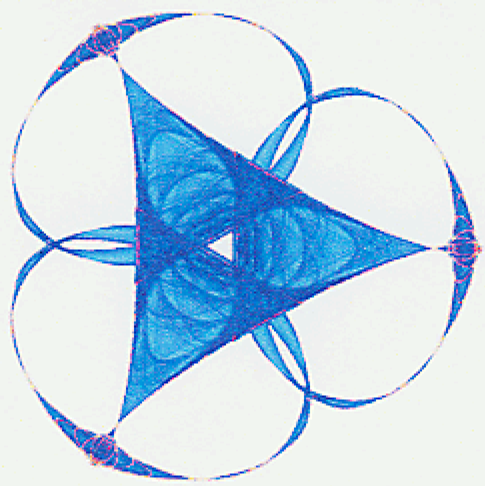
ON PARTIALLY OBSERVED CONTROL OF MARKOV PROCESSES

By

Omar Hijab



IMA Preprint Series # 274

October 1986

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS

UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455

ON PARTIALLY OBSERVED CONTROL OF MARKOV PROCESSES

Omar Hijab*

Mathematics Department
Temple University
Philadelphia, PA
19122

* Work supported by the NSF under grant #DMS848885 and the Institute for Mathematics and Its Applications (University of Minnesota).

ABSTRACT

In this paper it is shown that to each partially observed control problem corresponds a completely observed separated problem equivalent to it in the sense that the corresponding value functions are equal. This result, established in certain cases previously by Bismut and Fleming, is established here in general. It is also shown that the corresponding measure-valued martingale problem is well-posed for constant controls.

AMS 1980 Subject Classifications: 60G35, 49A10, 93E11, 93E20.

Key Words: Markov Processes, Partially Observed Control, Measure-Valued Diffusions, Martingale Problem, Filtering.

1. INTRODUCTION

Let $x(\cdot)$ denote a controlled state process evolving on some state space X and let observations be taken according to

$$(1.1) \qquad y(t) = \int_0^t c(x(s),u(s))ds + \eta(t), \; t \geq 0,$$

where $u(\cdot)$ is the control process and $\eta(\cdot)$ is a Brownian motion representing the noise in the signal-plus-noise model (1.1). In this paper we study the problem of choosing a control $u(\cdot)$ minimizing a cost criterion of the form

$$(1.2) \qquad v^u = E(\int_0^\infty e^{-t} L(x(t),u(t))dt)$$

over the set of all controls $u(\cdot)$ that depend on the past of the corresponding observations $y(\cdot)$. Because of this constraint, this problem is often referred to as partially observed to contrast with the completely observed situation in which the controller has complete access to the state, i.e. minimizes (1.2) over the set of all controls $u(\cdot)$ that depend on the past of the corresponding state process $x(\cdot)$. The partially observed nature of the above problem makes it difficult. At present, even the existence of a minimizing control is not established in the general setting. For the problem of existence, see Bismut (3), Borkar (5), Christopeit(6), Fleming and Pardoux (14), and Haussmann (15). There are special cases of the above problem that are completely and explicitly solvable. These are, however, very few in number.

Because of the difficulty of a direct approach, one is led to introduce a "separated" completely observed control problem designed in such a way that if $\hat{v}^u$ denote its cost criterion one ought to have

$$(1.3)\qquad \inf_u v^u = \inf_u \hat{v}^u.$$

Indeed using the techniques of Nonlinear Filtering one is naturally led to such a completely observed problem whose state space is the space M(X) of probability measures on X. The motivation for doing this is that the completely observed nature of the separated problem may allow its analysis along the lines of finite dimensional completely observed problems as in the work of Beneš (1), El Karoui, Hûû Nguyen, and Jeanblanc-Piqué (7), Fleming and Nisio (12), Haussmann (16), Krylov (17), Kushner (20), and Lions (21). Indeed the separated problem has been studied from this point of view by Beneš and Karatzas (2), Bismut(4), Fleming (10), Fleming (11), and Fleming and Nisio (13). In particular Bismut (4) and Fleming (10) have established (1.3) under certain hypotheses. The aim of this paper is to establish (1.3) in general. In particular the case when $x(\cdot)$ is a diffusion is not covered by either of the above results but is covered by the result presented here.

The paper proceeds as follows: In Section 2 we state our assumptions (which are that the martingale problem for $x^u(\cdot)$ is well-posed for constant controls u together with some continuity and boundedness assumptions on the coefficients). In Sections 3 and 4 we state the two problems precisely and in Section 8 we state and prove the main result. Sections 5-7, 9-12 deal with the various Lemmas appearing in the proof of the main result. In Section 13, we show that, for u constant and c in the domain of the state generator A^u, the martingale problem on $C([0,\infty);M(X))$ associated to the filtering equation is well-posed. This is related to a result of Kurtz and Ocone (19).

During the writing of this paper, a manuscript of El Karoui, Hûû Nguyen, and Jeanblanc-Picqué (8) was brought to my attention. In it they establish several results including (1.3). Nevertheless as their results and point of view are somewhat different than ours, we feel that the presentation given here is worthwhile.

I would like to thank W. H. Fleming for helpful discussions concerning aspects of this paper.

2. ASSUMPTIONS

We take the state space to be Polish, i.e. complete, separable, and metric. We have in mind (see below) $X = \mathbb{R}^d$ or $X = \{0,1,\ldots,d\}$. Controls take values in a separable metric space U; observations take values in $\mathbb{R}$. The following are standing assumptions throughout the paper.

(A0) There is a linear subspace $\mathbb{D} \subset C_b(X)$ on which is defined a linear operator $A^u : \mathbb{D} \to C_b(X)$ for each u in U. For ϕ in $\mathbb{D}$ we assume that $\psi(x,u) = A^u(\phi)(x)$ is in $C_b(X \times U)$ and satisfies the uniformity

$$(2.1) \qquad \sup\{|\psi(x,u) - \psi(x,u')| \mid x \in X,\ d(u,u') < \delta\} \to 0 \text{ as } \delta \downarrow 0.$$

We assume that $1 \in \mathbb{D}$, $A^u(\mathbb{D}) \subset \mathbb{D}$, and that the smallest collection of functions on X containing $\mathbb{D}$ and closed under bounded pointwise convergence is all of $B(X)$.

(A1) We are given functions c and L in $C_b(X \times U)$. We assume that if ψ denotes c or L, then (2.1) holds.

(A2) For each u in U, the martingale problem for $(A^u, \mathbb{D})$ is well-posed on the space $D([0,\infty);X)$ of all cadlag paths in X.

As examples of situations in which the above assumptions hold, we have in mind the following.

(i) $X = \mathbb{R}^d$, $\mathbb{D} = C_0^\infty(\mathbb{R}^d) \oplus \{\text{constants}\}$,

$$A^u = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x,u)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x,u)\frac{\partial}{\partial x_i}$$

with a, b, c in $C_b(\mathbb{R}^d \times U) \cap C^{\infty,0}(\mathbb{R}^d \times U)$, a,b uniformly continuous on U uniformly on compact subsets of $\mathbb{R}^d$, and c satisfying (2.1).

(ii) $X = \{0,1,\ldots,d\}$. Here $A^u = (a_{ij}(u))$ is an infinitesimally stochastic matrix and a_{ij} are assumed to be bounded and uniformly continuous on U. Here $\mathbb{D} = B(X)$.

The martingale problem point of view is used systematically throughout the paper. Stroock and Varadhan's book (25) is sufficient for any of the background material needed here. In what follows the control problems are defined in terms of martingale problems in a manner similar to what is done in the thesis of Sheu (22). Another useful reference is the recent text of Ethier and Kurtz (9).

3. PARTIALLY OBSERVED PROBLEM

To model (1.1) correctly we introduce the joint state-observations generator

$$G^u = G_o^u + c(x,u)\frac{\partial}{\partial y} = A^u + \frac{1}{2}\frac{\partial^2}{\partial y^2} + c(x,u)\frac{\partial}{\partial y} .$$

The linear operator G^u acts on the space $\mathbb{D} \otimes C_o^\infty(\mathbb{R})$ of all sums of products $\phi\psi$ with ϕ in $\mathbb{D}$, ψ in $C_o^\infty(\mathbb{R})$. Let $E=D([0,\infty);X)\times C([0,\infty);\mathbb{R})$. For $q = (\alpha,\beta)$ in E, set $x(t,q) = \alpha(t)$, $y(t,q) = \beta(t)$. Set $Y_t = \sigma[y(s),\ 0 \leq s \leq t]$, $Y = Y_\infty$, $F_t = \sigma[x(s),y(s),\ 0 \leq s \leq t]$, $F = F_\infty$. (E,F) is the basic probability space.

