INDUCED CLASS FUNCTIONS ARE CONDITIONAL EXPECTATIONS

JONATHAN D.H. SMITH*

Abstract. The induced class function determined by a class function on a refinement subscheme of an association scheme is interpreted as the conditional expectation of a certain random variable.

1. Introduction. A recurrent theme in the development of the theory of association schemes has been to regard the theory as "group theory without groups" [B1,B2; preface]. One problem suggested by this theme had been to give a way of inducing class functions from a subscheme, generalizing the induction of class functions from a subgroup of a group. An approach emerged as a by-product of the related development of the theory of quasigroups [J1, §2] [J2] [B2(App.1), B3; §3.3]. The purpose of the current note is to point out that this induction process is in fact a special case of the concept of conditional expectation that J. Doob had developed by the early 1950's [Bi, §34] [Do, Ch. I]. The combinatorial definitions are presented in the second section. The third section summarises the probability-theoretical concepts. The fourth section then presents the main result, Theorem 4.3, which interprets induced class functions as conditional expectations. The finite, combinatorial setting avoids many of the subtleties inherent in the probability theory. However, the formulation of Theorem 4.3 opens up a number of ready generalisations to various infinite cases. The theorem is also of interest in relating aspects of group theory, combinatorics, and probability theory.

2. Induced class functions of association schemes. Let Q be a finite nonempty set. Then an *association scheme* (Q,Γ) on Q [B1, B2, §2.2] is a partition $\Gamma = \{C_1, C_2, \ldots, C_s\}$ of the direct square $Q \times Q$ such that:

(A1) $\qquad C_1 = \{(x,x) \in Q^2 | x \in Q\};$

(A2) $\qquad \forall C_i \in \Gamma, \{(y,x)|(x,y) \in C_i\} \in \Gamma;$

(A3) $\qquad \forall C_i \in \Gamma, \forall C_j \in \Gamma, \forall C_k \in \Gamma, \exists c_{ijk} \in \mathbb{N}.$

$\qquad \forall (x,y) \in C_k, \quad |\{z \in Q|(x,z) \in C_i, (z,y) \in C_j\}| = c_{ijk},$ and

(A4) $\qquad \forall 1 \leq i,j,k \leq s, \quad c_{ijk} = c_{jik}.$

Example 2.1. [B1, B2, Ex 2.1(2)]. Let Q be a finite group, with conjugacy classes $C'_1 = \{1\}, C'_2, \ldots, C'_s$. Define $C_i = \{(x,y) \in Q \times Q | x^{-1}y \in C'_i\}$ and $\Gamma = \{C_1, \ldots, C_s\}$. Then (Q,Γ) is an association scheme, called the *group conjugacy class scheme* of the group Q.

*Department of Mathematics, Iowa State University, Ames, Iowa 50011, USA

Let $(Q, \Gamma = \{C_1, \ldots, C_s\})$ be an association scheme. A complex-valued function $f : Q \times Q \to \mathbb{C}$ is said to be a *class function* on (Q, Γ) if its restrictions to each C_i in Γ are constant. The set of class functions on (Q, Γ) is denoted by $\mathbb{C}\Gamma$; it carries the $\mathbb{C}$-algebra structure induced from the $\mathbb{C}$-algebra structure of $\mathbb{C}$. If Q is the finite group of Example 2.1, then a *group class function on* Q is a function $f' : Q \to \mathbb{C}$ whose restrictions to each group conjugacy class C_i' are constant. Such a group class function f' yields a class function f on (Q, Γ) with $f(x, y) = f'(x^{-1}y)$. Conversely, a class function f on (Q, Γ) yields a group class function $f' : Q \to \mathbb{C}$ with $f'(x) = f(1, x)$.

An association scheme $(P, \Delta = \{D_1, \ldots, D_t\})$ is said to be a *(refinement) subscheme* of an association scheme $(Q, \Gamma = \{C_1, \ldots, C_s\})$ [J2] if

$$\text{(2.2)} \qquad \begin{cases} (i) P \subseteq Q & \text{and} \\ (ii) \forall D_i \in \Delta, \exists C_j \in \Gamma \ . \ D_i \subseteq C_j. \end{cases}$$

For example, if P is a subgroup of a finite group Q, then the group conjugacy class scheme of P is a refinement subscheme of the group conjugacy class scheme of Q. A group class function g': $P \to \mathbb{C}$ determines a group class function $g' \uparrow_P^Q$: $Q \to \mathbb{C}$ by means of the Frobenius formula [Se, 7.2]

$$\text{(2.3)} \qquad g' \uparrow_P^Q (s) = \frac{1}{|P|} \sum_{\substack{u \in Q \\ u^{-1} s u \in P}} g'(u^{-1} s u).$$

The function $g' \uparrow_P^Q$ is called the group class function on Q *induced* from the group class function g' on P.

