COMPUTATIONAL METHODS FOR GLOBAL ANALYSIS OF HOMOCLINIC AND HETEROCLINIC ORBITS: A CASE STUDY

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Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study.

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Abstract: In earlier papers we have developed a numerical method for the computation of branches of heteroclinic orbits for a system of autonomous ordinary differential equations in $\mathbb{R}^n$. *The idea of the method is to reduce a boundary value problem on the real line to a boundary value problem on a finite interval by using linear approximation of the unstable and stable manifolds. In this paper we extend our algorithm to incorporate higher order approximations of the unstable and stable manifolds. This approximation is especially useful if we want to compute accurately center manifolds. An efficient procedure of switching between the periodic approximation of homoclinic orbits and higher order approximation of homoclinic orbits provides additional flexibility to the method. The algorithm is applied to a model problem: the DC Josephson Junction. Computations are done using the software package AUTO.*

1. Introduction.

Global analysis of *homoclinic and heteroclinic* orbits, i.e., orbits of an infinite period connecting fixed points of a system of autonomous ordinary differential equations, is important in analysis of dynamical systems. Applications include the problem of finding traveling wave front solutions of constant speed to nonlinear parabolic partial differential equations, etc...

In earlier papers [4, 5, 7] we have developed an accurate, robust, and systematic method for computing branches of heteroclinic orbits. Specifically, we have considered the problem of finding a branch of solutions of the system of autonomous ordinary differential equations

$$(1.1) \qquad \begin{aligned} &a)\ \ x'(t) - f(x(t),\lambda) = 0,\ \ x(\cdot),\ f(\cdot,\cdot) \in \mathbb{R}^n,\ \lambda \in \mathbb{R}^{n_\lambda},\\ &b)\ \ \lim_{t\to-\infty} x(t) = x_-,\ \ \lim_{t\to\infty} x(t) = x_+. \end{aligned}$$

The method utilizes linear approximation of the unstable (for $t < T_-$, $-T_- > 0$, large) and stable (for $t > T_+$, $T_+ > 0$, large) manifolds, under the assumption that solutions of (1.1) decay exponentially to their limits at $\pm\infty$. Since every translation of a solution of

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Eq. (1.1) is also a solution, we need to add a constraint. The equation

$$\int_{-\infty}^{\infty} \left(f(x(t),\lambda) - f\left(q(t),\lambda^0\right)\right) \cdot \frac{\partial}{\partial t} f(x(t),\lambda) dt = 0 \tag{1.2}$$

seems to be, computationally, the most appropriate way to do this. It is obtained by requiring that the current solution $x(t)$ be as "close" as possible to the previously computed one $q(t)$ (see [4] for the discussion). Our principal result, Theorem 2 in [7] can be summarized as follows:

Let $\left(q,\lambda^0\right)$ *be a solution of (1.1), (1.2). Assume that* $n_\lambda = 2-(n_- + n_+ - n) \geq 0$, *where* n_- and n_+ *are dimensions of unstable and stable manifolds, respectively, Under appropriate assumptions on* f *and appropriate transversality conditions, in a neighborhood of* $\left(q,\lambda^0\right)$ *there exists an unique solution branch* $(x(s),\lambda(s))$, $(x(0),\lambda(0)) = \left(q,\lambda^0\right)$, *of (1.1), (1.2) and for sufficiently large* $|T_-|$, T_+ *an unique branch* $(x(s),\lambda_T(s))$ *of approximate solutions. Here* s *is the pseudo-arclength continuation parameter (employed by AUTO).*

Moreover, we have an error estimate

$$\|\lambda(s) - \lambda_T(s)\|_{\mathbb{R}^{n_\lambda}} + \|x(s) - x_T(s)\|_{W^1_\infty(\mathbb{R})} \leq \mathrm{C}\left(\mathrm{e}^{-2|\mathrm{T}_-|\mu_0} + \mathrm{e}^{2\mathrm{T}_+\mu_1}\right), \tag{1.3}$$

for some $\mu_0 > 0 > \mu_1$.

Similar results were obtained in Beyn [1, 2]. A different approach to continuation of heteroclinic orbits, which makes use of solution of appropriate initial value problems, was developed in Rodriguez-Luis *et al.* [12].

In this paper we extend our method to higher order approximation of the unstable and stable manifolds. This is especially useful in the case when during the continuation process the exponential rate of decay is lost, i.e. a center manifold appears. We also include an efficient procedure of switching between the periodic approximation of homoclinic orbits (already incorporated into AUTO) and the higher order approximation of homoclinic orbits. This switching provides flexibility needed to accurately locate a homoclinic orbit when nearby periodic orbits are known and a periodic orbit when a nearby homoclinic orbit is known. The algorithm is applied to a model problem: the DC Josephson Junction. This problem has been chosen since it exhibits typical nontrivial behavior, is of practical interest and has been studied theoretically.

In Section 2 we describe our numerical method in the case $n = 2$, $n_\lambda = 2$. In Section 3 we apply our method to the model problem: in Section 3.1 we derive equations for the higher order approximation of the unstable and stable manifolds, in Section 3.2 we collect known theoretical results about the problem, and in Section 3.3 we describe the solution algorithm and the numerical results. Computations are done using the software package AUTO.

Higher order approximation of the unstable and stable manifolds in combination with the shooting method was also developed earlier in Hassard [9, 10] for the computation of heteroclinic orbits. Hassard's work helped us to develop the boundary value approach which is more powerful and simpler implement.

Rigorous numerical analysis of our method, including the case of a center manifold, will be given elsewhere [8].

2. Numerical Method.

We consider the following problem: given a solution $\left(q(t), \lambda^0\right) \equiv (x(0,t), \lambda(0))$, find branches of solutions $(x(s,t), \lambda(s))$, for the pseudo-arclength continuation, with the parameter s, of the problem

$$
(2.1)\quad
\begin{aligned}
&a)\ x'(t) - f(x(t), \lambda) = 0,\ \ x(\cdot),\ f(\cdot,\cdot) \in \mathbb{R}^2,\ \lambda \in \mathbf{\Lambda} \subset \mathbb{R}^2,\\
&b)\ \lim_{t\to-\infty} x(t) = x_-,\ \ \lim_{t\to\infty} x(t) = x_+,\\
&c)\ \int_{-\infty}^{\infty} \left(f(x(t),\lambda) - f\left(q(t), \lambda^0\right)\right) \cdot \frac{\partial}{\partial t} f(x(t), \lambda) dt = 0.
\end{aligned}
$$

Assume that $f \in C^k$, $k \geq 4$ and $f_\lambda(x, \lambda)$ is uniformly bounded in $\mathbb{R}^2 \times \mathbf{\Lambda}$. We assume that equation

$$
(3.11)\qquad f(x, \lambda) = 0
$$

can be solved to find $x_- = x_-(\lambda)$, $x_+ = x_+(\lambda)$, smooth functions of λ. (The latter condition can easily be relaxed).

Define the eigenpairs $\left(w_i^-, \mu_i^-\right)$, $\left(w_i^+, \mu_i^+\right) \in \mathbb{R}^2 \times \mathbb{R}$ respectively, by

$$
(2.2)\quad
\begin{aligned}
&a)\ f_x(x_-(\lambda), \lambda) w_i^- = \mu_i^- w_i^-,\ \ i = -1, 0,\\
&b)\ f_x(x_+(\lambda), \lambda) w_i^+ = \mu_i^+ w_i^+,\ \ i = -1, 1;
\end{aligned}
$$

$$
(2.3)\quad
\begin{aligned}
&a)\ \left|w_i^-\right| = 1,\ i = -1, 0,\\
&b)\ \left|w_i^+\right| = 1,\ i = -1, 1.
\end{aligned}
$$

We assume that at the beginning of the continuation process

$$
(2.4)\quad
\begin{aligned}
&a)\ Re\,\mu_{-1}^- < 0 < \mu_0^-,\\
&b)\ \mu_{-1}^+ < 0 < Re\,\mu_1^+,
\end{aligned}
$$

and that during the continuation process

$$
(2.5)\quad
\begin{aligned}
&a)\ Re\,\mu_{-1}^- < 0 \leq \mu_0^-,\\
&b)\ \mu_{-1}^+ < 0 < Re\,\mu_1^+.
\end{aligned}
$$

The assumptions (2.4), (2.5) say that at the beginning of the continuation process the heteroclinic orbit in question connects to saddles x_- and x_+. However, during the continuation process the positive eigenvalue μ_0^- corresponding to the unstable manifold of x_- can become zero, i.e. a center manifold appears.