A map $u : [0,\infty) \times E \to U$ is an <u>admissible control</u> if u is <u>Y_t-progressively measurable</u> and the martingale problem for $(G^u, \mathbb{D} \otimes C_o^\infty(\mathbb{R}))$ is well-posed on (E,F) (Section 5). Let $\{P_{x,y}^u\}$ denote the solution to this martingale problem and set $P_m^u = \int P_{x,o}^u\, m(dx)$ for m in $M(X)$. Set

$$v^u(m) = E_m^u\left(\int_o^\infty e^{-t} L(x(t),u(t))dt\right),$$

$$v(m) = \inf \{ v^u(m) \mid u \text{ admissible} \}.$$

This defines the value of the partially observed problem starting from m.

4. SEPARATED PROBLEM

Let $\hat{E} = C([0,\infty);M(X)\times \mathbb{R})$, $\hat{q} = (\alpha,\beta)\in\hat{E}$, $\mu(t,\hat{q}) = \alpha(t)$, $\nu(t,\hat{q}) = \beta(t)$, $\hat{F}_t = \sigma[\mu(s),\nu(s),\ 0 \le s \le t]$, $t \ge 0$, $\hat{F} = \hat{F}_\infty$. In what follows $c^u(x) = c(x,u)$ and for any ϕ,ϕ' in $B(X)$, μ in $M(X)$, $\langle\phi,\phi'\rangle_\mu = \mu(\phi\phi')-\mu(\phi)\mu(\phi')$ is the covariance. Let $N_t = \sigma[\nu(s), 0 \le s \le t]$. A feedback is a N_t-progressively measurable map $f : [0,\infty) \times \hat{E} \to U$. A feedback $f(\cdot)$ is admissible if for each m in M(X) there is a unique measure $\hat{P}^f_m$ satisfying $\hat{P}^f_m(\mu(0)=m,\nu(0)=0) = 1$ and

$$(4.1)\quad \hat{P}^f_m\Big(\mu(t)(\phi) = \mu(0)(\phi) + \int_0^t \mu(s)(A^{f(s)}\phi)ds + \int_0^t \langle c^{f(s)},\phi\rangle_{\mu(s)} d\nu(s),\ t \ge 0\Big) = 1$$

for all ϕ in $\mathbb{D}$. As we shall see below (Section 7), $f(\cdot)$ is admissible iff a certain martingale problem is well-posed. Set $\hat{L}(\mu,u) = \int L(x,u)\mu(dx)$ and

$$\hat{V}^f(m) = \hat{E}^f_m\Big(\int_0^\infty e^{-t}\hat{L}(\mu(t),f(t))dt\Big),$$

$$\hat{V}(m) = \inf\{\hat{V}^f(m) \mid f \text{ admissible}\}.$$

This defines the value of the separated problem starting from m.

5. PARTIALLY OBSERVED PROBLEM: PIECEWISE CONSTANT CONTROLS ARE ADMISSIBLE

In this section we establish that any piecewise constant control is admissible. By piecewise constant we mean there is a $\delta > 0$ such that $u([t/\delta]\delta) = u(t)$ for all $t \geq 0$. Recall that G_o^u is the joint state-observations generator when there are no observations $(c = 0)$.

5.1. Lemma Let Q solve the martingale problem for G_o^u. Then the restriction of Q to Y is Wiener measure W. Let Q_q denote a regular conditional probability distribution (r.c.p.d.) of Q given Y. Then $(\Omega_X = D([0,\infty);X))$

$$(5.1) \qquad Q = \int Q_q \, W(dq)$$

and for W-a.a. q the marginal Q'_q on Ω_X of Q_q solves the martingale problem for $A^{u(q)}$. Conversely, if $q \to Q'_q$ is a Y-measurable map of E into $M(\Omega_X)$ such that, for W-a.a. q, Q'_q solves the martingale problem for $A^{u(q)}$, then with $Q_q = Q'_q \times \delta_q$ on $F = X \times Y$ (5.1) defines a solution Q to the martingale problem for G_o^u. In particular the martingale problem for G_o^u is well-posed if and only if the martingale problem for $A^{u(q)}$ is well-posed for W-a.a. q.

proof. In the statement of the Lemma, $X = X_\infty$ where $X_t = \sigma[x(s),\ 0 \leq s \leq t]$ on Ω_X. Set for ϕ in $\mathbb{D}$ and ψ in $C_o^\infty(\mathbb{R})$ and u progressively measurable

$$(5.2) \qquad X_{\phi\psi}^u(t) = \phi(x(t))\psi(y(t)) - \int_o^t G_o^{u(s)}(\phi\psi)(x(s),y(s))ds, \quad t \geq 0.$$

Suppose now that Q solves the martingale problem for G_o^u. Then $X_{\phi\psi}^u(\cdot)$ is a (E,F_t,Q) martingale for all ϕ and ψ. Choosing $\phi = 1$ implies that $y(\cdot)$ is a (E,F_t,Q) Brownian motion and thus the marginal of Q on Y is W.

Then by definition (5.1) follows. Suppose we establish that for ϕ in $\mathbb{D}$, $0 \leq s < T$, Φ bounded and F_s-measurable, $0 \leq t_1 < t_2 < \ldots < t_n$, $\psi_1, \psi_2, \ldots, \psi_n$ in $C_o^\infty(\mathbb{R})$,

$$(5.3) \qquad E^Q((X_\phi^u(T) - X_\phi^u(s))\Phi\psi_1(y(t_1))\psi_2(y(t_2))\ldots\psi_n(y(t_n))) = 0.$$

Then because u(•) is Y-measurable for each $0 \leq s < T$, $\phi \in \mathbb{D}$, Φ bounded and X_s-measurable

$$(5.4) \qquad E^{Q_q'}((X_\phi^{u(q)}(T) - X_\phi^{u(q)}(s))\Phi) = 0 \text{ for W-a.a. } q.$$

Now by choosing a countable dense (under bounded pointwise convergence) subset $\mathbb{D}_s(\Omega_X)$ of $B(\Omega_X, X_s)$, (5.4) can be made to hold for all rational $0 \leq s < T$ and Φ in $\mathbb{D}_s(\Omega_X)$ and all q not in a fixed W-null set. Since the paths of x(•) are right continuous, it follows that for all q not in a fixed W-null set, $X_\phi^{u(q)}(\cdot)$ is a (Ω_X, X_t, Q_q') martingale for all ϕ in $\mathbb{D}$. Thus Q_q' solves the martingale problem for $\Lambda^{u(q)}$ for all q not in a W-null set as soon as (5.3) is established.

To establish (5.3), note first that, since y(•) is a (E, F_t, Q) Brownian motion, by conditioning on F_T in (5.3), we see that we can assume that $0 \leq t_1 < t_2 < \ldots < t_n \leq T$. Since Φ is in F_s, it is enough to establish (5.3) when $s < t_1 < t_2 < \ldots < t_n \leq T$. Writing $X(T) - X(s)$ as $(X(T)-X(t_n)) + (X(t_n)-X(s))$, we see that it is enough to establish (5.3) when $s < t_1 < t_2 < \ldots < t_n = T$. This we do by induction on $n \geq 1$. To establish (5.3) when $n = 1$, let P_t denote the heat semigroup on Schwartz space $S(\mathbb{R})$. Since Q solves the martingale problem for G_o^u, standard procedure shows that

$$\phi(x(t))P_{T-t}\psi(y(t)) - \int_0^t (P_{T-r}\psi)(y(r))A^{u(r)}(\phi)(x(r))dr, \; 0 \leq t \leq T,$$

is a (E,F_t,Q) martingale. In particular choosing $\phi = 1$ we have $P_{T-t}\psi(y(t))$, $0 \leq t \leq T$, is a martingale. It follows that

$$E^Q((X^u_\phi(T) - X^u_\phi(s))\Phi\psi(y(T))) = E^Q(\phi(x(s))(P_{T-s}\psi(y(s)) - \psi(y(T)))\Phi)$$

$$+ \int_s^t E^Q(A^{u(r)}(\phi)(x(r))(P_{T-r}\psi(y(r)) - \psi(y(T)))\Phi)dr = 0,$$

establishing the case $n = 1$. To establish the inductive step, let $s < t_1 < t_2 < \ldots < t_n < t_{n+1} = T$, $\psi_1, \psi_2, \ldots, \psi_n, \psi_{n+1}, \Phi$ be given as above and set $\Phi' = \Phi\psi_1(y(t_1))\ldots\psi_n(y(t_n))$, $\psi = \psi_{n+1}$, $s' = T' = t_n$, $a = T - t_n$. Then

$$E^Q((X^u_\phi(T) - X^u_\phi(s))\Phi\psi_1(y(t_1))\ldots\psi_n(y(t_n))\psi_{n+1}(y(t_{n+1})))$$

$$= E^Q((X^u_\phi(T) - X^u_\phi(s'))\Phi'\psi(y(T))) + E^Q((X^u_\phi(T') - X^u_\phi(s))\Phi'\psi(y(T)))$$

$$= 0 + E^Q((X^u_\phi(T') - X^u_\phi(s))\Phi'(P_a\psi)(y(T'))) = 0$$

where the first term vanishes by the case $n = 1$ and the second vanishes by the induction hypothesis.

