GLOBAL ESTIMATES FOR SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

For polynomial $P(\xi)$ in $\mathbf{R}^n$ with constant complex coefficients, the operator defined by $R(z)f = \mathcal{F}^{-1}((P(\cdot) - z)^{-1}\hat{f})$, where $\wedge$ denotes the Fourier transform and $\mathcal{F}^{-1}$ denotes its inverse, is not bounded from L^2 to L^2 when z is in the spectrum of $P(D)$. What are suitable spaces B and C so that $R(z)$ is bounded from B to C? When $P(\xi)$ is **simply characteristic**, we prove that the operator $R(z)$ is bounded from B_s to B^*_{1-s}, $0 \leq s \leq 1$, where B_s is a space smaller than L^2 and B^*_{1-s} is a space larger than L^2.

1 Introduction

In recent years, a method to obtain uniqueness results (and even give a theoretical inversion procedure) for a class of inverse problems in potential scattering has been developed very successfully. Sylvester and Uhlmann treated an inverse boundary value problem from electric impedance tomography (see [S-U]). Nachman and Ablowitz, Beals and Coifman, Novikov and Henkin studied some problems in inverse scattering (see [N-A], [B-C] and [N-H]). These have yielded a breakthrough on some problems for which only linearized approximation had been treatable before and led to the solution of a great number of related inverse problems.

Most of these works treat problems which can, by one device or another, be reduced to problems for the Schrödinger operator $-\Delta + q$. The crucial ingredients are estimates on $R(z)f = \mathcal{F}^{-1}((P(\cdot) - z)^{-1}\hat{f})$ for z in the spectrum of $P(D)$ and a family of solutions $\phi(x, \zeta)$ of $(-\Delta + q)\phi = E\phi$ which behave like so called inhomogeneous plane waves $\exp(ix \cdot \zeta)$ for large values of the complex vector ζ. One motivation for the study of such solutions comes from the observation that it is possible to have $\Delta e^{ix\cdot\zeta} = Ee^{ix\cdot\zeta}$ with $\zeta^2 = E$ and the energy E fixed, while ζ is made arbitrarily large. Then fixed energy uniqueness follows from a complex version of the Born limit, first observed in [N-A].

The direct scattering theory for operators of the form $P_0(D) + V(x, D)$, where $P_0(D)$ is a partial differential operator in $\mathbf{R}^n$ with real constant coefficients which is simply characteristic (see Definition 1.1 below) and V is a short-range perturbation, has been developed by Agmon and Hörmander (see [H]) and many other authors. One of the key ingredients of such a theory is an estimate for the resolvent $(P_0(D) - z)^{-1}$ which remains valid as z approaches the real axis. This clearly can't happen on the space L^2 when z approaches the spectrum of $P_0(D)$, but if $R_0(z) = (P_0(D) - z)^{-1}$ is viewed as an operator from a suitable space X smaller than L^2 to another space Y larger than L^2, then its norm can be shown to remain bounded, independent of the distance from z to the real axis (if z stays in a bounded subset K contained in the complex plane $\mathbf{C}$),

$$\|(P_0(D) - z)^{-1}f\|_Y \leq C\|f\|_X. \tag{1}$$

The result of the form (1) is known as the "limiting absorption principle". To see what spaces X and Y are appropriate, we note, for example, that when $P_0(D) = -\Delta$ in $\mathbf{R}^3$, the operator $(P_0(D) - z^2)^{-1}$, where $z \in \mathbf{C}\backslash\{0\}$, corresponds to the convolution by

$$G_0^+(x - y) = \frac{e^{iz|x-y|}}{4\pi|x - y|}.$$

Even for $f \in C_0^\infty$,

$$G_0^+ * f = O(1/|x|) \quad \text{as} \quad |x| \to \infty.$$

So $G_0^+ * f$ is in general not in L^2, but it can be shown that $(1+x^2)^{-\frac{\delta}{2}}(G_0^+ * f)$ does belong to L^2 whenever $(1 + x^2)^{\frac{\delta}{2}}f$ is in L^2 if $\delta > 1/2$. This motivates the introduction of the weighted L^2 spaces:

$$L^2_\delta = \{v \in L^2(\mathbf{R}^n) : \quad \int_{\mathbf{R}^n}(1 + |x|^2)^\delta |v(x)|^2 dx < \infty\}.$$

Estimates of the form (1), with $X = L^2_\delta$, $Y = L^2_{-\delta}$, $\delta > 1/2$ and P_0 simply characteristic, were first proved by Agmon in [Ag]. In the paper [A-H], Agmon and Hörmander showed that the following class of spaces B_s and their duals $B^*_s(-\infty < s < \infty)$:

$$B_s = \{v \in L^2_{loc}(\mathbf{R}^n) : \quad \sum_{j=1}^{\infty} R_j^s (\int_{\Omega_j} |v|^2 dx)^{1/2} < \infty\} \tag{2}$$

$$B^*_s = \{u \in L^2_{loc}(\mathbf{R}^n) : \quad \sup_{j \geq 1} R_j^{-s} (\int_{\Omega_j} |u|^2 dx)^{1/2} < \infty\} \tag{3}$$

where

$$R_0 = 0, \quad R_j = 2^{j-1}, \quad j = 1, 2, \cdots$$

$$\Omega_j = \{x \in \mathbf{R}^n : R_{j-1} < |x| < R_j\}, \quad j = 1, 2, \cdots$$

capture quite precisely the behavior of the resolvent operator at infinity. The relationship between L^2_δ and B_s is:

$$L^2_\delta \subset B_s \subset L^2_s \quad \text{and} \quad L^2_{-s} \subset B^*_s \subset L^2_{-\delta}$$

for $\delta > s \geq 0$.

Definition 1.1 *Let $P(\xi)$ be a real valued polynomial of degree m in $\xi \in \boldsymbol{R}^n$ such that*

$$\Lambda(P_0) = \{\eta \in \boldsymbol{R}^n : P_0(\xi + \eta) \equiv P_0(\xi)\} = \{0\}.$$

P_0 will be called simply characteristic if

$$\tilde{P}_0(\xi) \leq C(\sum_{|\alpha| \leq 1} |P_0^{(\alpha)}(\xi)| + 1), \quad \xi \in \boldsymbol{R}^n, \tag{4}$$

where

$$\tilde{P}_0(\xi) = \sum_{0 \leq |\alpha| \leq m} |P_0^{(\alpha)}(\xi)| \ .$$

To study fixed energy inverse problems in potential scattering for a general class of differential operators $P_0(D)$, we want solutions $\phi(x, \zeta)$ of $(P_0(D) + q)\phi = \lambda\phi$ which behave like $e^{ix\cdot\zeta}$ with $\zeta \in \mathbf{C}^n$, $P_0(\zeta) = \lambda$. The construction of such solutions requires a generalized limiting absorption estimate for

$P_0(D,\zeta) = P_0(D+\zeta)$. The first such estimate was obtained by Sylvester and Uhlmann ([S-U]) for the Laplacian $-\Delta$ at zero energy ($\zeta^2 = 0$):

$$\|(-\Delta - 2i\zeta\cdot\nabla)^{-1} f\|_{L^2_{-\delta}} \le \frac{C}{|\zeta|}\|f\|_{L^2_{1-\delta}}, \quad 0 < \delta < 1. \tag{5}$$

Our main estimates in this paper are given in the following theorem. The Theorem can be viewed as an analogue for complex polynomials of the Agmon-Hörmander's estimates. Since in our situation, the characteristic variety has codimension 2, the proof is quite different and the estimates are better than those valid in scattering theory (where $R(z)$ is bounded from $B_{1/2}$ to $B^*_{1/2}$). We will denote by $\mathcal{F}$ or $\hat{}$ the Fourier transform and $\mathcal{F}^{-1}$ or $\vee$ its inverse.

Theorem 1.2 *Assume that P is* ***simply characteristic*** *(see the definition 1.3 below) and let K be a compact subset of $\boldsymbol{C}$ containing no* ***critical value*** *of P in the sense given in the definition 1.4 below. If $f \in B_s$, $0 \le s \le 1$, it follows that $R(z)f = \mathcal{F}^{-1}((P(\cdot) - z)^{-1}\hat{f})$ belongs to B^*_{1-s}, for $z \in K$ and we have the bound:*

$$\|R(z)f\|_{B^*_{1-s}} \le C(s, c_P) \sup_{\xi\in\boldsymbol{R}^n} \frac{1}{\tilde{P}(\xi)}\|f\|_{B_s}, \qquad z \in K \quad ,$$

where c_P is in condition (6) in the definition below.