Define the matrices

$$
(2.6)\quad
\begin{aligned}
&a)\ P_- \equiv P_-(\lambda) = [\,w_1^- \quad w_0^-\,],\\
&b)\ P_+ \equiv P_+(\lambda) = [\,w_1^+ \quad w_{-1}^+\,].
\end{aligned}
$$

Upon changing variables according to

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_- + P_- \begin{bmatrix} y \\ v \end{bmatrix}, \quad y, v \in \mathbb{R}, \tag{2.7}$$

we can rewrite (2.1) as

$$\begin{aligned} y' &= \mu_0^- y + h_-(y, v, \lambda), \\ v' &= \mu_{-1}^- v + g_-(y, v, \lambda), \end{aligned} \tag{2.8}$$

where

$$\begin{bmatrix} \mu_0^- y + h_-(y, v, \lambda) \\ \mu_{-1}^- v + g_-(y, v, \lambda) \end{bmatrix} = P_-^{-1} f\left(x_-(\lambda) + P_- \begin{bmatrix} y \\ v \end{bmatrix}, \lambda \right). \tag{2.9}$$

Similarly, upon changing variables according to

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_+ + P_+ \begin{bmatrix} y \\ v \end{bmatrix}, \quad y, v \in \mathbb{R}, \tag{2.10}$$

we can rewrite (2.1) as

$$\begin{aligned} y' &= \mu_1^+ y + h_+(y, v, \lambda), \\ v' &= \mu_{-1}^+ v + g_+(y, v, \lambda), \end{aligned} \tag{2.11}$$

where

$$\begin{bmatrix} \mu_1^+ y + h_+(y, v, \lambda) \\ \mu_{-1}^+ v + g_+(y, v, \lambda) \end{bmatrix} = P_+^{-1} f\left(x_+(\lambda) + P_+ \begin{bmatrix} y \\ v \end{bmatrix}, \lambda \right). \tag{2.12}$$

Here $h_\pm$ and $g_\pm$ are real valued, and satisfy $h_\pm(0,0,\lambda) = 0$, $D_{(y,u)} h_\pm(0,0,\lambda) = 0$, $g_\pm(0,0,\lambda) = 0$, $D_{(y,u)} g_\pm(0,0,\lambda) = 0$. Denote by $V(y,\lambda)$ a function that represents the unstable manifold in a neighborhood of $y = 0$. It satisfies the equation

$$\begin{gathered} V_y(y,\lambda)\left[\mu_0^- y + h_-(y, V(y,\lambda), \lambda)\right] = \mu_{-1}^- V(y,\lambda) + g_-(y, V(y,\lambda), \lambda). \\ V(0,\lambda) = 0, \quad V_y(0,\lambda) = 0. \end{gathered} \tag{2.13}$$

In a neighborhood of $x_-(\lambda)$ a solution of (2.1), which is a "slowly growing" solution, can be written in the form

$$x_{sl} = x_-(\lambda) + P_-(\lambda) \begin{bmatrix} y \\ V(y,\lambda) \end{bmatrix}, \quad y, \ V(y,\lambda) \in \mathbb{R}, \tag{2.14}$$

where $y = y(t, \lambda, \epsilon_-)$ solves

$$\begin{aligned} y' &= \mu_1^- y + h_-(y, V(y), \lambda), \quad -\infty < t < T_-, \\ y(T_-) &= \epsilon_-, \end{aligned} \tag{2.15}$$

for some ϵ_-. Similarly, in a neighborhood of $x_+(\lambda)$ a solution of (2.1) can be written in the form

$$x = x_+(\lambda) + P_+(\lambda) \begin{bmatrix} Y(v,\lambda) \\ v \end{bmatrix}, \tag{2.16}$$

where $Y(v,\lambda)$ is a function that represents the stable manifold in a neighborhood of $y=0$ and satisfies an equation similar to (2.13); and $v = v(t,\lambda,\epsilon_+)$ solves

$$\begin{gathered} v' = \mu_{-1}^{+} v + g_+(Y(v,\lambda), v, \lambda),\ t > T_+, \\ v(T_+) = \epsilon_+, \end{gathered} \tag{2.17}$$

for some ϵ_+. By (2.1), (2.14) and (2.16) for sufficiently large $-T_-$, T_+ the problem (2.1) can be written in the equivalent form as

$$\begin{aligned} &a)\ x'(t) - f(x(t),\lambda) = 0,\ (\lambda_1,\lambda_2)\in\Lambda, \qquad T_- < t < T_+, \\ &b)\ x(t) = x_-(\lambda) + P_-(\lambda)\begin{bmatrix} y \\ V(y,\lambda) \end{bmatrix},\ y = Y(t,\lambda,\epsilon_-),\ t \le T_-, \\ &c)\ x(t) = x_+(\lambda) + P_+(\lambda)\begin{bmatrix} Y(v,\lambda) \\ v \end{bmatrix},\ v = V(t,\lambda,\epsilon_+),\ t \ge T_+, \\ &d)\ \int_{T_-}^{T_+} \left(f(x(t),\lambda) - f\left(q(t),\lambda^0\right)\right)\cdot \frac{\partial}{\partial t} f(x(t),\lambda)dt = 0. \end{aligned} \tag{2.18}$$

We next derive the approximate problem. Let $V^m(y,\lambda)$ be the Taylor polynomial of order m about $y=0$ for $V(y,\lambda)$ which is obtained by solving

$$\begin{gathered} V_y^m(y,\lambda)\left[\mu_0^- y + h_-(y, V^m(y,\lambda),\lambda)\right] = \mu_{-1}^- V^m(y,\lambda) + g_-(y, V^m(y,\lambda),\lambda), \\ V^m(0,\lambda) = 0, \quad V_y^m(0,\lambda) = 0, \end{gathered} \tag{2.19}$$

where $y = y_m(t,\lambda,\epsilon_-)$ solves

$$\begin{aligned} &y' = \mu_0^- y + h_-(y, V^m(y),\lambda), \quad -\infty < t < T_-, \\ &y(T_-) = \epsilon_-. \end{aligned} \tag{2.20}$$

Then an approximation to a "slowly growing" solution (2.14) of (2.1) is given by

$$x_{sl}^m = x_-(\lambda) + P_-(\lambda)\begin{bmatrix} y \\ V^m(y,\lambda) \end{bmatrix},\ y = y_m(t,\lambda,\epsilon_-),\ V^m(y,\lambda)\in\mathbb{R}. \tag{2.21}$$

For the stable manifold we use linear approximation. Let $Y^1(v,\lambda)$ be the Taylor polynomial of order 1 about $y=0$ for $Y(v,\lambda)$. Then

$$Y^1(v,\lambda) \equiv 0.$$

And $v = v(t,\lambda,\epsilon_+)$ is obtained from linear differential equation

$$\begin{aligned} &v' = \mu_{-1}^{+} v, \quad T_+ < t < \infty, \\ &v(T_+) = \epsilon_+. \end{aligned} \tag{2.22}$$

Hence the problem (2.18) can be approximated by

$$\begin{aligned} &a)\ x'(t) - f(x(t),\lambda) = 0,\ (\lambda_1,\lambda_2)\in\Lambda, \qquad T_- < t < T_+, \\ &b)\ x(t) = x_-(\lambda) + P_-(\lambda)\begin{bmatrix} y \\ V^m(y,\lambda) \end{bmatrix},\ y = Y(t,\lambda,\epsilon_-),\ t \le T_-, \\ &c)\ x(t) = x_+(\lambda) + P_+(\lambda)\begin{bmatrix} 0 \\ v \end{bmatrix},\ v = V(t,\lambda,\epsilon_+),\ t \ge T_+, \\ &d)\ \int_{T_-}^{T_+} \left(f(x(t),\lambda) - f\left(q(t),\lambda^0\right)\right)\cdot \frac{\partial}{\partial t} f(x(t),\lambda)dt = 0. \end{aligned} \tag{2.23}$$