For some time, it was not clear how to extend the Frobenius formula (2.3) to define an appropriate concept of induced class function for association schemes. A definition was given in [J1, §2] within the context of quasigroup conjugacy class association schemes. The definition is also described in [B2 (App. 1), B3; §3.3] (albeit without explanation of the relevant convention used to extend a function to the power set of its domain). Let (P, Δ) be a refinement subscheme of $(Q, \Gamma = \{C_1, \ldots, C_s\})$. For each C_i, set

$$\text{(2.4)} \qquad B_i = \bigcup \{D \in \Delta | D \subseteq C_i\}.$$

The induction function $\uparrow_P^Q$: $\Gamma P \to \Gamma Q$; $f \mapsto f^Q$ is then defined by

$$\text{(2.5)} \qquad f^Q(x, y) = \frac{|Q \times Q|}{|C_i| \cdot |P \times P|} \sum_{(z,t) \in B_i} f(z, t)$$

for $(x, y) \in C_i$ [J1, (2.2)][Sm, 552]. In [J1, §3] [Sm, §5.5], it is verified that (2.5) does indeed subsume the Frobenius formula (2.3), via the correspondence between group class functions f' and class functions f on the group conjugacy class scheme. In Section 4 below, it is shown how to interpret (2.5), and thus also the Frobenius formula (2.3), as giving a conditional expectation of a certain random variable. The next section summarises the requisite concepts from probability theory.

3. Conditional expectations. Let Ω be a set. A *σ-field* on Ω is a set F of subsets of Ω, among which is Ω itself, closed under complementation and countable unions. A *probability measure* on the σ-field F is a function $\mu : F \to [0, \infty]$ such that $\mu(\Omega) = 1$ and $\mu(\cup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$ for a family $(A_i | i \in \mathbb{N})$ of mutually disjoint elements A_i of F. A *probability space* is a triple (Ω, F, μ) consisting of a set Ω, a σ-field on Ω, and a probability measure μ on F [Bi, §2][Do, §I.2]. A *complex random variable* X on (Ω, F, μ) is a function $X : \Omega \to \mathbb{C}$ such that for any real number r, the sets $\{\omega \in \Omega | Re X(\omega) \leq r\}$ and $\{\omega \in \Omega | \text{Im } X(\omega) \leq r\}$ lie in F [Do, §I.3]. If these sets all lie in a σ-field G contained in F, then the random variable X is said to be *G-measurable* [Bi, §13]. The random variable X is said to be *integrable* if the Lebesque integrals $\int_{\Omega} Re X d\mu$ and $\int_{\Omega} \text{Im } X d\mu$ exist (so that $\int_A X d\mu = \int_A Re X d\mu + i \int_A \text{Im } X d\mu$ exists for any A in F [KT, Th. 5.5]).

Given an integrable random variable X on a probability space (Ω, F, μ), a *conditional expectation* $E(X|G)$ of X with respect to a σ-field G contained in F is a random variable $E(X|G)$ satisfying the conditions

$$(3.1) \qquad \begin{cases} (i) \quad E(X|G) \text{ is integrable;} \\ (ii) \quad E(X|G) \text{ is } G\text{-measurable, and} \\ (iii) \quad \forall A \in G, \displaystyle\int_A E(X|G) d\mu = \int_A X d\mu \end{cases}$$

[Bi, §34][Do, §I.7]. Conditional expectations satisfying (3.1) exist (e.g. as a consequence of the Radon-Nikodym Theorem); they are not necessarily unique, but may differ on a set of measure zero. There is a useful necessary and sufficient condition for a random variable to be a conditional expectation [Bi, Th. 34.1]:

PROPOSITION 3.2. *Let $(M, \cap)$ be a subsemilattice of the meet semilattice reduct $(G, \cap)$ of the σ-field G. Suppose that G is the smallest σ subfield of F containing M. Then an integrable random variable Y is a conditional expectation $E(X|G)$ if and only if it is G-measurable, and*

$$(3.3) \qquad \forall A \in M, \quad \int_A Y d\mu = \int_A X d\mu. \qquad \square$$

4. Induced class functions are conditional expectations. Let $(Q, \Gamma = \{C_1, \ldots, C_s\})$ be an association scheme. Define Ω to be the direct square $Q \times Q$. The power set F of Ω is a σ-field. Define the normalised counting measure

$$(4.1) \qquad \mu : F \to [0, 1]; \quad A \mapsto |A|/|Q \times Q|.$$

Then (Ω, F, μ) is a probability space. Since the members of Γ are mutually disjoint, $M = \Gamma \cup \{\emptyset\}$ forms a subsemilattice $(M, \cap)$ of $(F, \cap)$. Let G be the smallest σ-subfield of F containing M, namely $G = \{A_1 \cup \cdots \cup A_r | \{A_1, \ldots, A_r\} \subseteq \Gamma\}$. Any complex-valued function $X : \Omega \to \mathbf{C}$ is an integrable random variable, with $\int_A X d\mu = |Q \times Q|^{-1} \sum_{a \in A} X(a)$ for A in F. Further, X is G-measurable if and only if its restriction to each C_i in Γ is constant. Thus the G-measurable random variables on (Ω, F, μ) are precisely the class functions on (Q, Γ).

