Omega_0\}.$$

It follows from (1.1) that $\boldsymbol{\omega}(\boldsymbol{\Phi}(\mathbf{x},t),t)$ solves the equation

$$\frac{d\boldsymbol{\omega}}{dt} = (\nabla \mathbf{u}) \cdot \boldsymbol{\omega}.$$

Throughout the paper, the initial vorticity $\boldsymbol{\omega}_0$ is always assumed to be a C^α patch, i.e.,

$$\boldsymbol{\omega}_0 = \boldsymbol{\varpi}_0 \chi_{\bar{\Omega}_0}, \ \boldsymbol{\varpi}_0 \in C^\alpha\left(R^3\right), \ \partial\Omega_0 \in C^{1+\alpha}, \ \nabla \cdot \boldsymbol{\omega}_0 = 0, \tag{2.4}$$

where $0 < \alpha < 1$, $\chi_{\bar{\Omega}_0}$ is the characteristic function for the closure of the bounded domain Ω_0. Note that the last assumption in (2.4) is in the weak sense. By differentiating (2.3) in $\mathbf{x}$, we find that $\nabla\boldsymbol{\Phi}$ also satisfies the same ODE for $\boldsymbol{\omega}$. By uniqueness, we obtain

$$\boldsymbol{\omega}(\boldsymbol{\Phi}(\mathbf{x},t),t) = \nabla\boldsymbol{\Phi}(\mathbf{x},t)\, \boldsymbol{\omega}_0(\mathbf{x}). \tag{2.5}$$

It follows immediately that $\boldsymbol{\omega}(\mathbf{x},t)=0$ if $\mathbf{x}\notin\bar{\Omega}_t$. This means that $\boldsymbol{\omega}(\mathbf{x},t)$ remains as a patch over $\bar{\Omega}_t$. From (2.1), (2.3) and (2.5), we thus formally arrive at the following non-local integro-differential equation for $\boldsymbol{\Phi}$ in $\bar{\Omega}_0\times[0,T)$

$$
\begin{aligned}
(2.6)\qquad \frac{\partial\boldsymbol{\Phi}(\mathbf{x},t)}{\partial t} &= \int_{\Omega_t}\mathbf{K}(\boldsymbol{\Phi}(\mathbf{x},t)-\mathbf{y})\nabla\boldsymbol{\Phi}\left(\boldsymbol{\Phi}^{-1}(\mathbf{y},t),t\right)\boldsymbol{\omega}_0\left(\boldsymbol{\Phi}^{-1}(\mathbf{y},t),t\right)d\mathbf{y},\\
\boldsymbol{\Phi}(\mathbf{x},0) &= \mathbf{x}\in\bar{\Omega}_0,
\end{aligned}
$$

where $\boldsymbol{\Phi}^{-1}$ is the inverse of the mapping $\mathbf{x}\mapsto\boldsymbol{\Phi}(\mathbf{x},t)$ for fixed t. This equation can also be rewritten equivalently as

$$
(2.7)\qquad \boldsymbol{\Phi}(\mathbf{x},t)=\mathbf{x}+\int_0^t\int_{\Omega_t}\mathbf{K}(\boldsymbol{\Phi}(\mathbf{x},t)-\mathbf{y})(\nabla\boldsymbol{\Phi}\cdot\boldsymbol{\omega}_0)\left(\boldsymbol{\Phi}^{-1}(\mathbf{y},t),t\right)d\mathbf{y}dt.
$$

In this paper, we shall use these formulations. It will be showed that (2.6) is equivalent to (1.1). To study the equation (2.6), we need to introduce some function spaces and notations.

For any subset $G\subseteq R^3$, integer $m\geq 0$, and $0<\alpha<1$, denote by $C^{m+\alpha}(G)$ the set of all functions $f(\mathbf{x})$ defined in G such that f is of the class $C^{m+\alpha}(G)$. If $f(\mathbf{x},t)$ is defined in G_t for $t<T$, where $G_t\subseteq R^3$ depends on t, we sometimes use the notation $f\in C^{m+\alpha}_{\mathbf{x}}(G_t)$ to specify that $f(\cdot,t)\in C^{m+\alpha}(G_t)$ for any fixed t. Denote by $|f(t)|_{m+\alpha}$ the Hölder semi-norm defined as

$$
|f(t)|_{m+\alpha}=\sup_{\mathbf{x},\mathbf{y}\in G_t,\,|\beta|=m}\frac{\left|D^\beta f(\mathbf{x},t)-D^\beta f(\mathbf{y},t)\right|}{|\mathbf{x}-\mathbf{y}|^\alpha},
$$

where $\beta=(\beta_1,\beta_2,\beta_3)$ is a multi-index, $D^\beta=D_1^{\beta_1}D_2^{\beta_2}D_3^{\beta_3}$, D_i is the differential operator in x_i. In this paper, all the derivatives are spatial unless particularly indicated. We also denote by

$$
\|f(t)\|_{m+\alpha}=\sup_{\mathbf{x}\in G_t,\,|\beta|\leq m}\left|D^\beta f(\mathbf{x},t)\right|+|f(t)|_{m+\alpha}
$$

the $(m+\alpha)-$Hölder norm in $\mathbf{x}$. For clarity, sometimes we shall also use the notations $|f(t)|_{m+\alpha,G_t}$ and $\|f(t)\|_{m+\alpha,G_t}$ to specify the dependence on the domain G_t. In the case

$G = \Omega_0 \times (0,T)$ with $\Omega_0 \subseteq R^3$, we denote by $C^{m+\alpha,\alpha}_{\mathbf{x},t}(G)$ the set of all functions $f(\mathbf{x},t)$ such that $f(\cdot,t) \in C^{m+\alpha}(\Omega_0)$ for any fixed t and $f(\mathbf{x},\cdot) \in C^{\alpha}((0,T))$ for any fixed $\mathbf{x}$. For convenience, we introduce the following notations

$$\begin{aligned} |f(t)|_{\inf,\partial G_t} &= \inf_{\mathbf{x}\in\partial G_t} |f(\mathbf{x},t)|, \ |f(t)|_{0,\partial G_t} = \sup_{\mathbf{x}\in\partial G_t} |f(\mathbf{x},t)|, \\ |f(t)|_{\alpha,\partial G_t} &= \sup_{\mathbf{x},\mathbf{y}\in\partial G_t} \frac{|f(\mathbf{x},t) - f(\mathbf{y},t)|}{|\mathbf{x}-\mathbf{y}|^{\alpha}}. \end{aligned}$$

We shall also denote by L^p and Lip the usual L^p-spaces and spaces of Lipschitz functions, respectively.

DEFINITION 2.1. *We call $\mathbf{\Phi}(\mathbf{x},t)$ a $C^{1+\alpha}$ solution of (2.6) in $\bar{\Omega}_0 \times [0,T)$ if $\mathbf{\Phi}(\mathbf{x},t)$, $D_t\mathbf{\Phi}(\mathbf{x},t) \in C^{1+\alpha}\left(\bar{\Omega}_0\right)$, $\mathbf{\Phi}^{-1}(\mathbf{x},t) \in C^{1+\alpha}\left(\bar{\Omega}_t\right)$, for $t < T$, and (2.6) holds.*

Notice that if $\boldsymbol{\omega}_0$ is smooth with a compact support, it is easy to verify that (2.6) is equivalent to (1.1). We now verify that for the discontinuous initial data, our formulation (2.6) is also equivalent to the vorticity equation (1.1).

PROPOSITION 2.1. *Suppose that the initial datum $\boldsymbol{\omega}_0(\mathbf{x},t)$ is solenoidal and satisfies (2.4). Let $\mathbf{\Phi}(\mathbf{x},t)$ a $C^{1+\alpha}$ solution of (2.6) in $\bar{\Omega}_0 \times [0,T)$. Define the vorticity $\boldsymbol{\omega}(\mathbf{x},t)$ through (2.5) and the velocity $\mathbf{u}(\mathbf{x},t)$ by (2.1), respectively for $(\mathbf{x},t) \in R^3 \times [0,T)$. Then $\boldsymbol{\omega}$ is a weak solution of (1.1) with $\mathbf{u} \in Lip(R^3 \times [0,T))$. Consequently, there exists a pressure p such that $(\mathbf{u},p)$ forms a weak solution of the incompressible Euler system*

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla p, \\ \nabla\cdot\mathbf{u} &= 0, \end{aligned}$$

with the initial velocity $\mathbf{u}_0$ being defined through the Biot-Savart law (2.1) by $\boldsymbol{\omega}_0$.

Proof. The proof is elementary. For completion, we still provide the proof that consists of verifying the following three statements.

(1) The velocity $\mathbf{u}$ is Lipschitz in $\mathbf{x}$ and t, and is solenoidal.

Since $\nabla\mathbf{\Phi}(\mathbf{x},t)$ and $\nabla\mathbf{\Phi}_t(\mathbf{x},t)$, as functions of $\mathbf{x}$, are $\alpha-$Hölder continuous in $\bar{\Omega}_0$, from (2.5) one sees that $\boldsymbol{\omega}(\mathbf{x},t) \in C^{\alpha}\left(\bar{\Omega}_t\right), \partial\Omega_t \in C^{1+\alpha}$. From (2.1), $\mathbf{u}$ can be extended to

R^3. By (2.1), we write the velocity gradient as, for $\mathbf{x} \in \bar{\Omega}_t$,

$$
\begin{aligned}
\nabla \mathbf{u}(\mathbf{x},t) &= \int_{\Omega_t} (\boldsymbol{\omega}(\mathbf{y},t) - \boldsymbol{\omega}(\mathbf{x},t)) \times \frac{\boldsymbol{\sigma}(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y} + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x},t) \times \mathbf{I} \\
&\quad + \boldsymbol{\omega}(\mathbf{x},t) \times P_v \int_{\Omega_t} \frac{\boldsymbol{\sigma}(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y},
\end{aligned}
\tag{2.8}
$$

where P_v means the principle value, $\mathbf{I}$ is the identity matrix, and

$$
\frac{\boldsymbol{\sigma}(\mathbf{x})}{|\mathbf{x}|^3} = \nabla\nabla \frac{1}{4\pi |\mathbf{x}|} = \frac{1}{4\pi |\mathbf{x}|^5}\left(3\mathbf{x} \times \mathbf{x} - |\mathbf{x}|^2 \mathbf{I}\right),
$$

where we have used the convention for the cross product $\mathbf{a} \times (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (\mathbf{a} \times \mathbf{b}_1, \mathbf{a} \times \mathbf{b}_2, \mathbf{a} \times \mathbf{b}_3)$ between a vector and a matrix. For convenience, we shall drop the symbol P_v. All singular integrals in this paper should be understood as their principle values. By Lemma 3.2 in §3, we find that $\mathbf{u}(\mathbf{x},t) \in C^{1+\alpha}\left(\bar{\Omega}_t\right)$, $\mathbf{u}(\cdot,t) \in Lip(R^3)$.

From (2.8), it is easy to check that $\nabla \cdot \mathbf{u}(\mathbf{x},t) = 0$ for $\mathbf{x} \notin \partial\Omega_t$. Since $\mathbf{u}$ is continuous across $\partial\Omega_t$, it follows that $\mathbf{u}$ is solenoidal (weakly) in R^3. By differentiating (2.6) in $\mathbf{x}$ and using (2.8), it follows that $\nabla\boldsymbol{\Phi}$ satisfies

$$
\begin{aligned}
\frac{\partial \nabla \boldsymbol{\Phi}(\mathbf{x},t)}{\partial t} &= \int_{\Omega_t} \Big(\nabla\boldsymbol{\Phi}\left(\boldsymbol{\Phi}^{-1}(\mathbf{y},t),t\right)\boldsymbol{\omega}_0\left(\boldsymbol{\Phi}^{-1}(\mathbf{y},t)\right) \\
&\quad - \nabla\boldsymbol{\Phi}(\mathbf{x},t)\,\boldsymbol{\omega}_0(\mathbf{x})\Big) \times \frac{\boldsymbol{\sigma}(\boldsymbol{\Phi}(\mathbf{x},t)-\mathbf{y})}{|\boldsymbol{\Phi}(\mathbf{x},t)-\mathbf{y}|^3}\, d\mathbf{y}\,\nabla\boldsymbol{\Phi}(\mathbf{x},t) \\
&\quad + \nabla\boldsymbol{\Phi}(\mathbf{x},t)\,\boldsymbol{\omega}_0(\mathbf{x}) \times \int_{\Omega_t} \frac{\boldsymbol{\sigma}(\boldsymbol{\Phi}(\mathbf{x},t)-\mathbf{y})}{|\boldsymbol{\Phi}(\mathbf{x},t)-\mathbf{y}|^3}\, d\mathbf{y}\,\nabla\boldsymbol{\Phi}(\mathbf{x},t) \\
&\quad + \frac{1}{2}\left(\nabla\boldsymbol{\Phi}(\mathbf{x},t)\,\boldsymbol{\omega}_0(\mathbf{x}) \times \mathbf{I}\right)\nabla\boldsymbol{\Phi}(\mathbf{x},t) \\
&= \nabla\mathbf{u}(\boldsymbol{\Phi}(\mathbf{x},t),t)\,\nabla\boldsymbol{\Phi}(\mathbf{x},t).
\end{aligned}
\tag{2.9}
$$