Finally, to obtain the equations for our solution algorithm, we rewrite (2.23) as a boundary value problem on a finite interval, taking into account (2.19)–(2.22):

$$\text{(2.24)}\qquad \begin{aligned} &a)\ \ x'(t) - f(x(t),\lambda) = 0, \ = (\lambda_1,\lambda_2) \in \Lambda, \qquad T_- < t < T_+,\\ &b)\ \ x(T_-) = x_-(\lambda) + P_-(\lambda)\begin{bmatrix} \epsilon_- \\ V^m(\epsilon_-,\lambda)\end{bmatrix},\\ &c)\ \ x(T_+) = x_+(\lambda) + P_+(\lambda)\begin{bmatrix} 0 \\ \epsilon_+\end{bmatrix},\\ &d)\ \ \int_{T_-}^{T_+} \left(f(x(t),\lambda) - f(q(t),\lambda^0)\right)\cdot \frac{\partial}{\partial t} f(x(t),\lambda)dt = 0. \end{aligned}$$

Equations (2.24) represent $n = 2$ coupled differential equations subject to $n_c = 2n+1 = 5$ constraints, of which d) is an integral condition. In addition to the vector function variable $x(t) \in \mathbb{R}^2$ we have $n_v \equiv n + 2 = 4$ scalar variables: $\lambda \in \mathbb{R}^2$, ϵ_-, $\epsilon_+ \in \mathbb{R}$. Formally we need $n_v = n_c - n$ for a single heteroclinic connection. Here we shall be interested in computing an entire *branch* (one dimensional continuum) of orbits, in which case $n_v = n_c - n + 1$. This requirement is obviously satisfied here.

3. Example.

3.1. Derivation of the equations.

The DC Josephson Junction equations, in dimensionless form, are [13]:

$$\text{(3.1)}\qquad x' = f(x,\rho,\beta),$$

with $x = (x_1,x_2)^T$, $\lambda = (\rho,\beta)$,

$$\begin{aligned} f(x,\rho,\beta) &= (f_1(x_1,x_2,\rho,\beta), f_2(x_1,x_2,\rho,\beta))^T \\ &= \left(x_2, \frac{1}{\beta}(-x_2 - \sin x_1 + \rho)\right)^T. \end{aligned}$$

Assume that $\rho \le 1$, $\rho^\star = 1$. If $\rho < 1$ then (3.1) has a heteroclinic orbit connecting two saddles: $x_- = (\pi - \arcsin\rho, 0)^T$ and $x_+ = (3\pi - \arcsin\rho, 0)^T$. We note that

$$\text{(3.2)}\qquad f_x(x_-,\rho,\beta) = f_x(x_+,\rho,\beta) = \begin{bmatrix} 0 & 1 \\ \frac{\sqrt{1-\rho^2}}{\beta} & -\frac{1}{\beta}\end{bmatrix},$$

which has eigenvalues

$$\text{(3.3)}\qquad \mu_0^- \equiv \mu_0 = \frac{-1+\Delta}{2\beta},\ \mu_{-1}^+ \equiv \mu_{-1} = \frac{-1-\Delta}{2\beta}, \qquad \Delta \equiv \sqrt{1 + 4\beta\sqrt{1-\rho^2}}.$$

Corresponding right eigenvectors and the matrices P_- and P_+ are

$$\text{(3.4)}\qquad \begin{aligned} &w_{-1} = \frac{1}{\sqrt{1+\mu_{-1}^2}}\begin{bmatrix} 1 \\ \mu_{-1}\end{bmatrix},\ w_0 = \frac{1}{\sqrt{1+\mu_0^2}}\begin{bmatrix} 1 \\ \mu_0 \end{bmatrix},\\ &P \equiv P_- = P_+ = [\,w_0 \quad w_{-1}\,] = \begin{bmatrix} \frac{1}{\sqrt{1+\mu_0^2}} & \frac{1}{\sqrt{1+\mu_{-1}^2}} \\ \frac{\mu_0}{\sqrt{1+\mu_0^2}} & \frac{\mu_{-1}}{\sqrt{1+\mu_{-1}^2}}\end{bmatrix}. \end{aligned}$$

Upon changing the variables according to

$$x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \pi - \arcsin\rho \\ 0 \end{bmatrix} + P\begin{bmatrix} y \\ v \end{bmatrix}, \quad y, v \in \mathbb{R}, \tag{3.5}$$

we rewrite (3.1) in the form (2.8) as

$$\begin{aligned} y' &= \mu_0 y + h(y, v, \rho, \beta), \\ v' &= \mu_{-1} v + g(y, v, \rho, \beta), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} \begin{bmatrix} \mu_0 y + h(y,v,\rho,\beta) \\ \mu_{-1} v + g(y,v,\rho,\beta) \end{bmatrix} &= P^{-1} f(x_1, x_2, \rho, \beta) \\ &= \frac{1}{\mu_{-1} - \mu_0} \begin{bmatrix} \mu_{-1}\sqrt{1+\mu_0^2} & -\sqrt{1+\mu_0^2} \\ -\mu_0\sqrt{1+\mu_{-1}^2} & \sqrt{1+\mu_{-1}^2} \end{bmatrix} \begin{bmatrix} x_2 \\ \frac{1}{\beta}(\rho - x_2 - \sin x_1) \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} \sqrt{1+\mu_0^2}\left(\frac{\Delta-1}{2} x_2 + \rho - \sin x_1\right) \\ \sqrt{1+\mu_{-1}^2}\left(\frac{\Delta+1}{2} x_2 - \rho + \sin x_1\right) \end{bmatrix} \\ &= \begin{bmatrix} \mu_0 y + \frac{\rho(y+v)^2}{2\Delta(1+\mu_0^2)} - \frac{\sqrt{1-\rho^2}(y+v)^3}{6\Delta(1+\mu_0^2)^{3/2}} - \frac{\rho(y+v)^4}{24\Delta(1+\mu_0^2)^2} + O\left((y+v)^5\right) \\ \mu_{-1} v - \frac{\rho(y+v)^2}{2\Delta(1+\mu_0^2)} + \frac{\sqrt{1-\rho^2}(y+v)^3}{6\Delta(1+\mu_0^2)^{3/2}} + \frac{\rho(y+v)^4}{24\Delta(1+\mu_0^2)^2} + O\left((y+v)^5\right) \end{bmatrix}. \end{aligned} \tag{3.7}$$

Note that in the last equality we used the Taylor expansion

$$\begin{aligned} \sin x_1 &= \sin\left(\pi - \arcsin\rho + \frac{y+v}{\sqrt{1+\mu_0^2}}\right) = \sin\left(\arcsin\rho - \frac{y+v}{\sqrt{1+\mu_0^2}}\right) \\ &= \sin(\arcsin\rho) - [\cos(\arcsin\rho)]\frac{y+v}{\sqrt{1+\mu_0^2}} - [\sin(\arcsin\rho)]\frac{(y+v)^2}{2(1+\mu_0^2)} \\ &\quad -[\cos(\arcsin\rho)]\frac{(y+v)^3}{6(1+\mu_0^2)^{3/2}} - [\sin(\arcsin\rho)]\frac{(y+v)^4}{24(1+\mu_0^2)^2} + O\left((y+v)^5\right) \\ &= \rho + \sqrt{1-\rho^2}\frac{y+v}{\sqrt{1+\mu_0^2}} - \rho\frac{(y+v)^2}{2(1+\mu_0^2)} + \sqrt{1-\rho^2}\frac{(y+v)^3}{6(1+\mu_0^2)^{3/2}} \\ &\quad -\rho\frac{(y+v)^4}{24(1+\mu_0^2)^2} + O\left((y+v)^5\right), \end{aligned} \tag{3.8}$$

