On TP_2 and Log-Concavity

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Summary. Inter-relations between the TP_2 property and log-concavity of density functions have been investigated. The general results are then applied to noncentral chi-square density functions and beta density functions.

Results on Density Functions

Definition 1. A function $f: R^2 \to R$ is said to be TP_2 [5] if, for $x_1 < x_2$, $y_1 < y_2$

(1) $$f(x_1,y_2)\ f(x_2,y_1) \le f(x_1,y_1)\ f(x_2,y_2).$$

We shall say that $1/f$ is TP_2, if (1) holds for f with the inequality reversed.

Let X be a positive random variable having the p.d.f. $f(\cdot,\theta,\lambda)$ with respect to Lebesgue measure; $\theta > 0$, $\lambda \ge 0$.

Definition 2. The p.d.f. $f(x,\theta,\lambda)$ is said to have the reproductive property (RP) in θ, if there exists a distribution function $G(\cdot,s)$ on R^+ $(s > 0)$ such that

(2) $$\int_0^x f(x-y,\theta,\lambda)\ G(dy,s) = f(x,\ \theta+s,\lambda)$$

Theorem 1. Suppose $f(x,\theta,\lambda)$ has the RP in θ. Then

(i) $f(x,\theta,\lambda)$ TP_2 in $(x,\lambda) \Rightarrow 1/f(x,\theta,\lambda)$ TP_2 in (θ,λ).

(ii) $f(x,\theta,\lambda)$ TP_2 in $(x,\theta) \Rightarrow f(x,\theta,\lambda)$ log-concave in θ.

Proof (i) for $0 < x_1 < x_2$, $\lambda_1 < \lambda_2$ we have

(3) $$f(x_2,\theta,\lambda_1)\ f(x_1,\theta,\lambda_2) \le f(x_2,\theta,\lambda_2)\ f(x_1,\theta,\lambda_1)$$

Write $x_1 = x_2 - y$. Integrating (3) with respect to $G(dy,s)$ we get

(4) $$f(x_2,\theta,\lambda_1)\, f(x_2,\theta+s,\lambda_2) \le f(x_2,\theta,\lambda_2)\, f(x_2,\theta+s,\lambda_1),$$

which shows that $1/f(x,\theta,\lambda)$ is TP_2 in (θ,λ).

(ii) For $0 < x_1 < x_2$, $\theta_1 < \theta_2$, we have

(5) $$f(x_1,\theta_2,\lambda)\, f(x_2,\theta_1,\lambda) \le f(x_2,\theta_2,\lambda)\, f(x_1,\theta_1,\lambda).$$

Write $x_1 = x_2 - y$. Integrating (5) with respect to $G(dy,s)$ we get

(6) $$f(x_2,\theta_2+s,\lambda)\, f(x_2,\theta_1,\lambda) \le f(x_2,\theta_2,\lambda)\, f(x_2,\theta_1+s,\lambda),$$

which shows that $f(x,\theta,\lambda)$ is log-concave in θ.

Definition 3. The p.d.f. $f(x,\theta,\lambda)$ is said to have the mixture property (MP) in (θ,λ) if there exists a non-negative random variable K with the distribution $H(\cdot,\tau)$ with $\tau > 0$ such that

(7) $$\int_0^\infty f(x,\ \theta+k,\lambda)\, H(dk,\tau) = f(x,\theta,\lambda+\tau).$$

Suppose H in Definition 3 possesses a density function h with respect to a σ-finite measure ν.

Theorem 2. Suppose $f(x,\theta,\lambda)$ has the MP in (θ,λ). Then

(i) $f(x,\theta,0)$ TP_2, $h(k,\tau)$ TP_2 $\Rightarrow$ $f(x,\theta,\tau)$ TP_2 in (x,τ).

(ii) $1/f(x,\theta,\lambda)$ TP_2 in (θ,λ) $\Rightarrow$ $f(x,\theta,\lambda)$ log-concave in λ.