Conversely, suppose that Q'_q solves the martingale problem for $A^{u(q)}$. Set $Q_q = Q'_q \times \delta_q$ on $F = X \times Y$ and define Q by (5.1). Then (5.4) holds and hence (5.3) holds. Moreover $y(\cdot)$ is a (E,F_t,Q) Brownian motion and, for A in F_s, $q \to Q_q(A)$ is not just Y-measurable but is Y_s-measurable since <u>$u(\cdot)$ is Y_t-progressively measurable</u>. Thus for $\psi \in C_0^\infty(\mathbb{R})$ and $0 \leq s < T$,

$\Psi \in Y_s$, $\Phi \in X_s$, bounded

$$(5.5)\quad E^Q((X^u_\psi(T) - X^u_\psi(s))\Phi\Psi) = E^W((X^u_\psi(T) - X^u_\psi(s))E^{Q\cdot}(\Phi)\Psi) = 0.$$

Thus $X^u_\psi(\cdot)$ is a (E,F_t,Q) martingale. Hence

$$E^Q((X^u_{\phi\psi}(T) - X^u_{\phi\psi}(s))\Phi\Psi) = E^Q((X^u_\phi(T) - X^u_\phi(s))\psi(y(T))\Phi\Psi)$$

$$+ E^Q((X^u_\psi(T) - X^u_\psi(s))\phi(x(s))\Phi\Psi)$$

$$+ \int_s^T E^Q(\Big\{ A^{u(r)}(\phi)(x(r))(X^u_\psi(T)-X^u_\psi(r)) + \tfrac{1}{2}\Delta(\psi)(y(r))(X^u_\phi(s)-X^u_\phi(r))\Big\}\Phi\Psi)dr$$

$$+ \int_s^T E^Q(\Big\{A^{u(r)}(\phi)(x(r))\Big(\int_r^T \tfrac{1}{2}\Delta(\psi)(y(v))dv\Big) - \tfrac{1}{2}\Delta\psi(y(r))\Big(\int_s^r A^{u(v)}(\phi)(x(v))dv\Big)\Big\}\Phi\Psi)dr$$

$$= I + II + III + IV = 0.$$

Term I and the second half of term III vanish because of (5.3), term II and the first half of term III because of (5.5), and term IV because one can interchange the order of integration. This completes the proof of the Lemma.

5.2. Lemma Let Q solve the martingale problem for G^u_o and set

$$R(t) = \exp(\int_o^t c(x(s),u(s))dy(s) - \frac{1}{2}\int_o^t c(x(s),u(s))^2 ds),\ t \geq 0.$$

Then P solves the martingale problem for G^u iff $P \ll Q$ on F_t and $dP/dQ = R(t)$ on F_t for all $t \geq 0$. In particular the martingale problem for G^u is well-posed if and only if the martingale problem for G^u_o is well-posed.

proof. ((25), page 153).

In particular we conclude the following.

5.3. Corollary Let $u : [0,\infty) \times E \to U$ be Y_t-progressively measurable. Then $u(\cdot)$ is an admissible control if and only if the martingale problem for $A^{u(q)}$ is well-posed for W-a.a. q.

Since the well-posedness of the martingale problem for A^u, u constant, implies the well-posedness of the martingale problem for A^u, u piecewise constant, we conclude that piecewise constant controls are admissible. Note that Corollary 5.3 is a very useful result. For example if we assume that the martingale problem for A^u is well-posed for every measurable function $u(\cdot)$ of time, a rather mild assumption, then 5.3 implies that all Y_t-progressively measurable maps are admissible.

6. FILTERING

In this section we state well-known results from filtering theory in the form that we need them. In this section $u(\cdot)$ is an admissible control and m is in $M(X)$. Let $R(\cdot)$, Q, Q_q, W be as in Section 5 with Q, Q_q starting at m, $Q_q(x(0)\varepsilon A) = m(A)$ for all q.

6.1. Lemma For $t \geq 0$ and ϕ in $B(X)$,

$$(6.1) \qquad E_m^u(\phi(x(t))|Y_t) = E^Q(R(t)\phi(x(t)))/E^Q(R(t))$$

almost surely-P_m^u.

This Bayes formula follows in the usual manner from Lemmas 5.1 and 5.2.

6.2. Lemma There is a right continuous Y_t-progressively measurable π_m^u: $[0,\infty) \times E \to M(X)$ such that

$$(6.2) \qquad \pi_m^u(t)(\phi) = E_m^u(\phi(x(t))|Y_t), \; P_m^u\text{-a.s.},$$

for all $t \geq 0$ and ϕ in $B(X)$.

proof. Let A be the set of all (t,q) for which $E^{Q_q}(\sup_{0\leq s\leq t} R(s))$ and $E^{Q_q}(\sup_{0\leq s\leq t} R(s)^{-1})$ are finite. Then A is a Y_t-progressively measurable subset of $[0,\infty) \times E$. For (t,q) in A, define $\pi_m^u(t,q)(\phi)$ to be the right-hand-side of (6.1) evaluated at q. Let $A_q = \{t \mid (t,q)\varepsilon A\}$, $A_t = \{q \mid (t,q)\varepsilon A\}$. Then for each q, A_q is an interval of the form $[0,a)$ or $[0,a]$. In either case $\pi_m^u(\cdot,q)$ is right continuous on A_q. Moreover $\pi_m^u(\cdot)$ is Y_t-progressively measurable on A; therefore $\pi_m^u(\cdot)$ can be extended to all of $[0,\infty) \times E$ so as to fulfill the requirements ((25), 4.6.8). Also since $W(A_t) = 1$, (6.2) holds.

This completes the proof.

Now let

$$\nu_m^u(t) = y(t) - \int_0^t \pi_m^u(s)(c^{u(s)})ds, \ t \geq 0,$$

be the innovations process. Here $c^u(x) = c(x,u)$. By (6.1)

$$(6.3) \qquad E_m^u\left(\int_0^T (c(x(t),u(t)) - \pi_m^u(t)(c^{u(t)}))w(t)dt\right) = 0$$

for all bounded Y_t-progressively measurable $w(\cdot)$. Applying Ito's rule to $\exp(ic\nu_m^u(\cdot))$, $i = \sqrt{-1}$, c in $\mathbb{R}$, and using (6.3), it follows that $\exp(ic\nu_m^u(t)+c^2t/2)$, $t \geq 0$, is a (E,Y_t,P_m^u) martingale for all c and hence $\nu_m^u(\cdot)$ is a (E,Y_t,P_m^u) Brownian motion. Applying Ito's rule to the right hand side of (6.1), standard computations then yield the usual filtering equations $(\pi = \pi_m^u)$

$$(6.4) \quad P_m^u\Big(\pi_m^u(t)(\phi) = m(\phi) + \int_0^t \pi_m^u(s)(A^{u(s)}\phi)ds$$

$$+ \int_0^t \langle c^{u(s)},\phi\rangle_{\pi(s)} d\nu_m^u(s), \ t \geq 0 \Big) = 1,$$

for all ϕ in $\mathbb{D}$. Hence the distribution $\hat{P}_m^u$ of $(\nu_m^u(\cdot),\pi_m^u(\cdot))$ solves the martingale problem for $\hat{A}_1^u$ on $(\hat{E},\hat{F})$ <u>for u∈U constant</u>. (see Lemma 7.2).