Hence in view of (2.13)- (2.15) and (2.19)- (2.21) the "slowly growing" solution and its approximation, correspondingly, have the form:

$$x_{sl} = \begin{bmatrix} \pi - \arcsin\rho \\ 0 \end{bmatrix} + P\begin{bmatrix} y \\ V(y,\rho,\beta) \end{bmatrix}, \quad y = y(t,\rho,\beta,\epsilon_-), \ V(y,\rho,\beta) \in \mathbb{R}, \tag{3.9}$$

$$(3.10)\qquad x_{sl}^m = \begin{bmatrix} \pi - \arcsin\rho \\ 0 \end{bmatrix} + P\begin{bmatrix} y \\ V^m(y,\rho,\beta) \end{bmatrix}, \quad y = y_m(t,\rho,\beta,\epsilon_-),\ V^m(y,\rho,\beta) \in \mathbf{R},$$

where

$$(3.11)\qquad V^m(y,\rho,\beta) = A(\rho,\beta)y^2 + B(\rho,\beta)y^3 + C(\rho,\beta)y^4 + O(y^5)$$

is the Taylor polynomial of order m about $y = 0$ for $V(y,\rho,\beta)$ which is obtained, according to (2.19), by solving

$$(3.12)\qquad V_y^m[\mu_0 y + h(y, V^m, \rho, \beta)] = \mu_{-1} v + g(y, V^m, \rho, \beta),$$

with $y = y_m(t,\rho,\beta,\epsilon_-)$. Substituting (3.7) and (3.11) into (3.12), we have

$$(3.13)\qquad \begin{aligned} &(2Ay + 3By^2 + 4Cy^3)\Big[\mu_0 y + \frac{\rho}{2\Delta}(y + Ay^2 + By^3 + Cy^4)^2 \\ &- \frac{\sqrt{1-\rho^2}}{6\Delta}(y + Ay^2 + By^3 + Cy^4)^3 - \frac{\rho}{24\Delta}(y + Ay^2 + By^3 + Cy^4)^4 + O(y^5)\Big] \\ &= \mu_{-1}(Ay^2 + By^3 + Cy^4) - \frac{\rho}{2\Delta}(y + Ay^2 + By^3 + Cy^4)^2 \\ &+ \frac{\sqrt{1-\rho^2}}{6\Delta}(y + Ay^2 + By^3 + Cy^4)^3 + \frac{\rho}{24\Delta}(y + Ay^2 + By^3 + Cy^4)^4 + O(y^5). \end{aligned}$$

Equating the coefficients, we obtain the equations

$$(3.14)\qquad \begin{aligned} 2A\mu_0 &= \mu_{-1}A - \frac{\rho}{2\Delta}, \\ 3B\mu_0 + 2A\frac{\rho}{2\Delta} - \frac{\sqrt{1-\rho^2}}{6\Delta} &= \mu_{-1}B - \frac{\rho}{2\Delta}2A + \frac{\sqrt{1-\rho^2}}{6\Delta}, \end{aligned}$$

whose solutions are

$$(3.15)\qquad \begin{aligned} A &= \frac{\rho}{2\Delta(\mu_{-1} - 2\mu_0)} = \frac{\rho\beta}{\Delta(1-3\Delta)} < 0, \\ B &= \frac{\sqrt{1-\rho^2} - 6A\rho}{3\Delta(3\mu_0 - \mu_{-1})} = \frac{\sqrt{1-\rho^2} - 6A\rho}{3\Delta(2\Delta - 1)}\beta > 0. \end{aligned}$$

Substituting these solutions into (3.11) and then (3.11) into (3.10) gives

$$(3.16a)\qquad x_{sl}^2 = \begin{bmatrix} \pi - \arcsin\rho \\ 0 \end{bmatrix} + P\begin{bmatrix} Y_2 \\ Y_2 + AY_2^2 \end{bmatrix}, \quad A = \frac{\rho\beta}{\Delta(1-3\Delta)},$$

$$(3.16b)\qquad \begin{aligned} x_{sl}^3 &= \begin{bmatrix} \pi - \arcsin\rho \\ 0 \end{bmatrix} + P\begin{bmatrix} Y_3 \\ Y_2 + AY_3^2 + BY_3^3 \end{bmatrix}, \\ A &= \frac{\rho\beta}{\Delta(1-3\Delta)}, \quad B = \frac{\sqrt{1-\rho^2} - 6A\rho}{3\Delta(2\Delta - 1)}\beta. \end{aligned}$$

Here $y = y_m(t,\rho,\beta,\epsilon_-)$ is obtained, according to (2.20), by solving

$$(3.17)\qquad \begin{aligned} &y' = \mu_0 y + h(y, V^m(y,\rho,\beta),\rho,\beta), \quad -\infty < t < T_-,\\ &y(T_-) = \epsilon_-. \end{aligned}$$

To solve (3.17) approximately we use the Taylor expansion for the right hand side. Taking into account (3.7) and (3.11) we obtain

$$(3.18)\qquad \begin{aligned} &y' = \mu_0 y + \frac{\rho}{2\Delta}(y^2 + 2Ay^3) - \frac{\sqrt{1-\rho^2}}{6\Delta} y^3 + O(y^4), \quad -\infty < t < T_-,\\ &y(T_-) = \epsilon_-. \end{aligned}$$

In particular, in the case $m = 2$ neglecting the higher order terms and using (3.14), the above reduces to

$$(3.19)\qquad \begin{aligned} &y' = \mu_0 y + \frac{\rho}{2\Delta} y^2, \quad -\infty < t < T_-,\\ &y(T_-) = \epsilon_-, \end{aligned}$$

whose solution is an approximation to y_2 :

$$(3.20)\qquad y_2(t,\rho,\beta,\epsilon_-) \approx \left(\frac{\rho}{2\Delta} \frac{e^{\mu_0(T_- - t)} - 1}{\mu_0} + \frac{1}{\epsilon_-} \right)^{-1}, \quad \mu_0 = \frac{2\sqrt{1-\rho^2}}{1+\Delta}.$$

The boundary conditions (2.24b) and (2.24c), respectively, take the form:

(3.21)

$$b)\ \ x(T_-) = x_-(\lambda) + P(\lambda)\begin{bmatrix} \epsilon_- \\ V^m(\epsilon_-,\lambda) \end{bmatrix} \equiv \begin{bmatrix} \pi - \arcsin\rho \\ 0 \end{bmatrix} + \epsilon_- w_0 + \left(A\epsilon_-^2 + B\epsilon_-^3 + \ldots\right) w_{-1},$$

$$c)\ \ x(T_+) = x_+(\lambda) - P(\lambda)\begin{bmatrix} 0 \\ \epsilon_+ \end{bmatrix} \equiv \begin{bmatrix} 3\pi - \arcsin\rho \\ 0 \end{bmatrix} - \epsilon_+ w_{-1},$$

3.2. Known theoretical results (Schecter [13] and Chow *et al.* [3]).

We identify x_1 and $x_1 + 2\pi$, so that (3.1) defines a vector field on a cylinder. For an equilibrium x_- of (3.1) we denote by μ_{-1} and μ_0 the eigenvalues of $f_x(x_-,\rho,\beta)$. We shall be interested in the situation when during the continuation procedure $\mu_{-1} < 0$ and $\mu_0 \geq 0$. For $(\rho,\beta) = (1,\beta^\star)$ for some (unknown) $\beta^\star$ (3.1) has an equilibrium $(\pi/2, 0)$ of saddle-node type, with a separatrix loop Γ, where $\mu_{-1} = \mu_{-1}^\star < 0$ and $\mu_0 = \mu_0^\star \equiv 0$. Denote $\nu_1 = \rho - 1$, $\nu_2 = \beta - \beta^\star$. The bifurcation diagram [13] is depicted on Figure 1.