Since $\mathbf{u}(\cdot,t) \in Lip(R^3)$, the solution $\boldsymbol{\Phi}$ can be extended to R^3. The extended solution is also Lipschitz. Hence the Jacobian $J(\boldsymbol{\Phi}) = 1$. We can now write (2.1) as

$$
\begin{aligned}
\mathbf{u}(\mathbf{x},t) &= \int_{R^3} \boldsymbol{\omega}(\mathbf{y},t) \times \nabla\left(\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\right) dy \\
&= \int_{\Omega_0} \boldsymbol{\omega}(\boldsymbol{\Phi}(\mathbf{y},t),t) \times \left(\frac{\boldsymbol{\Phi}(\mathbf{y},t)-\mathbf{x}}{4\pi|\mathbf{x}-\boldsymbol{\Phi}(\mathbf{y},t)|^3}\right) d\mathbf{y}.
\end{aligned}
$$

It follows

$$\begin{aligned}
\mathbf{u}_t(\mathbf{x},t) &= \int_{\Omega_0} \frac{d}{dt}\boldsymbol{\omega}(\boldsymbol{\Phi}(\mathbf{y},t),t)\times\left(\frac{\mathbf{x}-\boldsymbol{\Phi}(\mathbf{y},t)}{4\pi|\mathbf{x}-\boldsymbol{\Phi}(\mathbf{y},t)|^3}\right)d\mathbf{y}\\
&+P_v\int_{\Omega_t}\boldsymbol{\omega}(\mathbf{y},t)\times\left(\frac{\sigma(\mathbf{y}-\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^3}\right)\boldsymbol{\Phi}_t\left(\boldsymbol{\Phi}^{-1}(\mathbf{y},t),t\right)d\mathbf{y}\\
&+\frac{1}{2}\boldsymbol{\omega}(\mathbf{x},t)\times\boldsymbol{\Phi}_t\left(\boldsymbol{\Phi}^{-1}(\mathbf{x},t),t\right).
\end{aligned}$$

This formula can be easily checked if $\boldsymbol{\omega}(\mathbf{y},t)$ is smooth in t and $\mathbf{x}$. We then use smooth function to approximate $\boldsymbol{\omega}(\mathbf{y},t)$ to obtain the above expression. Since, from (2.5), $\boldsymbol{\omega}(\boldsymbol{\Phi}(\mathbf{y},t),t)=\nabla\boldsymbol{\Phi}\boldsymbol{\omega}_0$ is Lipschitz in t, and $\boldsymbol{\Phi}_t\in C^{1+\alpha}\left(\bar{\Omega}_0\right)$, we find (by Lemma 3.2 in §3) that $\mathbf{u}_t$ is bounded.

(2) $\boldsymbol{\omega}$ is a weak solution of (1.1).

Since $\boldsymbol{\omega}(\boldsymbol{\Phi}(\mathbf{x},t),t)=\nabla\boldsymbol{\Phi}\boldsymbol{\omega}_0$ is differentiable in t, from (2.9), it solves

$$\frac{d\boldsymbol{\omega}}{dt}=(\nabla\mathbf{u})(\boldsymbol{\Phi}(\mathbf{x},t),t)\,\boldsymbol{\omega}\quad\text{for }\mathbf{x}\notin\partial\Omega_0.$$

This is equivalent to (1.1) if $\boldsymbol{\omega}$ is smooth. If it is not smooth, we multiply the equation by $\xi(\boldsymbol{\Phi},t)$, where ξ is any smooth function with a compact support, and then integrate over $R^3\times(0,T)$. It follows that $\boldsymbol{\omega}$ is a weak solution of (1.1).

(3) $\nabla\cdot\boldsymbol{\omega}=0$ and $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ in R^3.

For any test function ξ, we have

$$\begin{aligned}
&\int_{R^3}\boldsymbol{\omega}(\mathbf{x},t)\cdot\nabla\xi(\mathbf{x})\,d\mathbf{x}=\int_{R^3}\boldsymbol{\omega}(\boldsymbol{\Phi}(\mathbf{x},t),t)\cdot\nabla\xi(\boldsymbol{\Phi}(\mathbf{x},t))\,d\mathbf{x}\\
&=\int_{R^3}(\nabla\boldsymbol{\Phi}(\mathbf{x},t)\,\boldsymbol{\omega}_0(\mathbf{x}))\cdot\nabla\xi(\boldsymbol{\Phi}(\mathbf{x},t))\,d\mathbf{x}=\int_{R^3}\boldsymbol{\omega}_0(\mathbf{x})\cdot\nabla(\xi(\boldsymbol{\Phi}(\mathbf{x},t)))\,d\mathbf{x}=0.
\end{aligned}$$

Therefore $\nabla\cdot\boldsymbol{\omega}=0$ in R^3. By (2.8), it is now easy to see that $\boldsymbol{\omega}=\nabla\times\mathbf{u}$.

Finally, (1.1) implies that $\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}$ is irrotational. Hence $\mathbf{u}$ solves the Euler system. The proof is complete.

In the sequels, we shall always use the formulation (2.6). The main results are the follows.

THEOREM 2.1. *Assume (2.4). Then there exists a unique* $C^{1+\alpha}\left(\bar{\Omega}_0\right)$ *solution* $\mathbf{\Phi}(\mathbf{x}, t)$ *for (2.6) for* $0 < t < T$, *for some* $T > 0$. *Consequently, the vortex patch problem (1.1) admits a unique solution* $\boldsymbol{\omega}$ *that is of* $C^{\alpha}_{\mathbf{x}}\left(\bar{\Omega}_t\right)$ *with* $\partial\Omega_t \in C^{1+\alpha}$.

The proof of Theorem 2.1 relies only on some regularity results for the Newtonian potentials (see Lemma 3.1 and Lemma 3.2 in the next section), not on the special structure for vorticities. By differentiating (1.1), we find that $\nabla\boldsymbol{\omega}$ satisfies a differential equation analogous to (1.1). Consequently, by the same procedure leading to (2.6), we can see that $\nabla\mathbf{\Phi}(\mathbf{x}, t)$ satisfies a system of integro-differential equations analogous to (2.6). Theorem 2.1 immediately leads to $C^{m+1+\alpha}_{\mathbf{x}}\left(\bar{\Omega}_0\right)$ regularity for $\mathbf{\Phi}$ and $D_t\mathbf{\Phi}$, and $C^{m+\alpha}$ regularity for the vorticity $\boldsymbol{\omega}$.

COROLLARY 2.1. *If* $\boldsymbol{\varpi}_0 \in C^{m+\alpha}$, $\partial\Omega_0 \in C^{1+\alpha}$, *then* $\boldsymbol{\omega}$ *is of the class* $C^{m+\alpha}_{\mathbf{x}}\left(\bar{\Omega}_t\right)$, *and the particle trajectory* $\mathbf{\Phi}, D_t\mathbf{\Phi} \in C^{m+1+\alpha}_{\mathbf{x}}\left(\bar{\Omega}_0\right)$. *Furthermore, if* $\partial\Omega_0 \in C^{m+1+\alpha}$, *then* $\partial\Omega_t \in C^{m+1+\alpha}$.

The next result provides necessary and sufficient condition for blow-up of $C^{1+\alpha}$ solution.

THEOREM 2.2. *Assume (2.4). Then the* $C^{1+\alpha}$ *solution* $\mathbf{\Phi}$ *from Theorem 2.1 blows up at* $t = T$ *iff* $\theta(t)$ *defined in (1.5) is unbounded for* $t < T$. *If* $\theta(t)$ *is bounded for* $t < T$, *then the* $C^{1+\alpha}$ *solution can be extended to* $t \leq T + T_0$ *for some positive constant* T_0 *depending only on* $\theta(T)$. *Consequently, any* C^{α} *vortex patches* $\boldsymbol{\omega}$ *can be extended beyond* T *iff* $\theta(t)$ *is bounded for* $t < T$.

THEOREM 2.3. *Assume (2.4) with* $\alpha > 1/3$. *Suppose also that the initial vorticity is axisymmetric. Then there exists a unique* $C^{1+\alpha}\left(\bar{\Omega}_0\right)$ *solution* $\mathbf{\Phi}$ *for all* $t > 0$.

REMARK 2.1. *The global existence results for axisymmetric flows with smooth initial data were established previously in the several papers [1], [12], [13] and reference herein. We point out that in general* $\boldsymbol{\varpi}_0 \in C^{\alpha}$ *do not guarantee the condition (1.2). Therefore, this result is new even for solutions in* $H^p(R^3)$ *or* $C^{\infty}_0(R^3)$. *We also point out that the assertion of Theorem 2.3 remains true if we replace the condition* $\alpha > 1/3$

by the condition $|\boldsymbol{\omega}_0(\mathbf{x})| \leq cr^\gamma$ *for some* $\gamma > 1/3$.

3. Local Existence. In this section, we shall prove Theorem 2.1. Throughout the paper, we always assume (2.4) unless otherwise indicated. Let $\Omega \subseteq R^3$ be a $C^{1+\alpha}$ domain. Assume that there exists a function $\varphi \in C^{1+\alpha}(R^3)$ with $\nabla\varphi \neq 0$ on $\partial\Omega$ such that $\Omega = \{\mathbf{x} : \varphi(\mathbf{x}) < 0\}$. Set

$$\delta = \frac{|\nabla\varphi|_{\alpha,\partial\Omega}}{|\nabla\varphi(\mathbf{x})|_{\inf,\partial\Omega}}, \tag{3.1}$$

$$\mathbf{V}(\mathbf{x}) = \int_\Omega \frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,d\mathbf{x}.$$

We need the following lemmas [8, Lemma 3.1].

LEMMA 3.1. *Suppose that* $\partial\Omega \in C^{1+\alpha}$. *Then*

$$|\mathbf{V}|_{0,\Omega} \leq c(1+|\ln(\delta d(\Omega))|), \tag{3.2}$$

$$|\mathbf{V}|_{\alpha,\Omega} \leq c\delta(1+|\ln(\delta d(\Omega))|), \tag{3.3}$$

where $d(\Omega)$ *is the diameter of* Ω, *and* c *is a constant independent of* Ω.

LEMMA 3.2. *Let* $\boldsymbol{\omega}$ *be supported in* $\bar{\Omega}$ *with finite Hölder norm* $\|\boldsymbol{\omega}(t)\|_{\alpha,\Omega}$, *and* $\mathbf{u}$ *be defined in terms of* $\boldsymbol{\omega}$ *through (2.1) with* Ω_t *replaced with* Ω. *Then*

$$|\mathbf{u}|_{0,\Omega} \leq cd(\Omega)\,|\boldsymbol{\omega}|_{0,\Omega}, \tag{3.4}$$

$$|\nabla\mathbf{u}|_{0,\Omega} \leq c\,|\boldsymbol{\omega}|_{0,\Omega}\left(1+\left|\ln\left(\delta d(\Omega)^\alpha + \frac{\|\boldsymbol{\omega}\|_{\alpha,\Omega}\,d(\Omega)^\alpha}{|\boldsymbol{\omega}|_{0,\Omega}}\right)\right|\right), \tag{3.5}$$

$$|\nabla\mathbf{u}|_{\alpha,\Omega} \leq c\left(|\boldsymbol{\omega}|_{\alpha,\Omega} + |\boldsymbol{\omega}|_{0,\Omega}\,\delta\right)(1+|\ln(\delta d(\Omega)^\alpha)|), \tag{3.6}$$

$$|\nabla\mathbf{u}|_{0,R^3\setminus\Omega} \leq c\delta\,|\boldsymbol{\omega}|_{0,\Omega}\left(1+\left|\ln\left(\delta d(\Omega)^\alpha + \frac{\|\boldsymbol{\omega}\|_{\alpha,\Omega}\,d(\Omega)^\alpha}{|\boldsymbol{\omega}|_{0,\Omega}}\right)\right|\right), \tag{3.7}$$

where $d(\Omega)$ *is the diameter of* Ω, *and* c *is a constant independent of* Ω *and* $\boldsymbol{\omega}$.