6.3. <u>Lemma</u> Let $\hat{R}(t) = \exp\left(\int_0^t \hat{c}(s)dy(s) - \frac{1}{2}\int_0^t \hat{c}(s)^2 ds\right)$, $t \geq 0$, where $\hat{c}(t) = \pi_m^u(t)(c^{u(t)})$. Let $\mathbb{P}$ denote the Y_t-progressively measurable subsets of $[0,\infty) \times E$; set $leb = e^{-t}dt$ on $[0,\infty)$. Then $leb \times P << leb \times W$ on $\mathbb{P}$ and $d(leb \times P)/d(leb \times W) = \hat{R}$; moreover $E^{leb\ W}(\hat{R}\log\hat{R}) = \frac{1}{2}E^P\left(\int_0^\infty e^{-t}\hat{c}(t)^2 dt\right)$.

The proof is straightforward and is omitted.

7. SEPARATED PROBLEM: PIECEWISE CONSTANT FEEDBACKS ARE ADMISSIBLE

In this section we establish that any piecewise constant feedback is admissible. By piecewise constant we mean that there is a $\delta > 0$ such that $f(t) = f(\delta[t/\delta])$, $t \geq 0$. The following is due to Kunita but we include the proof for completeness.

7.1. Lemma (Pathwise Uniqueness- Kunita (18)) Let (Ω, M, W) be an arbitrary probability space equipped with an arbitrary filtration M_t and an arbitrary (Ω, M_t, W) Brownian motion $\nu(\cdot)$. Set $c(x) = c(x,u)$ and $A = A^u$ for some fixed but arbitrary u in U. Suppose that $\mu_1, \mu_2 : [0,\infty) \times \Omega \to M(X)$ are progressively measurable right continuous maps both satisfying

$$(7.1) \quad W\left(\mu(t)(\phi) = m(\phi) + \int_0^t \mu(s)(A\phi)ds + \int_0^t \langle c,\phi\rangle_{\mu(s)} d\nu(s),\ t \geq 0 \right) = 1$$

for all ϕ in $\mathbb{D}$. Then $W(\mu_1(t) = \mu_2(t),\ t \geq 0) = 1$.

proof. Let $P_t : C_b(X) \to C_b(X)$ denote the Feller semigroup generated by A. Then (Szpirglas (23)) $\mu(\cdot)$ satisfies (7.1) for all ϕ in $\mathbb{D}$ iff $\mu(\cdot)$ satisfies

$$(7.2) \quad W\left(\mu(t)(\phi) = m(P_t\phi) + \int_0^t \langle c, P_{t-s}\phi\rangle_{\mu(s)} d\nu(s),\ t \geq 0 \right) = 1$$

for all ϕ in $C_b(X)$ (Here one uses the fact $A(\mathbb{D}) \subset \mathbb{D}$). Set

$$\rho(t)(\phi) = E^W\left(\sup_{0\leq s\leq t} |\mu_1(s)(\phi) - \mu_2(s)(\phi)|^2\right).$$

Since both $\mu_1(\cdot)$, $\mu_2(\cdot)$ satisfy (7.1), Doob's inequality yields

$$\rho(t)(\phi) \leq 4E^W\left(\int_0^t (\langle c, P_{t-s}\phi\rangle_{\mu_1(s)} - \langle c, P_{t-s}\phi\rangle_{\mu_2(s)})^2 ds\right)$$

$$\leq 12\int_0^t (\rho(s)(cP_{t-s}\phi) + \rho(s)(c)|\phi|^2 + \rho(s)(P_{t-s}\phi)|c|^2)ds,$$

where $|\phi| = \sup_x|\phi(x)|$, $|c| = \sup_{x,u}|c(x,u)|$. By induction it follows that

$$(7.3) \qquad \rho(t)(\phi) \leq \text{constant} \times |\phi|^2(6|c|)^{2n}\, t^n/n! \to 0 \text{ as } n\uparrow\infty.$$

This completes the proof.

Now the Watanabe-Yamada technique can be applied to establish the uniqueness of $\hat{P}_m^f$, f in U constant, in (4.1). To see this we need to formulate admissibility in terms of a martingale problem on $(\hat{E},\hat{F})$.

To this end let $\hat{\mathbb{D}}$ denote the set of all constants and all functions in $C_b(M(X))$ of the form

$$(7.4) \qquad \Phi(\mu) = F(\mu(\phi_1),\ldots,\mu(\phi_n))$$

for some $n \geq 1$, $\phi_1,\phi_2,\ldots,\phi_n$ in $\mathbb{D}$, and F in $C^\infty(R^n)$. This set $\hat{\mathbb{D}}$ is frequently separating and satisfies properties analogous to those of $\mathbb{D}$. For example when X is compact, the Stone-Weierstrass theorem guarantees that the smallest collection of functions on M(X) closed under bounded pointwise convergence and containing $\hat{\mathbb{D}}$ is all of B(M(X)). For X not compact, it is enough to assume that there is a compactification $\bar{X}$ of X such that the functions in $\mathbb{D}$ extend continuously to $\bar{X}$ in such a way that $\mathbb{D}$ is dense in $C(\bar{X})$. Then by the Krein-Milman theorem, $M(\bar{X}) = \overline{M(X)}$; using this it follows that also in this case B(M(X)) is smallest collection of functions containing $\hat{\mathbb{D}}$ and closed under bounded convergence. The examples displayed in Section 2 fall into the above two categories.

For Φ in $\hat{\mathbb{D}}$ and u in U set

$$\hat{A}^u(\Phi)(\mu) = \sum_{i=1}^n \partial_i F(\mu)\mu(A^u\phi_i) + \tfrac{1}{2}\sum_{i,j=1}^n \partial_i\partial_j F(\mu)<c^u,\phi_i>_\mu <c^u,\phi_j>_\mu$$

with $<\phi,\phi'>_\mu = \mu(\phi\phi')-\mu(\phi)\mu(\phi')$. For Φ in $\hat{\mathbb{D}}$ and ψ in $C_o^\infty(\mathbb{R})$ set

$$\hat{A}_1^u(\Phi\psi)(\mu,y) = \hat{A}^u(\Phi)(\mu)\psi(y) + (\sum_{i=1}^n \partial_i F(\mu)<c^u,\phi_i>_\mu)\psi'(y) + \tfrac{1}{2}\Phi(\mu)\psi''(y).$$

Then $\hat{A}_1^u$ is well-defined since

$$\hat{A}_1^u(\Phi\psi)(m,y) = \frac{d}{dt}\Big|_{t=o} E_m^u(\Phi(\pi_m^u(t))\psi(\nu_m^u(t)))$$

with notation as in Section 6. Also note that $A^u(\mathbb{D}) \subset \mathbb{D}$ and $c^u\,\mathbb{D} \subset \mathbb{D}$ imply $\hat{A}^u(\hat{\mathbb{D}}) \subset \hat{\mathbb{D}}$. To relate $\hat{A}_1^u$ to the definition of admissibility given in Section 4, we use the following. Let $f(\cdot)$ be a feedback.

7.2. Lemma $\hat{P}$ solves the martingale problem for $(\hat{A}_1^f, \hat{\mathbb{D}}\times C_o^\infty(\mathbb{R}))$ on $(\hat{E},\hat{P})$ iff $\nu(\cdot)-\nu(0)$ is a $(\hat{E},\hat{F}_t,\hat{P})$ Brownian motion and

$$\hat{P}(\ \mu(t)(\phi) = \mu(0)(\phi) + \int_0^t \mu(s)(A^{f(s)}\phi)ds + \int_o^t <c^{f(s)},\phi>_{\mu(s)} d\nu(s),\ t \geq 0\ ) = 1$$

for all ϕ in $\mathbb{D}$.

The proof of 7.2 is completely analogous to that of 8.1.1 (25) and is omitted. In fact all one need note is that the development of the Watanabe-Yamada technique as presented in Chapter 8 of (25) is probabilistic, i.e. whether or not the diffusion is finite- or infinite-dimensional is irrelevant.

Given this remark, we conclude* that given m in M(X) and f in U, there is at most one solution $\hat{P}_m^f$ to the martingale problem for $\hat{A}_1^f$ starting from (m,0). Moreover by filtering (Section 6) we see that there is at least one such $\hat{P}_m^f$. It follows that constant feedbacks are admissible.

7.3. Corollary Piecewise constant feedbacks are admissible.

This follows in the standard manner by conditioning.

7.4. Corollary Let $u(\cdot)$ be a piecewise constant control and let $\pi_1(\cdot)$, $\pi_2(\cdot)$, be two solutions of (6.4). Then $P_m^u(\pi_1(t) = \pi_2(t),\ t \geq 0) = 1$.

This follows by applying Lemma 7.1 to (6.4) over each subinterval $k\delta \leq t < (k+1)$, $k = 0,1,2,\ldots$.

---

* See (25), 8.1.6.