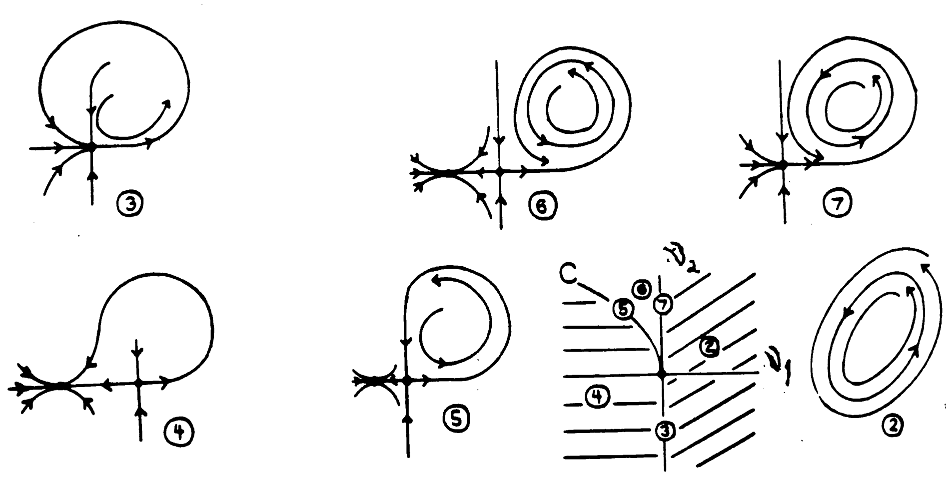

Figure 1.

The phase portrait in a neighborhood of Γ (Figure 1) is as follows:

1) $\nu_1 = 0,\ \nu_2 = 0$; a saddle-node and a separatrix loop.

2) $\nu_1 > 0$; no equilibria, a unique stable closed orbit near Γ.

3) $\nu_1 = 0,\ \nu_2 < 0$; a saddle-node.

4) $\nu_1 < 0,\ (\nu_1, \nu_2)$ below C; a saddle and a node.

5) $\nu_1 < 0,\ (\nu_1, \nu_2)$ on C; a saddle and a node; the saddle has a separatrix loop.

6) $\nu_1 < 0,\ (\nu_1, \nu_2)$ above C; a saddle and a node; there is a unique stable closed orbit near Γ.

7) $\nu_1 = 0,\ \nu_2 > 0$; a saddle-node and a unique stable closed orbit near Γ.

3.3. Solution algorithm and the numerical results.

Step 1. (Doedel and Kernevéz [6, pp. 50–54]). *A homotopy from a problem with known periodic solution to a periodic solution of period T of (3.1) in the region 2)* $(\beta > \beta^*,\ \rho > 1)$. The periodic approximation algorithm [6] is based on equations

(5.1) $$x' = Tf(x, \rho, \beta), \quad 0 < t < 1,$$

with $f(x,\rho,\beta) = \left(x_2, \frac{1}{\beta}(-x_2 - \sin x_1 + \rho)\right)^T$,

(5.2) $$\int_0^1 \left[\left(x_1(t) - x_1^0(t)\right) f_1\left(x^0, \rho^0, \beta^0\right) + x_2(t) f_2\left(x^0, \rho^0, \beta^0\right)\right] dt = 0,$$

where $\left(x^0(t), \rho^0, \beta^0\right)$ is a previously computed solution along a solution branch, and

(5.3) $$x_1(1) - x_1(0) - 2\pi = 0, \quad x_2(1) - x_2(0) = 0.$$

We introduce a homotopy parameter γ in (5.1) and consider the equations

(5.4) $$\begin{aligned} x_1' &= Tx_2, & 0 < t < 1, \\ x_2' &= \frac{T}{\beta}(\rho - \gamma \sin x_1 - x_2), & 0 < t < 1. \end{aligned}$$

Take $\rho = 2$, $\beta = 1$. Then for $\gamma = 0$ we see that $x_1(t) = 2\pi t$, $x_2(t) \equiv 2$, $T = \pi$ is a solution to (5.2), (5.3), (5.4), where in (5.2) f is replaced by the right hand side of (5.4), and $x^0 = x$. Freezing ρ and β we find a branch $(x(\cdot, s), T(s), \gamma(s))$ that passes through $\gamma = 1$. Thus the homotopy leads to a starting point for (5.1)- (5.3).

Step 2 [6, pp. 50–54]. *Computation of a branch $(x(s), T(s), \rho(s))$ of solutions of (5.1)-(5.3) in the direction of decreasing ρ and increasing T, while $\beta > \beta^\star$ is kept frozen; passing through the regions 7) and 6) and approaching the region 5).*

Step 3. *Switching from the periodic approximation to the tangent one.* See Figures 2, 3.

The tangent approximation algorithm is based on equations (5.1),

$$\int_0^1 \left(f(x,\rho,\beta) - f(x^0,\rho^0,\beta^0)\right) \cdot f_x(x,\rho,\beta) f(x,\rho,\beta)\, dt = 0, \tag{5.5}$$

where $\left(x^0(t), \rho^0, \beta^0\right)$ is a previously computed solution along a solution branch, and

$$a)\ x(0) = x_- + \epsilon_- w_0, \quad b)\ x(1) = x_+ - \epsilon_+ w_{-1}. \tag{5.6}$$

Here

$$\begin{aligned} x_- &= (\pi - \arcsin\rho, 0)^T, \quad \mathrm{x}_+ = (3\pi - \arcsin\rho, 0)^T, \\ \mu_{-1} &= \frac{-1-\Delta}{2\beta}, \quad \mu_0 = \frac{2\sqrt{1-\rho^2}}{1+\Delta}, \quad \Delta \equiv \sqrt{1 + 4\beta\sqrt{1-\rho^2}}, \\ w_{-1} &= \frac{1}{\sqrt{1+\mu_{-1}^2}} \begin{bmatrix} 1 \\ \mu_{-1} \end{bmatrix}, \qquad w_0 = \frac{1}{\sqrt{1+\mu_0^2}} \begin{bmatrix} 1 \\ \mu_0 \end{bmatrix}. \end{aligned} \tag{5.7}$$

The unknowns are $x(t)$, ρ, β, ϵ_-, ϵ_+.

A result in Lin [11, Th. 3.1, Th. 4.3, Lemma 4.5] can be interpreted as an estimate of a distance from a homoclinic orbit to its periodic approximation. Specifically, let (x, λ), $n_\lambda = 1$, be a homoclinic solution of (1.1). Under appropriate assumptions on f and appropriate transversality conditions, for sufficiently large T in a neighborhood of (x, λ) there exists an unique, up to time translations, branch of periodic solutions $(x_p(T), \lambda_p(T))$ of (1.1), parametrized by the period T. Moreover, we have the following estimate:

$$|\lambda - \lambda_p(T)| + \|x - x_p(T)\|_{L_\infty([-\frac{T}{2},\frac{T}{2}])} \le Ce^{-\frac{T}{2}\alpha}, \ \alpha = \min\{|\mu_1|, \mu_0\}. \tag{5.8}$$

where μ_0, μ_1 can be chosen the same as in (1.3). Note that $(\mu_0,\ \mu_1)$ in (5.8) can be identified with $(\mu_0,\ \mu_{-1})$ in (5.8). The estimates (5.8) and (1.3) provide us with the information necessary to design algorithms for switching between the periodic approximation of a homoclinic orbit and the tangent one.