Proof. The inequality (3.4) follows immediately from the fact that $|\mathbf{K}(\mathbf{x})| \leq c|\mathbf{x}|^{-2}$. To establish (3.5), we write (2.8) as, for $\mathbf{x} \in \bar{\Omega}$,

$$\begin{aligned} \nabla \mathbf{u}(\mathbf{x}) &= \int_{\Omega} (\boldsymbol{\omega}(\mathbf{y}) - \boldsymbol{\omega}(\mathbf{x})) \times \frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y} + \frac{1}{2}\boldsymbol{\omega}(\mathbf{x}) \times \mathbf{I} \\ &+ \boldsymbol{\omega}(\mathbf{x}) \times \mathbf{V}(\mathbf{x}-\mathbf{y}) = m_1 + m_2 + m_3. \end{aligned} \tag{3.8}$$

It is easy to see that $|m_2| \leq c|\boldsymbol{\omega}|_{0,\Omega}$. For $\varepsilon > 0$ to be chosen later on, let $B_\varepsilon(\mathbf{x})$ be the ball centered at $\mathbf{x}$ with the radius ε. Write

$$m_1 = \left(\int_{\Omega \backslash B_\varepsilon(\mathbf{x})} + \int_{B_\varepsilon(\mathbf{x})} \right) (\boldsymbol{\omega}(\mathbf{y}) - \boldsymbol{\omega}(\mathbf{x})) \times \frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y} = m_{11} + m_{12}.$$

Since $|\sigma(\mathbf{x})| \leq c$, we have

$$|m_{11}| \leq c|\boldsymbol{\omega}|_{0,\Omega} \int_{\Omega \backslash B_\varepsilon(\mathbf{x})} \frac{1}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{y} \leq c|\boldsymbol{\omega}|_{0,\Omega} \left| \ln \left(\frac{d(\Omega)}{\varepsilon} \right) \right|.$$

Since $\boldsymbol{\omega}(\mathbf{x}) \in C^\alpha$, we deduce that

$$|m_{12}| \leq c|\boldsymbol{\omega}|_{\alpha,\Omega} \int_{B_\varepsilon(\mathbf{x})} \frac{1}{|\mathbf{x}-\mathbf{y}|^{3-\alpha}} d\mathbf{y} \leq c|\boldsymbol{\omega}|_{\alpha,\Omega} \varepsilon^\alpha.$$

By choosing $\varepsilon^\alpha = |\boldsymbol{\omega}|_{0,\Omega} / \|\boldsymbol{\omega}\|_{\alpha,\Omega}$, we obtain from above that m_1 is bounded by

$$|m_1| \leq c|\boldsymbol{\omega}|_0 \left(1 + \left| \ln \left(\frac{\|\boldsymbol{\omega}\|_{\alpha,\Omega} d(\Omega)^\alpha}{|\boldsymbol{\omega}|_{0,\Omega}} \right) \right| \right). \tag{3.9}$$

The bounds on m_3 follows from Lemma 3.1:

$$|m_3| \leq c|\boldsymbol{\omega}|_{0,\Omega} \left| \int_{\Omega} \frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} d\mathbf{x} \right| \leq c|\boldsymbol{\omega}|_{0,\Omega} (1 + |\ln(\delta d(\Omega)^\alpha)|).$$

Combining these estimates, we arrive at

$$|\nabla \mathbf{u}|_{0,\Omega} \leq c|\boldsymbol{\omega}|_{0,\Omega} \left(1 + \left| \ln \left(\delta d(\Omega)^\alpha + \frac{\|\boldsymbol{\omega}\|_{\alpha,\Omega} d(\Omega)^\alpha}{|\boldsymbol{\omega}|_{0,\Omega}} \right) \right| \right), \tag{3.10}$$

We next show the C^α estimate. Notice that by [4, Appendix] and (3.2) that

$$|m_1|_\alpha \leq c|\boldsymbol{\omega}|_{\alpha,\Omega} (1 + |\ln(\delta d(\Omega)^a)|). \tag{3.11}$$

By (3.3),

$$(3.12) \qquad |m_3|_\alpha \leq c\left(\delta\,|\boldsymbol{\omega}|_{0,\Omega} + |\boldsymbol{\omega}|_{\alpha,\Omega}\right)\left(1 + |\ln(\delta d(\Omega)^\alpha)|\right).$$

The assertion (3.6) follows.

Finally, we show (3.7). Since $\bar{\Omega}$ is a $C^{1+\alpha}$ domain, we can find a $\tilde{\boldsymbol{\omega}} \in C_0^{1+\alpha}(R^3)$ that has the compact support and extends $\boldsymbol{\omega}$ in $\bar{\Omega}$ to R^3, i.e., $\tilde{\boldsymbol{\omega}}(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x})$ for $\mathbf{x} \in \bar{\Omega}$, and $\|\tilde{\boldsymbol{\omega}}\|_{\alpha,R^3} \leq c\delta\,\|\boldsymbol{\omega}\|_{\alpha,\Omega}$ [10]. For $\mathbf{x} \in R^3\backslash\bar{\Omega}$, we have

$$\begin{aligned}
\nabla\mathbf{u}(\mathbf{x}) &= \int_\Omega \boldsymbol{\omega}(\mathbf{y}) \times \frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,d\mathbf{y} \\
&= \int_{R^3} \tilde{\boldsymbol{\omega}}(\mathbf{y}) \times \frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,d\mathbf{y} - \int_{R^3\backslash\bar{\Omega}} \tilde{\boldsymbol{\omega}}(\mathbf{y}) \times \frac{\boldsymbol{\sigma}(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,\mathbf{dy}.
\end{aligned}$$

The first term in the above expression is obviously bounded. Since $\tilde{\boldsymbol{\omega}}$ is compactly supported in $R^3\backslash\bar{\Omega}$, by (3.5), the second term is also bounded by the right-hand side of (3.7). The assertion (3.7) thus follows.

Proof of Theorem 2.1:

For simplicity, we assume that there exists $\varphi_0 \in C^{1+\alpha}$ with $|\nabla\varphi_0|_{\inf,\partial\Omega_0} \neq 0$ such that $\Omega_0 = \{\varphi_0(\mathbf{x}) < 0\}$. For any $M, T > 0$ to be chosen later on, we define a set $B(M,T)$ of vector value functions in $\bar{\Omega}_0 \times [0,T)$ as follows:

$$\begin{aligned}
B(M,T) = \{\boldsymbol{\Phi}(\mathbf{x},t) \in R^3 : \quad & \|\boldsymbol{\Phi}(t)\|_{1+\alpha,\Omega_0},\ \|\boldsymbol{\Phi}(\mathbf{x},\cdot)\|_{\alpha,\Omega_0} \leq M, \\
& \boldsymbol{\Phi}(\mathbf{x},0) = \mathbf{x},\ |\nabla\boldsymbol{\Phi} - \mathbf{I}| \leq 1/2\}.
\end{aligned}$$

Next we define a mapping F from $B(M,T)$ to a functional space by

$$(3.13) \qquad F(\boldsymbol{\Phi})(\mathbf{x},t) = \mathbf{x} + \int_0^t \int_{\Omega_t} \mathbf{K}(\boldsymbol{\Phi}(\mathbf{x},t) - \mathbf{y})(\nabla\boldsymbol{\Phi}\boldsymbol{\omega}_0)\left(\boldsymbol{\Phi}^{-1}(\mathbf{y},t),t\right)d\mathbf{y}dt,$$

where $\Omega_t = \boldsymbol{\Phi}(\Omega_0,t)$, the matrix $\mathbf{K}(\mathbf{x})$ is defined in (2.2). Since $|\nabla\boldsymbol{\Phi} - \mathbf{I}| \leq 1/2$, $\boldsymbol{\Phi}^{-1}(\cdot,t)$ exists and maps Ω_t onto Ω_0. The mapping thus is well defined. For any $\boldsymbol{\Phi} \in B(M,T)$, the gradient $\nabla F(\boldsymbol{\Phi})$ can be expressed as

$$(3.14) \qquad \nabla F(\boldsymbol{\Phi})(\mathbf{x},t) = \mathbf{I} + \int_0^t \nabla\mathbf{u}(\boldsymbol{\Phi}(\mathbf{x},s),s)\,\nabla\boldsymbol{\Phi}(\mathbf{x},s)\,ds,$$

where $\nabla \mathbf{u}$ is defined in (3.8) with $\boldsymbol{\omega}$ defined by (2.5) and with Ω replaced with Ω_t (see also (2.9) for the complete expression). By apply Lemma 3.2 with $\boldsymbol{\omega}$ defined by (2.5), ,we deduce that

$$\begin{aligned} \|F(\boldsymbol{\Phi})\|_{1+\alpha,\Omega_0} &\leq c_0 + c\int_0^t c\|\boldsymbol{\Phi}\|_{1+\alpha,\Omega_0}\left(1+\delta_t|\boldsymbol{\omega}|_{0,\Omega_t}+\|\boldsymbol{\omega}\|_{\alpha,\Omega_t}\right) \\ &\cdot\ln\left(d(\Omega_t)^\alpha\frac{\|\boldsymbol{\omega}\|_{\alpha,\Omega_t}}{|\boldsymbol{\omega}|_{0,\Omega_t}}+d(\Omega_t)^\alpha\delta_t\right)dt, \end{aligned} \tag{3.15}$$

where c is a constant independent of $\boldsymbol{\Phi}$. $c_0=\sup_{\mathbf{x}\in\Omega_0}|\mathbf{x}|$, δ_t is defined in terms of Ω_t by

$$\delta_t=\frac{|\nabla\varphi(\mathbf{x},t)|_{\alpha,\partial\Omega_t}}{|\nabla\varphi(\mathbf{x},t)|_{\inf,\partial\Omega_t}}=\frac{|\nabla\varphi_0(\boldsymbol{\Phi}^{-1}(\mathbf{x},t))\nabla\boldsymbol{\Phi}^{-1}(\mathbf{x},t)|_{\alpha,\partial\Omega_t}}{|\nabla\varphi_0(\boldsymbol{\Phi}^{-1}(\mathbf{x},t))\nabla\boldsymbol{\Phi}^{-1}(\mathbf{x},t)|_{\inf,\partial\Omega_t}}, \tag{3.16}$$

since $\Omega_t=\{\varphi(\mathbf{x},t)<0\}$, where $\varphi(\mathbf{x},t)=\varphi_0(\boldsymbol{\Phi}^{-1}(\mathbf{x},t))$. It is easy to see that $\delta_t\leq c(M)$. We also notice that $d(\Omega_t)\leq 2|\boldsymbol{\Phi}|_0\leq 2M$, $\|\boldsymbol{\omega}\|_{\alpha,\Omega_t}\leq c(M)$, where the constant $c(M)$ depends on M only. Hence (3.15) results in, for $t<T$,

$$\|F(\boldsymbol{\Phi})(t)\|_{1+\alpha,\Omega_0}\leq c_0+c(M)T\ln(2+M),$$

Analogously we can derive

$$\|F(\boldsymbol{\Phi})(\mathbf{x},\cdot)\|_\alpha\leq c(M)T^{1-\alpha}$$

and

$$|\nabla F(\boldsymbol{\Phi})-\mathbf{I}|\leq c(M)T\ln(2+M).$$

We now choose $M=1+2c_0$, $T=c_0(c(M)\ln(2+c_0))^{-1}$. Then $F(\boldsymbol{\Phi})\in B(M,T)$. The mapping F maps $B(M,T)$ into itself.

For any $\boldsymbol{\Phi}$, $\tilde{\boldsymbol{\Phi}}\in B(M,T)$. Set $\Omega_t=\boldsymbol{\Phi}(\Omega_0,t)$, $\tilde{\Omega}_t=\tilde{\boldsymbol{\Phi}}(\Omega_0,t)$ and define

$$\rho(t)=\left|\boldsymbol{\Phi}(t)-\tilde{\boldsymbol{\Phi}}(t)\right|_{0,\Omega_0},\quad \nabla\rho(t)=\left|\nabla\boldsymbol{\Phi}(t)-\nabla\tilde{\boldsymbol{\Phi}}(t)\right|_{0,\Omega_0}.$$

By changing variables in the expressions for $F(\boldsymbol{\Phi})$ and $F\left(\tilde{\boldsymbol{\Phi}}\right)$, we obtain, from (3.13),

$$\begin{aligned} F(\boldsymbol{\Phi})(\mathbf{x},t) &= \mathbf{x}+\int_0^t\int_{\Omega_0}\mathbf{K}(\boldsymbol{\Phi}(\mathbf{x},t)-\boldsymbol{\Phi}(\mathbf{y},t))\nabla\boldsymbol{\Phi}(\mathbf{y},t)\boldsymbol{\omega}_0(\mathbf{y})J(\boldsymbol{\Phi})\,d\mathbf{y}dt, \\ F\left(\tilde{\boldsymbol{\Phi}}\right)(\mathbf{x},t) &= \mathbf{x}+\int_0^t\int_{\Omega_0}\mathbf{K}\left(\tilde{\boldsymbol{\Phi}}(\mathbf{x},t)-\tilde{\boldsymbol{\Phi}}(\mathbf{y},t)\right)\nabla\tilde{\boldsymbol{\Phi}}(\mathbf{y},t)\boldsymbol{\omega}_0(\mathbf{y})J\left(\tilde{\boldsymbol{\Phi}}\right)\,d\mathbf{y}\mathbf{dt}, \end{aligned}$$

where $J(\cdot)$ is the Jacobian. It follows, since $\mathbf{\Phi},\ \tilde{\mathbf{\Phi}} \in B(M,T)$,

$$\left|F(\mathbf{\Phi})(\mathbf{x},t) - F\left(\tilde{\mathbf{\Phi}}\right)(\mathbf{x},t)\right| \le c\int_0^t \nabla\rho(t)\,dt$$

$$+\int_0^t\int_{\Omega_0}\left|\mathbf{K}(\mathbf{\Phi}(\mathbf{x},t)-\mathbf{\Phi}(\mathbf{y},t)) - \mathbf{K}\left(\tilde{\mathbf{\Phi}}(\mathbf{x},t)-\tilde{\mathbf{\Phi}}(\mathbf{y},t)\right)\right| d\mathbf{y}dt. \tag{3.17}$$

We write, for $\varepsilon > 0$ to be determined later,

$$\int_{\Omega_0}\left|\mathbf{K}(\mathbf{\Phi}(\mathbf{x},t)-\mathbf{\Phi}(\mathbf{y},t)) - \mathbf{K}\left(\tilde{\mathbf{\Phi}}(\mathbf{x},t)-\tilde{\mathbf{\Phi}}(\mathbf{y},t)\right)\right| d\mathbf{y} \tag{3.18}$$

$$= \int_{\Omega_0\backslash B_\varepsilon(\mathbf{x})} + \int_{\Omega_0\cap B_\varepsilon(\mathbf{x})} = k_1 + k_2.$$

Set $\boldsymbol{\xi}(s) = \tilde{\mathbf{\Phi}}(\mathbf{x},t) - \tilde{\mathbf{\Phi}}(\mathbf{z},t) + s\left(\mathbf{\Phi}(\mathbf{x},t) - \mathbf{\Phi}(\mathbf{z},t) - \left(\tilde{\mathbf{\Phi}}(\mathbf{x},t) - \tilde{\mathbf{\Phi}}(\mathbf{z},t)\right)\right)$. Then

$$k_1 = \int_{\Omega_0\backslash B_\varepsilon}\left|\int_0^1 \frac{d}{ds}\mathbf{K}(\boldsymbol{\xi}(s))\,ds\right| d\mathbf{y} = \int_{\Omega_0\backslash B_\varepsilon}\left|\int_0^1 \nabla\mathbf{K}(\boldsymbol{\xi}(s))\frac{d}{ds}\boldsymbol{\xi}_s\,ds\right| d\mathbf{y}.$$