Definition 1.3 *Let $P(\xi) = P_1(\xi) + iP_2(\xi)$, $\xi \in \boldsymbol{R}^n$ be an mth order polynomial with complex coefficients. We define a* ***simply characteristic*** *polynomial P to be one which satisfies*

$$\tilde{P}(\xi) \le c_P(|P(\xi) - z_0| + |\nabla P(\xi)|) \tag{6}$$

for all $\xi \in \boldsymbol{R}^n$ and some $z_0 \in \boldsymbol{C}$ where

$$\tilde{P}(\xi) = \sum_{|\alpha|\le m} |P^{(\alpha)}(\xi)|$$

and

$$\sum_{|\alpha|=1} |P^{(\alpha)}(\xi)| \overset{def.}{=} |\nabla P(\xi)| \overset{def.}{=} \left[\sum_{i\ne j, 1\le i,j\le m} \left|\det\begin{pmatrix} \frac{\partial P_1}{\partial \xi_i} & \frac{\partial P_1}{\partial \xi_j} \\ \frac{\partial P_2}{\partial \xi_i} & \frac{\partial P_2}{\partial \xi_j} \end{pmatrix}\right|^2\right]^{1/4}.$$

Definition 1.4 *If* $\nabla P_1(\xi)$ *and* $\nabla P_2(\xi)$ *are not linearly independent at some point* $\xi \in \{\xi \in \boldsymbol{R}^n : P(\xi) - z = 0\}$, *we say that the value* z *is a* ***critical value*** *of* P.

Note that estimates in above theorem are from the space B_s to the space B^*_{1-s} for all $0 \le s \le 1$. Especially, the two end points $s = 0$ and $s = 1$ are included, i.e., $(P(D) - z)^{-1}$ is bounded from B_1 to B^*_0 and from B_0 to B^*_1. The end point estimates are more precise than weighted L^2 estimates:

$$\|R(z)f\|_{L^2_{\delta-1}} \le C_\delta \|f\|_{L^2_\delta},$$

for $0 < \delta < 1$. If $f \in B_1 \subset L^2_1$, then $R(z)f \in B^*_0$ shows that the asymptotic behavior of $R(z)f$ in average sense is $O(1/|x|^{\frac{n}{2}})$ as $|x| \to \infty$.

2 Estimates for the Model $\frac{1}{x_1+ix_2}$

Since $\nabla ReP(\xi)$ and $\nabla ImP(\xi)$ are linearly independent on the zero set of $P(\xi)$, we will eventually use a partition of unity in ξ space and a change of variables to reduce the problem to one where the symbol of P is $\xi_1 + i\xi_2$. We begin with the main estimate on this model operator.

We recall the definition of the spaces B_s and B^*_s introduced in Introduction.

$$B_s = \{v \in L^2_{loc}(\mathbf{R}^n) : \quad \sum_{j=1}^{\infty} R_j^s \Big(\int_{\Omega_j} |v|^2 dx\Big)^{1/2} < \infty\}.$$

The norm for $v \in B_s$ is

$$\|v\|_{B_s} = \sum_{j=1}^{\infty} R_j^s \Big(\int_{\Omega_j} |v|^2 dx\Big)^{1/2}.$$

The dual space of B_s is

$$B^*_s = \{u \in L^2_{loc}(\mathbf{R}^n) : \quad \sup_{j\ge1} R_j^{-s} \Big(\int_{\Omega_j} |u|^2 dx\Big)^{1/2} < \infty\}$$

and the norm for $u \in B^*_s$ is defined as

$$\|u\|_{B^*_s} = \sup_{j\ge1} R_j^{-s} \Big(\int_{\Omega_j} |u|^2 dx\Big)^{1/2}.$$

For $s > 0$, since

$$\|u\|_{B_s^*}^2 \leq \sup_{R\geq 1} R^{-2s} \int_{|x|<R} |u|^2 dx \leq \frac{2^{2s}}{(1-2^{-2s})} \|u\|_{B_s^*}^2$$

the norm $\|u\|_{B_s^*}$ is equivalent to $[\sup_{R\geq 1} R^{-2s} \int_{|x|<R} |u|^2 dx]^{1/2}$. To compare with the usual weighted-L^2 spaces, we note that

$$L_t^2 \subset B_s \subset L_s^2$$

and

$$L_{-s}^2 \subset B_s^* \subset L_{-t}^2 \tag{7}$$

for $t > s \geq 0$, where

$$L_s^2 = \{v \in L^2(\mathbf{R}^n): \quad \int_{\mathbf{R}^n} (1+|x|^2)^s |v(x)|^2 dx < \infty\}$$

$$L_{-s}^2 = \{u \in L_{loc}^2(\mathbf{R}^n): \quad \int_{\mathbf{R}^n} (1+|x|^2)^{-s} |u(x)|^2 dx < \infty\}.$$

Now we have

Theorem 2.1 *Let $f \in B_1(\boldsymbol{R}^n)$, $n \geq 2$. Define*

$$u = (\frac{1}{\xi_1 + i\xi_2} \hat{f}(\xi))^{\vee} = \frac{i}{2\pi} f * (\frac{1}{x_1 + ix_2}), \tag{8}$$

*where * represents convolution with respect to the first two variables. Then there is a constant $C > 0$ such that*

$$\|u\|_{B_0^*} \leq C \|f\|_{B_1}. \tag{9}$$

Proof: Write $x = (x', x'')$ where $x' = (x_1, x_2)$ and $x'' = (x_3, \cdots x_n)$. Then

$$u(x) = \frac{i}{2\pi} \int_{y_1,y_2} \frac{f(y, x'')}{(x_1 - y_1) + i(x_2 - y_2)} dy.$$

We need to show that there is a constant $C > 0$ such that

$$\|u\|_{B_0^*} \leq C \|f\|_{B_1}. \tag{10}$$

To prove (10), we'll need the following two two-dimensional estimates.

Lemma 2.2 *Assume $f \in L^1(\boldsymbol{R}^2) \cap L^2_{loc}(\boldsymbol{R}^2)$. Let u be defined by (8). Then*
(i)

$$\int_{|x|<R_j} |u(x)|^2 dx \le C[R_j^2 \int_{|y|\le R_{j+1}} |f(y)|^2 dy + \|f\|_{L^1}^2]$$

and for any integer m with $j-1 \ge m \ge 0$ we have

(ii)

$$\int_{R_{j-m-1}<|x_1|<R_j} \int_{|x_2|<R_j} |u(x)|^2 dx$$
$$\le C[R_j^2 \int_{R_{j-m-2}<|y_1|<R_{j+1}} \int_{|y_2|<R_{j+1}} |f(y)|^2 dy + (4^{m+2})\|f\|_{L^1}^2].$$

<u>Proof:</u> (i)

$$\begin{aligned}\int_{|x|<R_j} |u(x)|^2 dx &= \frac{1}{4\pi^2}\int_{|x|<R_j}[\int_{|y|\le R_{j+1}} \frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy \\ &\quad + \int_{|y|>R_{j+1}} \frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy]^2 dx \\ &\le \frac{1}{2\pi^2}[\int_{|x|<R_j} |\int_{|y|\le R_{j+1}} \frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2 dx \\ &\quad + \int_{|x|<R_j} |\int_{|y|>R_{j+1}} \frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2 dx]\end{aligned}$$

For the first integral, we have

$$\begin{aligned}&\int_{|x|<R_j} |\int_{|y|\le R_{j+1}} \frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2 dx \\ &= \underbrace{\int_{|x|<R_j} |\int_{|y|\le R_{j+1}}}_{|x-y|\le 3R_j} \frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2 dx \\ &\text{(by Young's inequality)} \\ &\le (\int_{|t|<3R_j} \frac{1}{|t|}dt)^2 \int_{|y|\le R_{j+1}} |f(y)|^2 dy \\ &\le (6\pi)^2 R_j^2 \int_{|y|\le R_{j+1}} |f(y)|^2 dy.\end{aligned}$$

For the second integral, we have

$$\begin{aligned}\int_{|x|<R_j}|\int_{|y|>R_{j+1}}\frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2dx &\overset{(\text{since}|x-y|\geq R_j)}{\leq} \frac{1}{R_j^2}\int_{|x|<R_j}(\int_{\mathbf{R}^2}|f(y)|dy)^2dx\\ &= \pi(\int_{\mathbf{R}^2}|f(y)|dy)^2.\end{aligned}$$