Let $\left(x_p^0(t), \rho_p^0, \beta_p^0, T^0\right)$ be a solution of (5.1)-(5.3). We obtain the tangent approximation of a homoclinic orbit from its periodic approximation by a homotopy as follows. Let

$$\begin{aligned} a)\ x(0) &= \gamma(x_- + \epsilon_- w_0) + (1-\gamma)x_p^0(0), \\ b)\ x(1) &= \gamma(x_+ - \epsilon_+ w_{-1}) + (1-\gamma)x_p^0(1). \end{aligned} \tag{5.9}$$

Then as γ varies continuously from 0 to 1, the solution $(x(s), \rho(s), \beta(s))$ of the system (5.1), (5.9) varies continuously from the solution of the system (5.1)–(5.3) to the solution of the system (5.1), (5.5), (5.6). The initial data for the homotopy is:

$$(5.10)\qquad \begin{aligned} &x(0) = x_p^0(0),\ x_- = x_-^0 \equiv \left(\pi - \arcsin \rho_p^0, 0\right)^T,\ w_0 = w_0\left(\rho_p^0, \beta_p^0\right), \\ &x(1) = x_p^0(1),\ x_+ = x_+^0 \equiv \left(3\pi - \arcsin \rho_p^0, 0\right)^T,\ w_{-1} = w_{-1}\left(\rho_p^0, \beta_p^0\right), \\ &\rho = \rho_p^0,\ \beta = \beta_p^0; \end{aligned}$$

and $T = T^0$, $\epsilon_- = \left|x_p^0(0) - x_-^0\right|$, $\epsilon_+ = \left|x_p^0(1) - x_+^0\right|$ are fixed.

Step 4. *Switching from the tangent approximation to a higher order approximation for the "slow" manifold.* The boundary condition for the second order approximation for the "slow" manifold is given by

$$(5.11)\qquad a)\ x(0) = x_- + \epsilon_- w_0 + A\epsilon_-^2 w_{-1}, \quad b)\ x(1) = x_+ - \epsilon_+ w_{-1}.$$

The boundary condition for the third order approximation for the "slow" manifold is given by

$$(5.12)\qquad a)\ x(0) = x_- + \epsilon_- w_0 + \left(A\epsilon_-^2 + B\epsilon_-^3\right) w_{-1}, \quad b)\ x(1) = x_+ - \epsilon_+ w_{-1},$$

etc... The second order approximation for the "slow" manifold is obtained by a homotopy from (5.1), (5.5), (5.6) to (5.1), (5.5), (5.11). The third order approximation is obtained similarly.

Step 5. *Computation of a branch $(x(s), \rho(s), \beta(s))$ of solutions of equations (5.1), (5.5) with one of the boundary conditions (5.6), (5.11) or (5.12) in the direction of increasing ρ and decreasing β, while T is frozen at a large value, and thus approaching the point $(\rho, \beta) = (1, \beta^\star)$.* See Figure 4, left. The figure agrees with the theoretical result [13]:

$$(5.13)\qquad \rho = 1 - l^2(\beta - \beta^*)^2 + o\Big((\beta - \beta^*)^2\Big) \text{ as } \rho \to 1.$$

Figure 5 depicts the orbits in the case of the periodic approximation and the tangent one. Note that though the phase plane representations of the orbits are visually indistinguishable, the values of β are quite different. In fact, one requires T=33.5, 135, 10600 in the case of the periodic approximation to obtain the same accuracy in β as for T=12.5, 25, 50, correspondingly, in the case of the tangent approximation.

To obtain a better approximation of $\beta^\star$, in view of (5.13), we make the following change of variables:

$$(5.14)\qquad \rho_1 = \sqrt{1 - \rho^2},$$

which implies $\mu_0 \approx \rho_1$, as $\rho_1 \to 0$. The results of computations are shown on Figure 6, left.

Step 6. *Investigating the dependence of the accuracy of approximation of β on the period T for fixed ρ.* The dependence of the accuracy of approximation of β on the period

T for fixed ρ close to 1 is shown on Figure 4, right, and for fixed ρ_1 close to 0 on Figure 6, right.

Finally, the following table shows how the error in the approximation of $\beta(\approx \beta^\star)$ for $\rho = 1 - .51 \times 10^{-6}$ $\left(\mu_0 = 10^{-3}\right)$ depends on T in the case of the linear, quadratic and cubic approximation of the slow (center) manifold. The table seems to indicate that the error in β decays exponentially with T in all three approximations, with the cubic approximation being slightly better than linear one and the quadratic approximation being slightly better than cubic one. One would expect the above errors to be proportional to powers of $\frac{1}{T}$. This exponential decay of the error is quite unexpected, and is perhaps, due to the specific problem. This might also explain why for this problem higher order approximations of the center manifold only slightly improve the error. Also from the table $\epsilon_- = O\left(\frac{1}{T}\right)$.

T	ϵ_- (first order approximation)	Relative error in the first order approximation	Relative error in the second order approximation	Relative error in the third order approximation
10	.341	1.538×10^{-1}	1.525×10^{-1}	1.533×10^{-1}
20	.154	1.931×10^{-2}	$1.903\times10o^{-2}$	1.924×10^{-2}
40	.0697	2.27×10^{-4}	2.20×10^{-4}	2.25×10^{-4}
60	.0445	2.574×10^{-6}	2.571×10^{-6}	2.571×10^{-6}
75	.0349	0	0	0

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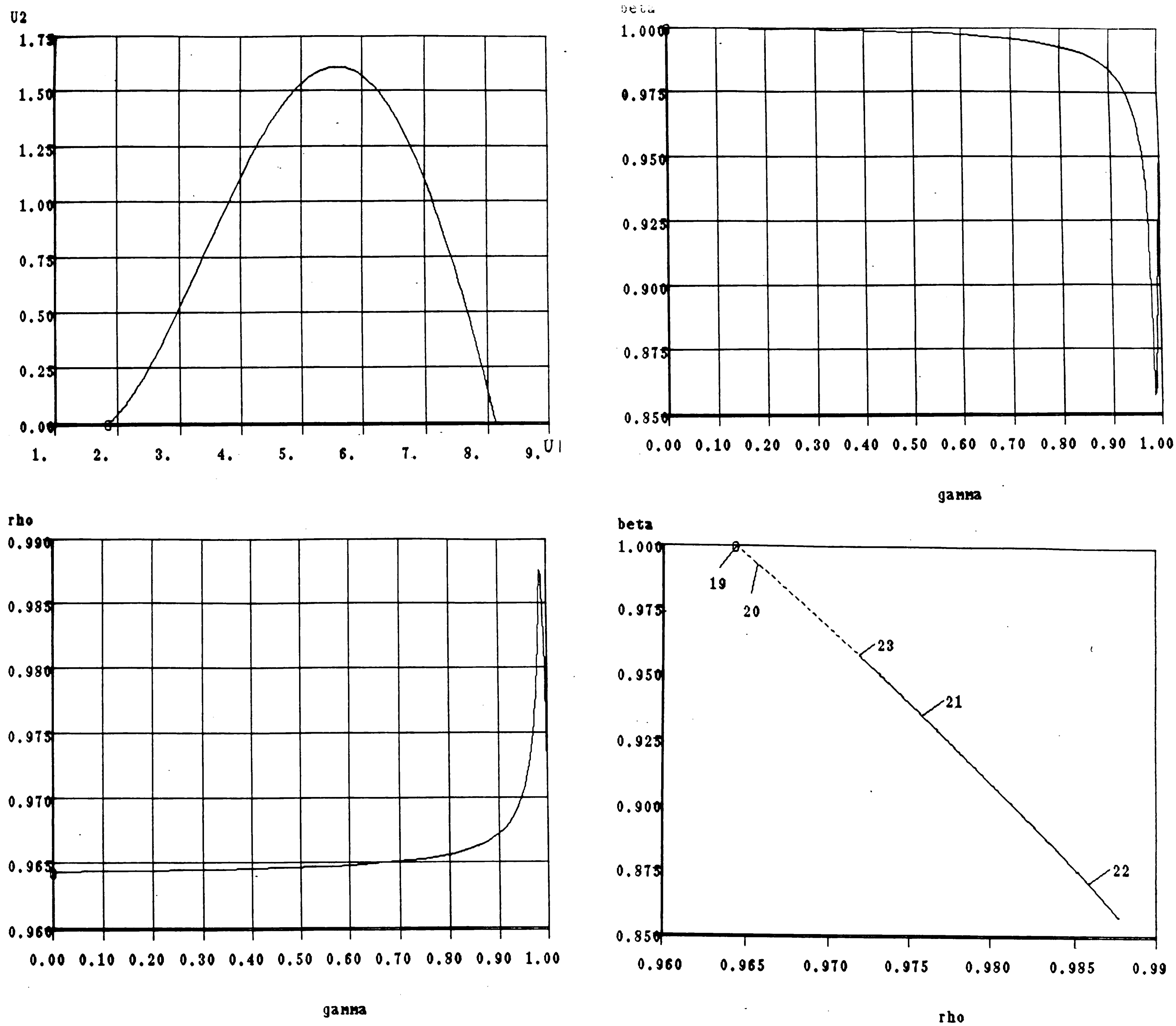