Notice that the matrix $|\nabla\mathbf{K}(\mathbf{x})| \le c|\mathbf{x}|^{-3}$, $|\boldsymbol{\xi}'(s)| \le c\rho(s)$. Therefore

$$|k_1| \le c\rho(t)\int_\varepsilon^{d(\Omega_0)} \frac{1}{r}\,dr = c\rho(t)(1+\ln\varepsilon). \tag{3.19}$$

Next, since $\nabla\mathbf{\Phi}^{-1}$ and $\nabla\tilde{\mathbf{\Phi}}^{-1}$ are bounded, we have

$$|\mathbf{K}(\mathbf{\Phi}(\mathbf{x},t)-\mathbf{\Phi}(\mathbf{y},t))| \le |\mathbf{\Phi}(\mathbf{x},t)-\mathbf{\Phi}(\mathbf{y},t)|^{-2} \le c|\mathbf{x}-\mathbf{y}|^{-2}.$$

Hence, k_2 can be estimated as

$$|k_2| \le c\int_0^\varepsilon \frac{1}{r^2}r^2\,dr \le c\varepsilon. \tag{3.20}$$

We now choose $\varepsilon = \rho(t)$. From (3.17)-(3.20), it follows

$$\left|F(\mathbf{\Phi})(\mathbf{x},t) - F\left(\tilde{\mathbf{\Phi}}\right)(\mathbf{x},t)\right| \le c\int_0^t (\nabla\rho(t) + \rho(t)|\ln\rho(t)|)\,dt. \tag{3.21}$$

Define a sequence $\mathbf{\Phi}_n(\mathbf{x},t)$ by

$$\mathbf{\Phi}_0 = \mathbf{x},\ \mathbf{\Phi}_{n+1}(\mathbf{x},t) = F(\mathbf{\Phi}_n)(\mathbf{x},t).$$

Since $\mathbf{\Phi}_n \in B(M,T)$, this sequence $\{\mathbf{\Phi}_n\}$ is precompact under the $C^{1+\gamma,\gamma}_{x,t}$-norm for any $\gamma < \alpha$. Hence we can select an subsequence, still denote it as $\{\mathbf{\Phi}_n\}$, and a function $\mathbf{\Phi} \in C^{1+\gamma,\gamma}_{x,t}(\Omega_0 \times [0,T))$ such that

$$\mathbf{\Phi}_n \longrightarrow \mathbf{\Phi} \text{ in } C^{1+\gamma,\gamma}_{x,t} - \text{norm}.$$

This implies that $\mathbf{\Phi} \in B(M,T)$ (by checking from the definitions) and by (3.21),

$$\begin{aligned} &\left|\mathbf{\Phi}_{n+1}(\mathbf{x},t) - F(\mathbf{\Phi})(\mathbf{x},t)\right| = \left|F(\mathbf{\Phi}_n)(\mathbf{x},t) - F(\mathbf{\Phi})(\mathbf{x},t)\right| \\ \leq\ & cT \sup_{0\leq t\leq T}\left(\left|\mathbf{\Phi}_n(t) - \mathbf{\Phi}(t)\right|_{0,\Omega_0}\right) \\ &+ \sup_{0\leq t\leq T}\left(\left|\nabla\mathbf{\Phi}_n(t) - \nabla\mathbf{\Phi}(t)\right|_{0,\Omega_0}\left|\ln\left|\nabla\mathbf{\Phi}_n(t) - \nabla\mathbf{\Phi}(t)\right|_{0,\Omega_0}\right|\right). \end{aligned}$$

Let $n \to \infty$, we find that $\mathbf{\Phi}$ is a fixed point for F, i.e., $F(\mathbf{\Phi}) = \mathbf{\Phi}$. The proof of existence is complete.

Since the corresponding velocity $\mathbf{u}$ is a Lipschitz solution for the Euler system. It follows (see [9]) that $\mathbf{u}$ is unique. Therefore, it is easy to see that $\mathbf{\Phi}$ is unique. The proof of Theorem 2.1 is complete.

Checking carefully the above proof, we find, by slightly modification, that the solution can be extended beyond T as long as $\|\boldsymbol{\omega}(t)\|_{\alpha,\Omega_t}$ and δ_t are bounded at $t = T$. More precisely, we have

COROLLARY 3.1. *Suppose that $\mathbf{\Phi}(\mathbf{x},t)$ is a $C^{1+\alpha}$ solution of (2.6) for $t < T$. If $\|\boldsymbol{\omega}(t)\|_{\alpha,\Omega_t}$ and δ_t are uniformly bounded for all $t < T$, then the solution can be extended to $t < T + T_0$ for some $T_0 > 0$.*

We also have the following results analogous (but weaker comparing) to the situations of smooth solutions. It will be used to prove Theorem 2.2.

COROLLARY 3.2. *Suppose that $\mathbf{\Phi}$ is a solution of (2.6) for $0 \leq t < T$ such that*

$$\lambda = \int_0^T \left|\nabla\mathbf{\Phi}(t)\right|^3_{0,\Omega_t} dt < \infty.$$

Then there exists a $T_0 > 0$ depending only on λ, T and the initial data such that the solution can be extended to $0 \leq t \leq T + T_0$.

Proof. For convenience, all the constants c below depend only on λ, T and the initial data. Since the fluid is incompressible, we only need to show that $|\nabla \mathbf{\Phi}(t)|_\alpha$ is uniformly bounded. By (3.13), we find

$$d(\Omega_t) \leq 2 |\mathbf{\Phi}(t)|_0 \leq c + c \int_0^t |\nabla \mathbf{\Phi}(t)|_0 \, d(\Omega_t) \, dt.$$

It follows from the Gronwall inequality that

$$d(\Omega_t) \leq c \exp\left(c \int_0^t |\nabla \mathbf{\Phi}(t)|_0 \, dt \right) \leq c(\lambda).$$

From Lemma 3.2, we find that

$$\begin{aligned} |\nabla \mathbf{u}(\mathbf{\Phi})(t)|_\alpha &\leq |\nabla \mathbf{u}(t)|_\alpha |\nabla \mathbf{\Phi}(t)|_0^\alpha \\ &\leq c |\nabla \mathbf{\Phi}(t)|_0^\alpha \left(|\boldsymbol{\omega}(t)|_\alpha + |\boldsymbol{\omega}(t)|_0 \delta_t \right) \ln(2 + \delta_t). \end{aligned}$$

From this estimate, (3.5) and (3.14) with $F(\mathbf{\Phi}) = \mathbf{\Phi}$, we deduce that, for $t < T$,

$$\begin{aligned} \|\nabla \mathbf{\Phi}(t)\|_\alpha &\leq c + \int_0^t \left(|\nabla \mathbf{\Phi}(t)|_0 |\nabla \mathbf{u}(\mathbf{\Phi})|_\alpha + |\nabla \mathbf{\Phi}(t)|_\alpha |\nabla \mathbf{u}(\mathbf{\Phi})|_0 \right) dt \\ &\leq c + c \int_0^t |\nabla \mathbf{\Phi}(t)|_0^3 \|\nabla \mathbf{\Phi}(t)\|_\alpha \ln\left(2 + \|\nabla \mathbf{\Phi}(t)\|_\alpha\right) dt, \end{aligned}$$

where we used the fact that $\boldsymbol{\omega}(\mathbf{x}, t) = (\nabla \mathbf{\Phi} \boldsymbol{\omega}_0)(\mathbf{\Phi}^{-1}(\mathbf{x}, t), t)$. It follows from the Gronwall inequality that $\|\nabla \mathbf{\Phi}(t)\|_\alpha \leq c(\lambda, T)$. Consequently, $\|\boldsymbol{\omega}(t)\|_{\alpha, \Omega_t}$ and δ_t are uniformly bounded. The assertion thus follows from Corollary 3.1.

4. Breakdown of C^α Vortex Patches. In this section we shall show that a singularity of a $C^\alpha(\Omega_t)$ vortex patch $\boldsymbol{\omega}$ could occur at $t = T$ only when $\theta(t)$ in (1.5) becomes infinity as $t \to T$.

LEMMA 4.1. *Let* $\mathbf{\Phi}$ *be a* $C^{1+\alpha}(\Omega_t)$ *solution of the system (2.6) and* $\mathbf{u}$ *the velocity field defined by (2.1). Denote by* $\varphi(\mathbf{x}, t) = \varphi_0(\mathbf{\Phi}^{-1}(\mathbf{x}, t))$. *Then for* $s \leq t$, $\mathbf{x} \in \bar{\Omega}_t$

$$\exp\left(-\int_s^t |\nabla \mathbf{u}(t)|_0 \, dt \right) \leq \left| \nabla \left(\mathbf{\Phi}\left(\mathbf{\Phi}^{-1}(\mathbf{x}, t), s \right) \right) \right|_{0, \Omega_t} \leq \exp\left(\int_s^t |\nabla \mathbf{u}(t)|_0 \, dt \right),$$

and for $\mathbf{x} \in \partial\Omega_t$

$$|\nabla\varphi_0|_{\inf,\partial\Omega_0} \exp\left(-\int_{0s}^{t} |\nabla \mathbf{u}(t)|_0\right) \leq |\nabla\varphi(\mathbf{x},t)| \leq |\nabla\varphi_0|_{0,\partial\Omega_0} \exp\left(\int_0^t |\nabla\mathbf{u}(t)|_0\right).$$

Proof. The first inequality follows by the method similar to that in the proof of [1, Lemma 1]. The only difference is that here the domain is time-dependent. However, the proof can be easily extended to our case.

From the definition $\varphi(\mathbf{\Phi}(\mathbf{x},t),t) = \varphi_0$, we find by differentiating in $\mathbf{x}$ that

$$\nabla\varphi(\mathbf{\Phi}(\mathbf{x},t),t) = \left((\nabla\mathbf{\Phi}(\mathbf{x},t))^{\top}\right)^{-1}\nabla\varphi_0,$$

where $A^{\top}$ is the transpose of A. By (2.9), we have

$$\frac{d}{dt}(\nabla\mathbf{\Phi}(\mathbf{x},t))^{-1} = -(\nabla\mathbf{\Phi}(\mathbf{x},t))^{-1}\nabla\mathbf{u}(\mathbf{\Phi}(\mathbf{x},t),t).$$

It follows that for any $\mathbf{x} \in \Omega_0$, $\nabla\varphi(\mathbf{\Phi}(\mathbf{x},t),t)$ solves

$$\frac{d}{dt}\nabla\varphi(\mathbf{\Phi}(\mathbf{x},t),t) = -(\nabla\mathbf{u})^{\top}\nabla\varphi(\mathbf{\Phi}(\mathbf{x},t)). \tag{4.1}$$

By the standard Gronwall lemma we derive

$$|\nabla\varphi_0(\mathbf{x})|\exp\left(-\int_0^t |\nabla\mathbf{u}(t)|_0\right) \leq |\nabla\varphi(\mathbf{\Phi}(\mathbf{x},t),t)| \leq |\nabla\varphi_0(\mathbf{x})|\exp\left(\int_0^t |\nabla\mathbf{u}(t)|_0\right).$$

Proof of Theorem 2.2:

For convenience, we shall sometimes drop all the subscripts $\Omega_t, \partial\Omega_t$. For any $\beta > 0$, denote by

$$\delta_{t,\beta} = \frac{|\nabla\varphi|_{\beta,\partial\Omega_t}}{|\nabla\varphi|_{\inf,\partial\Omega_t}}.$$

Suppose that $\|\boldsymbol{\omega}(t)\|_\beta$ and $\delta_{t,\beta}$ are bounded by a finite number M for $t < T$ for some $0 < \beta \leq \alpha$. From Corollary (3.1), we deduce that the solution $\mathbf{\Phi}$ can be extended, in the space $C^{1+\beta}$, to $t < T + T_0$ for some $T_0 > 0$ depending on M. In particular, $\mathbf{\Phi}$ is Lipschitz for $t < T+T_0$. It then follows from Corollary (3.2) that $\mathbf{\Phi}$ is actually the $C^{1+\alpha}$

solution. By the expressions for $\boldsymbol{\omega}$ and $\delta_{t,\alpha}$, we find that $\|\boldsymbol{\omega}(t)\|_{\alpha}$ and $\delta_{t,\alpha}$ are uniformly bounded. Therefore, it suffices to derive *à priòri* bounds on $\|\boldsymbol{\omega}(t)\|_{\beta}$ and $\delta_{t,\beta}$.