Combining the two terms, we have

$$\begin{aligned}\int_{|x|<R_j}|u(x)|^2dx &\leq \frac{1}{2\pi^2}[(6\pi)^2R_j^2\int_{|y|\leq R_{j+1}}|f(y)|^2dy+\pi\|f(\cdot)\|^2_{L^1(\mathbf{R}^2)}]\\ &\leq 18R_j^2\int_{|y|\leq R_{j+1}}|f(y)|^2dy+\frac{1}{2\pi}\|f(\cdot)\|^2_{L^1(\mathbf{R}^2)}.\end{aligned}$$

(ii)

$$\begin{aligned}&\int_{R_{j-m-1}\leq|x_1|\leq R_j}\int_{|x_2|\leq R_j}|u(x)|^2dx\\ &=\frac{1}{4\pi^2}\int_{R_{j-m-1}\leq|x_1|\leq R_j}\int_{|x_2|\leq R_j}|\int_{\mathbf{R}^2}\frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2dx\\ &=\frac{1}{4\pi^2}[\int_{R_{j-m-1}\leq|x_1|\leq R_j}\int_{|x_2|\leq R_j}|\underbrace{\int_{R_{j-m-2}\leq|y_1|\leq R_{j+1}}\int_{|y_2|\leq R_{j+1}}}_{\Omega_j^*}\\ &\quad+\int_{\mathbf{R}^2\backslash\Omega_j^*}\frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2dx]\end{aligned}$$

For the first integral, we have by Young's inequality

$$\begin{aligned}&\int_{R_{j-m-1}\leq|x_1|\leq R_j}\int_{|x_2|\leq R_j}|\underbrace{\int_{R_{j-m-2}\leq|y_1|\leq R_{j+1}}\int_{|y_2|\leq R_{j+1}}}_{\Omega_j^*}\frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2dx\\ &\leq(\int_{|t|\leq 6R_j}\frac{1}{|t|}dt)^2\int_{R_{j-m-2}\leq|y_1|\leq R_{j+1}}\int_{|y_2|\leq R_{j+1}}|f(y)|^2dy,\end{aligned}$$

since when $R_{j-m-1}<|x_1|<R_j$, $|x_2|\leq R_j$, $R_{j-m-2}<|y_1|<R_{j+1}$ and $|y_2|\leq R_{j+1}$,

$$|x-y|\leq|x_1-y_1|+|x_2-y_2|\leq|x_1|+|y_1|+|x_2|+|y_2|=6R_j.$$

For the second integral, we have

$$\begin{aligned}&\int_{R_{j-m-1}\le|x_1|\le R_j}\int_{|x_2|\le R_j}|\int_{\mathbf{R}^2\backslash\Omega_j^*}\frac{f(y)}{(x_1-y_1)+i(x_2-y_2)}dy|^2dx\\ &\le\frac{1}{R_{j-m-2}^2}\int_{R_{j-m-1}\le|x_1|\le R_j}\int_{|x_2|\le R_j}(\int_{\mathbf{R}^2\backslash\Omega_j^*}|f(y)|dy)^2dx\\ &\le\frac{R_j^2}{R_{j-m-2}^2}\|f(\cdot)\|_{L^1(\mathbf{R}^2)}^2,\end{aligned}$$

since when $R_{j-m-1}<|x_1|<R_j$, $|x_2|\le R_j$ and $y\in\mathbf{R}^2\backslash\Omega_j^*$,

$$|x-y|\ge|x_1-y_1|\ge R_{j-m-2}.$$

Combing the two terms, we have

$$\begin{aligned}&\int_{R_{j-m-1}\le|x_1|\le R_j}\int_{|x_2|\le R_j}|u(x)|^2dx\\ &\le(144/2)R_j^2\int_{R_{j-m-2}\le|y_1|\le R_{j+1}}\int_{|y_2|\le R_{j+1}}|f(y)|^2dy+\frac{1}{2\pi^2}\frac{R_j^2}{R_{j-m-2}^2}\|f(\cdot)\|_{L^1(\mathbf{R}^2)}^2\\ &=72R_j^2\int_{R_{j-m-2}\le|y_1|\le R_{j+1}}\int_{|y_2|\le R_{j+1}}|f(y)|^2dy+\frac{1}{2\pi^2}(4^{m+2})\|f(\cdot)\|_{L^1(\mathbf{R}^2)}^2.\end{aligned}$$

♯

The norm in B_1 is a majorant for the following mixed L^1, L^2 norms (compare with Theorem 14.1.2 in [H]):

Lemma 2.3 *Let* $n\ge3$ *and let* $x=(x',x'')$ *where* $x'=(x_1,x_2)$ *and* $x''=(x_3,\cdots x_n)$. *If* $f\in B_1(\boldsymbol{R}^n)$ *then*

(i) $\int_{\boldsymbol{R}^2}\|f(x',\cdot)\|_{L^2(\boldsymbol{R}^{n-2})}dx'\le\sqrt{\pi}\|f\|_{B_1}$

(ii) $(\int_{\boldsymbol{R}^{n-2}}\|f(\cdot,x'')\|_{L^1(\boldsymbol{R}^2)}^2dx'')^{1/2}\le\sqrt{\pi}\|f\|_{B_1}$.

Proof: (i) Let $f_j=f$ in Ω_j and $f_j=0$ elsewhere. Then, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\int_{\mathbf{R}^2}\|f_j(x',\cdot)\|_{L^2(\mathbf{R}^{n-2})}dx' &\le (\pi R_j^2)^{1/2}(\int_{\mathbf{R}^2}\|f_j(x',\cdot)\|_{L^2(\mathbf{R}^{n-2})}^2dx')^{1/2}\\ &\le \sqrt{\pi}R_j(\int_{\Omega_j}|f(x)|^2dx)^{1/2}.\end{aligned}$$

Thus, since $f = \sum_j f_j$, we obtain

$$\int_{\mathbf{R}^2} \|f(x',\cdot)\|_{L^2(\mathbf{R}^{n-2})}dx' \le \sum_j \int_{\mathbf{R}^2} \|f_j(x',\cdot)\|_{L^2(\mathbf{R}^{n-2})}dx' \le \sqrt{\pi}\|f\|_{B_1},$$

where the second inequality follows from the definition of the B_1 norm.

(ii) Minkowski's inequality for integrals shows that

$$\Big(\int_{\mathbf{R}^{n-2}} \|f(\cdot,x'')\|^2_{L^1(\mathbf{R}^2)}dx''\Big)^{1/2} \le \int_{\mathbf{R}^2} \|f(x',\cdot)\|_{L^2(\mathbf{R}^{n-2})}dx'$$

and (ii) follows from (i). $\sharp$

Now we are in the position to prove (10). For dimension $n = 2$, (10) follows from Lemma 2.2 (i) immediately. For $n \ge 3$, we cover Ω_j in the following way:

$$\begin{aligned}\Omega_j &= \{x \in \mathbf{R}^n : \quad R_{j-1} < |x| < R_j\} \\ &\subset \cup_{k=1}^n \{x \in \mathbf{R}^n : a_j < |x_k| < R_j, |x_l| < R_j, 1 \le l \le n, l \ne k\} = \textstyle\sum_{k=1}^n \diamond_k^j,\end{aligned} \tag{11}$$

where $a_j = R_{j-1}/\sqrt{n}$, and for each k,

$$\begin{aligned}\diamond_k^j &\overset{def}{=} \{x \in \mathbf{R}^n : a_j < |x_k| < R_j, |x_l| < R_j, 1 \le l \le n, l \ne k\} \\ &\subset \{x \in \mathbf{R}^n : a_j < |x| < b_j\}\end{aligned} \tag{12}$$

with $b_j = \sqrt{n}R_j$ (see Figure 1).