(iii) $f(x,\theta,\lambda)$ TP_2 in (x,θ) $\Rightarrow$ $f(x,\theta,\lambda)$ TP_2 in (x,λ).

(iv) $f(x,\theta,\lambda)$ log-concave in θ $\Rightarrow$ $f(x,\theta,\lambda)$ log-concave in λ.

Proof: (i) This follows from Karlin's Theorem [p. 17, 5].

(ii) For $\theta_1 \le \theta_2$, $\lambda_1 < \lambda_2$,

$$(8) \qquad f(x,\theta_2,\lambda_2)\, f(x,\theta_1,\lambda_1) \le f(x,\theta_2,\lambda_1)\, f(x,\theta_1,\lambda_2),$$

Write $\theta_2 = \theta_1 + k$. Integrating (8) with respect to $H(dk,\tau)$ we get

$$(9) \qquad f(x,\theta_1,\lambda_2+\tau)\, f(x,\theta_1,\lambda_1) \le f(x,\theta_1,\lambda_1+\tau)\, f(x,\theta_1,\lambda_2),$$

which shows that $f(x,\theta,\lambda)$ is log-concave in λ.

(iii) For $0 < x_1 < x_2$, $\theta_1 \le \theta_2$ we have

$$(10) \qquad f(x_2,\theta_1,\lambda)\, f(x_1,\theta_2,\lambda) \le f(x_2,\theta_2,\lambda)\, f(x_1,\theta_1,\lambda).$$

Write $\theta_2 = \theta_1 + k$. Integrating (10) with respect to $H(dk,\tau)$ we get

$$(11) \qquad f(x_2,\theta_1,\lambda)\, f(x_1,\theta_1,\lambda+\tau) \le f(x_2,\theta_1,\lambda+\tau)\, f(x_1,\theta_1,\lambda),$$

which shows that $f(x,\theta,\lambda)$ is TP_2 in (x,λ).

(iv) For $\theta_1 \le \theta_2$, $0 \le s$ we have

$$(12) \qquad f(x,\theta_2+s,\lambda)\, f(x,\theta_1,\lambda) \le f(x,\theta_2,\lambda)\, f(x,\theta_1+s,\lambda).$$

Write $\theta_2 = \theta_1 + k$ and integrate (12) with respect to $H(dk,\tau_1)\, H(ds,\tau_2)$. Then we get

$$(13) \qquad f(x,\theta_1,\lambda+\tau_1+\tau_2)\, f(x,\theta_1,\lambda) \le f(x,\theta_1,\lambda+\tau_1)\, f(x,\theta_1,\lambda+\tau_2).$$

The above shows that $f(x,\theta,\lambda)$ is log-concave in λ.

Theorem 3. Suppose $f(x,\theta,\lambda)$ is log-concave in x. Then

(14) $f(x,\theta,\lambda)$ has the RP in $\theta \Rightarrow f(x,\theta,\lambda)$ is TP_2 in (x,θ).

Proof For $0 < x_1 < x_2$, $0 < y$ we have

$$(15) \qquad f(x_1,\theta,\lambda)\, f(x_2+y,\theta,\lambda) \le f(x_2,\theta,\lambda)\, f(x_1+y,\theta,\lambda).$$

Write $x_1 = x_2 - z$. Integrating (15) with respect to $G(dz,s)$ we get

$$f(x_2,\theta+s,\lambda)\ f(x_2+y,\theta,\lambda) \leq f(x_2,\theta,\lambda)\ f(x_2+y,\theta+s,\lambda), \tag{16}$$

Which shows that $f(x,\theta,\lambda)$ is TP_2 in (x,θ).

<u>Define</u> $C(x,\theta,\lambda)$ by

$$f(x,\theta,\lambda)\ C(x,\theta,\lambda) = f(\lambda,\theta,x). \tag{17}$$

<u>Lemma 1</u>. If both $f(x,\theta,\lambda)$ and $C(x,\theta,\lambda)$ are log-concave in λ, then $f(x,\theta,\lambda)$ is log-concave in x.

The above lemma is a well-known fact; see [1,2].