From the equations (2.5) and (2.9), we find that

$$\boldsymbol{\omega}(\mathbf{x},t) = \boldsymbol{\omega}_0\left(\boldsymbol{\Phi}^{-1}(\mathbf{x},t)\right) + \int_0^t (\nabla\mathbf{u}\boldsymbol{\omega})\left(\boldsymbol{\Phi}\left(\boldsymbol{\Phi}^{-1}(\mathbf{x},t),s\right),s\right) ds.$$

Hence

$$\begin{aligned}|\boldsymbol{\omega}(t)|_{\beta} &\leq \left|\boldsymbol{\omega}_0\left(\boldsymbol{\Phi}^{-1}(\mathbf{x},t)\right)\right|_{\beta} + \int_0^t \left|(\nabla\mathbf{u}\boldsymbol{\omega})\left(\boldsymbol{\Phi}\left(\boldsymbol{\Phi}^{-1}(\mathbf{x},t),s\right),s\right)\right|_{\beta} ds \\ &= k_1(t) + \int_0^t k_2(s)\, ds.\end{aligned}$$

Since $|f(g)|_{\beta} \leq |f|_{\beta}\, |g|_0^{\beta}$, by Lemma 4.1, we deduce

$$k_1(t) \leq |\boldsymbol{\omega}_0|_{\beta} \exp\left(\beta\int_0^t |\nabla\mathbf{u}(\tau)|_0\, d\tau\right)$$

and

$$k_2(s) \leq \left(|\nabla\mathbf{u}\boldsymbol{\omega}(s)|_{\beta}\right)\exp\left(\beta\int_s^t |\nabla\mathbf{u}(\tau)|_0\, d\tau\right).$$

We now estimate $|\nabla\mathbf{u}\boldsymbol{\omega}(s)|_{\beta}$. Write

$$\nabla\mathbf{u}\boldsymbol{\omega}(s) = m_1\boldsymbol{\omega}(s) + m_2\boldsymbol{\omega}(s) + m_3\boldsymbol{\omega}(s).$$

where the notation m_i is the same as in (3.8). Notice that $\boldsymbol{\omega}(s)$ is divergence free in R^3. Hence $\boldsymbol{\omega}(\mathbf{x},s)$ is divergence free in Ω_s and tangent to $\partial\Omega_s$. It follows that (see [4])

$$\mathbf{V}\boldsymbol{\omega}(\mathbf{x},\mathrm{s}) = \int_{\Omega} \frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\left(\boldsymbol{\omega}(\mathbf{x},\mathrm{s}) - \boldsymbol{\omega}(\mathbf{y},\mathrm{s})\right) d\mathbf{y}.$$

Consequently, as in the proof of Lemma 3.2, we have

$$|m_3\boldsymbol{\omega}(\mathrm{s})|_{\beta} \leq c\left(|\mathbf{V}|_0 + 1\right)|\boldsymbol{\omega}(\mathrm{s})|_{\beta}. \tag{4.2}$$

After applying (4.2), we obtain

$$|m_3\boldsymbol{\omega}|_{\beta} = |(\boldsymbol{\omega}\times\mathbf{V})\,\boldsymbol{\omega}|_{\beta} = |\boldsymbol{\omega}\times(\mathbf{V}\boldsymbol{\omega})|_{\beta} \leq c\,|\boldsymbol{\omega}|_0\left(|\mathbf{V}|_0 + 1\right)|\boldsymbol{\omega}|_{\beta}.$$

Analogously, we can easily derive

$$|m_1\boldsymbol{\omega}+m_2\boldsymbol{\omega}|_\beta \leq c\,|\boldsymbol{\omega}|_0\,(|\mathbf{V}|_0+1)\,|\boldsymbol{\omega}|_\beta\,.$$

It follows

$$\begin{aligned} k_2 &\leq |(\nabla\mathbf{u})\,\boldsymbol{\omega}\,(s)|_\beta \exp\left(\beta\int_s^t |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right) \\ &\leq c\,|\boldsymbol{\omega}\,(s)|_0\,\|\boldsymbol{\omega}\,(s)\|_\beta\left(1+\left|\ln\left(\delta_{s,\beta}d\,(\Omega_s)+\frac{\|\boldsymbol{\omega}\,(s)\|_\beta}{|\boldsymbol{\omega}\,(s)|_0}d\,(\Omega_s)\right)\right|\right)\cdot \\ &\quad\cdot\exp\left(\beta\int_s^t |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right). \end{aligned}$$

Therefore,

$$\begin{aligned} |\boldsymbol{\omega}\,(t)|_\beta &\leq |\boldsymbol{\omega}_0|_\beta \exp\left(\beta\int_0^t |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right)+c\int_0^t |\boldsymbol{\omega}|_0\,\|\boldsymbol{\omega}\,(s)\|_\beta\,\cdot \\ &\quad\cdot\left(1+\left|\ln\left(\delta_{s,\beta}d\,(\Omega_s)+\frac{\|\boldsymbol{\omega}\|_\beta}{|\boldsymbol{\omega}|_0}d\,(\Omega_s)\right)\right|\right)\exp\left(\beta\int_s^t |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right)\,ds. \end{aligned}$$

Multiplying by $\exp\left(-\int_0^t |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right)$ in the above inequality, we find that

$$\begin{aligned} |\boldsymbol{\omega}\,(t)|_\beta \exp\left(-\int_0^t |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right) &\leq |\boldsymbol{\omega}_0|_\beta + c\int_0^t |\boldsymbol{\omega}\,(s)|_0\,\|\boldsymbol{\omega}\,(s)\|_\beta\,\cdot \\ &\quad\cdot\left(1+\left|\ln\left(\delta_{s,\beta}d\,(\Omega_s)+\frac{\|\boldsymbol{\omega}\|_\beta}{|\boldsymbol{\omega}|_0}d\,(\Omega_s)\right)\right|\right)\exp\left(-\int_0^s |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right)\,ds. \end{aligned} \tag{4.3}$$

Next, from (4.1), we have, for $\mathbf{x}\in\Omega_t$,

$$\nabla\varphi\,(\mathbf{x},t)=\nabla\varphi_0\left(\Phi^{-1}\,(\mathbf{x},t)\right)-\int_0^t \left((\nabla\mathbf{u})^\top\nabla\varphi\right)\left(\boldsymbol{\Phi}\left(\boldsymbol{\Phi}^{-1}\,(\mathbf{x},t)\,,s\right),s\right)\,ds.$$

Analogously, it follows that

$$\begin{aligned} |\nabla\varphi\,(t)|_\beta \exp\left(-\int_0^t |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right) &\leq |\nabla\varphi_0|_\beta \\ &+c\int_0^t \left(\left|(\nabla\mathbf{u})^\top\nabla\varphi\,(s)\right|_\beta\right)\exp\left(-\int_0^s |\nabla\mathbf{u}\,(\tau)|_0\,d\tau\right)\,ds. \end{aligned} \tag{4.4}$$

We claim that

$$
\begin{aligned}
\left|(\nabla \mathbf{u})^{\mathsf{T}} \nabla\varphi(s)\right|_{\beta} \leq\ & c\left(|\boldsymbol{\omega}(s)|_{\beta}|\nabla\varphi(s)|_{0}+|\boldsymbol{\omega}(s)|_{0}|\nabla\varphi(s)|_{\beta}\right)\cdot \\
& \cdot\left(1+\left|\ln\left(\delta_{s,\beta} d(\Omega_{s})+\frac{\|\boldsymbol{\omega}\|_{\beta}}{|\boldsymbol{\omega}|_{0}} d(\Omega_{s})\right)\right|\right). \qquad (4.5)
\end{aligned}
$$

Notice that the estimate (4.5) is much better than the general estimate

$$
\left|(\nabla \mathbf{u})^{\mathsf{T}} \nabla\varphi(s)\right|_{\beta} \leq |\nabla\mathbf{u}|_{\beta}|\nabla\varphi(s)|_{0}+|\nabla\mathbf{u}|_{0}|\nabla\varphi(s)|_{\beta}.
$$

Indeed, from the expression (3.8) for $\nabla\mathbf{u}$, we can write

$$
\begin{aligned}
(\nabla\mathbf{u}(\mathbf{x}))^{\mathsf{T}}\nabla\varphi(\mathbf{x}) &= \left(\int_{\Omega_t}(\boldsymbol{\omega}(\mathbf{y})-\boldsymbol{\omega}(\mathbf{x}))\times\frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{3}}\,d\mathbf{y}\right)^{\mathsf{T}}\nabla\varphi(\mathbf{x})+ \\
&+\frac{1}{2}(\boldsymbol{\omega}(\mathbf{x})\times\mathbf{I})^{\mathsf{T}}\nabla\varphi(\mathbf{x})+(\boldsymbol{\omega}(\mathbf{x})\times\mathbf{V}(\mathbf{x}-\mathbf{y}))^{\mathsf{T}}\nabla\varphi(\mathbf{x}) \qquad (4.6) \\
&= m_1+m_2+m_3.
\end{aligned}
$$

As in the proof of Lemma 3.2, we can easily find that $|m_1+m_2|_{\beta}$ is bounded by the right-hand side of (4.5).

To estimate m_3, we observe that one can write

$$
\begin{aligned}
m_3 \ =\ & \int_{\Omega_t}\left(\boldsymbol{\omega}(\mathbf{x})\times\frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{3}}\right)^{\mathsf{T}}(\nabla\varphi(\mathbf{x})-\nabla\varphi(\mathbf{y}))\,d\mathbf{y} \qquad (4.7) \\
& +\int_{\Omega_t}\left(\boldsymbol{\omega}(\mathbf{x})\times\frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{3}}\right)^{\mathsf{T}}\nabla\varphi(\mathbf{y})\,d\mathbf{y}.
\end{aligned}
$$

We now show

$$
\int_{\Omega_t}\left(\boldsymbol{\omega}(\mathbf{x})\times\frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{3}}\right)^{\mathsf{T}}\nabla\varphi(\mathbf{y})\,d\mathbf{y}=0. \qquad (4.8)
$$

To see this, we first write $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$, where σ_i is the i-th column vector. Note that

$$
\begin{aligned}
\frac{\sigma_i(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{3}} &= D_i\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^{3}}\right) \qquad (4.9) \\
&= D_i(\nabla g(\mathbf{x}-\mathbf{y}))=\nabla(D_i g(\mathbf{x}-\mathbf{y}))
\end{aligned}
$$

where $g(\mathbf{x}) = |\mathbf{x}|^{-1}$. Hence, by definition (see §2),

$$(\boldsymbol{\omega} \times \boldsymbol{\sigma})^{\mathsf{T}} = \begin{pmatrix} \boldsymbol{\omega} \times \sigma_1 \\ \boldsymbol{\omega} \times \sigma_2 \\ \boldsymbol{\omega} \times \sigma_3 \end{pmatrix},$$

A direct computation leads to

$$\frac{(\boldsymbol{\omega} \times \boldsymbol{\sigma})^{\mathsf{T}} \nabla \varphi}{|\mathbf{x}-\mathbf{y}|^3} = \frac{1}{|\mathbf{x}-\mathbf{y}|^3} \begin{pmatrix} \sigma_1 \times \nabla\varphi \\ \sigma_2 \times \nabla\varphi \\ \sigma_3 \times \nabla\varphi \end{pmatrix} \boldsymbol{\omega}.$$

Hence

$$\int_{\Omega_t} \left(\boldsymbol{\omega}(\mathbf{x}) \times \frac{\sigma(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} \right)^{\mathsf{T}} \nabla\varphi(\mathbf{y})\, d\mathbf{y}$$

$$= \int_{\Omega_t} \frac{1}{|\mathbf{x}-\mathbf{y}|^3} \cdot \begin{pmatrix} \sigma_1(\mathbf{x}-\mathbf{y}) \times \nabla\varphi(\mathbf{y}) \\ \sigma_2(\mathbf{x}-\mathbf{y}) \times \nabla\varphi(\mathbf{y}) \\ \sigma_3(\mathbf{x}-\mathbf{y}) \times \nabla\varphi(\mathbf{y}) \end{pmatrix} d\mathbf{y} \cdot \boldsymbol{\omega}(\mathbf{x}).$$

It remains to show that for $i = 1, 2, 3$,

$$\int_{\Omega_t} \frac{\sigma_i(\mathbf{x}-\mathbf{y}) \times \nabla\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y} = \mathbf{0}. \tag{4.10}$$

By direct computing and using (4.9), we find

$$\int_{\Omega_t} \frac{\sigma_i(\mathbf{x}-\mathbf{y}) \times \nabla\varphi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y} = \int_{\Omega_t} \begin{pmatrix} (\nabla D_i g(\mathbf{x}-\mathbf{y})) \cdot (\nabla\varphi \times \mathbf{e}_1) \\ (\nabla D_i g(\mathbf{x}-\mathbf{y})) \cdot (\nabla\varphi \times \mathbf{e}_2) \\ (\nabla D_i g(\mathbf{x}-\mathbf{y})) \cdot (\nabla\varphi \times \mathbf{e}_3) \end{pmatrix} d\mathbf{y},$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are orthogonal basis vectors and

$$\begin{aligned} \nabla\varphi \times \mathbf{e}_1 &= (0, D_3\varphi, -D_2\varphi), \\ \nabla\varphi \times \mathbf{e}_2 &= (-D_3\varphi, 0, D_1\varphi), \\ \nabla\varphi \times \mathbf{e}_2 &= (D_2\varphi, -D_1\varphi, 0). \end{aligned}$$

Notice that each of these vectors $\nabla\varphi \times \mathbf{e}_i$ is divergence free and tangent to $\partial\Omega_t$ (the normal vector is $\nabla\varphi$). Hence by the divergence theorem, it leads to

$$\int_{\Omega_t} (\nabla D_i g(\mathbf{x}-\mathbf{y})) \cdot (\nabla\varphi \times \mathbf{e}_j)\, d\mathbf{y} = \int_{\partial\Omega_t} D_i g(\mathbf{x}-\mathbf{y}) (\nabla\varphi \times \mathbf{e}_j) \cdot \frac{\nabla\varphi}{|\nabla\varphi|} = 0.$$

The identity (4.10) thus follows, so does (4.7). By using (4.7), we deduce that the last m_3 in (4.6) can be estimated as for m_1 :

$$\begin{aligned} |m_3|_\beta &\le c\left(|\boldsymbol{\omega}(s)|_\beta |\nabla\varphi(s)|_0 + |\boldsymbol{\omega}(s)|_0 |\nabla\varphi(s)|_\beta\right)\cdot \\ &\quad \cdot\left(1+\left|\ln\left(\delta_{s,\beta} d(\Omega_s) + \frac{\|\boldsymbol{\omega}\|_\beta}{|\boldsymbol{\omega}|_0} d(\Omega_s)\right)\right|\right). \end{aligned}$$

Combining the estimates for m_1, m_2, m_3, we derive

$$\begin{aligned} \left|(\nabla\mathbf{u})^\top \nabla\varphi(s)\right|_\beta &\le c\left(|\boldsymbol{\omega}(s)|_\beta |\nabla\varphi(s)|_0 + |\boldsymbol{\omega}(s)|_0 |\nabla\varphi(s)|_\beta\right) \\ &\quad \cdot\left(1+\left|\ln\left(\delta_{s,\beta} d(\Omega_s) + \frac{\|\boldsymbol{\omega}\|_\beta}{|\boldsymbol{\omega}|_0} d(\Omega_s)\right)\right|\right). \end{aligned}$$

Our previous claim (4.5) has been established.