Let m be the positive integer such that $2^{m-1} \le \sqrt{n} \le 2^m$. Then from (11),

$$\begin{aligned}\int_{\Omega_j} |u(x',x'')|^2 dx &\le \sum_{k=1}^n \int_{\diamond_k^j} |u(x',x'')|^2 dx \\ &= \sum_{k=1}^n \int_{a_j<|x_k|<R_j} \int_{|x_l|<R_j, l\ne k} |u(x',x'')|^2 dx_1 \cdots \hat{dx_k} \cdots dx_n dx_k.\end{aligned} \tag{13}$$

If $k = 1$ or 2, say $k = 1$, then

$$\begin{aligned}&\int_{a_j<|x_1|<R_j} \int_{|x_l|<R_j, l\ge 2} |u(x',x'')|^2 dx_1 \cdots dx_n \\ &\quad\le C_1 \int_{|x''|<R_j} \Big(\int_{R_{j-m-1}<|x_1|<R_j} \int_{|x_2|<R_j} |u(x',x'')|^2 dx'\Big)dx''\end{aligned} \tag{14}$$

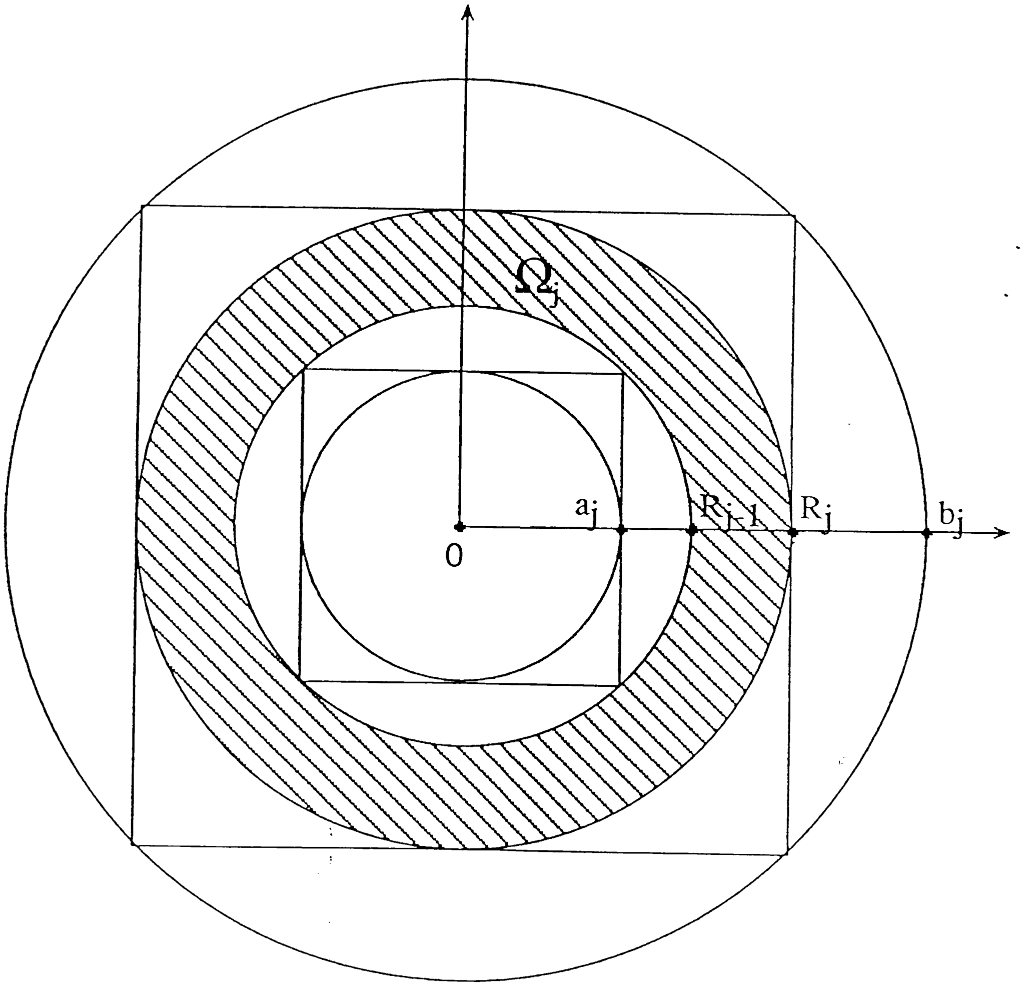

Figure 1: The graph of Ω_j

(by Lemma 2.2 (ii))

$$\le C_2 \int_{|x''|<R_j} [R_j^2 \int_{R_{j-m-2}<|y_1|<R_{j+1}} \int_{|y_2|<R_j} |f(y,x'')|^2 dy + \|f(\cdot,x'')\|^2_{L^1(\mathbf{R}^2)}] dx''$$

(by (12))

$$\le C_2 R_j^2 \int_{R_{j-m-2}<|(y,x'')|<\sqrt{n}R_{j+1}} |f(y,x'')|^2 dy dx'' + C_2 \int_{|x''|<R_j} \|f(\cdot,x'')\|^2_{L^1(\mathbf{R}^2)} dx''$$

(by Lemma 2.3 (ii))

$$\le C(4^{m+2})\|f\|^2_{B_1(\mathbf{R}^n)} + C\|f\|^2_{B_1(\mathbf{R}^n)}.$$

If $k \ge 3$, say $k = 3$, then

$$\int_{R_{j-1}/\sqrt{n}<|x_3|<R_j} \int_{|x_l|<R_j, l\neq 3} |u(x',x'')|^2 dx_1 \cdots \widehat{dx_3} \cdots dx_n dx_3 \tag{15}$$

$$\le C_3 \int_{R_{j-m-1}<|x_3|<R_j} \int_{|x_l|<R_j, l\neq 3,1,2} (\int_{|x'|<R_j} |u(x',x'')|^2 dx') dx''$$

(by Lemma 2.2 (ii))

$$\le C_3 \int_{R_{j-m-1}<|x_3|<R_j} \int_{|x_l|<R_j, l\neq 3,1,2} [18R_j^2 \int_{|y|<R_{j+1}} |f(y,x'')|^2 dy + \frac{1}{2\pi}\|f(\cdot,x'')\|^2_{L^1(\mathbf{R}^2)}] dx''$$

(by (12))

$$\le C_3 R_j^2 \int_{R_{j-m-1}<|(y,x'')|<\sqrt{n}R_{j+1}} |f(y,x'')|^2 dy dx'' + C_3 \int_{|x''|<R_j} \|f(\cdot,x'')\|^2_{L^1(\mathbf{R}^2)} dx''$$

$$\le C_3(4^{m+2})\|f\|^2_{B_1(\mathbf{R}^n)} + C_3\|f\|^2_{B_1(\mathbf{R}^n)},$$

by Lemma 2.3 (ii).

So for each k,

$$\int_{\diamond_k^j} |u(x',x'')|^2 dx \le C_k(4^{m+2}\|f\|^2_{B_1(\mathbf{R}^n)} + \|f\|^2_{B_1(\mathbf{R}^n)}), \tag{16}$$

with m depending only on the dimension n.

Returning to (13), we obtain

$$\|u\|_{B_0^*} = \sup_{j\ge 1}(\int_{\Omega_j} |u(x)|^2 dx)^{1/2} \le C\|f\|_{B_1(\mathbf{R}^n)}.$$