Combining the above results, we get the following:

<u>Theorem 4</u>. Suppose the following conditions hold:

(a) $f(x,\theta,\lambda)$ has the RP in θ, as defined in (2)

(b) $f(x,\theta,\lambda)$ has the MP in (θ,λ), as defined in (7)

(c) $C(x,\theta,\lambda)$, as defined in (17), is log-concave in λ.

Then the following are equivalent:

(i) $f(x,\theta,\lambda)$ is TP_2 in (x,λ).

(ii) $1/f(x,\theta,\lambda)$ is TP_2 in (θ,λ).

(iii) $f(x,\theta,\lambda)$ is log-concave in λ.

(iv) $f(x,\theta,\lambda)$ is log-concave in x.

(v) $f(x,\theta,\lambda)$ is TP_2 in (x,θ).

(vi) $f(x,\theta,\lambda)$ is log-concave in θ.

Moreover, all the above results (i) - (vi), under the conditions (a) - (c), are implied by the condition

(d) $f(x,\theta,0)$ is TP_2, $h(k,\tau)$ is TP_2.

Application: Noncentral chi-square distribution.

Suppose $f(x,\theta,\lambda)$ is the p.d.f. of the noncentral chi-square distribution with θ degrees of freedom and the noncentrality parameter λ. Then (2) holds with $G(\cdot,s)$ as the distribution of χ_s^2. Moreover, (7) holds if H is taken such that K/2 is distributed as Poisson with mean $\tau/2$. With this specification of h, condition (d) of Theorem 4 obtains. It can also be seen that $C(x,\theta,\lambda)$, as defined in (17), is given by

$$(18) \qquad C(x,\theta,\lambda) = (\lambda/x)^{\theta/2 - 1},$$

which is log-concave in λ if $\theta/2 \geq 1$. Hence (i) - (iii) of Theorem 4 hold when $\theta > 0$, and (iv) - (vi) hold when $\theta \geq 2$. It can be seen easily that $f(x,\theta,0)$ is log-concave in x when $\theta \geq 2$; also $f(x,\theta,0)$ is TP_2 in (x,θ), and $f(x,\theta,0)$ is log-concave in θ. Ghosh [4] gave an alternative proof of the TP_2 property of $f(x,\theta,\lambda)$ in (x,θ) when $\theta > 2$. Karlin [5] proved that $f(x,\theta,\lambda)$ is log-concave in x when $\theta > 2$.

Remark. The chain of arguments used in the above theorems can be used also for discrete random variables after minor modifications.

Results on C.D.F.'s.

Let X be a positive r.v. with the p.d.f. $f(x,\theta,\lambda)$ with respect to Lebesgue measure. The c.d.f. of X is given by

$$(19) \qquad F(C,\theta,\lambda) \equiv P[X \leq C] \equiv 1 - \bar{F}(C,\theta,\lambda).$$

Lemma 2.

(a) If $f(x,\theta,\lambda)$ satisfies (2), then so does $F(x,\theta,\lambda)$.

(b) If $f(x,\theta,\lambda)$ satisfies (7), then so does $F(x,\theta,\lambda)$.

(c) If $f(x,\theta,\lambda)$ is TP_2 in (x,θ) (or, in (x,λ)), then $F(x,\theta,\lambda)$ is also TP_2 in (x,θ)(or, in (x,λ)).

(d) If $f(x,\theta,\lambda)$ is log-concave in x, then $F(x,\theta,\lambda)$ is also log-concave in x.

The above results (b)-(d) also hold for $\bar{F}$.

Proof. The results (a) and (b) are trivial. The results (c) follows from Karlin's theorem [5] and the fact that the indicator function of the set $(-\infty,C]$ is TP_2 in (x,C). The result (d) follows from Prekopà's Theorem; see Das Gupta [1,2].

Remark. If f or F satisfies RP, then it trivially follows that $F(c,\theta,\lambda)$ is decreasing in θ; this fact also follows from the condition that $F(c,\theta,\lambda)$ is TP_2 in (c,θ).