It follows from (4.4) and (4.5) that

$$\begin{aligned} &|\nabla\varphi(t)|_\beta \exp\left(-\int_0^t |\nabla\mathbf{u}(\tau)|_0 \, d\tau\right) \le \\ &\le |\nabla\varphi_0|_\beta + c\int_0^t \left(|\boldsymbol{\omega}(s)|_\beta |\nabla\varphi(s)|_0 + |\boldsymbol{\omega}(s)|_0 |\nabla\varphi(s)|_\beta\right)\cdot \\ &\quad \cdot\left(1+\left|\ln\left(\delta_{s,\beta} d(\Omega_s) + \frac{\|\boldsymbol{\omega}\|_\beta}{|\boldsymbol{\omega}|_0} d(\Omega_s)\right)\right|\right) \exp\left(-\int_0^s |\nabla\mathbf{u}(\tau)|_0 \, d\tau\right) ds. \qquad (4.11) \\ &\le |\nabla\varphi_0|_\beta + c\int_0^t \left(\min\left(|\nabla\varphi(s)|_0, |\boldsymbol{\omega}(s)|_\beta\right) + |\boldsymbol{\omega}(s)|_0 + 1\right)\left(\|\boldsymbol{\omega}\|_\beta + \|\nabla\varphi\|_\beta\right)\cdot \\ &\quad \cdot\left(1+\left|\ln\left(\delta_{s,\beta} d(\Omega_s) + \frac{\|\boldsymbol{\omega}\|_\beta}{|\boldsymbol{\omega}|_0} d(\Omega_s)\right)\right|\right) \exp\left(-\int_0^s |\nabla\mathbf{u}(\tau)|_0 \, d\tau\right) ds. \end{aligned}$$

Notice that in (4.11), all the norms on $\nabla\varphi$ are taken in any subdomain of Ω_t containing $\mathbf{\Phi}(G,t)$, where G is the support of $\nabla\varphi_0$. We can choose φ_0 such that the support of $\nabla\varphi_0$ is as close to $\partial\Omega_0$ as we want. By an appropriate approximation argument, we find that (4.11) remains true if all the norms on $\nabla\varphi$ are understood to be over $\partial\Omega_t$.

Introduce the functions

$$\begin{aligned} a(t) &= |\boldsymbol{\omega}(t)|_0 + \min\left(|\nabla\varphi(t)|_{0,\partial\Omega_t}, |\boldsymbol{\omega}|_\beta\right), \\ b(t) &= \int_0^t |\nabla\mathbf{u}(\tau)|_0 \, d\tau, \end{aligned}$$

$$f(t) = \left(\|\boldsymbol{\omega}(t)\|_{\beta} + \|\varphi(t)\|_{1+\beta,} + e^{b(t)}\right) e^{-b(t)}.$$

Noticing by Lemma 4.1 that $\delta_{t,\beta} = |\nabla\varphi(t)|_{\beta,\partial\Omega_t} / |\nabla\varphi(t)|_{\text{inf},\partial\Omega_t} \leq ce^{b(t)} |\nabla\varphi(t)|_{\beta,\partial\Omega_t}$, we thus arrive at the following inequality

$$f(t) \leq c + c\int_0^t a(s) f(s) \left(1 + \ln\left|\left(\delta_{s,\beta} d(\Omega_s) + \frac{\|\boldsymbol{\omega}\|_{\beta}}{|\boldsymbol{\omega}|_0} d(\Omega_s)\right)\right|\right). \tag{4.12}$$

Since $d(\Omega_t) \leq 2|\boldsymbol{\Phi}|_0 \leq cb(t)$, $|\nabla\varphi|_{\text{inf},\partial\Omega_t} \geq ce^{b(t)}$, $ce^{-b(t)} \leq |\boldsymbol{\omega}(t)|_0 \leq ce^{b(t)}$ (by Lemma 4.1), it is easy to see that

$$\begin{aligned} &\left|\ln\left(\delta_{t,\beta} d(\Omega_t) + \frac{\|\boldsymbol{\omega}(t)\|_{\beta}}{|\boldsymbol{\omega}(t)|_0} d(\Omega_t)\right)\right| \\ \leq\ &\ln\left(\left(\|\boldsymbol{\omega}(t)\|_{\beta} + \|\nabla\varphi(t)\|_{\beta} + e^{b(t)}\right)^3\right) \leq c\ln f + cb(t). \end{aligned} \tag{4.13}$$

Therefore, we obtain from (4.12)

$$f(t) \leq c + c\int_0^t a(s) f(s) (\ln f(s) + b(s))\, ds,$$

where c is a constant depending only on the initial data. Denote by $g(t)$ the function on the right-hand side of the above inequality:

$$g(t) = c + c\int_0^t a(s) f(s) (\ln f(s) + b(s))\, ds.$$

Then

$$g' = ca(f\ln f + bf) \leq cag(\ln g + b).$$

It follows

$$\frac{d}{dt}\ln g(t) \leq ca(\ln g + b).$$

By the standard Gronwall inequality, and recalling the definition

$$\theta(t) = \int_0^t a(s)\, ds,$$

we obtain, since $b(t)$ is increasing,

$$\begin{aligned} \ln f(t) &\leq \ln g(t) \\ \text{(4.14)} \qquad &\leq c\exp\left(\int_0^t a(s)\,ds\right)\left(1+\int_0^t \exp\left(-\int_0^s a(\tau)\,d\tau\right) a(s)\,b(s)\,ds\right) \\ &= ce^{\theta(t)}\left(1+\int_0^t e^{-\theta(s)} a(s)\,b(s)\,ds\right) \leq ce^{\theta(t)}\left(1+b(t)\,\theta(t)\right). \end{aligned}$$

Recalling the definitions of $b(t)$ and $f(t)$, and applying (4.13) and Lemma 3.2, we arrive at

$$\begin{aligned} \text{(4.15)} \qquad b(t) &= \int_0^t |\nabla \mathbf{u}(s)|_0\,ds \leq c\int_0^t |\boldsymbol{\omega}(s)|_0 \ln\left(\delta_{s,\beta} d(\Omega)^\beta + \frac{\|\boldsymbol{\omega}\|_\beta}{|\boldsymbol{\omega}|_0}\right) ds \\ &\leq c\int_0^t |\boldsymbol{\omega}(s)|_0 \left(\ln f(s) + b(s)\right) ds. \end{aligned}$$

Substituting (4.14) into (4.15), it yields

$$\begin{aligned} b(t) &\leq c\int_0^t |\boldsymbol{\omega}(s)|_0\, e^{\theta(s)}\,ds + c\int_0^t \left(|\boldsymbol{\omega}(s)|_0 \int_0^s a(\tau)\,d\tau e^{\theta(s)}\right) b(s)\,ds \\ &\leq c\int_0^t |\boldsymbol{\omega}(s)|_0\, e^{\theta(s)} \left(1+\theta(s)\,b(s)\right) ds. \end{aligned}$$

Notice that $\int_0^t |\boldsymbol{\omega}(s)|_0\,ds \leq \theta(t)$. From the above inequality, it is now easy to see that the boundedness of $\theta(t)$ guarantees that $b(t)$ is bounded for $t \leq T$. Consequently, from (4.14) and Lemma 4.1, $\delta_{t,\beta}$ and $\|\boldsymbol{\omega}(t)\|_{\beta,\Omega_t}$ are also bounded for $t \leq T$. By apply Corollary 3.1, one can extend the solution to $t < T + T_0$ for some $T_0 > 0$. The proof of Theorem 2.2 is complete.

For two dimensional constant vortex patch problem, we know that $\boldsymbol{\omega}$ is tangent to $\partial\Omega$, and $|\boldsymbol{\omega}|_{\alpha,\Omega_t} = 0$, $|\boldsymbol{\omega}(t)|_{0,\Omega_t} = |\boldsymbol{\omega}_0|_{0,\Omega_0}$. We thus obtain global existence of constant vortex patches. This result was derived in [4] and [5].

5. Axisymmetric flows. In this section, we consider the vortex patch problem for flows with axisymmetric structures. For simplicity we choose $x_3 - axis$ as the symmetric axis. As in [12], we call a flow is axisymmetric in $x_3 - axis$ if its velocity and vorticity have the following structures

$$\mathbf{u} = \mathbf{u}(r, x_3, t) = u_r(r, x_3, t)\,\mathbf{e}_r + u_3(r, x_3, t)\,\mathbf{e}_3,$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}(r, x_3, t) = \omega_\theta(r, x_3, t)\,\mathbf{e}_\theta,$$

where $r^2 = x_1^2 + x_2^2$, $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_3$ are the standard basis vectors under cylindrical coordinate system:

$$\mathbf{e}_r = \frac{1}{r}\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}, \quad \mathbf{e}_\theta = \frac{1}{r}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thanks to the axisymmetric structures, the vorticity equation (1.1) reduces to

$$\frac{d}{dt}\xi(\boldsymbol{\Phi}(\mathbf{x}, t), t) = 0$$

where $\xi = \omega_\theta / r$ is the vorticity strength with respect to the symmetric axis. Hence the vorticity has the relation

$$\xi(\boldsymbol{\Phi}(\mathbf{x}, t), t) = \xi_0(r, x_3), \text{ or } \boldsymbol{\omega}(\mathbf{y}, t) = \xi_0\left(\boldsymbol{\Phi}^{-1}(\mathbf{y}, t)\right) r_{\mathbf{y}} \mathbf{e}_\theta(\mathbf{y}), \tag{5.1}$$

where we used the notation $r_{\mathbf{z}}$ for the distance of z to the symmetric axis, $\xi_0(\mathbf{y}) = \omega_\theta^0(\mathbf{y}) / r_{\mathbf{y}}$. It follows that the integral equation (2.6) reduces to

$$\boldsymbol{\Phi}(\mathbf{x}, t) = \mathbf{x} + \int_0^t \int_{\Omega_t} \mathbf{K}(\boldsymbol{\Phi}(\mathbf{x}, s) - \mathbf{y})\, \xi_0\left(\boldsymbol{\Phi}^{-1}(\mathbf{y}, t)\right) r_{\mathbf{y}} \mathbf{e}_\theta(\mathbf{y})\, d\mathbf{y} ds. \tag{5.2}$$

Notice that since the fluid is incompressible, we have

$$\|\xi\|_{L^p(R^3)} = \|\xi_0\|_{L^p(R^3)} \quad \text{for } p \geq 1. \tag{5.3}$$

LEMMA 5.1. *Suppose that the flow is axisymmetric and initially $\omega_\theta^0(r, x_3) = \varpi_0(r, x_3)\chi_{\Omega_0}$ with $\varpi_0 \in C^\alpha(\Omega_0)$, $\partial\Omega_0 \in C^{\alpha+1}$ and $\alpha > 1/3$. Then the $C^{1+\alpha}(\Omega_0)$ solution $\boldsymbol{\Phi}$ of system (5.2) satisfies*

$$|\boldsymbol{\Phi}|_{0,\Omega_0} + |\mathbf{u}|_{0,\Omega_0} \leq c\left\|\omega_\theta^0\right\|_\alpha \exp\left(ce^{c\left(\left\|\omega_\theta^0\right\|_\alpha + 1\right)t}\right), \tag{5.4}$$

where c is a constant depending only on α and $d(\Omega_0)$.

Notice that the assumption implies that the initial domain Ω_0 is axisymmetric.