♯

Corollary 2.4 *If $f \in B_0$ and $u = f * (\frac{1}{x_1+ix_2})$, then*

$$\|u\|_{B_1^*} \leq C\|f\|_{B_0}.$$

<u>Proof:</u> Note that B_1^* is the dual space of B_1. So

$$\begin{aligned}
\|u\|_{B_1^*} &= \sup_{\|g\|_{B_1}=1} |\int u(x)\overline{g(x)}dx| \\
&= \frac{1}{2\pi}\sup_{\|g\|_{B_1}=1} |\int_{x''}\int_{x'}(\int_y \frac{f(y,x'')}{(x_1-y_1)+i(x_2-y_2)}dy)\overline{g(x',x'')}dx'dx''| \\
&= \frac{1}{2\pi}\sup_{\|g\|_{B_1}=1} |\int_{x''}\int_{x'}\int_y \frac{f(y,x'')\overline{g(x',x'')}}{(x_1-y_1)+i(x_2-y_2)}dydx'dx''| \\
&\quad \text{(by Fubini's Theorem)} \\
&= \frac{1}{2\pi}\sup_{\|g\|_{B_1}=1} |\int_{x''}\int_y(\int_{x'} \frac{\overline{g(x',x'')}}{(x_1-y_1)+i(x_2-y_2)}dx')f(y,x'')dydx''| \\
&\leq \frac{1}{2\pi}\sup_{\|g\|_{B_1}=1}\sum_{j=1}^{\infty}|\int\int_{\Omega_j}|f(y,x'')||\int_{x'}\frac{\overline{g(x',x'')}}{(x_1-y_1)+i(x_2-y_2)}dx'|dydx'' \\
&\leq \frac{1}{2\pi}\sup_{\|g\|_{B_1}=1}\sum_{j=1}^{\infty}(\int\int_{\Omega_j}|f(y,x'')|^2dydx'')^{1/2}\cdot \\
&\quad (\int\int_{\Omega_j}|\int_{x'}\frac{\overline{g(x',x'')}}{(x_1-y_1)+i(x_2-y_2)}dx'|^2dydx'')^{1/2} \\
&\leq \frac{1}{2\pi}\sup_{\|g\|_{B_1}=1}[\sum_{j=1}^{\infty}(\int\int_{\Omega_j}|f(y,x'')|^2dydx'')^{1/2}\cdot \\
&\quad \sup_{j\geq 1}(\int\int_{\Omega_j}|\int_{x'}\frac{\overline{g(x',x'')}}{(x_1-y_1)+i(x_2-y_2)}dx'|^2dydx'')^{1/2}] \\
&\quad \text{(by Theorem 2.1)} \\
&\leq [\sum_{j=1}^{\infty}|\int\int_{\Omega_j}|f(y,x'')|^2dydx'')^{1/2}]\sup_{\|g\|_{B_1}=1}(C\|\bar{g}\|_{B_1}) \\
&= C\|f\|_{B_0}.
\end{aligned}$$

Thus, $u = f * (\frac{1}{x_1+ix_2})$ is in B_1^*. ♯

Corollary 2.5 *Let* $0 < s < 1$. *If* $f \in B_{1-s}$, *then* $f * (\frac{1}{x_1+ix_2}) \in B_s^*$ *and*

$$\|f * (\frac{1}{x_1+ix_2})\|_{B_s^*} \leq C_s \|f\|_{B_{1-s}}.$$

<u>Proof:</u> First if $u \in B_0^*$, we have

$$\begin{aligned}
\|u\|^2_{L^2_{-s}} &= \int (1+|x|^2)^{-s}|u(x)|^2 dx \\
&= \sum_{j=1}^{\infty} \int_{\Omega_j} (1+|x|^2)^{-s}|u(x)|^2 dx \\
&\leq \sum_{j=1}^{\infty} R_{j-1}^{-2s} \int_{\Omega_j} |u(x)|^2 dx \\
&= 2^{2s} \sum_{j=1}^{\infty} R_j^{-2s} \int_{\Omega_j} |u(x)|^2 dx \\
&\leq 4\{\sum_{j \leq k} R_j^{-2s} \int_{\Omega_j} |u(x)|^2 dx + \sum_{j>k} R_j^{-2s} \int_{\Omega_j} |u(x)|^2 dx\} \\
&= 4\{\sum_{j \leq k} R_j^{2-2s} \frac{1}{R_j^2} \int_{\Omega_j} |u(x)|^2 dx + \sum_{j>k} R_j^{-2s} \int_{\Omega_j} |u(x)|^2 dx\} \\
&\leq C\{R_k^{2-2s} \frac{1}{1-2^{-(2-2s)}} \|u\|^2_{B_1^*} + R_k^{-2s} \|u\|^2_{B_0^*}\}.
\end{aligned}$$

for each $k \geq 1$, where C is independent of k. So

$$\|u\|_{L^2_{-s}} \leq \sqrt{C_s}/2\{R_k^{1-s}\|u\|_{B_1^*} + R_k^{-s}\|u\|_{B_0^*}\}.$$

Now let $f = \sum_{k=1}^{\infty} f_k$ where $f_k = f|_{\Omega_k}$. Then, with $u_k = f_k * \frac{1}{x_1+ix_2}$,

$$\begin{aligned}
\|u\|_{L^2_{-s}} &= \|\sum_{k=1}^{\infty} u_k\|_{L^2_{-s}} \\
&\leq \sum_{k=1}^{\infty} \|u_k\|_{L^2_{-s}} \\
&\leq C_s \sum_k \{R_k^{1-s}\|u_k\|_{B_1^*} + R_k^{-s}\|u_k\|_{B_0^*}\} \\
&\qquad (\text{ by Theorem 2.1 and Corollary 2.4 }) \\
&\leq C_s \sum_k R_k^{1-s}\|f_k\|_{B_0} + R_k^{-s}\|f_k\|_{B_1}
\end{aligned}$$

$$
\begin{aligned}
&= C_s \sum_k \{ R_k^{1-s} (\int_{\Omega_k} |f(x)|^2 dx)^{1/2} + R_k^{-s} R_k (\int_{\Omega_k} |f(x)|^2 dx)^{1/2} \} \\
&= C_s \sum_k R_k^{1-s} (\int_{\Omega_k} |f(x)|^2 dx)^{1/2} \\
&= C_s \|f\|_{B_{1-s}}.
\end{aligned}
$$

Since $L^2_{-s} \subset B^*_s$,

$$\|u\|_{B^*_s} \le C\|u\|_{L^2_{-s}} \le C_s\|f\|_{B_{1-s}}.$$

♯

Note that although the map $f \to f * (\frac{1}{x_1+ix_2})$ is bounded from L^2_δ to $L^2_{-1+\delta}$ for $0 < \delta < 1$, it is neither bounded from L^2_1 to L^2_0 nor from L^2_0 to L^2_{-1}. So the Agmon-Hörmander spaces B_s provide the appropriate end point substitutes for the weighted-L^2 spaces.

3 Estimate for General Distribution $(\frac{1}{H_1(\xi)+iH_2(\xi)})$

Now we generalize the result from the special model $\frac{1}{x_1+ix_2}$ to a general distribution $(\frac{1}{H_1(\xi)+iH_2(\xi)})^\vee(x)$. Before doing so, we need some additional definitions and lemmas.

Let c_1, c_2 $\cdots$ be a sequence of positive numbers such that for some constant M,

$$\frac{c_j}{M} \le c_{j+1} \le Mc_j \tag{17}$$

$j = 1, 2, \cdots$. Define

$$B_{\{c\}} = \{ v \in L^2_{loc}(\mathbf{R}^n) : \quad \sum_{j=1}^\infty c_j (\int_{\Omega_j} |v|^2 dx)^{1/2} < \infty \} \quad .$$

Then its dual space is

$$B^*_{\{c\}} = \{ u \in L^2_{loc}(\mathbf{R}^n) : \quad \sup_{j\ge1} c_j^{-1} (\int_{\Omega_j} |u|^2 dx)^{1/2} < \infty \} \quad .$$

Then we have

Lemma 3.1 *Let N be the smallest integer such that $2^N > M$. Then there is a constant C_M such that if*

$$T: \quad L^2_{-N} \to L^2_{-N}$$

is bounded and

$$T: \quad L^2_N \to L^2_N$$

is bounded with both norms $\leq A$, it follows that

$$T: \quad B_{\{c\}} \to B_{\{c\}}$$

is bounded with norm $\leq C_M A$.

Lemma 3.2 *Let $r \in C^N(\boldsymbol{R}^n)$ and assume that $D^\alpha r$ is bounded when $|\alpha| \leq N$. Then the operator $r(D) = \mathcal{F}^{-1} r \mathcal{F}$ is bounded in $B_{\{c\}}$ and*

$$\|r(D)u\|_{B_{\{c\}}} \leq C_M \sum_{|\alpha| \leq N} \sup |D^\alpha r| \|u\|_{B_{\{c\}}}, \qquad u \in B_{\{c\}},$$

where $\mathcal{F}$ is the Fourier transform operator.

Lemma 3.3 *Let X_1 and X_2 be open sets in $\boldsymbol{R}^n$ and Ψ a C^{N+1} diffeomorphism $X_1 \to X_2$. Choose $\chi \in C_0^N(X_1)$ and set*

$$Tu = \mathcal{F}^{-1}(\chi(\hat{u} \circ \Psi))$$

Then T is bounded in $B_{\{c\}}$, with a norm which can be estimated in terms of the maximum of the derivatives of χ of order $\leq N$ and of Ψ, Ψ^{-1} of order $\leq N+1$.

Proofs of the above three lemmas can be found in [H].

For the space B_s, $s \geq 0$, we can choose $\{c\}_s = \{R_j^s\} = \{2^{(j-1)s}\}$ and $M_s = 2^s$. Then

$$2^{(j-2)s} = \frac{c_j}{M_s} \leq c_{j+1} = 2^{js} = M_s c_j \ .$$

In Lemma 3.1, we can choose $N = 2$ if $s = 1$. If $0 \leq s < 1$, we can choose $N = 1$.