8. EQUALITY OF THE VALUE FUNCTIONS

Theorem $v(m) = \hat{v}(m)$ for all m in $M(X)$.

The proof proceeds along the lines of the proof of Fleming's result (10). Imposing the topology of convergence in probability on the set of all controls and the set of all feedbacks, one first verifies that the cost criteria are continuous in this topology (Lemmas A and B). Since any control and any feedback can be approximated arbitrarily closely by a piecewise constant control and feedback respectively, in evaluating the value functions the infima may be taken over the sets of piecewise constant controls and feedbacks respectively. One also checks that piecewise constant controls and feedbacks are admissible (Sections 5 and 7).

The second step is to relate the two problems as follows. An admissible control $u(\cdot)$ (admissible feedback $f(\cdot)$) is called regular at m if there is some admissible feedback $f(\cdot)$ (admissible control $u(\cdot)$ respectively) such that

$$u(t) = f(t,\nu_m^u), \ t \geq 0, \quad \text{a.s.-}P_m^u, \tag{8.1}$$

where $\nu_m^u(\cdot)$ is the innovations process associated to $u(\cdot)$. In Sections 9 and 10 we show that piecewise constant controls and feedbacks are regular at m for all m. Moreover for $f(\cdot)$ piecewise constant we show that there is a strong pathwise solution $\mu_m^f(\cdot)$ to (4.1) satisfying

$$\pi_m^u(t) = \mu_m^f(t,\nu_m^u), \ t \geq 0, \ \text{a.s.-}P_m^u \tag{8.2}$$

whenever (8.1) holds (Corollary 7.4).

Here $\pi_m^u(\cdot)$ is a right-continuous version (Section 6) of the conditional probability distribution

(8.3) $$\pi_m^u(t)(\phi) = E_m^u(\phi(x(t))|Y_t) \quad \text{a.s.-}P_m^u$$

for all $t \geq 0$. This yields

$$E_m^u(L(x(t),u(t))) = E_m^u(E_m^u(L(x(t),u(t))|Y_t))$$

$$= E_m^u(\hat{L}(\pi_m^u(t),u(t)))$$

$$= E_m^u(\hat{L}(\mu_m^f(t,\nu_m^u),f(t,\nu_m^u)))$$

$$= E^W(\hat{L}(\mu_m^f(t),f(t)))$$

$$= \hat{E}_m^u(\hat{L}(\mu(t),f(t))).$$

Thus $v^u(m) = \hat{v}^f(m)$, which completes the proof of the theorem.

In particular note that (8.1) and (8.2) combined imply that the "innovations conjecture" holds for $u(\cdot)$ piecewise constant.

9. PARTIALLY OBSERVED PROBLEM: PIECEWISE CONSTANT CONTROLS ARE REGULAR

Let $\Omega = C([0,\infty);\mathbb{R})$, $b(t,\omega) = \omega(t)$, $\omega\in\Omega$, and $M_t = \sigma[b(s),\ 0 \leq s \leq t]$. Let W denote Wiener measure on (Ω, M), $M = M_\infty$. We first show there is a pathwise measure-valued solution $\mu = \mu_m^u$ to

$$(9.1)\quad \mu(t)(\phi) = m(\phi) + \int_0^t \mu(s)(A^u\phi)ds + \int_0^t \langle c^u,\phi\rangle_{\mu(s)} db(s),\ t \geq 0,$$

for all u in U, m in M(X).

9.1. Lemma (Pathwise Existence - Kunita (18)) There is a $\mathbb{B}(M(X)) \times \mathbb{B}(U) \times M_t$-progressively measurable right continuous map $\mu : M(X) \times U \times [0,\infty) \times \Omega \to M(X)$ such that for all u in U, m in M(X), $\mu = \mu_m^u$ satisfies (9.1) for all ϕ in $\mathbb{D}$, W-almost surely.

proof. In the proof of 7.1, it was noted that (9.1) is equivalent to

$$(9.2)\quad \mu(t)(\phi) = m(P_t^u\phi) + \int_0^t \langle c^u, P_{t-s}^u\phi\rangle_{\mu(s)} db(s),\ t \geq 0.$$

For $k \geq \sup_{x,u}|c(x,u)|$ let $B_k(X)$ denote the algebra of ϕ in $B(X)$ satisfying $|\phi| \leq k$. Then $B_k(X)$ is closed under multiplication by c^u and application of P_t^u, for all $t \geq 0$ and all u in U. Let $1_k(a) = (a\wedge k)\vee(-k)$. Then $|1_k(a)| \leq k$ and we first solve the truncated equation

$$(9.3)\quad \mu(t)(\phi) = m(P_t^u\phi) + \int_0^t \langle c^u, P_{t-s}^u\phi\rangle_{\mu(s)}^k db(s),\ t \geq 0,$$

where $\langle\phi,\phi'\rangle_\mu^k = \mu(\phi\phi') - 1_k(\mu(\phi))1_k(\mu(\phi'))$. This last equation is solvable by successive approximations. To this end, set $\mu_o(t)(\phi) = m(P_t\phi)$ (we suppress dependence on m,u) and for $n \geq 0$ define $\mu_{n+1}(\cdot)(\phi)$ by "plugging in" $\mu_n(\cdot)$ into the right-hand-side of (9.3). It is important to note that, although the notation suggests it, $\mu_n(\cdot)$ is not a measure-valued process, but a family of scalar processes $\mu_n(\cdot)(\phi)$ parametrized by ϕ. Note also that for each ϕ, $\mu_n(\cdot)(\phi)$ is progressively measurable as required and is right continuous. Then for $\rho_n(t)(\phi) = E^W(\sup_{0\leq s\leq t}|\mu_{n+1}(s)(\phi)-\mu_n(s)(\phi)|^2)$ one obtains

$$\rho_n(t)(\phi) \leq 12\int_0^t (\rho_{n-1}(s)(cP_{t-s}\phi) + k^2\rho_{n-1}(s)(P_{t-s}\phi) + k^2\rho_{n-1}(s)(c))ds$$

much as in the proof of 7.1. By induction one obtains

$$\rho_n(t)(\phi) \leq \text{constant} \times (1+|\phi|^2)6^{2n}(1+k^2)^{2n}t^{n+1}/(n+1)!.$$

Thus

$$\lim_{n\to\infty}\sup_{m\geq n} W(\sup_{0\leq s\leq t}|\mu_n(s)(\phi)-\mu_m(s)(\phi)| \geq \varepsilon) = 0$$

for all $\varepsilon > 0$ and all $t \geq 0$ and so there exists a right continuous progressively measurable limit $\mu(\cdot)(\phi) = \mu^k(\cdot)(\phi)$ satisfying (9.3) for all ϕ in $B(X)$. Now by (6.4), for ϕ in $B_k(X)$, $\pi(\cdot)(\phi) = \pi_m^u(\cdot)(\phi)$ satisfies

$$(9.4)\quad \pi(t)(\phi) = m(P_t^u\phi) + \int_0^t \langle c^u, P_{t-s}^u\phi\rangle_{\pi(s)}^k d\nu_m^u(s),\ t \geq 0.$$

Moreover $\pi(\cdot)(\phi) = \mu^k(\cdot,\nu_m^u)(\phi)$ also satisfies (9.4). Set

$$\rho(t)(\phi) = E_m^u(\sup_{0\leq s\leq t}|\mu_m^u(s)(\phi)-\mu^k(s,\nu_m^u)(\phi)|^2).$$