Proof. By using the relation (5.1) for the vorticity field, we can write the velocity as

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= \int_{\Omega_t} \mathbf{K}(\mathbf{x}-\mathbf{y})\,\xi_0\left(\mathbf{\Phi}^{-1}(\mathbf{y},t)\right) r_{\mathbf{y}}\mathbf{e}_\theta(\mathbf{y})\,d\mathbf{y} \\ &= \int_{\Omega_t\cap B_1(\mathbf{x})} + \int_{\Omega_t\setminus B_1(\mathbf{x})} = k_1 + k_2. \end{aligned} \tag{5.5}$$

Since $\alpha > 1/3$, we can choose p such that $2(1+\alpha)^{-1} < p < 3/2$. Then $q = p(p-1)^{-1} = 1/(1-1/p) < 2(1-\alpha)^{-1}$. By the Hölder inequality, we obtain

$$\begin{aligned} |k_1| &\le \left(\int_{B_\varepsilon(\mathbf{x})} |\mathbf{K}(\mathbf{x}-\mathbf{y})\,r_{\mathbf{y}}|^p\,d\mathbf{y}\right)^{1/p} \left(\int_{\Omega_t} \left|\xi_0\left(\mathbf{\Phi}^{-1}(\mathbf{y},t)\right)\right|^q d\mathbf{y}\right)^{1/q} \\ &= k_{11}k_{12}. \end{aligned}$$

It follows from (5.3) that, since $|\xi_0(\mathbf{y})| \le |\omega_\theta^0|_\alpha (r_{\mathbf{y}})^{\alpha-1}$,

$$\begin{aligned} k_{12} &= \left(\int_{\Omega_t} \left|\xi_0\left(\mathbf{\Phi}^{-1}(\mathbf{y},t)\right)\right|^q d\mathbf{y}\right)^{1/q} = \left(\int_{\Omega_0} |\xi_0(\mathbf{y})|^q\,d\mathbf{y}\right)^{1/q} \\ &\le \left|\omega_\theta^0\right|_\alpha \left(\int_{\Omega_0} (r_{\mathbf{y}})^{q(\alpha-1)}\,d\mathbf{y}\right)^{1/q} \\ &\le d(\Omega_0)^{1/q}\left|\omega_\theta^0\right|_\alpha \left(\int_0^{d(\Omega_0)} \rho^{q(\alpha-1)+1}\,d\rho\right)^{1/q} \le c\left|\omega_\theta^0\right|_\alpha, \end{aligned} \tag{5.6}$$

where c is a constant depending only on α and $d(\Omega_0)$. Notice that in deducing the last inequality in (5.6), we have used cylindrical coordinate system and the fact $q < 2(1-\alpha)^{-1}$. Since $p < 3/2$ and $|\mathbf{K}(\mathbf{z})| \le |\mathbf{z}|^{-2}$, we deduce that for $\mathbf{x} \in \Omega_t$

$$\begin{aligned} k_{11} &\le \left(\int_{B_1(\mathbf{x})} \frac{(r_{\mathbf{y}})^p}{|\mathbf{x}-\mathbf{y}|^{2p}}d\mathbf{y}\right)^{1/p} \le \left(\int_{B_1(\mathbf{x})} \frac{(|\mathbf{x}| + r_{\mathbf{x}-\mathbf{y}})^p}{|\mathbf{x}-\mathbf{y}|^{2p}}d\mathbf{y}\right)^{1/p} \\ &\le c(|\mathbf{x}|+1)\left(\int_{B_1(\mathbf{x})} \frac{1}{|\mathbf{x}-\mathbf{y}|^{2p}}d\mathbf{y}\right)^{1/p} \end{aligned}$$

$$\leq \; c\left(|\mathbf{x}|+1\right)\left(\int_0^1 \frac{1}{\rho^{2p-2}}d\rho\right)^{1/p} \leq c\left(|\mathbf{\Phi}(t)|_0+1\right).$$

Hence

$$|k_1| \leq c\left\|\omega_\theta^0\right\|_\alpha \left(|\mathbf{\Phi}(t)|_0+1\right). \tag{5.7}$$

Next, we apply the Hölder inequality with $p=3/2$, $q=3$ to obtain

$$\begin{aligned} |k_2| \;&\leq\; \left(\int_{\Omega_t\backslash B_1(\mathbf{x})} \left|\mathbf{K}(\mathbf{x}-\mathbf{y})\, r_{\mathbf{y}}\right|^{3/2} d\mathbf{y}\right)^{2/3} \\ &\quad\times\left(\int_{\Omega_t\backslash B_1(\mathbf{x})} \left|\xi_0\left(\mathbf{\Phi}^{-1}(\mathbf{y},s)\right)\right|^3 d\mathbf{y}\right)^{1/3} \;=\; k_{21}k_{22}. \end{aligned}$$

As in estimating k_{12} above, since $3 < 2(1-\alpha)^{-1}$, we can derive that $k_{22} \leq c\left\|\omega_\theta^0\right\|_\alpha$.

Finally,

$$\begin{aligned} |k_{21}| \;&\leq\; \left(\int_{\Omega_t\backslash B_1(\mathbf{x})} \frac{(r_{\mathbf{y}})^{3/2}}{|\mathbf{x}-\mathbf{y}|^3}d\mathbf{y}\right)^{2/3} \\ &\leq\; \left(\int_{\Omega_t\backslash B_1(\mathbf{x})} \frac{|\mathbf{x}|^{3/2}}{|\mathbf{x}-\mathbf{y}|^3}d\mathbf{y}\right)^{2/3} + \left(\int_{\Omega_t\backslash B_1(\mathbf{x})} \frac{(r_{\mathbf{x}-\mathbf{y}})^{3/2}}{|\mathbf{x}-\mathbf{y}|^3}d\mathbf{y}\right)^{2/3} \\ &\leq\; |\mathbf{x}|\left(\int_1^{2|\Phi|_0} \frac{1}{\rho}d\rho\right)^{2/3} + \left(\int_1^{2|\Phi|_0} \rho^{1/2}d\rho\right)^{2/3} \\ &\leq\; c\,|\mathbf{\Phi}|_0\left(1+\ln\left(2+|\mathbf{\Phi}|_0\right)\right). \end{aligned}$$

Therefore,

$$|k_2| \leq c\left\|\omega_\theta^0\right\|_\alpha |\mathbf{\Phi}|_0\left(1+\ln\left(2+|\mathbf{\Phi}|_0\right)\right). \tag{5.8}$$

From (5.5)-(5.8), we arrive at

$$|\mathbf{u}(t)|_0 \leq c + c\left\|\omega_\theta^0\right\|_\alpha \left(|\mathbf{\Phi}(s)|_0+1\right)\ln\left(2+|\mathbf{\Phi}(s)|_0\right)\,ds.$$

Hence

$$|\mathbf{\Phi}(t)|_0 \leq c + \int_0^t |u(s)|_0\,ds \leq c + c\left\|\omega_\theta^0\right\|_\alpha \int_0^t \left(|\mathbf{\Phi}(s)|_0+1\right)\ln\left(2+|\mathbf{\Phi}(s)|_0\right)\,ds.$$

The assertion (5.4) follows from the Gronwall inequality.

LEMMA 5.2. *Under the assumptions of Lemma 5.1, we have*

$$|\mathbf{u}(\mathbf{x},t)| \le c \left\|\omega_\theta^0\right\|_0 \left(|\mathbf{\Phi}(t)|_0 + 1\right) r_{\mathbf{x}},$$

where c is a constant depending only on α and $d(\Omega_0)$.

Proof. For any $\mathbf{x} = (x_1, x_2, x_3) \in \Omega_t, 0 < r_{\mathbf{x}} < 1$, let $\varepsilon = 2r_{\mathbf{x}}$. By symmetry, $\mathbf{u}(0,0,x_3,t) = \mathbf{\Phi}(0,0,x_3,t) = \mathbf{0}$. We denote by $\tilde{\mathbf{x}} = (0,0,x_3)$. Notice that $|\mathbf{x} - \tilde{\mathbf{x}}| = r_{\mathbf{x}}$ and consequently $B_{\varepsilon/2}(\tilde{\mathbf{x}}) \subset B_\varepsilon(\mathbf{x}) \subset B_{3\varepsilon/2}(\tilde{\mathbf{x}})$. We now write

$$\begin{aligned}
\mathbf{u}(\mathbf{x},t) &= \mathbf{u}(r,x_3,t) - \mathbf{u}(0,x_3,t) \\
&= \int\limits_{\Omega_t \cap B_\varepsilon(\mathbf{x})} \mathbf{K}(\mathbf{x}-\mathbf{y})\, \xi_0\left(\mathbf{\Phi}^{-1}\right) r_{\mathbf{y}} \mathbf{e}_\theta(\mathbf{y})\, d\mathbf{y} \\
&\quad + \int\limits_{\Omega_t \cap B_\varepsilon(\mathbf{x})} \mathbf{K}(\tilde{\mathbf{x}}-\mathbf{y})\, \xi_0\left(\mathbf{\Phi}^{-1}\right) r_{\mathbf{y}} \mathbf{e}_\theta\, d\mathbf{y} \\
&\quad + \int\limits_{\Omega_t \setminus B_\varepsilon(\mathbf{x})} \left(\mathbf{K}(\mathbf{x}-\mathbf{y}) - \mathbf{K}(\tilde{\mathbf{x}}-\mathbf{y})\right) \xi_0\left(\mathbf{\Phi}^{-1}\right) r_{\mathbf{y}} \mathbf{e}_\theta\, d\mathbf{y} \\
&= m_1 + m_2 + m_3.
\end{aligned} \tag{5.9}$$

We proceed to estimate each term m_i. As in the proof of Lemma 5.1, we choose p, q such that $1/p + 1/q = 1$, $2(1+\alpha)^{-1} < p < 3/2$, $q < 2(1-\alpha)^{-1}$. Applying the Hölder inequality, we find that

$$\begin{aligned}
|m_1| &\le \left(\int\limits_{B_\varepsilon(\mathbf{x})} |\mathbf{K}(\mathbf{x}-\mathbf{y})\, r_{\mathbf{y}}|^p\, d\mathbf{y}\right)^{1/p} \left(\int\limits_{\Omega_t} \left|\xi_0\left(\mathbf{\Phi}^{-1}(\mathbf{y},t)\right)\right|^q d\mathbf{y}\right)^{1/q} \\
&= m_{11} m_{12}.
\end{aligned}$$

From (5.6), we find that $m_{12} \le c \|\omega_\theta^0\|_\alpha$. Since $r_{\mathbf{y}} \le |\mathbf{y} - \mathbf{x}| + r_{\mathbf{x}} \le 3r_{\mathbf{x}}$ for any $\mathbf{y} \in B_\varepsilon(\mathbf{x})$, the first term m_{11} can be estimated as

$$\begin{aligned}
m_{11} &\le \left(\int\limits_{B_\varepsilon(\mathbf{x})} |\mathbf{x}-\mathbf{y}|^{-2p} (r_{\mathbf{y}})^p\, d\mathbf{y}\right)^{1/p} \le c r_{\mathbf{x}} \left(\int\limits_{B_\varepsilon(\mathbf{x})} |\mathbf{x}-\mathbf{y}|^{-2p}\, d\mathbf{y}\right)^{1/p} \\
&\le c r_{\mathbf{x}} \left(\int_0^\varepsilon \rho^{2-2p} d\mathbf{y}\right)^{1/p} \le c r_{\mathbf{x}},
\end{aligned}$$

where we used the fact that $p < 3/2$. Hence $|m_1| \leq c\|\omega_\theta^0\|_\alpha r_{\mathbf{x}}$. By using the fact $B_\varepsilon(\mathbf{x}) \subset B_{3\varepsilon/2}(\tilde{\mathbf{x}})$ and noticing that $r_{\mathbf{y}} \leq 3r_{\mathbf{x}}$ for $\mathbf{y} \in B_\varepsilon(\mathbf{x})$, we can derive analogously that $|m_2| \leq c\|\omega_\theta^0\|_\alpha r_{\mathbf{x}}$. We have thus established

$$|m_1| + |m_2| \leq c\left\|\omega_\theta^0\right\|_\alpha r_{\mathbf{x}}. \tag{5.10}$$

It remains to estimate m_3. We observe from (2.2) that each non-zero element of the matrix $\mathbf{K}(\mathbf{x})$ has the form $x_i/|\mathbf{x}|^{-3}$. Since

$$\frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|^3} - \frac{\tilde{x}_i - y_i}{|\tilde{\mathbf{x}}-\mathbf{y}|^3} = \frac{x_i - \tilde{x}_i}{|\mathbf{x}-\mathbf{y}|^3} - (\tilde{x}_i - y_i)\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^3} - \frac{1}{|\tilde{\mathbf{x}}-\mathbf{y}|^3}\right),$$

it follows that

$$|\mathbf{K}(\mathbf{x}-\mathbf{y}) - \mathbf{K}(\tilde{\mathbf{x}}-\mathbf{y})| \leq c|\mathbf{x}-\tilde{\mathbf{x}}|\left(\frac{1}{|\mathbf{x}-\mathbf{y}|^3} + \frac{1}{|\tilde{\mathbf{x}}-\mathbf{y}|^3}\right), \tag{5.11}$$

where we have applied the well-known inequality $xy \leq c(x^p + y^q)$. Therefore, by using the fact that $\|\xi_0\|_{L^q} \leq c\|\omega_\theta^0\|_\alpha$ (see (5.6), $r_{\mathbf{y}} \leq r_{\mathbf{x}-\mathbf{y}} + r_{\mathbf{x}}$, and that $|\tilde{\mathbf{x}}-\mathbf{y}| \geq |\mathbf{x}-\mathbf{y}| - r_{\mathbf{x}}$, we derive that

$$\begin{aligned}
|m_3|^p &\leq c(r_{\mathbf{x}})^p \left(\int_{\Omega_t\setminus B_\varepsilon(\mathbf{x})} \left(\frac{1}{|\mathbf{x}-\mathbf{y}|^3} + \frac{1}{|\tilde{\mathbf{x}}-\mathbf{y}|^3}\right)\xi_0\left(\mathbf{\Phi}^{-1}\right) r_{\mathbf{y}}\, d\mathbf{y}\right)^p \\
&\leq c\left(\left\|\omega_\theta^0\right\|_\alpha r_{\mathbf{x}}\right)^p \int_{\Omega_t\setminus B_\varepsilon(\mathbf{x})} \left(\frac{r_{\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3} + \frac{r_{\mathbf{y}}}{|\tilde{\mathbf{x}}-\mathbf{y}|^3}\right)^p d\mathbf{y} \\
&\leq c\left(\left\|\omega_\theta^0\right\|_\alpha r_{\mathbf{x}}\right)^p \int_{\Omega_t\setminus B_\varepsilon(\mathbf{x})} \left(\frac{r_{\mathbf{x}-\mathbf{y}}^p + r_{\mathbf{x}}^p}{|\mathbf{x}-\mathbf{y}|^{3p}} + \frac{r_{\tilde{\mathbf{x}}-\mathbf{y}}^p + r_{\tilde{\mathbf{x}}}^p}{|\tilde{\mathbf{x}}-\mathbf{y}|^{3p}}\right) d\mathbf{y} \\
&\leq c\left(\left\|\omega_\theta^0\right\|_\alpha r_{\mathbf{x}}\right)^p \int_{\varepsilon/2}^{2d(\Omega_t)} \left(\frac{r_{\mathbf{x}}^p}{\rho^{3p}}\rho^2 + \frac{1}{\rho^{2p}}\rho^2\right) d\rho \\
&\leq c\left(\left\|\omega_\theta^0\right\|_\alpha r_{\mathbf{x}}\right)^p \left(1 + \varepsilon^{3-2p} + d(\Omega_t)^{3-2p}\right),
\end{aligned}$$

where we have also used the fact that $p < 3/2$. Notice that $0 < 3 - 2p < p$. Hence

$$|m_3| \leq c\left\|\omega_\theta^0\right\|_\alpha r_{\mathbf{x}}\left(1 + |\mathbf{\Phi}(t)|_0\right).$$

Combining this with (5.9) and (5.10), the assertion follows.