Theorem 3.4 *Let* $H(\xi) = H_1(\xi) + iH_2(\xi) \in C^3(\Omega)$ *where* Ω *is an open set in* $\boldsymbol{R}^n$*; assume that* $Re(\nabla H(\xi)) = \nabla H_1(\xi)$ *and* $Im(\nabla H(\xi)) = \nabla H_2(\xi)$ *are linearly independent when* $H(\xi) = 0$ *in* Ω*. Then for fixed* $\chi \in C_0^2(\Omega)$*, there exists a constant* C *such that when* $u \in B_1$*,* $v \in B_0$*,*

$$|\int \chi(\xi)H(\xi)^{-1}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi| \leq C\|u\|_{B_1}\|v\|_{B_0}. \tag{18}$$

Proof: First, if *supp*χ is sufficiently small and

$$|\nabla H(\xi)| = \left[\sum_{i\neq j}\left|\det\begin{pmatrix} \frac{\partial H_1}{\partial \xi_i} & \frac{\partial H_1}{\partial \xi_j} \\ \frac{\partial H_2}{\partial \xi_i} & \frac{\partial H_2}{\partial \xi_j} \end{pmatrix}\right|^2\right]^{1/4} \neq 0$$

on *supp*χ, we can suppose $\chi = \chi_1\chi_2$ where χ_1, $\chi_2 \in C_0^2(\Omega')$ for an open set $\Omega' \subset \Omega$ such that there is a C^3 diffeomorphism $\psi : \Omega'' \to \Omega'$ with $H(\psi(\eta)) = \eta_1 + i\eta_2$. Then

$$\begin{aligned} &\int \chi_1(\xi)\chi_2(\xi)H(\xi)^{-1}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi \\ &\quad = \int \frac{1}{\eta_1 + i\eta_2}\chi_1(\psi(\eta))\hat{u}(\psi(\eta))\chi_2(\psi(\eta))\overline{\hat{v}(\psi(\eta))}|\det(\psi'(\eta))|d\eta \ . \end{aligned}$$

From Lemma 3.3, $\mathcal{F}^{-1}(\chi_1 \circ \psi \cdot \hat{u} \circ \psi) \in B_1$ and $\mathcal{F}^{-1}(\chi_2 \circ \psi \cdot \hat{v} \circ \psi) \in B_0$. Then Theorem 2.1 and Corollary 2.4 imply that

$$\mathcal{F}^{-1}(\frac{1}{\eta_1 + i\eta_2}\chi_1 \circ \psi \cdot \hat{u} \circ \psi) \in B_0^*.$$

Therefore

$$\int \chi_1(\xi)\chi_2(\xi)H(\xi)^{-1}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi| \leq C\|u\|_{B_1}\|v\|_{B_0}.$$

In general, if $|\nabla H(\xi)| = 0$ at some points ξ in *supp*χ, write $\chi(\xi) = \chi_1(\xi) + \chi_2(\xi)$ where $|\nabla H(\xi)| \neq 0$ in *supp*χ_1 and $H(\xi) \neq 0$ in *supp*χ_2. Then

$$\begin{aligned} |\int \chi_2(\xi)H(\xi)^{-1}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi| &\leq C\int |\hat{u}(\xi)\overline{\hat{v}(\xi)}|d\xi \\ &\leq C\|u\|_{L_0^2}\|v\|_{L_0^2} \\ &\leq C\|u\|_{B_1}\|v\|_{B_0}. \end{aligned}$$

On $supp\chi_1$, using a partition of unity and the proof for small $supp\chi$, we have

$$|\int \chi_1(\xi)H(\xi)^{-1}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi| \leq C\|u\|_{B_1}\|v\|_{B_0}.$$

Combining the estimates for χ_1 and χ_2 yeilds (18). ♯

Theorem 3.4 says that $Tu = \mathcal{F}^{-1}(\chi H^{-1}\hat{u})$ is bounded from B_1 to B_0^* and also bounded from B_0 to B_1^*. Then by interpolation we have

Corollary 3.5 *The operator* $T_s u = \mathcal{F}^{-1}(\chi H^{-1}\hat{u})$, $u \in B_s$ *is bounded from* B_s *to* B_{1-s}^* *for each* $0 < s < 1$*; i.e.,*

$$\|\mathcal{F}^{-1}(\chi H^{-1}\hat{u})\|_{B_{1-s}^*} \leq C_s\|u\|_{B_s}, \quad u \in B_s.$$

4 Estimate for Complex Simply Characteristic Polynomials

Let $P(\xi) = P_1(\xi)+iP_2(\xi)$ be a polynomial with constant complex coefficients, and assume that

$$|\nabla P(\xi)| = \left[\sum_{i\neq j} |\det\begin{pmatrix} \frac{\partial P_1}{\partial \xi_i} & \frac{\partial P_1}{\partial \xi_j} \\ \frac{\partial P_2}{\partial \xi_i} & \frac{\partial P_2}{\partial \xi_j} \end{pmatrix}|^2\right]^{1/4} \neq 0$$

on $\{\xi \in \mathbf{R}^n : P(\xi) = 0\}$. Then for $\chi \in C_0^2(\mathbf{R}^n)$, we have shown in the previous section that

$$\|\mathcal{F}^{-1}(\chi P^{-1}\hat{f})\|_{B_{1-s}^*} \leq C_s\|f\|_{B_s}, \quad f \in B_s \tag{19}$$

for $0 \leq s \leq 1$.

Now we want to impose a stronger condition on P to control the behavior at large ξ and thus allow an estimate like (19) without the cut-off function χ. We need a notion of a simply characteristic complex polynomials analogous to the one introduced by Hörmander for real polynomials.

Definition 4.1 *Let* $P(\xi) = P_1(\xi)+iP_2(\xi)$ *be a polynomial of* $\xi \in \boldsymbol{R}^n$ *of order* m *with complex constant coefficients. We say* P *is* ***simply characteristic*** *if for some complex* z

$$\tilde{P}(\xi) \leq c_P(|P(\xi) - z| + |\nabla P(\xi)|) \tag{20}$$

for all $\xi \in \boldsymbol{R}^n$*, where*

$$\tilde{P}(\xi) \overset{def}{=} \sum_{|\alpha|\leq m, |\alpha|\neq 1} |P^{(\alpha)}(\xi)| + |\nabla P(\xi)| \tag{21}$$

and

$$|\nabla P(\xi)| \overset{def}{=} [\sum_{i\neq j} |\det \begin{pmatrix} \frac{\partial P_1}{\partial \xi_i} & \frac{\partial P_1}{\partial \xi_j} \\ \frac{\partial P_2}{\partial \xi_i} & \frac{\partial P_2}{\partial \xi_j} \end{pmatrix} |^2]^{1/4}. \tag{22}$$

Now let us analyze the condition (20). We assume here $\tilde{P}(\xi) \to \infty$ as $\xi \to \infty$.

1. Set $P_\xi(\eta) = \frac{P(\xi+\eta)}{\tilde{P}(\xi)}$. Then we have from (20)

   $$1 \leq C(|Q(0)| + |\nabla Q(0)|) \tag{23}$$

   if Q is any limit of P_ξ as $\xi \to \infty$. Since $Q_\theta(\eta) = \frac{Q(\eta+\theta)}{\tilde{Q}(\theta)}$ is also such a limit it follows that if $Q(\theta) = Q_1(\theta)+iQ_2(\theta) = 0$ at θ, then $|\nabla Q(\theta)| \neq 0$. This means that $\nabla Q_1(\theta)$ and $\nabla Q_2(\theta)$ are linearly independent on the zero set of Q.

2. If condition (20) is not valid for any C when $|\xi|$ large, then we can find a limit Q with $|\nabla Q(0)| = 0$ and this contradicts (23). Thus the validity of (20) for large $|\xi|$ is independent of $z \in \mathbf{C}$ and means precisely that the limits of P_ξ have the property that $\nabla Q_1(\xi)$ and $\nabla Q_2(\xi)$ are linearly independent on the zero set $\{\xi \in \mathbf{R}^n : Q(\xi) = 0\}$. For a **simply characteristic** polynomial we conclude that (20) is uniformly valid when z belongs to a compact set which does not contain any value $z \in \mathbf{C}^n$ such that

   (I): $\nabla P_1(\xi)$ and $\nabla P_2(\xi)$ are not linearly independent at some point $\xi \in \{\xi \in \mathbf{R}^n : P(\xi) - z = 0\}$.