Theorem 5.

(a) Suppose $f(x,\theta,\lambda)$ or $F(x,\theta,\lambda)$ satisfies the RP in θ, as given in (2). Then

(i) $F(x,\theta,\lambda)$ TP_2 in $(x,\lambda) \Rightarrow 1/F(x,\theta,\lambda)$ TP_2 in (θ,λ).

(ii) $F(x,\theta,\lambda)$ TP_2 in $(x,\theta) \Rightarrow F(x,\theta,\lambda)$ log-concave in θ.

(iii) $F(x,\theta,\lambda)$ log-concave in $x \Rightarrow F(x,\theta,\lambda)$ TP_2 in (x,θ).

(b) If $f(x,\theta,\lambda)$ or $F(x,\theta,\lambda)$ satisfies the MP in (θ,λ) as given in (7), then

(i) $1/F(x,\theta,\lambda)$ TP_2 in $(\theta,\lambda) \Rightarrow f(x,\theta,\lambda)$ log-concave in λ.

(ii) $F(x,\theta,\lambda)$ TP_2 in $(x,\theta) \Rightarrow F(x,\theta,\lambda)$ TP_2 in (x,λ).

(iii) $F(x,\theta,\lambda)$ log-concave in $\theta \Rightarrow F(x,\theta,\lambda)$ log-concave in λ.

Proof. It follows from (a) that $\bar{F}(c,\theta)$ is TP_2. For $0 < c_1 < c_2$, $\theta_1 < \theta_2$ we have

$$(25) \qquad \bar{F}(c_2, \theta_1)\, \bar{F}(c_1,\theta_2) \leq \bar{F}(c_2,\theta_2)\, \bar{F}(c_1,\theta_1).$$

Write $c_2 = c_1 + y$. Integrating (25) with respect to $G(dy,\theta_2,\delta)$ we get

$$(26) \qquad \begin{aligned} \bar{F}(c_2,\theta_1)\, \bar{F}(c_2,\theta_2+\delta) &\leq \bar{F}(c_2,\theta_2) \int_0^\infty \bar{F}(c_2-y,\theta_1)\, G(dy,\theta_1,\delta) \\ &\leq F(c_2,\theta_2) \int_0^\infty \bar{F}(c_2-y,\theta_1)\, G(dy,\theta_1,\delta) \\ &= \bar{F}(c_2,\theta_2)\, \bar{F}(c_2,\theta_1+\delta). \end{aligned}$$

Application to Beta Distribution

Suppose $U \sim \beta_{m,n}$ and $V \sim \beta_{\delta,m+n}$ are independently distributed. Then $UV \sim \beta_{m+\delta,n}$. Write $X = -\log U$, $Y = -\log V$, and $\theta = m$. Let $f(\cdot,\theta)$ be the density of X and $G(\cdot,\theta,\delta)$ be the c.d.f. of Y. Then the conditions (a)-(c) of Theorem 5 hold. Hence $P[U \leq c]$ is log-concave in m.

Remark. Some of the above results relating to chi-square distribution are given in the Ph.D. dissertation of Sarkar. Furthermore, following the ideas of Das Gupta and Perlman [3], Sarkar [6] has shown that $\chi^2_{m,\alpha}$ is log-concave in $m > 0$, where $P[\chi^2_m > \chi^2_{m,\alpha}] = \alpha$.

Remark. The only basic result relating the TP_2 properties and log-concavity available in the literature is the following [5]: A positive-valued function g is log-concave iff $g(x-y)$ is TP_2 in (x,y). This result follows easily from the development in Theorem 1; one has to consider a special G in Definition 2 which assigns the entire probability mean at a positive number.