Then, much as before,

$$\rho(t)(\phi) \leq 12\int_o^t (\rho(s)(c^u P^u_{t-s}\phi) + k^2\rho(s)(P^u_{t-s}\phi) + k^2\rho(s)(c))ds,$$

for all ϕ in $B_k(X)$, and hence we obtain an inequality similar to (7.3). Thus $\pi^u_m(\cdot)(\phi) = \mu^k(\cdot,\nu^u_m)(\phi)$ for all ϕ in $B_k(X)$ which yields the consistency of the $\mu^k(\cdot)$'s, $\mu^{k+1}(\cdot)(\phi) = \mu^k(\cdot)(\phi)$ a.s.-W for all ϕ in $B_k(X)$. A standard argument then produces for each j a process $\bar{\mu}^j(\cdot)(\phi)$ that satisfies (9.3) for all $k \geq j$ and all ϕ in $B_j(X)$. Letting $k\uparrow\infty$, we see that $\bar{\mu}^j(\cdot)(\phi)$ satisfies (9.2) for all ϕ in $B_j(X)$. Repeating this argument one extracts a process $\bar{\mu}(\cdot)(\phi)$ satisfying (9.2) for all ϕ in B(X). To produce a measure-valued process, let d be a totally bounded metric on X and let $U_d(X)$ denote the d-uniformly continuous functions on X. Then $U_d(X)$ is separating and separable. Let $A \subset M(X) \times U \times [0,\infty) \times \Omega$ be the set of all (m,u,t,ω) such that

$$\bar{\mu}(s,\omega)(a\phi+a'\phi') = a\bar{\mu}(s,\omega)(\phi) + a'\bar{\mu}(s,\phi)(\phi'),$$

$$\bar{\mu}(s,\omega)(\phi^+) \geq 0, \quad \phi^+ = \phi\vee 0, \qquad (\bar{\mu} = \bar{\mu}^u_m)$$

$$\bar{\mu}(s,\omega)(1) = 1$$

for all $0 \leq s \leq t$, all a,a' rational and ϕ,ϕ' running through a countable dense subset of $U_d(X)$. Then A is progressively measurable and we can define a progressively measurable right continuous measure-valued $\mu(\cdot)$ satisfying $\mu^u_m(t,\omega)(\phi) = \bar{\mu}^u_m(t,\omega)(\phi)$ on A for all ϕ in the countable dense subset of $U_d(X)$. Since (m,u,t,ω) in A and $s < t$ implies (m,u,s,ω) is in A, there is a right continuous progressively measurable extension μ : $M(X) \times U \times [0,\infty) \times \Omega \to M(X)$ solving (9.2) ((25), 4.6.8) and hence (9.1).

As a Corollary to the proof, one has the following.

9.2. Corollary For all u in U, m in M(X),

$$P^u_m(\pi^u_m(t) = \mu^u_m(t,\nu^u_m), \ t \geq 0) = 1.$$

9.3. Corollary Given a M_t-progressively measurable piecewise constant $f : [0,\infty) \times \Omega \to U$, there exists a right continuous M_t-progressively measurable $\mu : [0,\infty) \times \Omega \to M(X)$ satisfying

$$(9.5) \quad \mu(t)(\phi) = m(\phi) + \int_0^t \mu(s)(\Lambda^{f(s)}\phi)ds + \int_0^t \langle c^{f(s)},\phi\rangle_{\mu(s)} db(s), \ t \geq 0,$$

a.s.-W, for all ϕ in $\mathbb{D}$.

proof. Patch things on δ-intervals. Here the measurability of the solution to (9.1) with respect to m and u is crucial.

Now we can establish the regularity of piecewise constant controls. Let $u(\cdot)$ be piecewise constant, $u([t/\delta]\delta) = u(t)$, $t \geq 0$. We have to find a piecewise constant feedback $f : [0,\infty) \times \hat{E} \to U$ such that

$$(9.6) \qquad u(t) = f(t,\nu^u_m) \qquad \text{a.s. } P^u_m$$

for all $t \geq 0$. Let $N^u_t = \sigma[\nu^u_m(s), \ 0 \leq s \leq t]$. We prove (9.6) by induction on the intervals $k\delta \leq t < (k+1)\delta$, $k \geq 0$. For k=0 there is nothing to prove. Assume that (9.6) holds for $0 \leq t < N\delta$. Then for all $k < N$ there is a $N_{k\delta}$-measurable $f_k : \hat{E} \to U$ such that $u(k\delta) = f_k(\nu^u_m)$ a.s.-P^u_m. Now set $f(t) = f_k$ when $k\delta \leq t < (k+1)\delta$, $k < N-1$ and $f(t) = f_{N-1}$ for $t \geq (N-1)\delta$. Let $\mu(\cdot)$ denote the solution of (9.5). Then both $\pi^u_m(\cdot)$ and $\mu(\cdot,\nu^u_m)$ satisfy (6.4) up to time $N\delta$. Thus by Corollary 7.4 $\pi^u_m(\cdot) = \mu(\cdot,\nu^u_m)$ up to time $N\delta$. Thus by definition of $\nu^u_m(\cdot)$, $N^u_t = Y_t$ a.s. P^u_m, which yields the existence of a $M_{N\delta}$-measurable f_N satisfying $u(N\delta) = f_N(\nu^u_m)$, completing the proof.

10. SEPARATED PROBLEM: PIECEWISE CONSTANT FEEDBACKS ARE REGULAR

Let $f(\cdot)$ be a piecewise constant feedback. Fix m in $M(X)$. In this section we show there is a piecewise constant control $u(\cdot)$ satisfying

$$(10.1) \qquad u(t) = f(t,\nu_m^u), \; t \geq 0, \; \text{a.s.-}P_m^u.$$

We begin with a variation of 9.1.

10.1. Lemma With notation as in Section 9, there is a $\mathbb{B}(M(X)) \times \mathbb{B}(U) \times M_t$-progressively measurable right continuous map $\mu : M(X) \times U \times [0,\infty) \times \Omega \to M(X)$ such that for all u in U, m in $M(X)$, $\mu = \mu_m^u$ satisfies

$$\mu(t)(\phi) = m(\phi) + \int_0^t \mu(s)(A^u\phi)ds + \int_0^t \langle c^u,\phi\rangle_{\mu(s)} db(s) - \int_0^t \langle c^u,\phi\rangle_{\mu(s)} \mu(s)(c^u)ds, \; t \geq 0,$$

a.s.-W for all ϕ in $\mathbb{D}$.

The proof is completely analogous to that of 9.1 and is omitted. We return to the basic probability space (E,F).

10.2. Lemma Let $u(\cdot)$ be piecewise constant. Any two right continuous Y_t-progressively measurable maps $\pi_1,\pi_2 : [0,\infty) \times E \to M(X)$ satisfying

$$(10.2) \quad \pi(t)(\phi) = m(\phi) + \int_0^t \pi(s)(A^{u(s)}\phi)ds + \int_0^t \langle c^{u(s)},\phi\rangle_{\pi(s)} dy(s) - \int_0^t \langle c^{u(s)},\phi\rangle_{\pi(s)} \pi(s)(c^{u(s)})ds, \; t \geq 0,$$

a.s.-W for all ϕ in $\mathbb{D}$, are necessarily equal a.s.-W.

The proof is analogous to that of Corollary 7.4, and depends on the corresponding variation of Lemma 7.1.

10.3. Corollary Let $f(\cdot)$ be piecewise constant and fix m in $M(X)$. Then there are Y_t-progressively measurable maps $(\mu,\nu) : [0,\infty) \times E \to M(X) \times \mathbb{R}$ satisfying

$$(10.3) \quad \mu(t)(\phi) = m(\phi) + \int_0^t \mu(s)(A^{f(s,\nu)}\phi)ds + \int_0^t \langle c^{f(s,\nu)},\phi\rangle_{\mu(s)} dy(s) - \int_0^t \langle c^{f(s,\nu)},\phi\rangle_{\mu(s)} \mu(s)(c^{f(s,\nu)})ds, \quad t \geq 0,$$

and

$$(10.4) \quad \nu(t) = y(t) - \int_0^t \mu(s)(c^{f(s,\nu)})ds, \quad t \geq 0,$$

a.s.-W, for all ϕ in $\mathbb{D}$.

proof. Although the above two equations are coupled in general , we can construct a solution because for $f(\cdot)$ piecewise constant they are uncoupled on any δ-interval, given the solution up to the beginning of the δ-interval. To begin, since $y(\cdot)$ is a Brownian motion, $f(t,\nu) = f(0,\nu)$ is a nonrandom constant for $0 \leq t < \delta$. Thus by 10.1 $\mu(\cdot)$ exists on $0 \leq t < \delta$. Now use (10.4) to define $\nu(\cdot)$ on $0 \leq t < \delta$. Thus the solution $(\mu(\cdot),\nu(\cdot))$ exists on $0 \leq t \leq \delta$. Therefore $f(t,\nu) = f(\delta,\nu)$ is defined for $\delta \leq t < 2\delta$, and so applying 10.1 again on $\delta \leq t < 2\delta$, we see that $\mu(\cdot)$ exists on $\delta \leq t < 2\delta$. Use (10.4) again to define $\nu(\cdot)$ on $\delta \leq t < 2\delta$. Then the solution exists on $\delta \leq t < 2\delta$. Proceeding in this manner, the result follows. Note the measurability of $\mu(\cdot)$ in 10.1 with respect to m, u is crucial for this argument.