   We will call these values **critical values**.

3. If $z \in \mathbf{C}$ is not a **critical value** of P, then when we set $P_\xi(\eta) = \frac{P(\xi+\eta)-z}{\tilde{P}(\xi)}$, condition (20) means

   $$P_\xi(0) = 0 \quad , \quad |\nabla P_\xi(0)| \geq \frac{1}{C} \tag{24}$$

for $\xi \in \{\xi \in \mathbf{R}^n : P(\xi) - z = 0\}$. Combining (23) and (24) we see that the polynomials P_ξ and their limits as $\xi \to \infty$ either have a simple zero at $\eta = 0$ in the sense that $P_\xi(0) = 0$, $\nabla_\xi P_1(0)$ and $\nabla_\xi P_2(0)$ are linearly independent or do not equal 0 at $\eta = 0$. Furthermore, from (23), (24) and the uniform boundedness of the coefficients of P_ξ and their limits we find using the implicit function theorem that for some $r > 0$ independent of ξ, all zeros of $P_\xi(\eta)$ and of their limits with $|\eta| < r$ can for some i and j (after C^3 diffeomophism) be written in the form

$$\eta_i = h_1(\eta_1 \cdots \eta_{i-1}, \eta_{i+1} \cdots \eta_{j-1}, \eta_{j+1} \cdots \eta_n)$$

$$\eta_j = h_2(\eta_1 \cdots \eta_{i-1}, \eta_{i+1} \cdots \eta_{j-1}, \eta_{j+1} \cdots \eta_n)$$

where h_1, $h_2 \in C^3$.

Remark: If $P(\xi)$ is a hypoelliptic polynomial,

$$P^{(\alpha)}(\xi)/P(\xi) \to 0$$

when $\xi \to \infty$ in $\mathbf{R}^n$ for $\alpha \neq 0$. So hypoelliptic polynomials satisfy condition(20) for large ξ . Thus if $z \in \mathbf{C}$ is not a **critical value** of P, condition (20) is satisfied for all $\xi \in \mathbf{R}^n$.

Theorem 4.2 *Assume that P is* ***simply characteristic*** *and let K be a compact subset of $\mathbf{C}$ containing no* ***critical value*** *of P. If $f \in B_1$, it follows that*

$$R(z)f = \mathcal{F}^{-1}((P(\cdot) - z)^{-1}\hat{f})$$

is in B_0^* for $z \in K$ and we have the estimate

$$\|R(z)f\|_{B_0^*} \leq C \sup_{\xi \in \boldsymbol{R}^n} \frac{1}{\tilde{P}(\xi)} \|f\|_{B_1} \tag{25}$$

for $z \in K$ where C depends only on the dimension n and the constant c_P in condition (20).

Proof: It suffices to prove the theorem for some neighborhood K of 0 in $\mathbf{C}$ when 0 is not a **critical value**. Since P_ξ and their limits as $\xi \to \infty$ have uniformly bounded coefficients, the polynomials P_ξ and their limits as $\xi \to \infty$ form a compact set of polynomials either satisfying the hypotheses on H in Theorem 3.4 with $\Omega = \{\eta \in \mathbf{R}^n : |\eta| < \rho\}$, say, or else uniformly bounded from below in Ω, where the lower bound depends only on c_P in (20) and the compact set K. According to the analysis following Definition 4.1, we can choose $\rho > 0$ small enough such that in $|\eta| < \rho$, all real zeros of $P_\xi(\eta)$ and their limits as $\xi \to \infty$ with $|\eta| < \rho$ can be represented by

$$\eta_i = h_1(\eta_1 \cdots \eta_{i-1}, \eta_{i+1} \cdots \eta_{j-1}, \eta_{j+1} \cdots \eta_n)$$

$$\eta_j = h_2(\eta_1 \cdots \eta_{i-1}, \eta_{i+1} \cdots \eta_{j-1}, \eta_{j+1} \cdots \eta_n)$$

for some i and j (after C^3 diffeomophism $\psi : P_\xi(\psi(\beta)) = \beta_i + i\beta_j$). If $\chi \in C_0^\infty(\Omega)$ it follows from the proof of Theorem 3.4 that there is a constant C and a compact neighborhood K of 0 in $\mathbf{C}$, both independent of ξ, such that

$$|\int |\chi(\eta)|^2 (\frac{1}{P_\xi - z/\tilde{P}(\xi)}) \hat{f}(\eta)\overline{\hat{g}(\eta)} d\eta| \le C \|\mathcal{F}^{-1}(\chi \hat{f})\|_{B_1} \|\mathcal{F}^{-1}(\chi \hat{g})\|_{B_0}$$

if f, $g \in \mathcal{S}$ where $\mathcal{S}$ is the Schwartz space, and $z/\tilde{P}(\xi) \in K$. Let K' be a neighborhood of 0 in $\mathbf{C}$ contained in $\tilde{P}(\xi)K$ for all ξ. Making a translation of $\hat{f}$ and $\hat{g}$ and writing $\chi_\xi(\eta) = \chi(\eta - \xi)$ we have

$$\begin{aligned} &|\int \tilde{P}(\xi) \frac{1}{P(\eta) - z} \chi_\xi(\eta) \hat{f}(\eta) \overline{\chi_\xi(\eta)\hat{g}(\eta)} d\eta| \\ &= |\int |\chi(\eta)|^2 (\frac{1}{P_\xi - z/\tilde{P}(\xi)}) \hat{f}(\eta + \xi) \overline{\hat{g}(\eta + \xi)} d\eta| \\ &\le C \|\mathcal{F}^{-1}(\chi \hat{f}(\xi + \cdot))\|_{B_1} \|\mathcal{F}^{-1}(\chi \hat{g}(\xi + \cdot))\|_{B_0} \\ &\le C \|\mathcal{F}^{-1}(\chi_\xi \hat{f})\|_{B_1} \|\mathcal{F}^{-1}(\chi_\xi \hat{g})\|_{B_0} \end{aligned}$$

If we then write $\hat{f}_\xi = \chi_\xi \hat{f}$, $\hat{g}_\xi = \chi_\xi \hat{g}$, it follows that

$$|\int \frac{1}{P(\eta) - z} \chi_\xi(\eta) \hat{f}(\eta) \overline{\chi_\xi(\eta)\hat{g}(\eta)} d\eta| \le C \sup_{\xi \in \mathbf{R}^n} \frac{1}{\tilde{P}(\xi)} \|f_\xi\|_{B_1} \|g_\xi\|_{B_0}$$

If we integrate with respect to ξ and use the following lemma we obtain

$$|(R(z)f, g)| \le C \sup_{\xi \in \mathbf{R}^n} \frac{1}{\tilde{P}(\xi)} \|f\|_{B_1} \|g\|_{B_0} \tag{26}$$

which proves the theorem. ♯

Lemma 4.3 *Let* $\chi \in C_0^\infty(\boldsymbol{R}^n)$ *and set* $\chi(D-\eta)u = \mathcal{F}^{-1}\chi(\cdot-\eta)\hat{u}$, $u \in \mathcal{S}'$. *Then we have*

$$\int \|\chi(D-\eta)u\|^2_{B_{\{c\}}} d\eta \leq C_{M,\chi}\|u\|^2_{B_{\{c\}}}, \quad u \in B_{\{c\}}$$

for all $\{c\}$ *satisfying condition (17).*

This lemma is Theorem 14.1.7 in Hörmander's book [H].

From estimate (26) we also see that $R(z)$ is also bounded from B_0 to B_1^*. Then by interpolation we obtain that $R(z)$ is also bounded from B_s to B_{1-s}^* for $0 < s < 1$. Therefore we have

Theorem 4.4 *Supppose that* P *and* K *are as in the hypotheses of Theorem 4.2. If* $f \in B_s (0 \leq s \leq 1)$ *it follows that*

$$R(z)f = \mathcal{F}^{-1}((P(\cdot)-z)^{-1}\hat{f})$$

is in B_{1-s}^* *for* $z \in K$ *and we have estimate*

$$\|R(z)f\|_{B_{1-s}^*} \leq C(s, c_P) \sup_{\xi \in \boldsymbol{R}^n} \frac{1}{\tilde{P}(\xi)} \|f\|_{B_s}$$

for $z \in K$.