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for any d_1, d_2 and $0 \leq \alpha_1, \alpha_2 \leq 1$, $\alpha_1 + \alpha_2 = 1$, where F* is the c.d.f. corresponding to f*. The above inequality is equivalent to

$$(22) \qquad F(c_1^{\alpha_1} c_2^{\alpha_2}, \theta) \geq [F(c_1,\theta)]^{\alpha_1} [F(c_2,\theta)]^{\alpha_2}$$

for any positive c_1, c_2, where F is the c.d.f. of χ^2_θ . Thus from the "arithmetic mean geometric mean" inequality we get

$$(23) \qquad F(\alpha_1 c_1 + \alpha_2 c_2, \theta) \geq [F(c_1, \theta)]^{\alpha_1} [F(c_2,\theta)]^{\alpha_2},$$

which shows that $F(c,\theta)$ is log-concave in c for $\theta > 0$. Incidentally (22) also holds for $\bar{F}$ in place of F.

<u>More Results on p.d.f.'s</u>

Let X be a positive random variable with the p.d.f. $f(\cdot,\theta)$ with respect to Lebesgue measure.

<u>Definition 4</u>. The density $f(\cdot,\theta)$ is said to have the restricted reproductive property (RRP) in θ, if there exists a positive r.v. Y with the distribution $G(\cdot,\theta,\delta)$ such that

$$(24) \qquad \int_0^x f(x-y,\theta)\, G(dy,\theta,\delta) = f(x,\theta+\delta).$$

<u>Theorem 6</u>. Suppose the following conditions hold:

(a) $f(x,\theta)$ is TP_2.

(b) f satisfies the RRP, as given in (24).

(c) $G(\cdot,\theta,\delta)$, as given in Definition 4, is stochastic decreasing in θ.

Then $\bar{F}(x,\theta)$ is log-concave in θ.

The above results in (a) and (b) also hold if F is replaced by $\bar{F}$.

This theorem can be proved following the proofs of Theorems 1 and 2. However, we need to note some additional facts in order to prove the results for $\bar{F}$. If $F(x,\theta,\lambda)$ satisfies (2), we get

$$\int_0^c \bar{F}(c-y,\theta,\lambda)\, G(dy,s) = \bar{F}(c,\theta+s,\lambda) - \bar{G}(c,s). \tag{20}$$

So, in order for Theorem 5(a)(i) to hold for $\bar{F}$ we must have $\bar{F}(c,\theta,\lambda)$ increasing in λ; but this is implied by the condition that $\bar{F}(c,\theta,\lambda)$ is TP_2 in (c,λ). Also, for Theorem 5(a)(ii) to hold for $\bar{F}$ we need $\bar{F}(c,\theta,\lambda)$ increasing in θ; again this is implied by the condition that $\bar{F}(c,\theta,\lambda)$ is TP_2 in (c,θ).

Application to chi-square Distribution

(a) If $F(\cdot,\theta,\lambda)$ is the c.d.f. of the chi-square distribution with θ degrees of freedom and noncentrality parameter λ, then the results (i)-(iii) of Theorem 4 hold for F or $\bar{F}$ in place of f, and (iv) - (vi) hold for F or $\bar{F}$ in place of f when $\theta \geq 2$.

(b) If $f(\cdot,\theta,0)$ is the p.d.f. of χ^2_θ, then it is well-known that $f(x,\theta,0)$ is TP_2. Hence, following the proof of Theorem 1 (ii) it can be shown that both $F(c,\theta,0)$ and $\bar{F}(c,\theta,0)$ are log-concave in $\theta > 0$.

(c) It follows from Lemma 2 and the subsequent remark that both $F(c,\theta,\lambda)$ and $\bar{F}(c,\theta,\lambda)$ are log-concave in c when $\theta \geq 2$. However, a stronger result can be obtained when $\lambda = 0$ by appealing to Prekopa's Theorem.

Suppose $X \sim \chi^2_\theta$, and let f^* be the p.d.f. of $Y = \log X$. Then $f^*(y,\theta)$ is log-concave in y for $\theta > 0$. Using Prekopa's Theorem, we get

$$F^*(\alpha_1 d_1 + \alpha_2 d_2,\theta) \geq [F^*(d_1,\theta)]^{\alpha_1}\, [F^*(d_2,\theta)]^{\alpha_2}, \tag{21}$$