10.4. Corollary With the set-up of 10.3, set $u(t) = f(t,\nu)$, $t \geq 0$, where $\nu(\cdot)$ is the second component of some solution to (10.3), (10.4). Then $u(\cdot)$ is an admissible control and (10.1) holds.

proof Let $\pi_m^u(\cdot)$ be the process satisfying (6.4); since P_m^u and W are mutually absolutely continuous on Y_t, $\pi_m^u(\cdot)$ satisfies (10.2). By Lemma 10.2 $\pi_m^u(\cdot) = \mu(\cdot)$ and hence $\nu(\cdot) = \nu_m^u(\cdot)$ a.s.-W. The result follows.

11. PROOF OF LEMMA A

Throughout leb denotes the probability measure $e^{-t}dt$ on $0 \leq t < \infty$. A sequence of admissible controls u_k is said to converge to an admissible control u if this is so in (leb $\times$ W)-probability on $[0,\infty) \times E$. Since these processes are Y_t-progressively measurable, this makes sense.

Lemma A. $m_k \to m$ in $M(X)$ and $u_k \to u$ implies $v^{u_k}(m_k) \to v^u(m)$.

Before we prove Lemma A, we state some preliminary facts.

11.1. Lemma If u is an admissible control, there is a sequence u_k of piecewise constant controls such that $u_k \to u$.

For the proof see (17) page 143.

11.2. Lemma Let μ,ν be probability measures on some abstract space and suppose $\mu << \nu$ and $R = d\mu/d\nu$ satisfies $E^\nu(R\log R) \leq C$. Let h_C denote the universal function

$$h_C(b) = \inf_{1<a<\infty} \{ab + (C + (1/e))/\log(a)\}, \qquad b \geq 0.$$

Then $h_C(0^+) = h_C(0) = 0$, h_C is nondecreasing, and $\mu(B) \leq h_C(\nu(B))$ for all B.

The proof is straightforward, well-known, and omitted.

11.3. Lemma Let μ_k, μ, ν be probability measures on some Polish space and suppose $\mu_k << \nu$ with $R_k = d\mu_k/d\nu$ satisfying $E^\nu(R_k \log R_k) \leq C$. Assume that $\mu_k \to \mu$. Then $\mu << \nu$ and $R = d\mu/d\nu$ satisfies $E^\nu(R\log R) \leq C$.

The proof is omitted. The key fact used in the proof of 11.3 is the identity ((24) Lemma 3.38)

$$E^\nu(R\log R) = \sup \left\{ \int f d\mu - \log \int e^f d\nu \;\middle|\; f \text{ in } C_b \right\}$$

in the sense that if the right-hand side is finite, then $\mu << \nu$ and $R\log R$ is in $L^1(\nu)$ and the identity holds.

Given this identity, the rest of the proof of 11.3 is straightforward.

11.4. Lemma Suppose $\kappa \in B([0,\infty) \times X \times \mathbb{R} \times U)$ satisfies $\kappa(t,\cdot,\cdot,\cdot) \in C_b(X \times \mathbb{R} \times U)$ for all $t \geq 0$ and

$$\sup \{ |\kappa(t,x,y,u) - \kappa(t,x,y',u')| \mid t \geq 0,\ x \in X,\ |y-y'| < \delta,\ d(u,u') < \delta \} \to 0$$

as $\delta \downarrow 0$. Suppose that P_k solves the martingale problem for G^{u_k} and converges to some Q. Suppose that $u_k \to u$. Then

$$E^{P_k}\left(\int_0^\infty e^{-t}\kappa(t,x(t),y(t),u_k(t))dt\right) \to E^Q\left(\int_0^\infty e^{-t}\kappa(t,x(t),y(t),u(t))dt\right).$$

proof. We use Lemma 6.3 and Lemma 11.2 with $\mu = \text{leb} \times P_k$, $\nu = \text{leb} \times W$. Then for all ε there exists δ such that

$$(\text{leb} \times P_k)(|\kappa(\cdot,x,y,u_k)-\kappa(\cdot,x,y,u)| \geq \varepsilon)$$

$$\leq h_C((\text{leb} \times W)(d(u_k,u) \geq \delta)) \to 0 \text{ as } k \uparrow \infty.$$

Thus it is enough to show

$$(11.1)\quad E^{P_k}\left(\int_0^\infty e^{-t}\kappa(t,x(t),y(t),u(t))dt\right) \to E^Q\left(\int_0^\infty e^{-t}\kappa(t,x(t),y(t),u(t))dt\right).$$

Now if u were in $C_b([0,\infty) \times E)$ there would be no problem: (11.1) would follow immediately since in this case

$$\int_0^\infty e^{-t}\kappa(t,x(t),y(t),u(t))dt$$

is in $C_b(E)$ and $P_k \to Q$. For u not continuous, we proceed as follows: By Lusin's theorem for all δ we can find u^δ in $C_b([0,\infty) \times E)$ satisfying $(\text{leb} \times W)(u \neq u^\delta) \leq \delta$; then (11.1) holds with u^δ replacing u; to make sure that the damage done in (11.1) by replacing u with u^δ vanishes as $\delta \downarrow 0$, we use 11.2 and 11.3 once again to conclude

$$(\text{leb} \times P_k)(\kappa(\cdot,x,y,u) \neq \kappa(\cdot,x,y,u^\delta)) \leq h_C((\text{leb} \times W)(u \neq u^\delta)) \leq h_C(\delta) \to 0$$

$$(\text{leb} \times Q)(\kappa(\cdot,x,y,u) \neq \kappa(\cdot,x,y,u^\delta)) \leq h_C((\text{leb} \times W)(u \neq u^\delta)) \leq h_C(\delta) \to 0$$

as $\delta \downarrow 0$. The result follows.

We can now prove Lemma A. Let P_k, P solve the martingale problems for G^{u_k}, G^u starting from m_k, m respectively. Since the generators $G^u(\phi\psi)$ are uniformly bounded (notation as in Section 5), $\{P_k\}$ is a pre-compact family of measures. Let Q be any limit point; choosing

$$\kappa(t,x,y,u) = e^t G^u(\phi\psi)(x,y) 1_{t \leq T}$$

with $T > 0$, ϕ in $\mathbb{D}$, ψ in $C_0^\infty(\mathbb{R})$ fixed but arbitrary, Lemma 11.4 applies and so Q solves the martingale problem for G^u starting from m. By uniqueness it follows that $Q = P$ and $P_k \to P$. Now apply Lemma 11.4 again with $\kappa = L$. The result follows.

11.5. <u>Corollary</u> For u admissible v^u is in $C_b(M(X))$. The value function v is upper-semicontinuous and concave on M(X).

The continuity follows immediately from Lemma A. Since $v^u(m) = \int v^u(\delta_x)\, m(dx)$ and $x \to v^u(\delta_x)$ is in $C_b(X)$, we see that v^u is affine and hence v is concave.

12. PROOF OF LEMMA B

A sequence of admissible feedbacks f_k is said to converge to an admissible feedback f if this is so in (leb × W)-probability.

Lemma B. $m_k \to m$ in $M(X)$ and $f_k \to f$ implies $\hat{v}^{f_k}(m_k) \to \hat{v}^{f}(m)$.

To prove this, recall that

$$\hat{v}^{f}(m) = \hat{E}_m^f\left(\int_0^\infty e^{-t}\hat{L}(\mu(t), f(t))dt\right). \tag{12.1}$$

Lemma B follows exactly as Lemma did. We therefore state only the relevant Lemma.

12.1. Lemma Suppose $\kappa \in B([0,\infty) \times M(X) \times \mathbb{R} \times U)$ satisfies $\kappa(t,\cdot,\cdot,\cdot) \in C_b(M(X) \times \mathbb{R} \times U)$ for all $t \geq 0$ and

$$\sup\{ |\kappa(t,\mu,\nu,f) - \kappa(t,\mu,\nu',f')| \; | t \geq 0, \mu \in M(X), |\nu-\nu'| + d(f,f') < \delta \} \to 0$$

as $\delta \downarrow 0$. Suppose that $\hat{P}_k$ solves the martingale problem for $\hat{A}_I^{f_k}$ and converges to some $\hat{Q}$. Suppose that $f_k \to f$. Then

$$E^{\hat{P}_k}\left(\int_0^\infty e^{-t}\kappa(t,\mu(t),\nu(t),f_k(t))dt\right) \to E^{\hat{Q}}\left(\int_0^\infty e^{-t}\kappa(t,\mu(t),\nu(t),f(t))dt\right).$$

The proof here is simpler than that of Lemma 11.4 since $\nu(\cdot)$ here is a Brownian motion under $\hat{P}_k$. Because of this Lemmas 11.2 and 11.3 are unnecessary here. By first choosing $\kappa = e^{t}\hat{A}_I^f(\Phi\psi)(\mu,\nu)1_{t\leq T}$ and then choosing $\kappa = \hat{L}$, the result follows as in Section 11.