Remark:

1. If $\{P(\xi,\zeta)\}$ is a family of polynomials of ξ depending on a parameter ζ in a subset $M_\zeta \subset \mathbf{C}^n$ and condition (20) is valid for all $\zeta \in M_\zeta$, with the constant $c_P(\zeta)$ depending on ζ, then

$$|(R(z,\zeta)f, g)| \leq C(s, c_P(\zeta)) \sup_{\xi \in \mathbf{R}^n} \frac{1}{\tilde{P}(\xi,\zeta)} \|f\|_{B_s}\|g\|_{B_{1-s}} \qquad (27)$$

   for all $f \in B_s$, $g \in B_{1-s}$ and $0 \leq s \leq 1$.

2. If c_P in condition (20) is independent of $\zeta \in M_\zeta$, then

$$|(R(z,\zeta)f, g)| \leq C(s, c_P) \sup_{\xi \in \mathbf{R}^n} \frac{1}{\tilde{P}(\xi,\zeta)} \|f\|_{B_s}\|g\|_{B_{1-s}} \qquad (28)$$

   for all $f \in B_s$, $g \in B_{1-s}$ and $0 \leq s \leq 1$.

5 The Uniqueness of Solutions and The Behavior at Infinity

Now let us turn to the partial differential equation:

$$P(D)u = f \in B_s, \quad 0 \leq s \leq 1, \tag{29}$$

where we assume that the symbol $P(\xi) = P_1(\xi) + iP_2(\xi)$ of $P(D)$ is **simply characteristic** and 0 is not a **critical value**. Then from Theorem 4.4, we have

$$R(0)f = \mathcal{F}^{-1}((P(\cdot))^{-1}\hat{f}) \in B^*_{1-s}$$

with

$$\|R(0)f\|_{B^*_{1-s}} \leq C_s \sup_{\xi \in \mathbf{R}^n} \frac{1}{\tilde{P}(\xi)} \|f\|_{B_s}$$

for $0 \leq s \leq 1$. $R(0)f$ is a solution to (29); we can prove uniqueness of the solution to (29) in B^*_{1-s} if $0 < s \leq 1$.

Theorem 5.1 *Let $P(\xi) = P_1(\xi) + iP_2(\xi)$ be a polynomial. Suppose the zero set M of $P(\xi)$ is a C^1 submanifold of codimension 2. Then solution to (29) in B^*_{1-s} is unique if $0 < s \leq 1$. Moreover, if $f \in B_s$ and $0 < s < 1$ then*

$$\lim_{R \to \infty} \frac{1}{R^{2(1-s)}} \int_{|x|<R} |u(x)|^2 dx = 0.$$

Especially, if $f \in B_1$ then

$$\lim_{R \to \infty} \frac{1}{R^{2(1-s)}} \int_{|x|<R} |u(x)|^2 dx = 0$$

for all $0 < s < 1$.

Proof: To prove first statement, we need the following lemma proved in the paper [A-H].

Lemma 5.2 *Let $u \in \mathcal{S}' \cap L^2_{loc}$ and assume that*

$$\lim_{R \to \infty} \sup \frac{1}{R^k} \int_{|x|<R} |u(x)|^2 dx < \infty$$

If the restriction of the Fourier transform $\hat{u}$ to an open subset Ω of $\boldsymbol{R}^n$ is supported by a C^1 submanifold M of codimension k, then it is an L^2 density $\hat{u}_0 ds$ on M and

$$\int_M |\hat{u}_0|^2 ds \leq c \lim_{R\to\infty} \sup \frac{1}{R^k} \int_{|x|<R} |u(x)|^2$$

where c only depends on n.

Suppose we have u_1 and u_2 in B^*_{1-s} satisfying equation (29). Set $u = u_1 - u_2$. Then

$$P(D)u = 0$$

This implies that $\hat{u}$ is supported on $M = \{\xi \in \mathbf{R}^n : P(\xi) = 0\}$ which is a C^1 submanifold of codimension 2. Since $u \in B^*_{1-s}$, $(0 < s \leq 1)$

$$\begin{aligned}
&\lim_{R\to\infty} \sup \frac{1}{R^2} \int_{|x|<R} |u(x)|^2 dx \\
&\quad = \lim_{R\to\infty} \sup \frac{1}{R^{2s}} \frac{1}{R^{2(1-s)}} \int_{|x|<R} |u(x)|^2 dx \\
&\quad = \lim_{R\to\infty} \frac{1}{R^{2s}} \lim_{R\to\infty} \sup \frac{1}{R^{2(1-s)}} \int_{|x|<R} |u(x)|^2 dx \\
&\quad = 0
\end{aligned}$$

The last equality follows from

$$\sup_{R\geq 1} \frac{1}{R^{2(1-s)}} \int_{|x|<R} |u(x)|^2 dx \leq \frac{2^{2(1-s)}}{(1 - 2^{-2(1-s)})} \|u\|^2_{B^*_{1-s}} < \infty.$$

By Lemma 5.2,

$$\int_M |\hat{u}|^2 ds \leq c \lim_{R\to\infty} \sup \frac{1}{R^2} \int_{|x|<R} |u(x)|^2 dx = 0 \quad .$$

This proves first statement.

To prove second statement we note if $0 < s < 1$ and $f \in L^2_s$, then $R(0)f \in L^2_{-1+s}$; i.e.,

$$\int_{\mathbf{R}^n} (1 + |x|^2)^{-1+s} |\mathcal{F}^{-1}(P(\cdot)^{-1}\hat{f})(x)|^2 dx < \infty.$$

Then

$$\lim_{j\to\infty} \sup \int_{x\in\Omega_j} (1+|x|^2)^{-1+s} |\mathcal{F}^{-1}(P(\cdot)^{-1}\hat{f})(x)|^2 dx = 0.$$

This implies

$$\lim_{j\to\infty} \sup \frac{1}{R_j^{2(1-s)}} \int_{x\in\Omega_j} |\mathcal{F}^{-1}(P(\cdot)^{-1}\hat{f})(x)|^2 dx = 0.$$

For any $\epsilon > 0$, there is an integer $N > 0$ such that when $m > N$

$$\frac{1}{R_m^{2(1-s)}} \int_{\Omega_m} |u(x)|^2 dx < \epsilon$$

where $u(x) = \mathcal{F}^{-1}(P(\cdot)^{-1}\hat{f})(x)$. Then

$$\begin{aligned}
\frac{1}{R_m^{2(1-s)}} \int_{|x|<R_m} |u(x)|^2 dx &= \frac{1}{R_m^{2(1-s)}} \sum_{j=1}^{m} \int_{\Omega_j} |u(x)|^2 dx \\
&= \sum_{j=1}^{m} \frac{1}{(2^{2(1-s)})^{m-1}} \int_{\Omega_j} |u(x)|^2 dx \\
&= \sum_{j=1}^{m} \frac{1}{(2^{2(1-s)})^{m-j}} \frac{1}{(2^{2(1-s)})^{j-1}} \int_{\Omega_j} |u(x)|^2 dx \\
&= \sum_{j=1}^{N} \frac{1}{(2^{2(1-s)})^{m-j}} \frac{1}{R_j^{2(1-s)}} \int_{\Omega_j} |u(x)|^2 dx \\
&\quad + \sum_{j=N+1}^{m} \frac{1}{(2^{2(1-s)})^{m-j}} \frac{1}{R_j^{2(1-s)}} \int_{\Omega_j} |u(x)|^2 dx \\
&\leq \|u\|^2_{B^*_{1-s}} \sum_{j=1}^{N} \frac{1}{(2^{2(1-s)})^{m-j}} + \epsilon \sum_{j=N+1}^{m} \frac{1}{(2^{2(1-s)})^{m-j}} \\
&\leq \|u\|^2_{B^*_{1-s}} \frac{1}{(2^{2(1-s)})^{m-N}} \frac{1}{1-2^{-2(1-s)}} + \epsilon \frac{1}{1-2^{-2(1-s)}} \\
&\leq \frac{\epsilon}{1-2^{-2(1-s)}} (1 + \|u\|^2_{B^*_{1-s}})
\end{aligned}$$

for $m > N$ large enough. Therefore

$$\begin{aligned}
&\lim_{R\to\infty} \sup \frac{1}{R^{2(1-s)}} \int_{|x|<R} |u(x)|^2 dx \\
&\quad = \lim_{R\to\infty} \sup_{R'\geq R} \frac{1}{(R')^{2(1-s)}} \int_{|x|<R'} |u(x)|^2 dx = 0.
\end{aligned}$$

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