Figure 2. *Some orbits that connect the fixed points* $x_- = (\pi - \arcsin\rho, 0)^T$ *and* $x_+ = (3\pi - \arcsin\rho, 0)^T$*, of system (5.1), (5.9). Top left: phase plane representation of (5.1), (5.9), when* $\gamma = 0$ *(in the beginning of Step 3, periodic approximation),* $T=50$, $\rho=0.9643326$ *and* $\beta=1.0$*. Top right: a homotopy from the periodic approximation to the tangent one in* (γ, β) *plane, starting at the point* $(\gamma, \beta)=(0.0, 1.0)$ *and terminating at the point* $(\gamma, \beta)=(1.0, 0.9916909)$*. Bottom left: a homotopy from the periodic approximation to the tangent one in* (γ, ρ) *plane, starting at the point* $(\gamma, \rho)=(0.0, 0.9643326)$ *and terminating at the point* $(\gamma, \rho)=(1.0, 0.9718606)$*. Bottom right: a homotopy from the periodic approximation to the tangent one in* (ρ, β) *plane, starting at the point* $(\rho, \beta)=(0.9643326, 1.0)$ *with the label 19 and terminating at the point* $(\rho, \beta)=(0.9718606, 0.9916909)$ *with the label 23.*

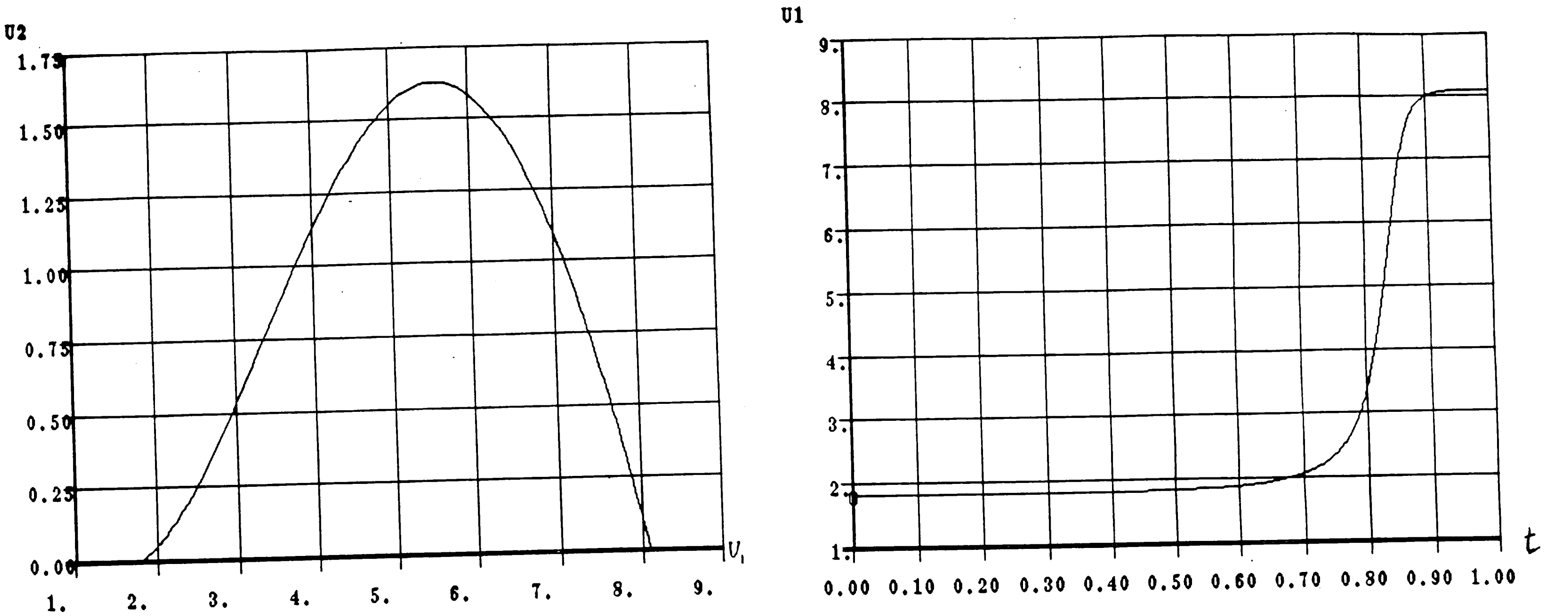


Figure 3. *Some orbits that connect the fixed points* $x_- = (\pi - \arcsin\rho, 0)^T$ *and* $x_+ = (3\pi - \arcsin\rho, 0)^T$, *of system (5.1), (5.9). Left: phase plane representation of (5.1), (5.9), when* $\gamma = 1$ *(in the end of Step 3, tangent approximation),* $T=50$, $\rho=0.9718606$ *and* $\beta=0.9916909$. *Right:* $(t,\ x_1)$ *plane representation in the above case.*

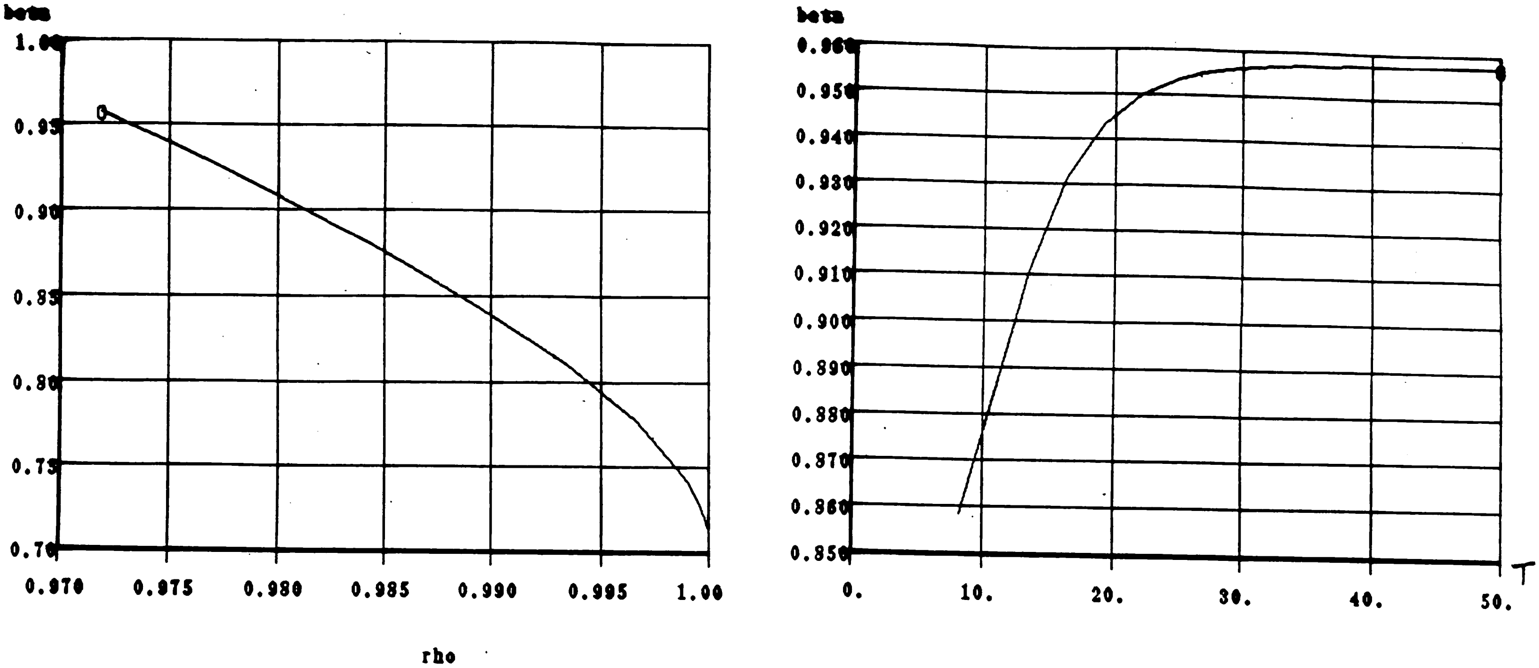
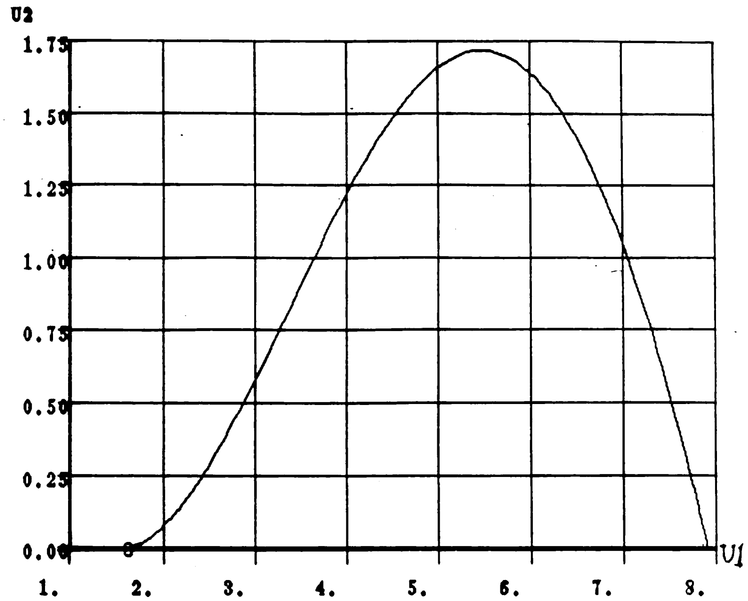