LEMMA 5.3. *Under the assumptions of Lemma 5.1, we have*

$$|\boldsymbol{\omega}(t)|_0 \le c\left|\omega_\theta^0\right|_0 \exp\left(c\left\|\omega_\theta^0\right\|_\alpha \int_0^t \exp\left(ce^{c\left(\left\|\omega_\theta^0\right\|_\alpha+1\right)s}\right) ds\right),$$

where c depends only on α *and* $d(\Omega_0)$.

Proof. By Lemma 5.2, we have the following estimate:

$$\begin{aligned} r_{\Phi(\mathbf{x},t)} &\le r_{\mathbf{x}} + \int_0^t \left|\mathbf{u}\left(\Phi(\mathbf{x},s), s\right)\right| ds \\ &\le r_{\mathbf{x}} + c\left\|\omega_\theta^0\right\|_\alpha \int_0^t \left(\left|\Phi(s)\right|_0 + 1\right) r_{\Phi(\mathbf{x},s)}\, ds. \end{aligned}$$

It follows that

$$r_{\Phi(\mathbf{x},t)} \le r_{\mathbf{x}} \exp\left(c\left\|\omega_\theta^0\right\|_\alpha \int_0^t \left|\Phi(s)\right|_0 ds\right). \tag{5.12}$$

By (5.1) and (5.12), we find

$$\left|\boldsymbol{\omega}\left(\Phi(\mathbf{x},t), t\right)\right| \le \left|\omega_\theta^0\right|_0 \frac{r_{\Phi(\mathbf{x},t)}}{r_{\mathbf{x}}} \le c\left|\omega_\theta^0\right|_0 \exp\left(c\left\|\omega_\theta^0\right\|_\alpha \int_0^t \left|\Phi(s)\right|_0 ds\right).$$

The assertion follows from Lemma 5.1.

LEMMA 5.4. *Under the assumptions of Lemma 5.1, the velocity is quasi-Lipschitz, and consequently the particle trajectory* $\Phi(\mathbf{x},t)$ *and its inverse* $\Phi^{-1}(\mathbf{x},t)$ *are in* $C^{\beta(t)}$ *with* $\beta(t) = e^{-c\left(|\omega(t)|_0 + |\omega_0(t)|_\alpha\right)}$, *i.e.,*

$$\begin{aligned} \left|\mathbf{u}(\mathbf{x},t) - \mathbf{u}(\tilde{\mathbf{x}},t)\right| &\le c\left(\left|\boldsymbol{\omega}(t)\right|_0 + \ln\left(d\left(\Omega_t\right)\right)\right)\left|\mathbf{x}-\tilde{\mathbf{x}}\right| \left|\ln\left|\mathbf{x}-\tilde{\mathbf{x}}\right|\right|, \\ c\left|\mathbf{x}-\tilde{\mathbf{x}}\right|^{1/\beta(t)} &\le \left|\Phi(\mathbf{x},t) - \Phi(\tilde{\mathbf{x}},t)\right| \le c\left|\mathbf{x}-\tilde{\mathbf{x}}\right|^{\beta(t)}, \end{aligned}$$

where c is a constant depending only on α *and* Ω_0.

Proof. By Lemma 5.3, $\beta(t)$ is bounded. For any $\mathbf{x}, \tilde{\mathbf{x}} \in \Omega_t$, set $\varepsilon = 2\left|\mathbf{x}-\tilde{\mathbf{x}}\right|$. We then write, as in (5.9),

$$\mathbf{u}(\mathbf{x},t) - \mathbf{u}(\tilde{\mathbf{x}},t) = \int_{\Omega_t \cap B_\varepsilon(\mathbf{x})} \mathbf{K}(\mathbf{x}-\mathbf{y})\, \boldsymbol{\omega}(\mathbf{y})\, d\mathbf{y}$$

$$
\begin{aligned}
&- \int\limits_{\Omega_t \cap B_\varepsilon(\mathbf{x})} \mathbf{K}(\tilde{\mathbf{x}} - \mathbf{y}) \boldsymbol{\omega}(\mathbf{y})\, d\mathbf{y} \\
&+ \int\limits_{\Omega_t \setminus B_\varepsilon(\mathbf{x})} \left(\mathbf{K}(\mathbf{x} - \mathbf{y}) - \mathbf{K}(\tilde{\mathbf{x}} - \mathbf{y})\right) \boldsymbol{\omega}\, d\mathbf{y} \\
= \; & m_1 + m_2 + m_3.
\end{aligned}
$$

The first term m_1 and m_2 can be estimated as

$$
\begin{aligned}
|m_1| &\leq |\boldsymbol{\omega}(t)|_0 \int\limits_{\Omega_t \cap B_\varepsilon(\mathbf{x})} |\mathbf{x} - \mathbf{y}|^{-2}\, d\mathbf{y} \leq c\, |\boldsymbol{\omega}(t)|_0\, \varepsilon, \\
|m_2| &\leq |\boldsymbol{\omega}(t)|_0 \int\limits_{\Omega_t \cap B_{3\varepsilon/2}(\tilde{\mathbf{x}})} |\tilde{\mathbf{x}} - \mathbf{y}|^{-2}\, d\mathbf{y} \leq c\, |\boldsymbol{\omega}(t)|_0\, \varepsilon.
\end{aligned}
$$

By (5.11), we have

$$
\begin{aligned}
|m_3| &\leq c\, |\boldsymbol{\omega}(t)|_0\, |\mathbf{x} - \tilde{\mathbf{x}}| \int\limits_{\Omega_t \setminus B_\varepsilon(\mathbf{x})} \left(\frac{1}{|\mathbf{x} - \mathbf{y}|^3} + \frac{1}{|\tilde{\mathbf{x}} - \mathbf{y}|^3} \right) d\mathbf{y} \\
&\leq c\, |\boldsymbol{\omega}(t)|_0\, |\mathbf{x} - \tilde{\mathbf{x}}| \left(\int\limits_{\Omega_t \setminus B_\varepsilon(\mathbf{x})} \frac{1}{|\mathbf{x} - \mathbf{y}|^3}\, d\mathbf{y} + \int\limits_{\Omega_t \setminus B_{\varepsilon/2}(\tilde{\mathbf{x}})} \frac{1}{|\tilde{\mathbf{x}} - \mathbf{y}|^3}\, d\mathbf{y} \right) \\
&\leq c\, |\boldsymbol{\omega}(t)|_0\, |\mathbf{x} - \tilde{\mathbf{x}}| \left| \ln \left(\frac{d(\Omega_t)}{|\mathbf{x} - \mathbf{y}|} \right) \right|.
\end{aligned}
$$

The first assertion follows. The second assertion follows from the estimate

$$
\begin{aligned}
&\left| \frac{d}{dt} |\boldsymbol{\Phi}(\mathbf{x}, t) - \boldsymbol{\Phi}(\tilde{\mathbf{x}}, t)| \right| \leq |\mathbf{u}(\boldsymbol{\Phi}(\mathbf{x}, s), s) - \mathbf{u}(\boldsymbol{\Phi}(\tilde{\mathbf{x}}, s), s)| \\
\leq \; & c\, |\boldsymbol{\omega}(t)|_0\, |\boldsymbol{\Phi}(\mathbf{x}, t) - \boldsymbol{\Phi}(\tilde{\mathbf{x}}, t)| \left| \ln \left(\frac{d(\Omega_t)}{|\boldsymbol{\Phi}(\mathbf{x}, t) - \boldsymbol{\Phi}(\tilde{\mathbf{x}}, t)|} \right) \right|
\end{aligned}
$$

and the generalized Gronwall inequality.

Proof of Theorem 2.3:

By Theorem 2.2, it suffices to show that for any fixed $T > 0$, the integral (1.5) is finite for some $\beta_T > 0$. By Lemma 5.3. $\boldsymbol{\omega}(t)$ is bounded for $t < T$. We only need to show that $|\boldsymbol{\omega}(t)|_{\beta_T}$ is bounded for $t < T$.

From (5.1), we know that $\boldsymbol{\omega}(\boldsymbol{\Phi}(\mathbf{x}.t), t) = \xi_0(\mathbf{x})\, r_{\boldsymbol{\Phi}} \mathbf{e}_\theta(\mathbf{x}) = \boldsymbol{\omega}_0(\mathbf{x})\, r_{\boldsymbol{\Phi}}/r_{\mathbf{x}}$. By Lemma 5.4, since $r_{\boldsymbol{\Phi}}$ is the component of $\boldsymbol{\Phi}$ in $\mathbf{e}_r$ direction, it follows $r_{\boldsymbol{\Phi}(\mathbf{x},t)} \in C^{\beta(t)}(\Omega_0)$

and $\left|r_{\Phi(\cdot,t)}\right|_{\beta(t)} \leq |\Phi(t)|_{\beta(t)} \leq c$, where $\beta(t) = e^{-c\left(|\omega(t)|_0 + |\omega_0(t)|_\alpha\right)}$. We claim that

$$\left|\frac{\omega_0(\mathbf{x})\, r_{\Phi(\cdot,t)}}{r_{\mathbf{x}}}\right|_{\alpha\beta(t)} \leq c(T). \tag{5.13}$$

To verify (5.13), we see first from (5.12) that for $t < T$,

$$\left|\frac{\omega_0(\mathbf{x})\, r_{\Phi(\cdot,t)}}{r_{\mathbf{x}}}\right| \leq c(T)\, r_{\mathbf{x}}^{\alpha},$$

where $c(T)$ depends on the initial data and T. Next, for any $r > \tilde{r} > 0$, since $\left|\omega_0(r, x_3)\, r_{\Phi(\tilde{r},x_3,t)}\right| + |\omega_0(\tilde{r}, x_3)| + r_{\Phi(\tilde{r},x_3,t)} \leq c(T)\, r^{\alpha}$, we obtain

$$\begin{aligned}
&\left|\frac{\omega_0(r, x_3)\, r_{\Phi(r,x_3,t)}}{r} - \frac{\omega_0(\tilde{r}, x_3)\, r_{\Phi(\tilde{r},x_3,t)}}{\tilde{r}}\right| \\
\leq\ & |\omega_0(r, x_3) - \omega_0(\tilde{r}, x_3)|\, \frac{r_{\Phi(r,x_3,t)}}{r} + |r - \tilde{r}|\, \frac{|\omega_0(\tilde{r}, x_3)|\, r_{\Phi(\tilde{r},x_3,t)}}{r\tilde{r}} \\
&+ \left|r_{\Phi(r,x_3,t)} - r_{\Phi(\tilde{r},x_3,t)}\right| \frac{|\omega_0(\tilde{r}, x_3)|}{r} \\
\leq\ & c(T)\, |r - \tilde{r}|^{\alpha} + \left|r_{\Phi(r,x_3,t)} - r_{\Phi(\tilde{r},x_3,t)}\right|^{\alpha} \frac{\left|r_{\Phi(r,x_3,t)} + r_{\Phi(\tilde{r},x_3,t)}\right|^{1-\alpha} |\omega_0(\tilde{r}, x_3)|}{r} \\
\leq\ & c(T)\, |r - \tilde{r}|^{\alpha\beta(t)}.
\end{aligned}$$

Analogously, we can derive, for any $x_3 \neq \tilde{x}_3$,

$$\left|\frac{\omega_0(r, x_3)\, r_{\Phi(r,x_3,t)}}{r} - \frac{\omega_0(r, \tilde{x}_3)\, r_{\Phi(r,\tilde{x}_3,t)}}{r}\right| \leq c(T)\, |x_3 - \tilde{x}_3|^{\alpha\beta(t)}.$$

Hence, (5.13) follows.

We now choose $\beta_T = \alpha \min_{t<T} \beta^2(t) > 0$. Then, since $\Phi^{-1} \in C_{\mathbf{x}}^{\beta(t)}$, we see that $\omega(\mathbf{x}, t) = \omega\left(\Phi\left(\Phi^{-1}(\mathbf{x}, t), t\right), t\right) \in C_{\mathbf{x}}^{\beta_T}$ and $|\omega(t)|_{\beta_T}$ is bounded by $c(T)$ for $t < T$, where $c(T)$ depends only on T and the initial data. The proof is complete.

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