Now let (P, Δ) be a refinement subscheme of (Q, Γ). For a complex-valued function $f : P \times P \to \mathbf{C}$, define the random variable $X_f : \Omega \to \mathbf{C}$ by

$$X_f(x,y) = \begin{cases} 0 \quad , & (x,y) \notin P \times P; \\ \frac{|Q \times Q|}{|P \times P|} f(x,y), & (x,y) \in P \times P. \end{cases} \tag{4.2}$$

THEOREM 4.3. *For a class function f in $\mathbf{C}\Delta$, the induced class function f^Q in $\mathbf{C}\Gamma$ is just the (unique) conditional expectation $E(X_f|G)$.*

Proof. Since the empty set is the only set A in F with $\mu(A) = 0$, conditional expectations are unique. Certainly the class function f^Q is integrable and G-measurable. Also $\int_\emptyset f^Q d\mu = \int_\emptyset X_f d\mu = 0$. Now consider C_i in Γ, with corresponding B_i defined by (2.4). Then $\int_{C_i} f^Q d\mu = |Q \times Q|^{-1} \sum_{(x,y) \in C_i} f^Q(x,y) = |P \times P|^{-1} \sum_{(z,t) \in B_i} f(z,t) = |Q \times Q|^{-1} \sum_{(z,t) \in B_i} X_f(z,t) = |Q \times Q|^{-1} \sum_{(x,y) \in C_i} X_f(x,y) = \int_{C_i} X_f d\mu$. The second equality here holds by (2.5); the third and fourth hold by (4.2). Condition (3.3) has thus been verified, so that f^Q is a conditional expectation $E(X_f|G)$ by Proposition 3.2. $\square$

By virtue of Theorem 4.3, a number of properties of induction correspond with standard results about conditional expectations. For example, the fact that $\mathbf{C}\Delta \uparrow_P^Q$ is an ideal of $\mathbf{C}\Gamma$ [J1, Prop. 6.3 (iv)][Se, §9.1][Sm, 562(d)] may be viewed as a special case of the relation $E(XY|G) = XE(Y|G)$ a.e. [Bi, Th. 34.3] for a G-measurable random variable X and integrable random variables Y and XY. As a second example, suppose that (N, Θ) is a refinement subscheme of the refinement subscheme (P, Δ) of (Q, Γ). Then the "transitivity" or "functoriality" property $\uparrow_N^Q = \uparrow_N^P \uparrow_P^Q : \mathbf{C}\Theta \to \mathbf{C}\Gamma$ [J1, Prop. 2.3.][Se, §7.1] [Sm, 553] is a consequence of the relation $E(E(X|G_2)|G_1) = E(X|G_1)$ a.e. [Bi, Th. 34.4] for an integrable random variable X and σ-subfields $G_1 \subseteq G_2$. Nevertheless, the chief utility of Theorem 4.3 will probably lie in the generalisations to infinitary cases that it suggests.

REFERENCES

[B1] E. BANNAI AND T. ITO, *Algebraic Combinatorics I*, Benjamin/Cummings Lecture Note Series, No. 58, Menlo Park CA (1984).

[B2] E. BANNAI AND T. ITO, *Algebraic Combinatorics* (Russian), Mir, Moscow, 1987.

[B3] E. BANNAI AND T. ITO, *Current research on algebraic combinatorics*, Graphs and Combinatorics, 2 (1986), pp. 287–308.

[Bi] P. BILLINGSLEY, *Probability and Measure*, J. Wiley, New York NY, 1979.

[Do] J.L. DOOB, *Stochastic Processes*, J. Wiley, New York NY, 1953.

[J1] K.W. JOHNSON AND J.D.H. SMITH, *Characters of finite quasigroups II: induced characters*, Europ. J. Combinatorics, 7 (1986), pp. 131–137.

[J2] K.W. Johnson and J.D.H. Smith, *A note on character induction in association schemes*, Europ. J. Combinatorics, 7 (1986), p. 139.

[KT] J.F.C. Kingman and S.J. Taylor, *Introduction to Measure and Probability*, Cambridge University Press, Cambridge UK, 1966.

[Se] J.-P. Serre (tr. L.L. Scott), *Linear Representations of Finite Groups, Springer Graduate Texts in Mathematics No. 42*, Springer-Verlag, New York NY, 1977.

[Sm] J.D.H. Smith, *Representation Theory of Infinite Groups and Finite Quasigroups, Séminaire de Mathématiques Supérieures No. 101*, Université de Montréal, Montréal PQ, 1986.