Figure 4. *Left (Step 5): a branch of solutions of (5.1), (5.5), (5.6) with* $T = 20$, *fixed, starting at the point* $(\rho,\ \beta)=(0.9718606,\ 0.9916909)$ $(\mu_0 \approx 0.24)$ *and terminating at the point* $(\rho,\ \beta)=(0.99999949,\ 0.7028165)(\mu_0 \approx 0.001)$. *Right (Step 6):* $(T,\ \rho)$ *plane representation of a branch of approximate solutions of (5.1), (5.5), (5.6) parametrized by the period* T *for fixed* $\rho=0.9718606$ $(\mu_0 \approx 0.24)$.



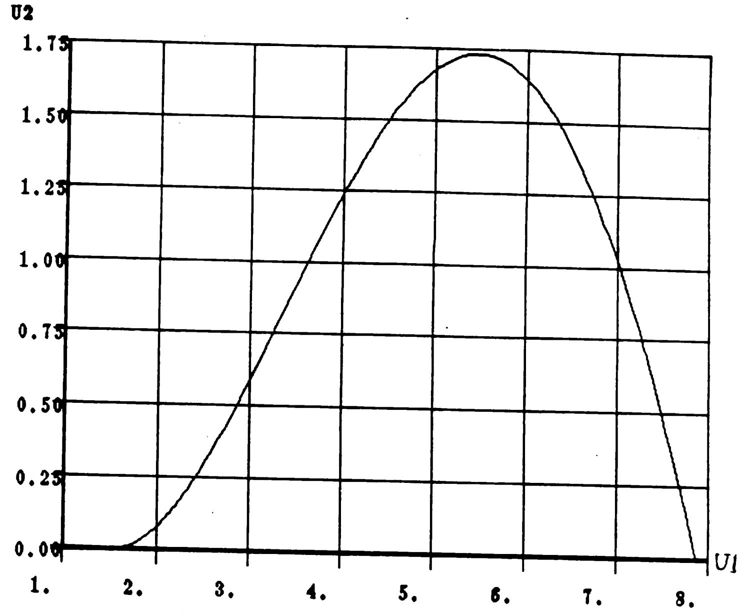


Figure 5. *Phase plane representation of orbits that connect the fixed points* $x_- = (\pi - \arcsin\rho, 0)^T$ *and* $x_+ = (3\pi - \arcsin\rho, 0)^T$, *with* T=*50,* ρ=*0.99999949* ($\mu_0 \approx 10^{-3}$), *fixed. Left: tangent approximation (system (5.1), (5.5), (5.6)), here* β=*0.7028165. Right: periodic approximation (system (5.1)- (5.3)), here* β=*0.7424534.*

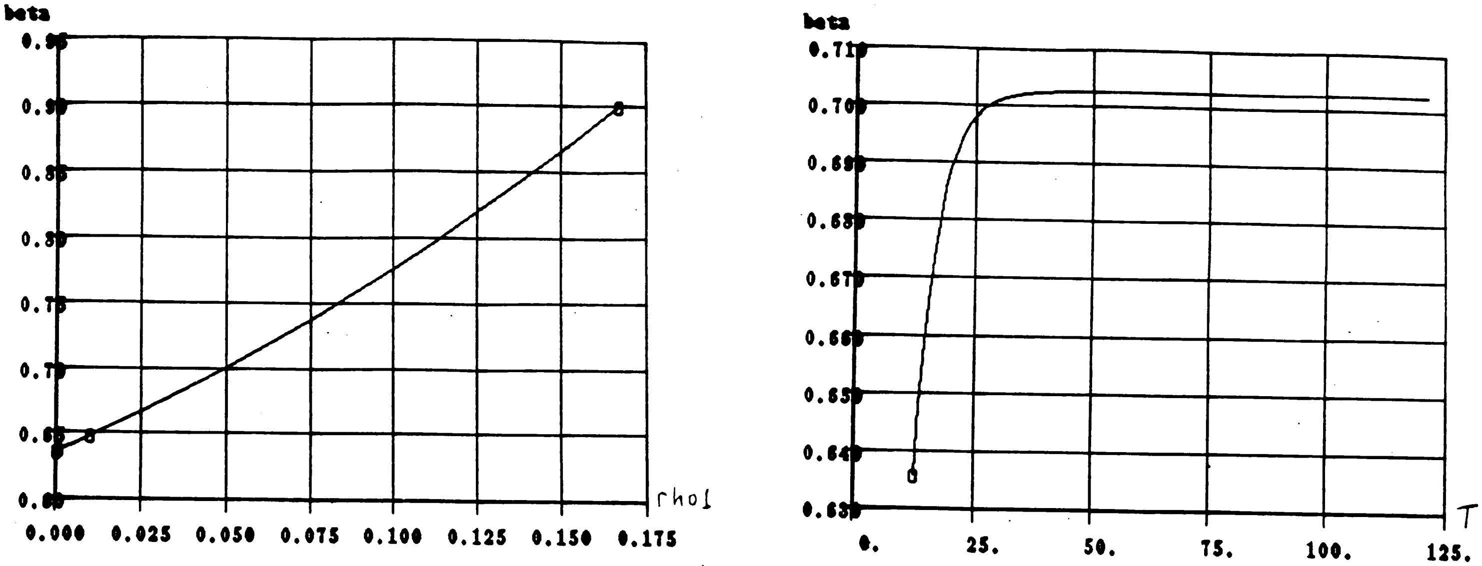


Figure 6. *Left (Step 5): a branch of solutions of (5.1), (5.5), (5.6) with* $T = 20$, *fixed, starting at the point* (ρ_1, β)*=(0.167, 0.9004192) and terminating at the point* $(\rho_1, \beta) = (10^{-6}, 0.6890448)$ *Right (Step 6): a branch of solutions of (5.1), (5.5), (5.6) with* $\rho_1 = 10^{-6}$ $(\mu_0 \approx 10^{-6})$, *fixed, starting at the point* $(T, \beta) = (10, 0.5951997)$ *and terminating at the point* $(T, \beta) = (121.3, 0.7025648)$.

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