13. WELL-POSEDNESS OF THE NONLINEAR FILTERING MARTINGALE PROBLEM

Throughout this section f denotes a fixed element in U, and so dependence on f is suppressed. For any Polish space P, set $\Omega_P = C([0,\infty);P)$. Let $\hat{A}: \mathbb{D}(M(X)) \to C_b(M(X))$, $\hat{A}_1: \mathbb{D}(M(X)) \times C_o^\infty(\mathbb{R}) \to C_b(M(X) \times \mathbb{R})$ be the operators defined in Section 7. Then (Corollary 7.3) the martingale problem for $\hat{A}_1$ on $\Omega_{M(X)} \times \Omega_{\mathbb{R}}$ is well-posed. In this section we establish the well-posedness of the martingale problem for $\hat{A}$ on $\Omega_{M(X)}$ provided c lies in $\mathbb{D} = \mathbb{D}(X)$. This proceeds along standard lines (Chapter 8 of (25)).

Let $\hat{P}$ solve the martingale problem for $\hat{A}_1$ and let P denote the first marginal of $\hat{P}$. Then it is self-evident ($\hat{A}(\Phi\psi) = \hat{A}_1(\Phi\psi)$ when $\psi = 1$) that P solves the martingale problem for $\hat{A}$. This proves existence of solutions to the martingale problem for $\hat{A}$. To prove uniqueness all one need verify is the following.

13.1. Lemma Let P solve the martingale problem for $\hat{A}$ on $\Omega_{M(X)}$. Then there is a solution $\hat{P}$ to the martingale problem for $\hat{A}_1$ on $\Omega_{M(X)} \times \Omega_{\mathbb{R}}$ whose first marginal is P.

This follows from the following variation of ((25), 4.5.1).

13.2. Lemma Let P be as in 13.1. Then there is a probability space (Ω,M,Q), a nondecreasing family of sub σ-algebras M_t, a (Ω,M_t,Q) Brownian motion $\nu(\cdot)$ and a progressively measurable right continuous map $\mu : [0,\infty) \times \Omega \to M(X)$ satisfying

$$(13.1) \qquad \mu(t)(\phi) = \mu(0)(\phi) + \int_0^t \mu(s)(A\phi)ds + \int_0^t \langle c,\phi\rangle_{\mu(s)} d\nu(s), \quad t \geq 0, \text{ a.s.-}Q$$

for all ϕ in $\mathbb{D}$. Moreover the distribution of $\mu(\cdot)$ on $\Omega_{M(X)}$ is P and the distribution $\hat{P}$ of $(\mu(\cdot),\nu(\cdot))$ on $\Omega_{M(X)} \times \Omega_{\mathbb{R}}$ solves the martingale problem for $\hat{A}_1$.

proof. Set $\Omega = \hat{E}$, $M_t = \hat{F}_t$ (Section 4), $Q = P \times W$ (W = Wiener measure), $\mu(\cdot)$, $\beta(\cdot)$ the canonical maps. Then $\mu(\cdot)$ and $\beta(\cdot)$ are independent. Now for each $\phi_1,\ldots,\phi_d$ in $\mathbb{D}$

$$(13.2) \qquad (\mu(\phi_1),\ldots,\mu(\phi_d)) : [0,\infty) \times \Omega \rightarrow \mathbb{R}^d$$

is an (Ω, M_t, Q) Ito process ((25) Chapter 4) with drift and covariance coefficients

$$b_i(t) = \mu(t)(\phi_i), \quad a_{ij}(t) = \langle c,\phi_i\rangle_{\mu(t)} \langle c,\phi_j\rangle_{\mu(t)}$$

(see Section 7). In particular

$$\nu(t) = \int_0^t \alpha(s)^{-1} 1_{\alpha(s)\neq 0} d\gamma(s) + \int_0^t 1_{\alpha(s)=0} d\beta(s)$$

$$\alpha(t) = \langle c,c\rangle_{\mu(t)}$$

$$\gamma(t) = \mu(t)(c) - \int_0^t \mu(s)(Lc)ds$$

is a well-defined progressively measurable right continuous process. Using the fact that (13.2) is an Ito process, one checks that $\nu(\cdot)$ is a Brownian motion and that (13.1) holds. This completes the proof.

REFERENCES

(1) V. E. Beneš, "Existence of Optimal Stochastic Controls", SIAM J. Control 9 (1971) 446-472.

(2) ___________ and I. Karatzas, "Filtering of Diffusions Controlled Through their Conditional Measures" Stochastics 13 (1984) 1-23.

(3) J. M. Bismut, "Partially Observed Diffusions and Their Control", SIAM J. Control 20 (1982) 302-309.

(4) ___________, "Sur Une Problème de Contrôle Stochastique avec Observations Partielle", Z. Wahrsch. Verw. Gebiete 49 (1979) 63-95.

(5) V. S. Borkar, "Existence of Optimal Controls for Partially Observed Diffusions", Stochastics 11 (1983) 103-141.

(6) N. Christopeit, "Existence of Optimal Stochastic Controls under Partial Observations", Z. Wahrsch. Verw. Gebiete 51 (1980) 201-213.

(7) N. El Karoui, D. Hưu Nguyen, and M. Jeanblanc-Piqué, "Compactification Methods in the Control of Degenerate Diffusions: Existence of an Optimal Markovian Control", Preprint.

(8) ____, Existence of an Optimal Markovian Control for the Control under Partial Observations", Preprint.

(9) S. N. Ethier and T. G. Kurtz, Markov Processes: Characterization and Convergence, Wiley 1986.

(10) W. H. Fleming, "Measure-Valued Processes in the Control of Partially-Observable Stochastic Systems", Appl. Math. Optim. 6 (1980) 271-285.

(11) ___________, "Nonlinear Semigroup for Controlled Partially Observed Diffusions", SIAM J. Control 20 (1982) 286-301.

(12) ___________ and N. Nisio, "On the Existence of Optimal Stochastic Controls", J. Math. Mech. 15 (1966) 777-794.

(13) ______________ and N. Nisio, "On Stochastic Relaxed Controls for Partially Observed Diffusions", Nagoya Math. Journal 93 (1984) 71-108.

(14) ______________ and E. Pardoux, "Existence of Optimal Controls for Partially Observed Diffusions", SIAM J. Control 20 (1982) 261-283.

(15) U. G. Haussmann, "On the Existence of Optimal Controls for Partially Observed Diffusions", SIAM J. Control 20 (1982) 385-407.

(16) ________________,"Existence of Optimal Markovian Controls for Degenerate Diffusions", appears in Proceedings of the 3rd Bad Honnef Conference on Stochastic Differential Systems,Springer-Verlag Lecture Notes in Control #78, 1985.

(17) N. V. Krylov, Controlled Diffusion Processes, Springer-Verlag, New York 1980.

(18) H. Kunita, "Asymptotic Behaviour of the Nonlinear Filtering Errors of Markov Processes", J. Multivariate Analysis 1 (1971) 365-393.

(19) T. G. Kurtz and D. L. Ocone, "A Martingale Problem for Conditional Distributions and Uniqueness for the Nonlinear Filtering Equations" in Stochastic Differential Systems, Lecture Notes in Control #69, Springer-Verlag.

(20) H. J. Kushner, Probability Methods for Approximations in Stochastic Control and Elliptic Equations, Academic Press, New York 1977.

(21) P. L. Lions,"Control of Diffusion Processes in R^n", Comm. Pure & Applied Math. 34 (1981) 121-147.

(22) S-J. Sheu,"Optimal Control & Its Application to Large Deviation Theory" Ph.D. Dissertation, Brown University 1983.

(23) J. Szpirglas,"Sur l'Equivalence d'Équations Différentielles a Valeurs Mesures Intervenant dans le Filtrage Markovien Nonlineare", Ann. Inst. Henri Poincaré (Section B) 14 (1978) 33-59.

(24) D. W. Stroock, An Introduction to the Theory of Large Deviations, Springer-Verlag Universitexts, New York, 1984.

(25) ____________ and S. R. S. Varadhan, Multidimensional Diffusion Processes, Springer-Verlag, New York, 1979.

(26) T. Yamada and S. Watanabe,"On the Uniqueness of Solutions of Stochastic Differential Equations", J. Math. Kyoto Univ. 11 (1971) 155-167.